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by

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Estimates of the Distance to the Set of Divergence Free Fields and Applications to Quantitative Analysis of Incompressible Fluids

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Abstract

We are concerned with computable estimates of the distance to the set of divergence free fields, which are necessary for quantitative analysis of mathematical models of incompressible media (e.g., Stokes, Oseen, and Navier–Stokes problems). These estimates are connected with the so-called Inf-Sup condition (or Aziz–Babuška-Ladyzhenskaya–Solonnikov inequality) and require sharp estimates of the respective constant, which are known only for a very limited amount cases. We consider a way to bypass this difficulty and show that for a vide class of domains (and different boundary conditions) computable estimates of the distance to the set of divergence free field can be presented in the form, which includes the LBB constant for a certain basic problem. In the last section, we apply these estimates to problems in the theory of viscous incompressible fluids and deduce fully computable bounds of the distance to generalized solutions.

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1 Introduction

Let Ω be a bounded connected domain in $\mathbb{R}^d$ ($d \geq 2$) with Lipschitz boundary Γ. We consider estimates of the distance between a function

$$v \in V := \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d) := \{v \in W^{1,q}(\Omega, \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma\} \qquad 1 < q < +\infty$$

and the space $\overset{\circ}{S}{}^{1,q}(\Omega, \mathbb{R}^d)$, which is the closure (with respect to the norm of V) of smooth divergence free fields having compact supports in Ω. Also, we consider similar estimates for vector valued functions vanishing only on a measurable part $\Gamma_D \subset \Gamma$ and the set of divergence free fields satisfying the same boundary condition.

Throughout the paper, $\{f\}_\Omega$ denotes the mean value of f in Ω, $\| \cdot \|_\omega$ denotes the L^2 norm of a scalar or vector valued function over the set ω (if ω coincides with Ω, then the subindex is omitted).

If $q = 2$, then an estimate of the distance between $v \in \overset{\circ}{W}{}^{1,2}(\Omega, \mathbb{R}^d)$ and the set of divergence free fields is based on the following principal result.

Theorem 1 *For any $f \in L^2(\Omega)$ such that $\{f\}_\Omega = 0$, there exists a function $w_f \in \overset{\circ}{W}{}^{1,2}(\Omega, \mathbb{R}^d)$ such that*

$$\text{div} w_f = f \quad \text{and} \quad \|\nabla w_f\| \leq \kappa_\Omega \|f\|, \tag{1.1}$$

where κ_Ω is a positive constant depending on Ω.

We refer to [2, 11] for the proof of Theorem 1, which has several important applications. It was used by O. A. Ladyzhenskaya and V. A. Solonnikov for proving existence of a generalized solution to the Stokes problem (see, e.g., [10, 11]).

Theorem 1 implies the key relation in the mathematical theory of incompressible fluids (which is often called the Inf–Sup (or LBB) condition): there exists a positive constant c_Ω such that

$$\inf_{\substack{p \in L^2(\Omega) \\ \{p\}_\Omega = 0,\, p \neq 0}} \quad \sup_{\substack{w \in V_0 \\ w \neq 0}} \frac{\int_\Omega p \, \text{div} w \, dx}{\|p\| \, \|\nabla w\|} \geq c_\Omega. \tag{1.2}$$

In view of (1.1), the condition (1.2) holds with $c_\Omega = (\kappa_\Omega)^{-1}$.

Also, (1.2) can be justified by means of the Nečas inequality [13]:

$$\|p\|^2 \leq \|p\|^2_{-1,\Omega} + \sum_{i=1}^{d} \left\| \frac{\partial p}{\partial x_i} \right\|^2_{-1,\Omega} \qquad \forall\, p \in L^2(\Omega),$$

where $\|\zeta\|^2_{-1,\Omega} := \sup_{\eta \in \overset{\circ}{H}{}^1(\Omega)} (\zeta, \eta)/\|\eta\|_{H^1}$. For domains with Lipschitz boundaries a simple proof can be found in [3].

In [1] and [6], the LBB condition was introduced, proved, and used in order to justify the convergence of the so–called mixed approximation methods, in which a boundary–value problem is reduced to a saddle–point problem for a certain Lagrangian. Conditions analogous to (1.2) written for various pairs of finite dimensional spaces are often used for proving stability and convergence of numerical methods developed for viscous incompressible fluids (see, e.g., [12]).

Theorem 1 can be extended to L^q spaces ($1 < q < +\infty$). These generalizations were obtained in ([4, 5]).

Theorem 2 *Let $f \in L^q(\Omega)$. If $\{f\}_\Omega = 0$, then there exists $v_f \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ such that*

$$\mathrm{div} v_f = f \quad \text{and} \quad \|\nabla v_f\|_{q,\Omega} \leq \kappa_{\Omega,q} \|\mathrm{div} v_f\|_{q,\Omega}, \tag{1.3}$$

where $\kappa_{\Omega,q}$ ($\kappa_{\Omega,2} = \kappa_\Omega$) is a positive constant, which depends only on Ω.

This theorem implies an estimate of the distance between a vector function $v \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ and the subspace $\overset{\circ}{S}{}^{1,q}(\Omega, \mathbb{R}^d) \subset \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ containing divergence free functions if the distance is measured in terms of the quantity

$$d(v, \overset{\circ}{S}{}^{1,q}(\Omega, \mathbb{R}^d)) := \inf_{v_0 \in \overset{\circ}{S}{}^{1,q}(\Omega, \mathbb{R}^d)} \|\nabla(v - v_0)\|_{q,\Omega}.$$

Lemma 1 *For any $v \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$,*

$$d(v, \overset{\circ}{S}{}^{1,q}(\Omega, \mathbb{R}^d)) \leq \kappa_{\Omega,q} \|\mathrm{div} v\|_{q,\Omega}. \tag{1.4}$$

This result directly follows from Theorem 2 if we set $f = \mathrm{div} v$. Then, a function $v_f \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ exists such that (1.3) holds. We set

$$v_0 := v - v_f \in \overset{\circ}{S}{}^{1,q}(\Omega)$$

and obtain

$$\|\nabla(v - v_0)\|_{q,\Omega} = \|\nabla v_f\|_{q,\Omega} \leq \kappa_{\Omega,q} \|\mathrm{div} v\|_{q,\Omega}.$$

Hence, the distance between $v \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ and the set of divergence free fields is easily estimated from above provided that the constant $\kappa_{\Omega,q}$ (or a suitable upper bound of it) is known. Regrettably, the latter requirement generates a very difficult problem. Even for the most simple case $q = 2$ estimates of the constant are known only for a restricted amount of special (simple) domains (see, e.g., [7, 14, 15, 22]). In particular, for $d = 2$ it is known that the constant c_Ω can be expressed throughout the constant L in the inequality $\|u\|^2 \leq L\|v\|^2$, which holds for an analytic function $u + iv$ provided that $\{u\}_\Omega = 0$ (see [8]). It was shown (see [22]) that $c_\Omega = \frac{1}{\sqrt{1+L}} \leq \frac{1}{\sqrt{2}}$. For star shaped domains estimation of the constant L is based on simple geometrical properties of Ω and, in particular, leads to the conclusion that $c_\Omega = \frac{1}{\sqrt{2}}$ for the circle, $\sin\frac{\pi}{8} \leq c_\Omega \leq \sqrt{\frac{\pi-2}{2\pi}}$ for the square and $\sin\frac{\pi}{16} \leq c_\Omega \leq \sqrt{\frac{\pi-2\sqrt{2}}{2\pi}}$ for the isosceles right triangle. Analogous constants can be found analytically or computed numerically for certain basic three dimensional domains.

However, in general, the constants $\kappa_{\Omega,q}$ are unknown. Moreover, so far we do not know any method able to compute guaranteed and realistic bounds of these constants for arbitrary three dimensional Lipshitz domains or, at least, for polygonal 3D domains. This fact imposes the question, which often arises in quantitative analysis of incompressible media: *how to estimate the distance between a function $v \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ and the set of divergence free fields for a sufficiently wide class of domains?* Moreover, it is necessary to answer the same question in the case where the functions are vanishing only on a part of the boundary. Below we show that the estimates can be obtained provided that estimates of the respective constants associated with some basic (elementary) domains are known or precomputed.

In Section 2, we deduce estimates of the distance to the set of divergence free fields for functions vanishing on a part Γ_D of the boundary and show that regardless of the particular form of Γ_D the corresponding estimate holds with the same constant as for $\Gamma_D = \Gamma$ provided that the function has divergence with zero mean (this result generalizes Lemma 6.2.1 in [19]). Then, a more sophisticated estimate is derived, which provides an upper bound of the distance to the set of divergence free fields without this zero mean condition. Section 3 presents the estimate based on domain decomposition. It can be useful for polygonal domains, which can be decomposed into simplicial and polyhedral cells. If the constants $\kappa_{\Omega,q}$ for these cells are known, then Lemma 4 shows that the distance to the set of divergence free fields is easy to estimate. Finally, in Section 4 we discuss applications of these results to a posteriori estimates for problems in the theory of viscous incompressible fluids.

2 Estimates for functions vanishing on a part of the boundary

Assume that Γ consists of two measurable non-intersecting parts Γ_D and Γ_N, $\text{meas}_{d-1}\Gamma_D > 0$, and

$$v \in W^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d) := \{v \in W^{1,q}(\Omega, \mathbb{R}^d) \ | \ v = 0 \text{ on } \Gamma_D\}.$$

We define the set

$$K_\mu(\Omega, \Gamma_D) := \left\{ w \in W^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d) \ \middle| \ \int\limits_\Omega \text{div} w \, dx = \mu \in \mathbb{R} \right\}.$$

Our goal is to find an upper bound of

$$d(v, S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)) := \inf_{v_0\in S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v-v_0)\|_{q,\Omega}, \tag{2.1}$$

where

$$S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d) = \left\{ v\in W^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d) \mid \operatorname{div} v = 0 \right\},$$

and to show that the estimate (1.4) holds for the functions vanishing only on Γ_D with the same constant $\kappa_{\Omega,q}$ as in (1.4).

Lemma 2 *Let $v\in K_0(\Omega,\Gamma_D)$. Then,*

$$d(v, S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)) \;\leq\; \kappa_{\Omega,q}\|\operatorname{div} v\|_{q,\Omega}. \tag{2.2}$$

Indeed, the function $f=\operatorname{div} v$ has zero mean, so that Theorem 2 guarantees existence of $v_f\in \overset{\circ}{W}{}^{1,q}(\Omega,\mathbb{R}^d)$ such that (1.3) holds. Since

$$v_0 := v - v_f \in S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d),$$

we arrive at (2.2).

Now we consider estimates of the distance, which use the same constant $\kappa_{\Omega,q}$ and hold without the condition $\int\limits_\Omega \operatorname{div} v\, dx = 0$.

We begin with the most interesting case $q=2$ and first of all deduce an upper bound of the quantity

$$\inf_{\widetilde{v}\in K_0(\Omega,\Gamma_D)} \|\nabla(\widetilde{v}-v)\|. \tag{2.3}$$

Since any function $\widetilde{v}\in K_0(\Omega,\Gamma_D)$ can be represented in the form $\widetilde{v}=v-\widetilde{w}$, where $\widetilde{w}\in K_\mu(\Omega,\Gamma_D)$, this task leads to the auxiliary variational problem

$$\inf_{\widetilde{w}\in K_\mu(\Omega,\Gamma_D)} J(\widetilde{w}), \quad J(\widetilde{w}) := \frac{1}{2}\|\nabla\widetilde{w}\|^2, \tag{2.4}$$

which is equivalent to the minimax problem

$$\inf_{w\in W^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \sup_{\lambda\in\mathbb{R}} \left\{ \frac{1}{2}\|\nabla w\|^2 + \lambda\left(\int\limits_\Omega \operatorname{div} w dx - \mu\right)\right\}.$$

The corresponding dual problem generated by the functional

$$G(\lambda) = \inf_{w\in W^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \left\{ \frac{1}{2}\|\nabla w\|^2 + \lambda\int\limits_\Omega \operatorname{div} w dx\right\} - \lambda\mu, \tag{2.5}$$

which contains a well posed convex minimization problem. Let u_* denote the minimizer of this problem for $\lambda = 1$. It meets the integral identity

$$\int\limits_{\Omega} \nabla u_* : \nabla w \, dx + \int\limits_{\Gamma_N} n \cdot w \, ds = 0 \quad \forall w \in W^{1,2}_{0,\Gamma_D}(\Omega, \mathbb{R}^d) \tag{2.6}$$

and solves the problem

$$\begin{aligned} \Delta u_* &= 0 && \text{in}\, \Omega, \\ u_* &= 0 && \text{on}\, \Gamma_D, \\ \nabla u_* \, n + n &= 0 && \text{on}\, \Gamma_N. \end{aligned}$$

It is easy to see that λu_* is the minimizer of the problem (2.5) and

$$\|\nabla u_*\|^2 + \int\limits_{\Omega} \mathrm{div} u_* \, dx = 0. \tag{2.7}$$

We obtain

$$G(\lambda) = \frac{1}{2}\lambda^2 \|\nabla u_*\|^2 + \lambda \left(\lambda \int\limits_{\Omega} \mathrm{div} u_* \, dx - \mu \right) = -\frac{1}{2}\lambda^2 \|\nabla u_*\|^2 - \lambda\mu.$$

Therefore, $\sup\limits_{\lambda} G(\lambda)$ is attained at $\lambda = \lambda_* := -\frac{\mu}{\|\nabla u_*\|^2}$. By (2.6) we conclude that $\|\nabla u_*\| \neq 0$ so that λ_* is a finite real number and

$$G(\lambda_*) = \frac{1}{2} \frac{\mu^2}{\|\nabla u_*\|^2}.$$

Note that

$$\lambda_* \int\limits_{\Omega} \mathrm{div} u_* \, dx = \mu. \tag{2.8}$$

Hence, $\lambda_* u_* \in K_\mu(\Omega, \Gamma_D)$. Since

$$J(\lambda_* u_*) = \frac{1}{2}\|\nabla \lambda_* u_*\|^2 = \frac{1}{2} \frac{\mu^2}{\|\nabla u_*\|^2}$$

we see that the values of the primal and dual functionals associated with the auxiliary problem coincide and, therefore, $\lambda_* u_*$ is the minimizer of the auxiliary problem (2.4).

We set in (2.3) $\widetilde{v} = v_* := v - \lambda_* u_*$ and find that

$$\inf_{\widetilde{v} \in K_0(\Omega, \Gamma_D)} \|\nabla(\widetilde{v} - v)\| = \frac{1}{\|\nabla u_*\|} \left| \int\limits_{\Omega} \mathrm{div} v \, dx \right|. \tag{2.9}$$

Now

$$\inf_{v_0\in S^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v-v_0)\| \le \inf_{v_0\in S^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v_*-v_0)\| + \|\lambda_*\nabla u_*\|$$

$$\le \kappa_\Omega\|\mathrm{div}v - \lambda_*\,\mathrm{div}u_*\| + \frac{1}{\|\nabla u_*\|}\left|\int\limits_\Omega \mathrm{div}v\,dx\right|. \quad (2.10)$$

In view of (2.8), we arrive at the following result.

Lemma 3 *For any* $v\in W^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)$,

$$d(v,S^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)) \le$$

$$\frac{\kappa_\Omega}{|\{\mathrm{div}u_*\}_\Omega|}\|\{\mathrm{div}u_*\}_\Omega \mathrm{div}v - \mathrm{div}u_*\{\mathrm{div}v\}_\Omega\| + \frac{1}{\|\nabla u_*\|}\left|\int\limits_\Omega \mathrm{div}v\,dx\right|. \quad (2.11)$$

It is easy to see that this estimate converts into (2.2) if $\{\mathrm{div}v\}_\Omega = 0$.

Corollary 1 *(2.10) implies a somewhat different estimate:*

$$d(v,S^{1,2}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)) \le \kappa_\Omega\|\mathrm{div}v\| + C_*\left|\int\limits_\Omega \mathrm{div}v\,dx\right|, \quad (2.12)$$

where

$$C_* = \frac{1}{\|\nabla u_*\|}\left(\kappa_\Omega\frac{\|\mathrm{div}u_*\|}{\|\nabla u_*\|}+1\right).$$

A similar estimate can be derived for $q\in(1,+\infty)$. Let u_* be the minimizer of the problem

$$\inf_{w\in W^{1,q}_0(\Omega,\Gamma_D)}\left\{\frac{1}{q}\|\nabla w\|^q + \lambda\int\limits_\Omega \mathrm{div}w dx\right\}, \quad (2.13)$$

which meets the integral identity

$$\int\limits_\Omega\left(|\nabla u_*|^{q-2}\nabla u_* : \nabla w + \mathrm{div}w\right)dx = 0 \qquad \forall w\in W^{1,q}_0(\Omega,\Gamma_D).$$

Then,

$$\|\nabla u_*\|^q_{q,\Omega} + \int\limits_\Omega \mathrm{div}u_*\,dx = 0.$$

We set $v_* = v - \lambda_* u_*$, where

$$\lambda_* = \frac{\int\limits_\Omega \mathrm{div} v\, dx}{\int\limits_\Omega \mathrm{div} u_*\, dx} = -\frac{\int\limits_\Omega \mathrm{div} v\, dx}{\|\nabla u_*\|^q_{q,\Omega}}.$$

We obtain

$$\inf_{v_0 \in S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v - v_0)\|_{q,\Omega} \leq \inf_{v_0 \in S^{1,q}_{0,\Gamma_D}(\Omega,\mathbb{R}^d)} \|\nabla(v_* - v_0)\|_{q,\Omega} + \|\lambda_* \nabla u_*\|_{q,\Omega}$$

$$\leq \kappa_{\Omega,q} \|\mathrm{div} v - \lambda_* \,\mathrm{div} u_*\|_{q,\Omega} + \frac{1}{\|\nabla u_*\|^{q-1}_{q,\Omega}} \left| \int\limits_\Omega \mathrm{div} v\, dx \right|$$

$$\leq \kappa_{\Omega,q} \|\mathrm{div} v\|_{q,\Omega} + C_{*,q} \left| \int\limits_\Omega \mathrm{div} v\, dx \right|, \quad (2.14)$$

where

$$C_{*,q} = \frac{1}{\|\nabla u_*\|^{q-1}_{q,\Omega}} \left(\kappa_{\Omega,q} \frac{\|\mathrm{div} u_*\|_{q,\Omega}}{\|\nabla u_*\|_{q,\Omega}} + 1 \right).$$

Remark 1 *The constant C_* depends on the solution u_* of the auxiliary boundary value problem (2.6) (or problem (2.13)). In general, this function is unknown. It can replaced by a finite dimensional approximation $u_{*,h}$, which solves the problem*

$$\int\limits_\Omega (\nabla u_{*,h} : \nabla w_h + \mathrm{div} w_h)\, dx = 0 \quad \forall w \in K^h_0(\Omega, \Gamma_D),$$

where K^h_0 is a certain finite dimensional subspace of $K_0(\Omega, \Gamma_D)$. Then, repeating above arguments, we find that

$$\inf_{\widetilde{v} \in K_0(\Omega,\Gamma_D)} \|\nabla(\widetilde{v} - v)\| \leq \|\nabla(v - u_{*,h})\| = \frac{1}{\|\nabla u_{*,h}\|} \left| \int\limits_\Omega \mathrm{div} v\, dx \right| \quad (2.15)$$

and (2.12) holds with the fully computable constant

$$C_{*,h} = \frac{1}{\|\nabla u_{*,h}\|} \left(\kappa_\Omega \frac{\|\mathrm{div} u_{*,h}\|}{\|\nabla u_{*,h}\|} + 1 \right).$$

By applying known argumentation of the approximation theory one can prove that $u_{*,h}$ tends to u_* provided that standard regularity assumptions on the structure of subspaces $K_0(\Omega, \Gamma_D)$ are satisfied. Then, $C_{*,h}$ tends to C_*.

3 Estimates based upon decomposition of Ω

A generalization of (2) is obtained if Ω is divided into a collection of non-overlapping Lipschitz subdomains Ω_i, $i = 1, 2, ...N$ and the corresponding constants $\kappa_{\Omega_i,q}$ are known.

Lemma 4 *Let* $v \in W^{1,q}_{0,\Gamma_D}(\Omega, \mathbb{R}^d)$ *and*

$$\{\mathrm{div} v\}_{\Omega_i} = 0 \qquad i = 1, 2, ..., N. \tag{3.1}$$

Then,

$$d^q(v, S^{1,q}_0(\Omega, \Gamma_D)) \leq \sum_{i=1}^{N} \kappa^q_{\Omega_i,q} \|\mathrm{div} v\|^q_{\Omega_i}. \tag{3.2}$$

Proof. Since $\int\limits_{\partial\Omega_i} v \cdot n\, ds = 0$, we know that (see, e.g., [9], III.3) there exists a vector field $u^{(i)} \in W^{1,q}(\Omega_i, \mathbb{R}^d)$ such that $\mathrm{div} u^{(i)} = 0$ in Ω_i and $u^{(i)} = v$ on $\partial\Omega_i$ in the sense of traces.

In all Ω_i, we define $w^{(i)} := v - u^{(i)}$. Note that $w_i \in W^{1,q}_0(\Omega_i, \mathbb{R}^d)$. Now, we set $g_i = \mathrm{div} w^{(i)}$ and apply Theorem 2, which guarantees existence of $w^{(i)}_g \in W^{1,q}_0(\Omega_i, \mathbb{R}^d)$ such that

$$\mathrm{div}\, w^{(i)}_g = g_i = \mathrm{div} v \qquad \text{in } \Omega_i, \tag{3.3}$$

$$\|\nabla w^{(i)}_g\|_{q,\Omega_i} \leq \kappa_{\Omega_i,q} \|g\|_{q,\Omega_i}. \tag{3.4}$$

Let w_g be the vector valued function that coincides with $w^{(i)}_g$ in each Ω_i. It is continuous and belongs to $W^{1,q}_0(\Omega, \mathbb{R}^d)$. From (3.4), it follows that

$$\|\nabla w_g\|^q_{q,\Omega} \leq \sum_{i=1}^{N} \kappa^q_{q,\Omega_i} \|g\|^q_{q,\Omega_i}. \tag{3.5}$$

We set $v_0 = v - w_g \in S^{1,2}_{0,\Gamma_D}(\Omega, \mathbb{R}^d)$ and find that

$$\|\nabla(v - v_0)\|_{q,\Omega} = \|\nabla w_g\|_{q,\Omega} = \sum_{i=1}^{N} \kappa^q_{q,\Omega_i} \|g\|^q_{q,\Omega_i} \|\nabla(w_g - v + v)\|_{q,\Omega},$$

which implies (3.2). □

For $q = 2$, Lemma 4 has been proved in [20]. It gives the following answer to the question stated in the introduction: if a domain is decomposed into a set of "simple" subdomains (for which the constants $\kappa_{\Omega,q}$ are known), then an upper bound of the distance is easy to compute provided that mean values of the divergence in each subdomain are zero.

It should be noted that satisfaction of a certain amount of integral conditions (3.1) can be performed without essential difficulties unlike the methods based on constructing a sufficiently wide subspace of divergence free functions and computing the estimate directly (especially in the three dimensional case). Indeed, if v does not satisfy (3.1), then the corresponding correction can be done be changing N parameters in the representation of this function. Since

$$\int\limits_{\Omega_i} \operatorname{div} v \, dx = \int\limits_{\partial\Omega_i} v \cdot n_i \, ds \qquad i = 1, 2, ..., N,$$

where n_i is the outward normal to the boundary $\partial\Omega_i$, changing the parameters should be done such that all the boundary integrals vanish. If N is not very large, then this requirement do not lead to essential difficulties (especially if v is presented by edge based approximations such as, e.g., Raviart–Thomas elements).

Moreover, we can deduce fully computable estimates of the distance, which are valid without the conditions or (3.1). Indeed, let $\mu_i = \int\limits_{\Omega_i} \operatorname{div} v \, dx$ and $w \in W_0^{1,q}(\Omega, \Gamma_D)$ be a "correction function" such that

$$\int\limits_{\Omega_i} \operatorname{div} w \, dx = \mu_i \qquad \text{for } i = 1, 2, ..., N.$$

Then

$$d(v, S_0^{1,q}(\Omega, \Gamma_D)) \leq \; d(v - w, S_0^{1,q}(\Omega, \Gamma_D)) + \|\nabla w\|_{q,\Omega}$$

and (3.2) yields a simple estimate

$$d(v, S_0^{1,q}(\Omega, \Gamma_D)) \leq \left(\sum_{i=1}^{N} \kappa_{\Omega_i}^q \|\operatorname{div}(v - w)\|_{\Omega_i}^q \right)^{1/q} + \|\nabla w\|_{q,\Omega}. \tag{3.6}$$

This estimate provides an upper bound of the distance to the set of divergence free fields for any $w \in W_0^{1,q}(\Omega, \Gamma_D)$. In order to obtain the best estimate, w should be selected in such a way that the right hand side of (3.6) be minimal. For this purpose, we should use a generalized version of the method exposed in Lemma 3.

4 Estimates of the distance to the exact solutions of boundary value problems

Finally, we consider applications of the above derived estimates to quantitative analysis of mathematical models arising in the theory of viscous incompressible fluids. For the sake of simplicity, we consider only stationary models with Dirichl'et boundary conditions (i.e., $u = u_0$ on Γ, where u_0 is a given divergence free vector function). For this class of problems, a generalized solution u is defined as a divergence free field satisfying the integral identity

$$\int\limits_{\Omega} (\nu \nabla u - \eta(u)) : \nabla w \, dx = \int\limits_{\Omega} f \cdot w \, dx \qquad \forall w \in \overset{\circ}{S}{}^{1,2}(\Omega, \mathbb{R}^d), \tag{4.1}$$

where ν is a positive constant (viscosity), $\eta(u) = 0$ for the Stokes problem, $\eta(u) = \mathrm{a} \otimes u$ for the Oseen problem (where a is a certain bounded divergence free vector function), and $\eta(u) = u \otimes u$ for the Navier–Stokes problem (in the latter case, we assume that u is a certain weak solution).

Let $v \in \overset{\circ}{S}{}^{1,2}(\Omega, \mathbb{R}^d)$ be a function satisfying the Dirichlét boundary conditions, which we view as an approximation of the generalized solution u. In order to get an estimate of the error $e = u - v$, we rewrite (4.1) as follows:

$$\int\limits_{\Omega} (\nu \nabla (u - v) + \eta(v) - \eta(u)) : \nabla w \, dx = \mathcal{L}_v(w), \tag{4.2}$$

where

$$\mathcal{L}_v(w) = \int\limits_{\Omega} (f \cdot w - \nu \nabla v + \eta(v) : \nabla w) \, dx$$

is the residual functional associated with v. This relation yields the general error identity

$$\mathbf{m}\,(e) := \sup_{w \in \overset{\circ}{S}{}^{1,2}(\Omega, \mathbb{R}^d)} \frac{\int\limits_{\Omega} (\nu \nabla (u - v) + \eta(v) - \eta(u)) : \nabla w \, dx}{\|\nabla w\|} = |\, \mathcal{L}_v \,| . \tag{4.3}$$

Here, $\mathbf{m}\,(e)$ is the error measure and $|\, \mathcal{L}_v \,|$ is the norm of the residual functional. Since the latter quantity contains all the available information concerning the quality of the exact solution, we see that $\mathbf{m}\,(e)$ is the measure to be used (see also a discussion in [20]).

For the Stokes problem, $\mathbf{m}\,(e) = \nu \|\nabla e\|$. For the Oseen problem, we have

$$\int\limits_{\Omega} (\mathrm{a} \otimes w) : \nabla w \, dx = - \int\limits_{\Omega} \mathrm{Div}(\mathrm{a} \otimes w) \cdot w \, dx =$$
$$= - \int\limits_{\Omega} (\mathrm{a} \cdot \nabla w) \cdot w \, dx = -\frac{1}{2} \int\limits_{\Omega} \mathrm{a} \cdot \nabla (|w|^2) \, dx = 0. \tag{4.4}$$

Therefore,

$$\sup_{w \in \overset{\circ}{S}{}^{1,2}(\Omega, \mathbb{R}^d)} \frac{\int\limits_{\Omega} (\nu \nabla e : \nabla w - (\mathrm{a} \otimes e) : \nabla w) \, dx}{\|\nabla w\|} \geq \frac{\int\limits_{\Omega} \nu \nabla e : \nabla e \, dx}{\|\nabla e\|}$$

and $\mathbf{m}(e) \geq \nu\|\nabla e\|$.

In general, such a simple bound does not take place for the Navier–Stokes problem. We can only prove (see [18]) that $\mathbf{m}\,(e)$ is bounded from below by $\mu\|\nabla e\|$ (where μ is a positive multiplier) provided that ∇v is sufficiently small.

The residual functional can be decomposed into two physically meaningful parts by means of known methods based on suitable integration by parts relations (see [19]). Let $q \in L^2(\Omega)$ and

$$\tau \in H(\Omega, \mathrm{Div}) := \left\{\tau \in L^2(\Omega, \mathbb{M}^{2\times 2}) \;|\; \mathrm{Div}\tau \in L^2(\Omega, \mathbb{R}^2)\right\}.$$

Then,

$$\mathcal{L}_v(w) \;=\; \int\limits_\Omega (f - \mathrm{Div}\tau)\cdot w + \int\limits_\Omega (\nu\nabla v + \eta(v) - \tau - q\mathbb{I}) \,:\, \nabla w\, dx. \quad (4.5)$$

Hence, we find that

$$|\mathcal{L}_v| \leq \|\nu\nabla v + \eta(v) - \tau - q\mathbb{I}\| + C_{F\Omega}\|f - \mathrm{Div}\tau\|, \quad (4.6)$$

where $C_{F\Omega}$ is a constant in the Friedrich's type inequality

$$\|v\| \;\leq\; C_{F\Omega}\|\nabla v\| \qquad \forall v \in W_0^{1,2}(\Omega, \mathbb{R}^d).$$

It is easy to see that (4.3) and (4.6) yield an upper bound for $\mathbf{m}\,(e)$ for any $v \in S_0^{1,2}(\Omega, \mathbb{R}^d)$.

Using the results of Sections 2 and 3, we can extend this estimate to functions, which do not satisfy the divergence free condition.

Let $v \in W_0^{1,2}(\Omega, \mathbb{R}^d)$ but $v \notin S_0^{1,2}(\Omega, \mathbb{R}^d)$. Note that

$$\mathbf{m}\,(e) := \sup_{w\in S_0^{1,2}(\Omega,\mathbb{R}^d)} \frac{\int\limits_\Omega (\nu\nabla(u - v_0) + \eta(v_0) - \eta(u)) : \nabla w\, dx}{\|\nabla w\|} + \Upsilon(v - v_0), \quad (4.7)$$

where v_0 is an arbitrary function in $S_0^{1,2}(\Omega, \mathbb{R}^d)$ and Υ is a nonnegative functional defined by the relation $\Upsilon(v - v_0) := \nu\|\nabla(v_0 - v)\| + \|\eta(v) - \eta(v_0)\|$. In view of (4.6), we find that

$$\mathbf{m}\,(e) \leq \|\nu\nabla v + \eta(v) - \tau - q\mathbb{I}\| + C_{F\Omega}\|f - \mathrm{Div}\tau\| + 2\inf_{v_0\in S_0^{1,2}(\Omega,\mathbb{R}^d)} \Upsilon(v - v_0). \quad (4.8)$$

For the Stokes problem $\Upsilon(v - v_0) = \nu\|\nabla(v - v_0)\|$ and we obtain

$$\nu\|\nabla(u - v)\| \leq \|\tau + \widetilde{p}\mathbb{I} - \nu\nabla v\| + C_{F\Omega}\|\mathrm{Div}\tau + f\| + 2\nu d(v, S^{1,2}(\Omega, \mathbb{R}^d)), \quad (4.9)$$

where q is an approximation of the pressure p and τ is an approximation of the stress $\sigma = \nu\nabla u - p\mathbb{I}$.

If the constant κ_Ω is known, then $d(v, S^{1,2}(\Omega, \mathbb{R}^d) \leq \kappa_\Omega \|\mathrm{div} v\|$ and we obtain a fully computable upper bound of the error (cf. [17, 16, 19]). If the constant κ_Ω is unknown, then we can split Ω is a union of "simple" non-overlapping domains Ω_i for which the respective constants κ_{Ω_i} are known. Let v satisfy the conditions

$$\int\limits_{\Omega_i} \mathrm{div}\widetilde{u}\, dx = 0, \qquad i = 1, 2, ..., N.$$

Then,

$$\nu\|\nabla(u-v)\| \leq \|\tau + \widetilde{p}\mathbb{I} - \nu\nabla v\| + C_{F\Omega}\|\mathrm{Div}\tau + f\| + 2\nu\left(\sum_{i=1}^{N} \kappa^2_{\Omega,i}\|\mathrm{div} v\|^2_{\Omega_i}\right)^{1/2}. \tag{4.10}$$

Analogously, for the Oseen problem

$$\inf_{v_0 \in S_0^{1,2}(\Omega,\mathbb{R}^d)} \Upsilon(v - v_0) \leq C_{\mathrm{Os}}\, d(v, S^{1,2}(\Omega, \mathbb{R}^d)) \leq C_{\mathrm{Os}}\kappa_\Omega\|\mathrm{div} v\|, \tag{4.11}$$

where $C_{\mathrm{Os}} = (\nu + \|\mathrm{a}\|_{\infty,\Omega} C_{F\Omega})$. If the constant κ_Ω is unknown, then instead of (4.11) we can use the estimate

$$\inf_{v_0 \in S_0^{1,2}(\Omega,\mathbb{R}^d)} \Upsilon(v - v_0) \leq C_{\mathrm{Os}}\left(\sum_{i=1}^{N} \kappa^2_{\Omega,i}\|\mathrm{div} v\|^2_{\Omega_i}\right)^{1/2}, \tag{4.12}$$

which together with (4.8) yields an error majorant.

For the Navier–Stokes problem, we need more sophisticated estimates. First, we note that $\|\eta(v) - \eta(v_0)\|^2 \leq 2\int\limits_{\Omega}(|v|^2|v - v_0|^2 + |v - v_0|^4)\, dx$, which due to embedding of H^1 to L^4 (which holds the respective inequality with the constant $\mu(\Omega)$) yields the estimate

$$\inf_{v_0 \in S_0^{1,2}(\Omega,\mathbb{R}^d)} \Upsilon(v - v_0)$$
$$\leq \mu(\Omega) d(v, S^{1,2}(\Omega, \mathbb{R}^d))\left(2\|v\|^2_{4,\Omega} + \mu^2(\Omega) d^2(v, S^{1,2}(\Omega, \mathbb{R}^d))^2\right)^{1/2}. \tag{4.13}$$

Then, (4.8) yields the corresponding error majorant, in which the term related to the distance to the set of divergence free field is either estimated by a single global constant of by means of a collection of local constants $\kappa_{\Omega,i}$.

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