Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Costratification in terms of coherent states

by

Erik Fuchs

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Erik Fuchs

Institute for Theoretical Physics, University of Leipzig, Germany[*]

Abstract

Following the Hamiltonian approach on a finite spatial lattice, I construct a quantum gauge phase space with singularities and its quantum counterpart by the tool of Kähler quantization. Since the reduced phase space is a stratified Kähler space it is possible to construct a corresponding costratification on the quantum level which consists of a family of Hilbert subspaces corresponding to the strata in the classical phase space stratification. By means of Hall coherent states a new method for constructing a generating set of the costratified Hilbert space will be given, where each Hilbert subspace of the costratification corresponds to a certain subfamily of coherent states. Although the construction is applicable to all existing strata, it will be done explicitly for the case of point and toral stratum. For the simplest non abelian toy model with structure group $SU(2)$ and at least two spatial plaquettes, these are the only occurring non generic strata.

[*] erik.fuchs@itp.uni-leipzig.de

I. INTRODUCTION

To study quantum gauge theories in the framework of Hamiltonian lattice gauge theory, there are effectively two approaches. The first would be to quantize the unreduced system and to reduce the symmetries on the quantum level while the second would be to reduce first and to quantize the reduced system. The classical reduced phase space is in general a stratified symplectic space (see e.g. [1], appendix B), i.e it is not a smooth symplectic manifold, but it is a union of symplectic pieces, called strata. In the framework of Hamiltonian lattice gauge theory with structure group $SU(3)$ (i.e. quantum chromodynamics) and canonical quantization, the first approach was studied for example in [2] and [3], where the field algebra and the algebra of observables and its representations were investigated. It turns out, that the field algebra of pure Yang-Mills theory is a tensor product of certain crossed product C^*-algebras, which admits a unique (up to isomorphism) irreducible representation (generalized Schrödinger representation) on the Hilbert space H of L^2 functions on the product of copies of the gauge group K. To implement the singularities, one uses the Segal-Bargman transform for compact Lie groups as developed by Hall (see [4]), which yields a unitary isomorphism from H to the Hilbert space $\tilde{H}$ of holomorphic square integrable functions on the corresponding product of copies of the complecification of K. Within this approach, the notion of a Hilbert space costratification was developed in [5]. This structure assigns to each classical phase space stratum a Hilbert subspace of $\tilde{H}$. Together they constitute a partially ordered family such that the associated projection operators reverse the classical partial ordering. This structure seems to be the appropriate counterpart to the classical stratification.

The Hamiltonian approach to lattice gauge theory on a finite spatial lattice leads to a finite dimensional phase space $T^*(K \times \ldots \times K)$ consisting of finitely many copies of the structure group K, where the symmetry, i.e. the gauge transformations, is given by the lift of the diagonal conjugation on $K \times \ldots \times K$. The stratification structure of this Hamiltonian system with symmetry was investigated for example in [6, 7]. Since this phase space also possesses a compatible complex structure, it is Kähler and the costratification procedure can be applied. This was done in [8] for the structure group $K = SU(2)$ and with one copy of K. In [9], the structure of the observable algebra was studied in this context. The costratified Hilbert space structure was explicitly constructed by means of representation functions of irreducible representations of the semi simple Lie group $SU(2)$. This can be also understood via a holomorphic version of the Peter-Weyl theorem (see [10]). But by using the methods of [8] for the more copy case, one would have to solve a lot of difficulties at first. For example the algebra of the direct generalization of the complete orthonormal system of the reduced Hilbert space to the more copy case is far more complicated as in the one copy case. Also the stratification becomes more complicated (even in the $SU(2)$ case). For at least two copies the so called toral stratum occurs, with orbit type given by the maximal torus of $SU(2)$. For this continuous stratum the explicit construction of the corresponding Hilbert subspace would need a complete description of the stratum via invariant relations and the knowledge of structure coefficients of the algebra of the representation functions spanning the symmetry reduced Hilbert space. These problems could be avoided by the more geometric approach of coherent states.

This paper is a step towards generalizing the construction given in [8] to finitely many copies of a simply connected compact Lie group, by using so called Hall coherent states (see [4]). These states are already arising from the explicit construction done in [8] and represent maximal localized states on the occurring point strata. Using these states it is possible to give a description of the Hilbert subspaces corresponding to the point strata and of the tunneling probability between these Hilbert subspaces. By averaging these states over the diagonal conjugation it turns out that every Hilbert subspace of the costratification structure is given by the closure of the span of a certain subfamily of these averaged states. For the toral stratum of the diagonal conjugation action this subfamily will be explicitly described by means of a complexified maximal toral subgroup of K.

The paper is organized as follows. In Section 2 the model will be introduced with a brief description of the Kähler structure and the Kähler quantized Hilbert space with the definition of the costratification structure. Section 3 contains a description of the point stratum in the finitely many copy case and introduces the Hall coherent states as the corresponding one dimensional costratified Hilbert spaces. Finally in Section 4 a method to describe the costratified Hilbert spaces of all occurring strata will be developed in terms of generalised coherent states. This construction will be made more explicit in the case of the toral stratum.

II. STRATIFICATION AND COSTRATIFICATION

A. Classical picture

We are starting with a hamiltonian gauge model with a compact simply connected gauge group K (e.g. $SU(2)$). The time is separated from space and space is discretized by a finite cubic lattice with nodes Λ^0, links Λ^1 and plaquette Λ^2. The gauge fields are represented via parallel transport by the set $K^{\Lambda^1} := \{f : \Lambda^1 \to K\}$ and the set of gauge transformations is given by $K^{\Lambda^0} := \{t : \Lambda^0 \to K\}$. The action of the transformations on the gauge fields is given in the following way. Let $f_{i,j}$ represent the value of a gauge field $f \in K^{\Lambda^1}$ on a link connecting the nodes $i, j \in \Lambda^0$ and let t_i, t_j be the values of the gauge transformation on those nodes. Then the action can be written as $(f_{i,j}, t_i, t_j) \mapsto (t_i f_{i,j} t_j^{-1})$.

By the choice of a maximal connected subset of the lattice, called tree, we can reduce the freedom of our gauge fields to the off tree links. Every gauge field can be represented by a single element $k \in K$ acting on the gauge fields by diagonal conjugation. Let N be

the number of off tree links and $(k_1, \ldots, k_N) \in K^N$ represent a gauge field on the off tree links, then the action is given by $((k_1, \ldots, k_N), k) \to (kk_1k^{-1}, \ldots, kk_Nk^{-1})$.
This describes the configuration space of the gauge model. The phase space action is obtained by lifting the K-action to the cotangent bundle of K^N, which is diffeomorphic to $K^N \times \mathfrak{k}^N$ by left translation. The K-action $\psi : K \times (K^N \times \mathfrak{k}^N) \to K^N \times \mathfrak{k}^N$ is given by

$$\psi(k, (k_1, \ldots, k_N), (X_1, \ldots, X_N)) = ((kk_1k^{-1}, \ldots, kk_Nk^{-1}), (\operatorname{Ad}_k X_1, \ldots, \operatorname{Ad}_k X_N)) \quad (1)$$

and the corresponding momentum mapping $J : K^N \times \mathfrak{k}^N \to \mathfrak{k}^N$ is

$$J((k_1, \ldots, k_N), (X_1, \ldots, X_N)) = \sum_{i=1}^{N} (\operatorname{Ad}_{k_i} X_i - X_i)\,. \quad (2)$$

The canonical symplectic form $\omega = -d\theta$ on $K^N \times \mathfrak{k}^N$ is given by the potential 1-form

$$\theta_{(k,X)}(L'_k V, W) = \langle X, V \rangle \quad k \in K^N,\, X, V, W \in \mathfrak{k}^N. \quad (3)$$

In (3) $\langle \cdot, \cdot \rangle$ denotes a positive definite invariant inner product on $\mathfrak{k}^N$, e.g. the negative Killing form, and L'_k is the left translation on $T(K^N)$. The resulting tuple $(K^N \times \mathfrak{k}^N,\, \omega,\, K,\, \psi,\, J)$ is a Hamiltonian K-manifold, which is the classical Hamiltonian gauge model. To reduce the existing symmetry there is the tool of singular symplectic reduction. The resulting space $J^{-1}(0)/K = \cup_{(H),i} P_{(H),i}$ is a stratified space, i.e. a disjoint union of symplectic pieces $P_{(H),i}$, where i enumerates the connected pieces and (H) denotes the orbit type of its elements defined as follows:

Definition 1. Let $\psi : K \times M \to M$ be a smooth proper action of the Lie group K on the manifold M. Let $O_m = \{n \in M | \exists\, k \in K \,:\, n = \psi(k, m)\}$ be the orbit of m and G_m denote its isotropy group. If G_m is conjugate to the closed subgroup $H \subset K$ then we say O_m is of orbit type (H).

Remark 2. For the action (1) there are always at least three strata corresponding to the orbit types (K), (T) and $(Z(K))$. T is a maximal torus of K and $Z(K)$ denotes the

center of K.
Those strata are called bottom or point stratum, toral stratum and top stratum respectively. If $K = SU(2)$ those are the only existing strata.

B. Quantum picture

At first we will recall that the phase space of a compact connected group K has an natural complex structure compatible with its canonical symplectic structure. For that purpose we mention the polar decomposition

$$K^N \times \mathfrak{k}^N \to (K^N)^{\mathbb{C}} \quad (k, X) \mapsto k\exp(iY), \tag{4}$$

yielding a diffeomorphism of the cotangent bundle $T^*K^N \cong K^N \times \mathfrak{k}^N$ to the complexification of the Lie group K^N, written as $(K^N)^{\mathbb{C}}$. In the case of $K = SU(2)$ the above diffeomorphism maps to $SL(2,\mathbb{C})^N$. Via this diffeomorphism the cotangent bundle becomes an analytic manifold. Additionally the canonical symplectic structure of T^*K^N is compatible with the induced complex structure and we obtain a Kähler manifold, whose Kähler potential turns out to be

$$\kappa(k\exp(iX)) = |X|^2 = \langle X, X\rangle \,. \tag{5}$$

For reference see e.g. [11]. By the bi-invariance of the diffeomorphism (4) the K-action ψ on $K^N \times \mathfrak{k}^N$ is transported to a diagonal K-action (also called ψ) on $(K^N)^{\mathbb{C}}$
So we have a Kähler structure on the unreduced phase space and it is possible to use the tool of half-form Kähler quantization on $T^*K^N \cong (K^N)^{\mathbb{C}}$. For reference see e.g. [12]. The result is the Hilbert space $\mathcal{H}L^2(G, \mu_\hbar)$ of holomorphic functions on $G = (K^N)^{\mathbb{C}}$, which are square integrable with respect to the scalar product

$$\langle\psi_1, \psi_2\rangle = \frac{1}{\mathrm{vol}(K^N)} \int_G \overline{\psi_1}\psi_2\mu_\hbar. \tag{6}$$

The measure $\mu_\hbar$ is given as

$$\mu_\hbar = e^{-\kappa/\hbar}\eta\varepsilon,$$

where ε is the symplectic volume form on G, κ is the Kähler potential as above and η is a K-bi-invariant function on G given by

$$\eta(k\exp(iX)) = \sqrt{\det\left(\frac{\sin\mathrm{ad}(X)}{\mathrm{ad}(X)}\right)}.$$

The function η is known as the half-form correction. By the form of the measure $\mu_\hbar$ it is easily seen, that the measure and hence the scalar product (6) is invariant under left and right translation by K.

So we have got the unreduced Hilbert space of our lattice gauge model. Reduction after quantization yields $\mathcal{H}L^2(G,\mu_\hbar)^K$, the subspace of $\mathcal{H}L^2(G,\mu_\hbar)$ of K-invariant functions with respect to diagonal conjugation.

Now we see the lack of structure in terms of reduction after quantization. The previous stratified structure given by the orbit type decomposition in the classical case does not seem to be present in the quantum case. But in the case of a Kähler structure it was shown in [5] that there is a stratified structure on $\mathcal{H}L^2(G,\mu_\hbar)^K$ coinciding with the notion of quantization after reduction. This structure is the costratified Hilbert space on a stratified space. In our context the following definition of the costratified Hilbert space is sufficient.

Definition 3. Let $\mathcal{P}_{(H),i}$ be the connected components in the orbit type decomposition relative to G (see e.g. [1] appendix B for definition and properties). Then we call

$$\mathcal{V}_{(H),i} := \left\{ f \in \mathcal{H}L^2(G,\mu_\hbar)^K \mid f|_{\mathcal{P}_{(H),i}} = 0 \right\} \tag{7}$$

the vanishing space of $\mathcal{P}_{(H),i}$. The orthogonal complements of $\mathcal{V}_{(H),i}$ in $\mathcal{H}L^2(G,\mu_\hbar)^K$ will be denoted by $\mathcal{H}_{(H),i}$. The collection of these Hilbert subspaces is called the costratified

Hilbert space structure, assigning to each stratum $P_{(H),i}$ of the classical level a Hilbert subspace $\mathcal{H}_{(H),i}$.

By the above definition it is easily seen, that the Hilbert space corresponding to the top stratum coincides with the hole Hilbert space $\mathcal{H}L^2\left(G,\mu_\hbar\right)^K$.

C. Remarks on the Hilbert space $\mathcal{H}L^2\left(G,\mu_\hbar\right)$

In [4] the Segal-Bargman transform was generalized for compact Lie groups K. Hall proved, that there exists an unitary isomorphism between $L^2(K,dx)$ and $\mathcal{H}L^2(G,\nu_\hbar)$, where $G=K^{\mathbb{C}}$, dx denotes the Haar measure on K and $\nu_\hbar$ is a K-bi-invariant measure on G derived by its heat kernel. To make this statement clear we will give a few necessary definitions.

Definition 4. Let Δ_K and Δ_G be the Casimir operators on K and G. The corresponding heat equations are:

$$\begin{aligned} 0 &= \left(\Delta_K - \frac{1}{2}\frac{\partial}{\partial t}\right)u \qquad u\in C^2(K) \\ 0 &= \left(\Delta_G - \frac{1}{4}\frac{\partial}{\partial t}\right)v \qquad v\in C^2(G). \end{aligned}$$

The fundamental solutions of these equations will be labeled by ρ_t and σ_t and called heat kernels on K or G respectively. Also we will denote the analytic continuation of ρ_t to G as ρ_t. Now we can define the heat kernel measure ν_t on G by:

$$\nu_t := \int_K \sigma_t(x^{-1}g)dx$$

By theorem 2 of [4] the following has to hold.

Theorem 5. *The map* $C_t : L^2(K, dx) \to \mathcal{H}L^2(G, \nu_t)$ *given by*

$$C_t(f)(g) := \int_K f(x)\rho_t(x^{-1}g)dx, \quad f \in L^2(K, dx),\ d \in G$$

is a unitary isomorphism of Hilbert spaces.

Let $\mathbb{C}_\lambda(K)$ denote the space of complex representation functions of the irreducible representation (T_λ, V_λ) with highest weight λ. These functions can be represented by $f_{\lambda,A}(x) = \mathrm{tr}(T_\lambda(x)A)$. where $A \in \mathrm{End}(V_\lambda)$. The well known Peter-Weyl-Theorem states:

$$L^2(K, dx) \cong \hat{\oplus}_{\lambda \in \hat{K}} \mathbb{C}_\lambda(K).$$

The right side is the completion of the direct orthogonal sum of the function spaces $\mathbb{C}_\lambda(K)$. An important step in the proof of theorem 2 of [4] was the following property of C_t:

$$C_t(f_{\lambda,A})(g) = e^{-\varepsilon_\lambda t/2}\,\mathrm{tr}(T_\lambda(g)A),$$

where ε_λ is minus the eigenvalue of the Casimir operator Δ_K with respect to the eigenvector $f_{\lambda,A}$, given by $\varepsilon_\lambda = |\lambda + \rho|^2 - |\rho|^2$ (ρ denotes the Weyl vector, which is half the sum over the positive roots of K.). $T_\lambda(g)$ denotes the analytic continuation of T_λ to G. Hence we can conclude by theorem 5 and the Peter-Weyl-Theorem, that the Hilbertspace $\mathcal{H}L^2(G, \nu_t)$ decomposes as following:

$$\mathcal{H}L^2(G, \nu_t) \cong \hat{\oplus}_{\lambda \in \hat{G}} \mathbb{C}_\lambda(G), \tag{8}$$

where the right side summation is over all finite dimensional irreducible holomorphic represenations of G.

A connection between the Hilbertspaces $\mathcal{H}L^2\,(G, \mu_\hbar)$ and $\mathcal{H}L^2(G, \nu_t)$ was analysed in [11]. Theorem 2.5 in [11] states, that the heat kernel measure ν_t and the half-form corrected measure $\mu_\hbar$ are related by a constant factor, i.e.:

$$\nu_\hbar = c_\hbar \mu_\hbar, \qquad c_\hbar := (\pi\hbar)^{-\dim K/2} e^{-|\rho|^2\hbar}.$$

These facts can be summarised in the following theorem:

Theorem 6 (Holomorphic Peter-Weyl-Theorem)**.** *The Hilbert space $\mathcal{H}L^2(G,\mu_\hbar)$ decomposes into the direct sum:*

$$\mathcal{H}L^2(G,\mu_\hbar) \cong \hat{\oplus}_{\lambda\in\hat{G}}\mathbb{C}_\lambda(G). \tag{9}$$

The sum ranges over the finite dimensional irreducible representations $\hat{G}$ of G, which are identified by its highest weights λ. The summand $\mathbb{C}_\lambda(G)$ is the space of all representation functions of a finite dimensional irreducible representation $T_\lambda : G \to \mathrm{End}(V_\lambda)$, generated by functions of the form $v(T_\lambda(\cdot)w)$, $v \in V_\lambda^*$, $w \in V_\lambda$.
Furthermore there is a unitary isomorphism between $L^2(K,dx)$ and $\mathcal{H}L^2(G,\mu_\hbar)$, induced by the map

$$\mathbb{C}_\lambda(G) \ni \phi^{\mathbb{C}} \mapsto C_{\hbar,\lambda}^{\frac{1}{2}}\phi \in \mathbb{C}_\lambda(K). \tag{10}$$

ϕ is the restriction of $\phi^{\mathbb{C}}$ to $K \subset G$, $\mathbb{C}_\lambda(K)$ denotes the space of representation functions of T_λ restricted to K and the factor $C_{\hbar,\lambda}$ is given by

$$C_{\hbar,\lambda} = (\hbar\pi)^{\dim G/2} e^{\hbar|\lambda+\rho|^2} \tag{11}$$

Also the map

$$\mathrm{End}\, V_\lambda \ni A \mapsto \sqrt{\dim V_\lambda}\, \mathrm{tr}\,(T_\lambda(\cdot)A) \in \mathbb{C}_\lambda(G) \tag{12}$$

is an isomorphism of the $G\times G$-representations $\mathrm{End}\, V_\lambda$ *and* $\mathbb{C}_\lambda(G)$.

A detailed proof of theorem 6 can be found in [10].

III. THE POINT STRATA

Since K is compact and simply connected it is also semi simple. Hence its center is discrete and finite. The point strata consists of all elements $g = (g_1,\ldots,g_N) \in G$

which fulfill $hg_lh^{-1} = g_l,\ l = 1, \ldots, N,\ \forall h \in K$. Since the polar decomposition (4) is an diffeomorphism we obtain for each factor $g_l = k_l \exp(iX_l)$:

$$hk_lh^{-1} = k_l \qquad \mathrm{Ad}_h X_l = X_l \quad \forall h \in K.$$

So k_l has to be in $Z(K)$ and X_l has to be in $Z(\mathfrak{k})$ which is zero since K is semi simple. Hence each point stratum $\mathcal{P}_{(K),i}$ is a discrete point $g \in K^N \subset G$ and its components are in the center of K. So the vanishing spaces $\mathcal{V}_{(K),i}$ simply consist of the invariant functions vanishing on single points on G and the corresponding Hilbert subspaces are of a simple form:

Theorem 7. *Let $\mathcal{P}_{(K),i} \subset G$ be given by an element $g \in G$. Then the corresponding Hilbert subspace $\mathcal{H}_{(K),g}$ of the costratified Hilbert space has dimension one. Let ρ_t denote the heat kernel on K^N analytically continued to G. Then the function $v_g(x) := \rho_{2\hbar}(g^{-1}x)$ spans $\mathcal{H}_{(K),g}$*

At first, we will construct the Hilbert subspace $H_g := \mathcal{V}_g^{\perp},\ g \in G$, with

$$\mathcal{V}_g := \{f \in H | f(g) = 0\}\,. \tag{13}$$

The first step is, to define a simple but important projector.

Definition 8. Let $H = \mathcal{H}L^2\left(G, \mu_\hbar\right),\ f \in H$ and $g \in G$, then we define the linear map $P_g : H \to H$ by

$$P_g(f) = f - f(g)1_G \tag{14}$$

and call it the vanishing projection.

P_g obviously projects H to $\mathcal{V}_g$. That it is bounded follows from the fact that the functions $f \in H$ are holomorphic.

Lemma 9. *The vanishing projection P_g is bounded and therefore continuous.*

Proof. By the definition of P_g and the triangle inequality we obtain $||P_g f|| \leq ||f|| + |f(g)| \cdot ||1_{K^{\mathbb{C}}}||$. By the analyticity of f we will estimate the value of f at g by an upper value given by a multiple of its Hilbert space norm which will prove the lemma by the above inequality.

Let $\kappa : G \supset U \to V \subset \mathbb{C}^n$ be an analytic chart with $g \in U$. We will denote the local representative of f as $\tilde{f} := f \circ \kappa^{-1}$ and the local representative of g as $z_g := \kappa(g)$. $\tilde{f} : V \to \mathbb{C}$ is an analytic function on an open subset $V \subset \mathbb{C}^n$. The pull back of the measure $\mu_\hbar$ on H will be denoted as $\tilde{\mu}_\hbar dz' := (\kappa^{-1})^* \mu_\hbar$ where dz' is the Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\tilde{\mu}_\hbar : V \to \mathbb{R}^+$. Let $R > 0$ be the radius of a polydisc $D^n_R(z_g) := \left\{ z = (z^1, \ldots, z^n) \in V \middle| \, \left| z^i_g - z^i \right| < R, \ 1 \leq i \leq n \right\}$ such that $\overline{D^n_R(z_g)} \subset V$.

$$\tilde{f}(z_g) = \left(\frac{1}{\pi R^2} \right)^n \int_{D^n_R(z_g)} \tilde{f}(z') dz'$$

By this relation we obtain:

$$\begin{aligned}
|f(g)| = \left| \tilde{f}(z_g) \right| &\leq \left(\frac{1}{\pi R^2} \right)^n \int_{D^n_R} \left| \tilde{f}(z') \right| dz' \\
&\overset{\text{Hölder}}{\leq} \left(\frac{1}{\pi R^2} \right)^n \cdot \left(\pi R^2 \right)^{\frac{n}{2}} \left(\int_{D^n_R(z_g)} \left| \tilde{f}(z') \right|^2 dz' \right)^{\frac{1}{2}} \\
&\leq \underbrace{\left(\frac{1}{\pi R^2} \right)^{\frac{n}{2}} \left(\inf_{D^n_R} \tilde{\mu}_\hbar \right)^{-\frac{1}{2}}}_{=c_{R,t}} \left(\int_{D^n_R(z_g)} \left| \tilde{f}(z') \right|^2 \tilde{\mu}_\hbar(z') dz' \right)^{\frac{1}{2}} \\
&\leq c_{R,t} \left(\int_U |f(x)|^2 \mu_\hbar(x) \right)^{\frac{1}{2}} \qquad\qquad (D^n_R(z_g) \subset V = \kappa(U)) \\
&\leq c_{R,t} \, ||f||
\end{aligned}$$

So we have got the wanted estimate and the proof is complete. □

Corollary 10. *The dimension of H_g is one. Furthermore let $\{v_n\}$ be a complete orthonormal system on H, then $v \in H_g \Leftrightarrow \langle P_g v_n, v \rangle = 0 \; \forall \, v_n$ holds true.*

Proof. Since P_g is a projection, we can decompose H into the direct sum $H = P_g H \oplus (\mathrm{id} - P_g) H = \mathcal{V}_g \oplus \ker P_g$. $\ker P_g$ is a closed subspace of H consisting of the constant functions on G and therefore its dimension is one. Also $H = \mathcal{V}_g \oplus H_g$ holds, since $H_g = \mathcal{V}_g^{\perp}$. So the dimension of H_g has to be one.

$\{v_n\}$ is a complete orthonormal system on H. By continuity of P_g the span of the vectors $P_g v_n$ is dense in $\mathcal{V}_g$ and hence $\langle P_g v_n, v\rangle = 0 \;\Leftrightarrow\; v \in H$. $\square$

Remark 11. By the definition of P_g one sees that it can be restricted to a projection operator $P_g^K : H^K \to H^K$ with $H^K = \mathcal{H}L^2\left(G, \mu_\hbar\right)^K$ projecting on $\mathcal{V}_{(K),g}$. Hence the statement of corollary 10 can be applied to $\mathcal{H}_g$ and $\dim \mathcal{H}_g = 1$ has to hold, too.

To construct the vector spanning H_g we need a complete orthogonal system on $H = \mathcal{H}L^2\left(G, \mu_\hbar\right)$. By the holomorphic Peter-Weyl theorem 6 the following vectors form a complete orthogonal system on H.

Definition 12. Let $\hat{G}$ be the set of isomorphism classes of finite dimensional irreducible representations $T_\lambda : G \to \mathrm{End}(V_\lambda)$ of G with highest weights λ and $d_\lambda = \dim V_\lambda$. Let $C_{ij}^\lambda \in \mathrm{End}(V_\lambda), i, j = 1, \ldots, d_\lambda$ be the standard basis of $\mathrm{End}(V_\lambda)$ with respect to a chosen orthonormal basis on V_λ. Then we define the complete orthogonal system

$$v_{ij}^\lambda(g) := \sqrt{d_\lambda}\,\mathrm{tr}\left(T_\lambda(g) C_{ij}^\lambda\right) \qquad g \in G. \tag{15}$$

By the isomorphism induced by (10) the norm $\left|\left|v_{ij}^\lambda\right|\right|$ can be calculated via Schur-Weyl orthogonality on $L^2(K^N, dx)$:

$$\left|\left|v_{ij}^\lambda\right|\right|^2 = C_{\hbar,\lambda} \qquad (C_{\hbar,\lambda} \text{ given by (11)}) \tag{16}$$

By corollary 10 we can construct the $v_g \in H$, which spans H_g by evaluating the defining relations $\left\langle P_g v_{ij}^\lambda, v_g\right\rangle = 0$ on an arbitrary vector $w_g = \sum_{\lambda\in\hat{G}} \sum_{i,j=1}^{d_\lambda} a_\lambda^{ij}(g) v_{ij}^\lambda$:

$$0 = \left\langle P_g v_{lm}^\delta, w_g\right\rangle = \left\langle v_{kl}^\delta - v_{kl}^\delta(g)\cdot 1_G, \sum_{\lambda\in\hat{G}} \sum_{i,j=1}^{d_\lambda} a_\lambda^{ij}(g) v_{ij}^\lambda \right\rangle$$

$$= \left|\left|v^\delta_{kl}\right|\right|^2 a^{kl}_\delta(g) - \overline{v^\delta_{kl}(g)} \cdot a^{11}_{\lambda_0}(g) \left|\left|v^{\lambda_0}_{11}\right|\right|^2 \qquad \left(1_G = v^{\lambda_0}_{11}\right)$$

$$\Rightarrow \qquad a^{kl}_\delta(g) \propto \frac{\overline{v^\delta_{kl}(g)}}{\left|\left|v^\delta_{kl}\right|\right|^2} \qquad \forall\ \delta \in \hat{G},\ 1 \leq k, l \leq d_\delta$$

Hence the vector

$$w_g := \sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d_\lambda} \frac{\overline{v^\lambda_{ij}(g)}}{\left|\left|v^\lambda_{ij}\right|\right|^2} v^\lambda_{ij} \tag{17}$$

spans $H_g = \mathcal{V}_g^\perp$ for arbitrary $g \in G$.

Proposition 13. *The vectors w_g of the form* (17) *are square integrable and holomorphic on G and hence elements of $H = \mathcal{H}L^2(G, d\mu_t)$.*

Proof. At first we will check the L^2 property. Since the v^λ_{ij} are orthogonal it is sufficient to show that the following series converges absolutely:

$$\sum_\lambda \sum_{ij} \frac{\left|v^\lambda_{ij}(g)\right|^2}{\left|\left|v^\lambda_{ij}\right|\right|^2} \tag{18}$$

For that purpose we have to find an upper bound of the absolute value of $v^\lambda_{ij}(g)$ with fixed $g \in G$, decreasing fast enough with the highest weights of the representations T_λ. The Lie group G is the complexification of the simple connected, compact group K^N, hence it is diffeomorph to $K^N \times \mathfrak{k}^N$ via the polar decomposition $(k, X) \mapsto k \exp(iX)$. Furthermore a Cartan subalgebra (CSA) $\mathfrak{h}$ of G can be chosen to be of the form $\mathfrak{h} = \mathfrak{h}_\mathbb{R} + i\mathfrak{h}_\mathbb{R}$, where $\mathfrak{h}_\mathbb{R}$ is a maximal abelian subalgebra of $\mathfrak{k}^N$. By lemma III.5.15 of [13] the Lie subalgebra $\mathfrak{h}_\mathbb{R}$ is a CSA of $\mathfrak{k}^N$ and $\mathfrak{k}^N$ can be decomposed in its conjugates, i.e:

$$\mathfrak{k}^N = \bigcup_{y \in K^N} \mathrm{Ad}_y\, \mathfrak{h}_\mathbb{R}.$$

So for all $X \in \mathfrak{k}^N$ exist elements $y \in K^N$ and $Y \in \mathfrak{h}_\mathbb{R}$ such that $X = \mathrm{Ad}_y\, Y$. Hence every element $g \in G$ can be written as:

$$g = k \exp(iX) = k \exp(i\, \mathrm{Ad}_y\, Y) = ky \exp(iY) y^{-1}$$

$$= k_1 \exp(iY) k_2 \qquad k_1, k_2 \in K,\ Y \in \mathfrak{h}_{\mathbb{R}}.$$

Since K^N is compact, the restriction of T_λ to K^N acts on the representation space V_λ unitarily with respect to a chosen scalar product $\langle \cdot, \cdot \rangle_\lambda$. So the corresponding operator norm of $||T_\lambda(k)||_\lambda$ equals one for all $k \in K^N$ and we get:

$$||T_\lambda(g)||_\lambda = ||T_\lambda(k_1 \exp(iY) k_2)||_\lambda = ||T_\lambda(k_1) \exp(iT'_\lambda(Y)) T_\lambda(k_2)||_\lambda = ||\exp(iT'_\lambda(Y))||_\lambda$$

By the unitarity of $T_\lambda(k)$ on V_λ for all $k \in K^N$ the derivation $T'_\lambda(Y)$ is anti hermitian and hence diagonalizable. For every $Y \in \mathfrak{h}_{\mathbb{R}}$ the operator norm of $\exp(iT_\lambda(Y))$ equals the absolute value of the largest eigenvalue of this operator, which is the exponential of the largest absolute value of an eigenvalue of $T'_\lambda(Y)$. But all eigenvalues of $T'_\lambda(Y)$ are of the form $\nu(Y)$, where ν is a weight of T'_λ. There is an upper bound for this, given by:

$$|\nu(Y)| \leq C\,|\lambda|\,||Y|| \qquad Y \in \mathfrak{h}_{\mathbb{R}},\ C > 0.$$

$||Y||$ can be chosen to be the norm of Y induced by minus the Killing form on $\mathfrak{h}$ and $|\lambda|$ is the norm of the highest weight $\lambda \in \mathfrak{h}^*$ given by minus the dual of the Killing form. Hence we can write:

$$||T_\lambda(g)||_\lambda \leq \exp(C\,|\lambda|\,||Y||).$$

Now we will calculate the estimate, for that purpose let $\left\{e_l^\lambda\right\}_{l=1}^{d_\lambda}$ be the orthonormal basis in V_λ used to define the operators C_{ij}^λ:

$$\begin{aligned} \frac{\left|v_{ij}^\lambda(g)\right|}{\sqrt{d_\lambda}} &= \left|\operatorname{tr}\left(T_\lambda(g) C_{ij}^\lambda\right)\right| = \left|\sum_{l=1}^{d_\lambda} \left\langle e_l^\lambda, T_\lambda(g) C_{ij}^\lambda e_l^\lambda \right\rangle_\lambda\right| \leq \sum_{l=1}^{d_\lambda} \left|\left\langle e_l^\lambda, T_\lambda(g) C_{ij}^\lambda e_l^\lambda \right\rangle\right| \\ &\leq \sum_{l=1}^{d_\lambda} \left|\left|e_l^\lambda\right|\right|_\lambda \left|\left|T_\lambda(g) C_{ij}^\lambda e_l^\lambda\right|\right|_\lambda = \left|\left|T_\lambda(g) e_i^\lambda\right|\right|_\lambda \leq \exp\left(C\,|\lambda|\,||Y||\right) \end{aligned} \tag{19}$$

Thus by using the norm of the v_{ij}^λ given by (16) and the above estimate, the following series is an upper bound to the series (18):

$$(\hbar\pi)^{-\dim K^N/2} \sum_\lambda d_\lambda^3 \cdot \exp\left(-(\hbar\,|\lambda+\rho|^2 - 2C\,||Y||\,|\lambda|)\right) \tag{20}$$

Weyl's dimension formula states, that the dimension of V_λ increases only polynomially by the norm of λ and hence (20) converges.

Since the $Y \in \mathfrak{h}_{\mathbb{R}}$ in formula (19) is adjoint to $X \in \mathfrak{k}$ in the polar decomposition, $G \ni g = k\exp(iX)$ and the Killing form is invariant under the adjoint representation, its norm has to be bounded, if g is an element of a compact subset $U \subset G$. So let us consider the series of $w_g(x)$, where we set $x = k'\exp(iY') \in U$ and $||Y'|| \leq a \in \mathbb{R}^+$ and $g \in G$ as above. Then we obtain as upper bounds of the absolute values of the summands in the series of $w_g(x)$ for all $x \in U$:

$$(\hbar\pi)^{\dim K^N} d_\lambda^3 \cdot \exp\left(-\left(\hbar\left|\lambda+\rho\right|^2 - 2C\left(||Y||+a\right)|\lambda|\right)\right),$$

which tends to zero for all $x \in U$ while $|\lambda|$ is increasing. So the series defining w_g converges locally uniformly and hence is holomorphic. □

Proof of Theorem 7. Let $\{e_l^\lambda\}_{l=1}^{d_\lambda}$ be an orthonormal basis in V_λ with respect to a K^N-invariant scalar product $\langle\cdot,\cdot\rangle_\lambda$ as above. Then the vector w_g takes the form:

$$\begin{aligned}
w_g(x) &= \sum_\lambda C_{\hbar,\lambda}^{-1} \sum_{ij=1}^{d_\lambda} d_\lambda \overline{\operatorname{tr}\left(T_\lambda(g)C_{ij}\right)}\operatorname{tr}\left(T_\lambda(x)C_{ij}\right) \\
&= \sum_\lambda C_{\hbar,\lambda}^{-1} d_\lambda \sum_{i=1}^{d_\lambda} \left\langle T_\lambda(g)e_i^\lambda, T_\lambda(x)e_i^\lambda\right\rangle_\lambda \\
&\overset{g\in K^N}{=} \sum_\lambda d_\lambda C_{\hbar,\lambda}^{-1} \cdot \chi_\lambda\left(g^{-1}x\right) \\
&= (\hbar\pi)^{-\dim G/2} e^{-\hbar|\rho|^2} \sum_\lambda d_\lambda e^{-\hbar\varepsilon_\lambda} \chi_\lambda\left(g^{-1}x\right),
\end{aligned}$$

where $\varepsilon_\lambda = |\lambda+\rho|^2 - |\rho|^2$ is minus the eigenvalue of the Casimir operator Δ_{K^N} corresponding to the highest weight λ (as in subsection II C) and $\chi_\lambda(x) = \operatorname{tr}(T_\lambda(x))$ is the character of the representation T_λ. In [14] the heat kernel of the heat equation

$(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_{K^N})f = 0$ was derived in terms of the characters:

$$\rho_t(x) = \sum_{\lambda \in \hat{K^N}} d_\lambda e^{-\varepsilon_\lambda t/2} \chi_\lambda(x). \tag{21}$$

And hence $w_g(x) = (\hbar\pi)^{-\dim G/2} e^{-\hbar|\rho|^2} \rho_{2\hbar}(g^{-1}x)$ for all $g \in K^N$. By the correlation

$$w_g(\psi(h,x)) = w_{\psi(h^{-1},g)}(x), \tag{22}$$

the w_g are invariant by ψ for all $g \in Z(K^N)$. Hence the w_g are elements of $\mathcal{H}L^2\,(G, \mu_\hbar)^K$ and are especially orthogonal to the vanishing space $\mathcal{V}_{(K),g} \subset \mathcal{V}_g$. Since the dimension of $\mathcal{H}_{(K),g}$ is one we have:

$$\mathcal{H}_{(K),g} = \mathbb{C} \cdot w_g = \mathbb{C} \cdot \rho_{2\hbar}(g^{-1}\cdot) \qquad \{g\} = \mathcal{P}_{(K),i}. \tag{23}$$

$\square$

Remark 14. An important physical quantity which we can already compute is the tunneling probability between the point strata $H_{(K),g}$. For pure physical states ψ_1, ψ_2 which are normalized vectors of H this quantity is given by the formula $P_{\psi_1,\psi_2} = |\langle\psi_1, \psi_2\rangle|^2$. Since the summands in (21) are orthogonal it is sufficient to know the following scalar products:

$$\big\langle \chi_\lambda(g^{-1}\cdot), \chi_\lambda(h^{-1}\cdot) \big\rangle = \frac{C_{\hbar,\lambda}}{d_\lambda} \chi_\lambda(gh^{-1}).$$

The above holds true, because $g, h \in K^N$. Consequently the scalar product of two vectors $\rho_{2\hbar}(g^{-1}\cdot)$ and $\rho_{2\hbar}(h^{-1}\cdot)$ is given by:

$$\big\langle \rho_{2\hbar}(g^{-1}\cdot), \rho_{2\hbar}(h^{-1}\cdot) \big\rangle = e^{2|\rho|^2\hbar} \sum_{\lambda \in \hat{K^N}} d_\lambda C_{\hbar,\lambda}^{-1} \chi_\lambda(gh^{-1})$$

and there norm is:

$$\big|\big|\rho_{2\hbar}(g^{-1}\cdot)\big|\big|^2 = e^{2|\rho|^2\hbar} \sum_{\lambda \in \hat{K^N}} d_\lambda^2 C_{\hbar,\lambda}^{-1}.$$

We can see, that the norm of the coherent states concentrating on an element $g \in K^N$ is independent of this element. The tunneling probability $P_{g,h}$ between the two point strata $\{g\}$ and $\{h\}$ equals:

$$P_{g,h} = \left| \frac{\sum_{\lambda \in \hat{K^N}} d_\lambda C_{\hbar,\lambda}^{-1} \chi_\lambda (gh^{-1})}{\sum_{\lambda \in \hat{K^N}} d_\lambda^2 C_{\hbar,\lambda}^{-1}} \right|^2$$

IV. THE TORAL STRATUM

An essential part to construct a generating system of vectors in the toral stratum case will be the following statement.

Proposition 15. *Let $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of Hilbert subspaces of the Hilbert space H. Then the following holds true:*

$$\overline{\operatorname{span}\{V_\alpha^\perp\}_{\alpha \in \mathcal{A}}} = (\cap_{\alpha \in \mathcal{A}} V_\alpha)^\perp \tag{24}$$

Proof. See e.g. [15]. □

By the above proposition, it is possible to determine the toral stratum Hilbert space $H_{(T)}$ in terms of generalized coherent states.

Definition 16. Let $f \in H = \mathcal{H}L^2(G, \mu_\hbar)$,$g \in G$ and $\psi : K \times G \to G$ the diagonal conjugation. Then we define the linear map $P_D : H \to H$ by:

$$(P_D f)(g) = \int_K f(\psi(h,g))dh \qquad (dh \ldots \text{Haar measure on } K). \tag{25}$$

We will call P_D invariance projector.

Theorem 17. *Let $w_g \in H$ be as in* (17) *and $\mathcal{P}_T$ denote the toral stratum in the orbit type decomposition of G with respect to the K-action ψ. Then the following holds true:*

$$\overline{\operatorname{span}\left\{P_D w_g\right\}_{g\in\mathcal{P}_T}} = \mathcal{H}_{(T)}, \tag{26}$$

i.e. the family $\left\{P_D w_g\right\}_{g\in\mathcal{P}_T}$ generates a dense subspace in $\mathcal{H}_{(T)}$.

Remark 18. In the point stratum case, the strata were single points in $K^N \subset G$ with isotropy group K. Hence the w_g were invariant functions spanning $H_{(K),g}$ respectively, by relation (22). This is not true for the elements of $\mathcal{P}_T \subset G$, since the isotropy group has to be smaller than K. So we need a tool to restrict our functions to the subspace of invariant functions, which is the projector P_D in our case.

Lemma 19. *Let $\mathcal{V}_{\mathcal{P}_T} := \{f \in H|\, f(g) = 0\ \forall\, g \in \mathcal{P}_T\}$ and $H_{\mathcal{P}_T} := \mathcal{V}_{\mathcal{P}_T}^{\perp}$. Furthermore let $\mathcal{V}_g$ be as in* (13) *with $H_g = \mathcal{V}_g^{\perp}$. Then*

$$\overline{\operatorname{span}\left\{H_g\right\}_{g\in\mathcal{P}_T}} = H_{\mathcal{P}_T} \tag{27}$$

Proof. Obviously $\mathcal{V}_{\mathcal{P}_T} = \cap_{g\in\mathcal{P}_T}\mathcal{V}_g$ holds true. Hence (27) is just (24), where the subspace family is given by $\left\{\mathcal{V}_g\right\}_{g\in\mathcal{P}_T}$. □

Proposition 20. *The operator P_D is well defined on H, i.e. it is bounded and maps holomorphic functions on holomorphic functions on G. Also P_D is an orthogonal projector on the Hilbert subspace H^K.*

Proof. i) The norm of an element $P_D f$, $f \in H$ has the following form:

$$||P_D f||^2 = \langle P_D f, P_D f\rangle = \int_G \mu_\hbar(g) \overline{\int_K dh f(\psi(h,g)} \int_K dh' f(\psi(h',g)) \tag{28}$$

By the theorem of Tonelli the above integral exists and the integration order can be changed iff the following iterated integral is well defined:

$$\int_{K\times K} dhdh' \int_G \mu_\hbar(g) \left|\overline{f(\psi(h,g))} f(\psi(h',g))\right|$$

$$\overset{\text{Hölder}}{\leq} \int_{K\times K} dhdh' \left(\int_G \mu_\hbar(g) \left|\overline{f(\psi(h,g))}\right|^2 \int_G \mu_\hbar(g) \left|f(\psi(h',g))\right|^2 \right)^{\frac{1}{2}}$$

$$= \int_{K\times K} dhdh' \, ||f||^2 = ||f||^2$$

So the iterated integral is bounded and maps to $\mathbb{R}$. Therefore it has to be well defined. So we can change the order of integration in (28). This will lead to the calculation already made and we obtain:

$$||P_D f|| \leq ||f|| \tag{29}$$

ii) Analyticity of $P_D f$: At first we define $f_h(g) := (f \circ \psi)(h,g)$. Let $U \subset G$ be compact with non empty interior $\dot{U}$, then there is a constant $c_U \in \mathbb{R}^+$ such that $|f_h(g)| < c_U$ on $K \times U$, since $K \times U$ is compact and f_h is continuous in both arguments. Let $\kappa : U \subset W \to V \subset \mathbb{R}^{2n} = \mathbb{C}^n$ be a holomorphic chart on G and $\tilde{f}(h,\mathbf{z}) := f_h(\kappa^{-1}(\mathbf{z}))$ the local representative of f. The $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ are the holomorphic coordinates. Let $D^n_r(\mathbf{z}_0) = \left\{\mathbf{z} \in \mathbb{C}^n \middle| \; |z^i - z_0^i| < r\right\} = D^1_r(z_0^1) \times \ldots \times D^1_r(z_0^1)$ with $z_0 \in W$. Fix an $r > 0$ such that $\overline{D^n_r(\mathbf{z}_0)} \subset W$. Then for all $\mathbf{z} \in D^n_{\frac{r}{2}}(\mathbf{z}_0)$ the following consequence of the Cauchy integral formula holds true:

$$\left(\frac{\partial}{\partial z^k}\tilde{f}\right)(h,\mathbf{z}) = \frac{1}{(2\pi i)^n} \int\limits_{\partial D^1_{\frac{r}{2}}(z^1)\times\ldots\times\partial D^1_{\frac{r}{2}}(z^n)} \frac{\tilde{f}(h,\xi)}{(\xi^k - z^k)\cdot\prod_{l=1}^n(\xi^l - z^l)} d^n\xi$$

$$\Rightarrow \qquad \left|\frac{\partial}{\partial x^i}\tilde{f}(h,\mathbf{z})\right| = \left|\frac{\partial}{\partial y^i}\tilde{f}(h,\mathbf{z})\right| \leq 2\cdot\left|\frac{\partial}{\partial z^i}\tilde{f}(h,\mathbf{z})\right|$$

$$\leq \frac{2}{(2\pi)^n} \int\limits_{\partial D^1_{\frac{r}{2}}(z^1)\times\ldots\times\partial D^1_{\frac{r}{2}}(z^n)} \frac{\left|\tilde{f}(h,\xi)\right|}{(r/2)^{n+1}} d^n\xi \leq \frac{4}{r} c_U$$

By a standard result of analysis concerning differentiability of parameter dependent integrals the above inequality states, that the function $\int_K dh\tilde{f}(h,\mathbf{x}+i\mathbf{y})$ is differentiable on $\mathbb{R}$ and we can change the order of differentiation and integration. So it remains to show the differentiability on $\mathbb{C}$. This can be done by proving that the function $P_D f$ fulfills the Cauchy-Riemann equations, i.e. $dp(JX) = idp(X),\ p \in C^1(G)$ where J is the complex structure on G and $X \in \mathfrak{X}(G)$. Since we have already proven that we can change integration and differentiation in the real sense, we obtain:

$$d(P_D f)(JX) - id(P_D f)(X) = \int_K dh\, df_h(JX) - idf_h(X) = 0$$

since f_h is holomorphic.

iii) P_D maps into H^K, i.e. $\psi_h^* P_D f = P_D f$:

$$\begin{aligned}(\psi_h^* P_D f)(g) = (P_D f)(\psi_h(g)) &= \int_K (f \circ \psi_{h'})(\psi_h(g)) dh' \\ &= \int_K f(\psi_{h'h}(g)) dh' = \int_K f(\psi_{h'}(g)) dh' = (P_D f)(g)\end{aligned}$$

iv) P_D is a projection onto H^K, i.e. $P_D|_{H^K} = \mathrm{id}_{H^K}$: Let us consider a $f \in H^K$ i.e. $(f \circ \psi)(h, g) = f(g)\ \forall h \in K,\ g \in G$:

$$(P_D f)(g) = \int_K f(\psi(h, g)) dh = \int_K f(g) dh = f(g) \quad \text{(since } dh \text{ is normalized on } K\text{)}$$

v) At last we have to show, that P_D is self adjoint on H. By it's boundedness it is sufficient to prove that P_D is symmetric. This will be a consequence of the fact, that the unimodular function of the Haar measure on a compact Lie group K is constant one. This is equivalent to the already known fact, that the Haar measure on K is also right invariant. We have the characterization of the Haar measure as the unique left invariant measure μ on K with normalization $\mu(K) = 1$. Let us define another measure on K by $\tilde{\mu}(H) = \mu(H^{-1})$ for all Borel sets H. Obviously we have got $\tilde{\mu}(K) = 1$ and by the right invariance of μ we obtain:

$$\tilde{\mu}(gH) = \mu(H^{-1}g^{-1}) = \mu(H^{-1}) = \tilde{\mu}(H) \quad \forall\, g \in G.$$

So $\tilde{\mu}$ is left invariant on K and normalized to one. Hence it has to be the Haar measure μ. So the Haar measure on the compact Lie group K is invariant under inversion.

Now we can calculate the symmetry of P_D:

$$\begin{aligned}\langle P_D f, q\rangle &= \int_G \mu_\hbar(g)\overline{\int_K dhf\left(\psi(h,g)\right)} \cdot q(g) = \int_K dh \int_G \mu_\hbar(g)\overline{f(g)} \cdot q(\psi(h^{-1},g)) \\ &= \int_G \mu_\hbar(g)\overline{f(g)} \int_K dh\, q(\psi(h,g)) = \langle f, P_D q\rangle \quad \forall f,q \in H\end{aligned}$$

□

Proof of Theorem 17. Since the family $\{w_g\}_{g\in\mathcal{P}_T}$ is dense in $H_{\mathcal{P}_T}$ by lemma 19, it is sufficient to prove $P_D H_{\mathcal{P}_T} = \mathcal{H}_{(T)}$. For that purpose we use that $\mathcal{V}_{(T)} = \mathcal{V}_{\mathcal{P}_T} \cap H^K$ by definition. Hence $V_{(T)} \subset P_D\mathcal{V}_{\mathcal{P}_T}$ has to hold. By the invariance of $P_{(T)}$ under ψ we can conclude that $P_D\mathcal{V}_{\mathcal{P}_T} \subset \mathcal{V}_{(T)}$ is true and we obtain:

$$P_D\mathcal{V}_{\mathcal{P}_T} = \mathcal{V}_{(T)}. \tag{30}$$

Since P_D is an orthogonal projection the following holds:

$$\langle q, P_D f\rangle = \langle P_D q, f\rangle \qquad q,f \in H$$

If we set $q \in H_{\mathcal{P}_T}$, then the left side is zero for all $f \in \mathcal{V}_{(T)}$ and hence $P_D q \in \mathcal{H}_{(T)}$ by the right side. So we have got $P_D H_{\mathcal{P}_T} \subset \mathcal{H}_{(T)}$.

Also if we set $q \in \mathcal{H}_{(T)}$, then the left side is zero for all $f \in \mathcal{V}_{\mathcal{P}_D}$ by (30) and hence we obtain $P_D q \in H_{\mathcal{P}_D}$ by the right side, which leads to $P_D\mathcal{H}_{(T)} \subset H_{\mathcal{P}_D}$. By the projection property of P_D this is equivalent to $\mathcal{H}_{(T)} \subset H_{\mathcal{P}_D}$ and we conclude $\mathcal{H}_{(T)} = P_D\mathcal{H}_{(T)} \subset P_D H_{\mathcal{P}_D}$, which completes the proof. □

Remark 21. By the relation (22) we can reformulate the integral $P_D w_g$:

$$(P_D w_g)(x) = \int_K dh\, w_g(\psi(h,x)) = \int_K dh\, w_{\psi(h^{-1},g)}(x) = \int_K dh\, w_{\psi(h,g)}(x) \qquad g,x \in \mathcal{P}_T \tag{31}$$

Equation (31) states, that $P_D w_g = P_D w_{g'}$ if g and g' are in the same orbit of the diagonal conjugation. Since the isotropy groups of all elements in $\mathcal{P}_T$ are conjugate to the maximal torus T, there has to be a $g \in \mathcal{P}_T \subset G$ in every orbit of ψ, which is stabilized by T. By the polar decomposition (4) we obtain for $t \in T,\ t_D := (t, \ldots, t) \in T^N,\ g = k \exp(iX),\ k = (k_1, \ldots, k_N) \in K^N,\ X = (X_1, \ldots, X_N) \in \mathfrak{k}^N$:

$$
\begin{aligned}
& k \exp(iX) \stackrel{!}{=} \psi(t, k \exp(iX)) = \psi_t(k) \exp(i \operatorname{Ad}_{t_D} X)\ \forall t \in T \\
\Leftrightarrow\quad & t k_l t^{-1} = k_l \text{ and } \operatorname{Ad}_t X_l = X_l\ \forall\ l = 1, \ldots, N,\ t \in T \\
\Leftrightarrow\quad & t k_l = k_l t \text{ and } \operatorname{ad}_Y X_l = 0\ \forall\ l = 1, \ldots, N,\ Y \in \mathfrak{t}
\end{aligned}
$$

Consequently $k_l \in T$ and $X_l \in \mathfrak{t}$ have to hold true for all l by the maximality of T and its Lie algebra $\mathfrak{t}$ and hence $g \in (T^{\mathbb{C}})^N$ is true. This leads to:

Theorem 22. *Let $\mathcal{H}_{(T)}$ be the Hilbert subspace of the costratified Hilbert space corresponding to the toral stratum $\mathcal{P}_T$. Let $t := (t_1, \ldots, t_N) \in \left(T^{\mathbb{C}}\right)^N$ Then the vectors $\{P_D w_t\}_{t \in (T^{\mathbb{C}})^N}$ are an over-complete set spanning $\mathcal{H}_{(T)}$.*

Remark 23. The construction of the generating set in theorem 17 is valid for all ψ-invariant subsets of G by simply substituting the stratum $\mathcal{P}_T$ in relation 26 by this set. So the theorem 17 can be generalized to the case of an arbitrary stratum $\mathcal{P}_{(S),i}$ by stating

$$
\mathcal{H}_{(S),i} = \overline{\operatorname{span}\{P_D w_g\}_{g \in \mathcal{P}_{(S),i}}},
$$

for every orbit type (S), especially for the orbit type $(Z(K^N))$ whose corresponding Hilbert subspace is $\mathcal{H}L^2(G, \mu_\hbar)^K$ itself.

V. OUTLOOK

The choice of a minimal generating set for the Hilbert subspace $\mathcal{H}_{(T)}$ is still an open question, since the symmetry of the diagonal conjugation on the set of vectors $P_D w_g$

is not fully reduced by theorem 22. Also the problem of non orthogonality of coherent states remains in this description.
Another step will be, to examine the Hamiltonian of the $SU(n)$-gauge model on this Hilbert space and its costratification structure, to determine its eigenspaces and eigenvalues and study the dynamical relation between the eigenspaces and the strata as done in [8] for the case $N = 1$.

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