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Entanglement Detection Using Mutually Unbiased Measurements

by

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We study the entanglement detection by using mutually unbiased measurements and provide a quantum separability criterion that can be experimentally implemented for arbitrary d-dimensional bipartite systems. We show that this criterion is more effective than the criterion based on mutually unbiased base measurements. For isotropic states our criterion becomes both necessary and sufficient.

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I. INTRODUCTION

Entanglement is one of the most appealing features in the quantum world and has been extensively investigated in the past decades [1, 2]. Being useful resources, quantum entangled states play key roles in many quantum information prossing tasks, such as quantum cryptography [3], quantum teleportation [4], and dense coding [5]. An important problem proposed is how to distinguish quantum entangled states from the states without entanglement, i.e., separable states. For bipartite pure states, Schmidt decomposition tells us that a state is separable if and only if it is of Schmidt rank one. But for mixed states, the problem becomes formidably complicated. There have been various separability criteria such as positive partial transposition criterion [6–8], realignment criterion [9–13], covariance matrix criterion [14], and correlation matrix criterion [15, 16]. Recently, Li *et al.* [21] present a generalized form of the correlation matrix criterion for both bipartite and multipartite quantum systems, which is more effective than the previous criteria.

Although numerous mathematical tools have been employed in entanglement detection of given known quantum states, experimental implementation of entanglement detection for unknown quantum states has fewer results [17–20]. In Ref. [22], the authors connected the separability criteria to mutually unbiased bases (MUBs) [23] in two-qudit, multipartite and continuous-variable quantum systems. Based on the correlation functions, the criterion employs local measurements only, and can be implemented experimentally. For two-qudit systems, the criterion is shown to be very powerful in detecting entanglement of particular states. If d is a prime power the criterion is both necessary and sufficient for the separability of isotropic states [24]. However, when d is not a prime power, the criterion becomes less effective. Generally the applications of MUBs are subject to the maximum number $N(d)$ of MUBs. It has been shown that $N(d)$ is no more than $d+1$, and $N(d) = d+1$ when d is a prime power [23]. But when d is a composite number, $N(d)$ is still unknown. Even for $d = 6$, we do not know whether or not there exist four MUBs [25–28].

Recently, Kalev and Gour generalize the concept of MUBs to mutually unbiased measurements (MUMs) [29]. These measurements, containing the complete set of MUBs as a special case, need not to be rank one projectors. Unlike the existence of MUBs which depends on the dimension of the system, there always exists a complete set of $d+1$ MUMs, and can be explicitly constructed. MUMs have also many useful applications in quantum information processing, such as quantum state tomography [23, 30, 31] and entropic uncertainty relation [32, 33].

In this paper, we study the separability problem by using mutually unbiased measurements. We provide a separability criterion for two-qudit systems. We show that this criterion is necessary and sufficient for the separability of all isotropic states in any dimensions.

II. MUBS AND MUMS

Let us first review some basic definitions of mutually unbiased bases. Two orthonormal bases $\mathcal{B}_1 = \{|b_i\rangle\}_{i=1}^d$ and $\mathcal{B}_2 = \{|c_j\rangle\}_{j=1}^d$ of $\mathbb{C}^d$ are said to be mutually unbiased if

$$|\langle b_i|c_j\rangle| = \frac{1}{\sqrt{d}}, \quad \forall\, i, j = 1, 2, \cdots, d.$$

A set of orthonormal bases $\{\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_m\}$ in $\mathbb{C}^d$ is called a set of mutually unbiased bases if every pair of bases in the set are mutually unbiased. Since each $\mathcal{B}_k$ can be written as d rank one projectors summing to the identity operator, these MUBs describe m POVM (projective) measurements on a quantum system of dimension d. Thus, if a physical system is prepared in an eigenstate of basis $\mathcal{B}_k$ and measured in basis $\mathcal{B}_{k'}$, then all the measurement outcomes are equally probable.

Let $\mathcal{B}_k = \{|i_k\rangle\}_{i=1}^d, k = 1, 2, \cdots, m$, be any m MUBs. For any two-qudit state ρ, define

$$I_m(\rho) = \sum_{k=1}^{m} \sum_{i=1}^{d} \langle i_k| \otimes \langle i_k|\rho|i_k\rangle \otimes |i_k\rangle.$$

It has been shown that if ρ is separable, then $I_m(\rho) \leq 1 + \frac{m-1}{d}$ [22]. For isotropic states

$$\rho_{iso} = \alpha|\Phi^+\rangle\langle\Phi^+| + \frac{1-\alpha}{d^2} I,$$

where

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle, \quad 0 \le \alpha \le 1,$$

one gets $I_m(\rho) = m(\alpha + \frac{1-\alpha}{d})$. Therefore if $\alpha > \frac{1}{m}$, then ρ_{iso} is entangled by the criterion. Thus, the power of the criterion depends on m. When d is a prime power, then $m = d+1$, the criterion becomes both necessary and sufficient for the separability of ρ_{iso}, as ρ_{iso} is entangled for $\alpha > \frac{1}{d+1}$, and separable for $\alpha \le \frac{1}{d+1}$ [34].

In Ref. [29], the authors introduced the concept of mutually unbiased measurements (MUMs). Two POVM measurements on $\mathbb{C}^d$, $\mathcal{P}^{(b)} = \{P_n^{(b)}\}_{n=1}^d$, $b = 1, 2$, are said to be mutually unbiased measurements if

$$\begin{aligned} \mathrm{Tr}(P_n^{(b)}) &= 1, \\ \mathrm{Tr}(P_n^{(b)}P_{n'}^{(b')}) &= \frac{1}{d}, \quad b \neq b', \\ \mathrm{Tr}(P_n^{(b)}P_{n'}^{(b)}) &= \delta_{n,n'}\,\kappa + (1-\delta_{n,n'})\frac{1-\kappa}{d-1}, \end{aligned} \tag{1}$$

where $\frac{1}{d} < \kappa \le 1$, and $\kappa = 1$ if and only if all $P_n^{(b)}$s are rank one projectors, i.e., $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are given by MUBs.

A general construction of $d+1$ MUMs has been presented in [29]. Let $\{F_{n,b} : n = 1, 2, \cdots, d-1, b = 1, 2, \cdots, d+1\}$ be a set of d^2-1 Hermitian, traceless operators acting on $\mathbb{C}^d$, satisfying $\mathrm{Tr}(F_{n,b}F_{n',b'}) = \delta_{n,n'}\delta_{b,b'}$. Define $d(d+1)$ operators

$$F_n^{(b)} = \begin{cases} F^{(b)} - (d+\sqrt{d})F_{n,b}, & n = 1, 2, \cdots, d-1; \\ (1+\sqrt{d})F^{(b)}, & n = d, \end{cases} \tag{2}$$

where $F^{(b)} = \sum_{n=1}^{d-1} F_{n,b}$, $b = 1, 2, \cdots, d+1$. Then one can construct $d+1$ MUMs explicitly [29]:

$$P_n^{(b)} = \frac{1}{d}I + tF_n^{(b)}, \tag{3}$$

with $b = 1, 2, \cdots, d+1, n = 1, 2, \cdots, d$, and t should be chosen such that $P_n^{(b)} \ge 0$. Any $d+1$ MUMs can be expressed in such form.

Corresponding to the construction of MUMs (3), the parameter κ is given by

$$\kappa = \frac{1}{d} + t^2(1+\sqrt{d})^2(d-1).$$

If the operators $F_{n,b}$ are chosen to be the generalized Gell-Mann operator basis, one can choose $\kappa = \frac{1}{d} + \frac{2}{d^2}$ optimally [29]. That is to say, there always exist $d+1$ MUMs for arbitrary d, in contrast to MUBs for which this is possible for prime power d. However, if one fixes the parameter κ, then the two values of t cannot guarantee the existence of operator basis such that $P_n^{(b)}$s are positive. For instance, suppose $\kappa = 1$ and there exist $d+1$ MUMs, then it must be a complete set of MUBs. But one does not know whether or not there exist $d+1$ MUBs when d is not a prime power.

III. MUMS BASED SEPARABILITY CRITERION

We now generalize the results in Ref. [22], and present a new separability criterion for two-qudit states by using mutually unbiased measurements.

Theorem 1 *Let ρ be a density matrix in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, $\{\mathcal{P}^{(b)}\}_{b=1}^{d+1}$ and $\{\mathcal{Q}^{(b)}\}_{b=1}^{d+1}$ be any two sets of $d+1$ MUMs on $\mathbb{C}^d$ with the same parameter κ, where $\mathcal{P}^{(b)} = \{P_n^{(b)}\}_{n=1}^d$, $\mathcal{Q}^{(b)} = \{Q_n^{(b)}\}_{n=1}^d$, $b = 1, 2, \cdots, d+1$. Define $J(\rho) = \sum_{b=1}^{d+1}\sum_{n=1}^{d}\mathrm{Tr}(P_n^{(b)} \bigotimes Q_n^{(b)}\rho)$. If ρ is separable, then $J(\rho) \le 1+\kappa$.*

[Proof]. We need only to consider pure separable state, $\rho = |\phi\rangle\langle\phi| \otimes |\psi\rangle\langle\psi|$, since $J(\rho)$ is a linear function. We have

$$\begin{aligned} J(\rho) &= \sum_{b=1}^{d+1}\sum_{n=1}^{d}\mathrm{Tr}(P_n^{(b)} \bigotimes Q_n^{(b)}\rho) \\ &= \sum_{b=1}^{d+1}\sum_{n=1}^{d}\mathrm{Tr}(P_n^{(b)}|\phi\rangle\langle\phi|)\mathrm{Tr}(Q_n^{(b)}|\psi\rangle\langle\psi|) \\ &\le \frac{1}{2}\sum_{b=1}^{d+1}\sum_{n=1}^{d}\{[\mathrm{Tr}(P_n^{(b)}|\phi\rangle\langle\phi|)]^2 + [\mathrm{Tr}(Q_n^{(b)}|\psi\rangle\langle\psi|)]^2\}. \end{aligned}$$

By using the relation

$$\sum_{b=1}^{d+1}\sum_{n=1}^{d}(\mathrm{Tr}(P_n^{(b)}\rho))^2 = 1+\kappa$$

for pure state ρ [29], we obtain $J(\rho) \le 1+\kappa$ □

Note that when $\kappa = 1$, our criterion reduces to the previous one in Ref.[22], which demonstrates that $I_{d+1}(\rho) \le 2$ for all separable states ρ, if there exists a complete set of MUBs in $\mathbb{C}^d$. However, the entanglement detection based on mutually unbiased measurements is more efficient than the one based on MUBs for some states.

Let $\{P_n^{(b)}\}_{n=1}^d$, $b = 1, 2, \cdots, d+1$, be $d+1$ MUMs with the parameter κ. Let $\overline{P}_n^{(b)}$ denote the conjugation of $P_n^{(b)}$. It is obvious that $\{\overline{P}_n^{(b)}\}_{n=1}^d$, $b = 1, 2, \cdots, d+1$, is another $d+1$ MUMs with the same parameter κ. Then we get

$$\begin{aligned} J(\rho_{iso}) &= \sum_{b=1}^{d+1}\sum_{n=1}^{d}\mathrm{Tr}(P_n^{(b)} \bigotimes \overline{P}_n^{(b)}\rho_{iso}) \\ &= (d+1)(\alpha\kappa + \frac{1-\alpha}{d}). \end{aligned}$$

If $\alpha > \frac{1}{d+1}$, then $J(\rho_{iso}) > 1+\kappa$, and ρ_{iso} must be entangled by the theorem. Therefore, this criterion is both necessary and sufficient for the separability of the isotropic states, namely, it can detect all the entanglement of the isotropic states (see FIG. 1). It should be emphasized that, unlike the previous criterion based on

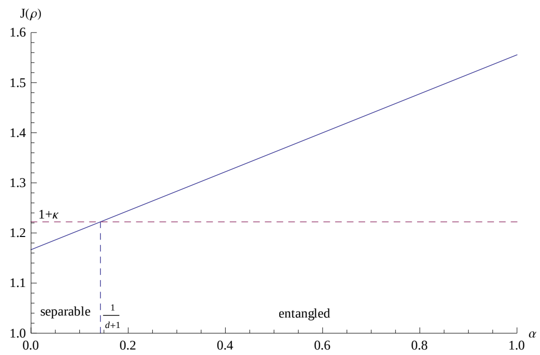

FIG. 1: For isotropic states ρ_{iso}, $J(\rho_{iso}) > 1+\kappa$ if $\alpha > \frac{1}{d+1}$, and ρ_{iso} is entangled. This result coincides with the fact that ρ_{iso} is entangled for $\alpha > \frac{1}{d+1}$, and separable for $\alpha \leq \frac{1}{d+1}$ [34].

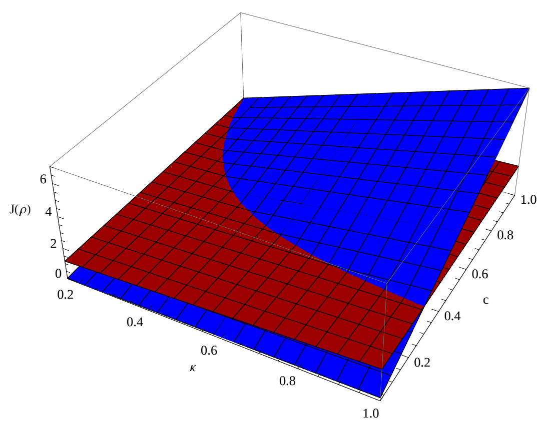

FIG. 2: For Bell-diagonal states, $J(\rho_{Bell})$ violates the criterion when $c > (1+\frac{1}{\kappa})/(d+1)$. The criterion detects more entanglement as κ increases.

MUBs, our criterion works perfectly for any dimension d.

For a complete set of mutually unbiased measurements, the value of the parameter κ is of great importance. Just as noted in Ref. [29], for a given choice of operator basis, the maximal value of κ suggests how close we can find a complete set of MUBs. Therefore, κ is expected to be as close as to 1 as possible.

The efficiency of entanglement detection also depends on the parameter κ. Consider the Bell-diagonal states,

$$\rho_{Bell} = \sum_{s,t=0}^{d-1} p_{s,t} |\Phi^+_{s,t}\rangle\langle\Phi^+_{s,t}|,$$

where $p_{s,t} \geq 0$, $\sum_{s,t=0}^{d-1} p_{s,t} = 1$, $|\Phi^+_{s,t}\rangle = (U_{s,t} \bigotimes I)|\Phi^+\rangle$, $U_{s,t} = \sum_{j=0}^{d-1} \zeta_d^{sj} |j\rangle\langle j \oplus t|$, $s,t = 0,1,\cdots,d-1$, are Weyl operators, $\zeta_d = e^{\frac{2\pi\sqrt{-1}}{d}}$ and $j \oplus t$ denotes $(j+t)$ mod d. For a proper choice of $\{\mathcal{P}^{(b)}\}_{b=1}^{d+1}$ and $\{\mathcal{Q}^{(b)}\}_{b=1}^{d+1}$ with the same parameter κ, it is straightforward to obtain

$$J(\rho_{Bell}) \geq c\kappa(d+1),$$

where $c = \max\{p_{s,t} : s,t = 0,1,\cdots,d-1\}$, $\frac{1}{d^2} \leq c \leq 1$. Thus if $c > (1+\frac{1}{\kappa})/(d+1)$, then $J(\rho_{Bell}) > 1+\kappa$ and ρ_{Bell} must be entangled by our criterion. It is obvious that in order to detect more entanglement of Bell-diagonal states, the parameter κ should be as large as possible (see FIG. 2).

It is interesting to see the connection between the separability criteria presented in [21] (and the references therein) and ours. In fact, given $d+1$ mutually unbiased measurements $\mathcal{P}^{(b)} = \{P_n^{(b)} = \frac{1}{d}I + tF_n^{(b)} : n = 1,2,\cdots d\}$, $b = 1,2,\cdots,d+1$, with the parameter $\kappa = \frac{1}{d} + t^2(1+\sqrt{d})^2(d-1)$, expanding a two-qudit state ρ in the operator basis adopted in $\{\mathcal{P}^{(b)}\}_{b=1}^{d+1}$, we have

$$\begin{aligned} J(\rho) &= \sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}(P_n^{(b)} \bigotimes P_n^{(b)} \rho) \\ &= \frac{d+1}{d} + \frac{2(d\kappa-1)}{d-1}\mathrm{Tr}(T), \end{aligned}$$

where T is the correlation matrix of ρ. Thus, if ρ is separable, then we have $\mathrm{Tr}(T) \leq \frac{d-1}{2d}$, which is just a special case of the inequality satisfied by separable states in Ref. [21].

However, for a general expression $J(\rho) = \sum_{b=1}^{d+1}\sum_{n=1}^{d} \mathrm{Tr}(P_n^{(b)} \bigotimes Q_n^{(b)} \rho)$, we do not know whether or not our criterion can be deduced to some inequalities contained in Ref.[21], since we do not know the connections between two complete sets of mutually unbiased measurements with the same parameter κ. Here we would like to point out that, unlike the criterion presented in Ref. [21], our criterion is based on mutually unbiased measurements, which provides an experimental way of entanglement detection.

IV. CONCLUSION AND DISCUSSIONS

We have studied the separability problem via mutually unbiased measurements. We have presented a new separability criterion for the separability of two-qudit states. It has been shown that the criterion based on mutually unbiased measurements is more efficient than the one based on mutually unbiased bases. For isotropic states, this criterion is both necessary and sufficient. It detects all the entangled isotropic states of arbitrary dimension d. The powerfulness of our criterion is due to that there always exists a complete set of mutually unbiased measurements, which is not the case for mutually unbiased bases when d is not a prime power.

Measurement based quantum separability criteria are of practically significance, as it provides experimental implementation in detecting entanglement of unknown quantum states. It may be also interesting to generalize our results to multipartite systems.

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