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Walter Hengartner & Gerhard Opfer

Abstract. Haar spaces are certain finite dimensional subspaces of $\mathrm{C}(K)$, where K is a compact set and $\mathrm{C}(K)$ is the Banach space of continuous functions defined on K having values in $\mathbb{C}$. We characterize thoses Haar spaces which are generated by shifts applied to a single, analytic function for $K \subset \mathbb{C}$. That means, that an arbitrary finite number of shifts generates Haar spaces by forming linear hulls. We have to distinguish two cases: (a) $K \neq \overline{K^\circ}$, (b) $K = \overline{K^\circ}$. It turns out, that in case (a), an analytic Haar space generator for dimensions one and two is already a universal Haar space generator for all dimensions. The geometrically simplest case, that in case (b), K is convex with smooth boundary turns out to be the most difficult case. There is one numerical example in which the entire function $f := 1/\Gamma$ is interpolated in a shift generated Haar space of dimension four.

Keywords. Complex Haar spaces, complex approximation, shift generated Haar spaces

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1 Introduction

Let K be a compact subset of the complex plane $\mathbb{C}$ and denote by $H(K)$ the linear space of all analytic functions on K, i.e., on a neighborhood of K. The famous Runge theorem [1885] states that each function $f \in H(K)$ can be uniformly approximated by rational functions. It is even possible to do it by rational functions whose poles are only of order one. The proof is very elementary. Indeed, just take the Riemann sums of the Cauchy integral which represents f on K. Later, Runge's theorem was generalized by Mergelyan [1952] to functions in the space $A(K) := H(K^\circ) \cap \mathrm{C}(K)$, where K° denotes the interior of K and $\mathrm{C}(K)$ the set of all continuous functions on K endowed with the topology of uniform convergence. In other words, $\mathrm{C}(K)$ is a normed linear space with $\|f\| := \max_{x\in K} |f(x)|$. Restrictions with respect to the components of $\mathbb{C}\backslash K$ are necessary. However, if $\mathbb{C}\backslash K$ has only a finite number of components, the approximation of $f \in A(K)$ by rational functions with simple poles is possible. The proof of Mergelyan's theorem is no more constructive. Therefore, any applied method is of importance which contains results on the existence and the uniqueness of the 'best' approximation of a continuous function. We begin with the definition of a so-called Haar space. Let $h_j \in \mathrm{C}(K), j = 1, 2, \ldots, N$ be given functions. By $\mathrm{V}{:=}\langle h_1, h_2, \ldots, h_N\rangle$ we denote the linear hull of the functions $h_j \in \mathrm{C}(K)$ with respect to $\mathbb{C}$ which are enclosed by the brackets $\langle \quad \rangle$. We say, that V is *generated* or *spanned* by the functions $h_j \in \mathrm{C}(K)$, $j = 1, 2, \ldots, N$. The dimension of V is N if and only if these functions are linearly independent.
The restriction of Haar spaces to one dimensional cases (either $\mathbb{R}$ or $\mathbb{C}$) was investigated by Mairhuber [1956], Curtis [1958], Schoenberg & Yang [1961], Hendersen & Ummel [1973]. Examples of Haar spaces can be found in Karlin & Studden [1966], Dunham [1974]. The problem of finding real Haar spaces generated by shifts was posed by Cheney & Light [2000, p. 76].

Definition 1.1 Haar [1918]. Let K be a compact subset of $\mathbb{C}$ and $V := \langle h_1, h_2, \ldots, h_N\rangle$ be an N-dimensional linear subspace of $\mathrm{C}(K)$ generated by $h_1, h_2, \ldots, h_N$. We call V a *Haar space for* K if each function $h \in V\backslash\{0\}$ vanishes at at most $N-1$ points of K.

The above stated Haar-condition is equivalent to one of the following two properties, cf. MEINARDUS [1967, p. 16/17].

1. For any selection of N pairwise distinct points $t_j \in K$ and any set of N numbers $\eta_j \in \mathbb{C}$, the interpolation problem

$$h(t_j) = \eta_j, \quad j = 1, 2, \ldots, N,$$

has a unique solution $h \in V$.

2. Let $V := \langle h_1, h_2, \ldots, h_N \rangle$ have dimension N. Then, the $(N \times N)$ matrix

$$\mathbf{M} := (h_j(t_k)), \quad j, k = 1, 2, \ldots, N,$$

is non-singular for any choice of pairwise distinct points $t_j \in K$, $j = 1, 2, \ldots, N$.

We now define the best approximation of a given $f \in \mathrm{C}(K)$.

Definition 1.2 Let V be an N-dimensional linear subspace of $\mathrm{C}(K)$. A function $\hat{h} \in V$ is called a *best approximation* of a given function $f \in \mathrm{C}(K)$ if $\rho_V(f) = \|f - \hat{h}\| \leq \|f - h\|$ holds for all $h \in V$, where $\rho_V(f) := \inf_{h \in V} ||f - h||$ is called the *minimal distance* of f and V.

The existence of a best approximation of $f \in \mathrm{C}(K)$ by elements of a finitely dimensional subspace V is easy to show. See e.g. MEINARDUS [1967, p. 1]. HAAR [1918] and KOLMOGOROFF [1948] found the following necessary and sufficient conditions for the uniqueness.

Theorem 1.3 *Let V be an N-dimensional linear subspace of* $\mathrm{C}(K)$. *Then, for each $f \in \mathrm{C}(K)$, there exists a unique best approximation of f in V if and only if V is a Haar space.*

Our next definition deals with Haar space generators.

Definition 1.4
1. Let $N \in \mathbb{N}$ be a fixed natural number. A function G defined on $\mathbb{C} \backslash \{0\}$ with values in $\mathbb{C}$ will be called an *N-dimensional Haar space generator for K*, if for each set of N pairwise distinct points $s_1, s_2, \ldots, s_N \in \mathbb{C} \backslash K$ and for all $z \in K$, the functions h_j defined by $h_j(z) := G(z - s_j)$, $j = 1, 2, \ldots, N$, span an N-dimensional Haar space for K.

2. The function G is called a *universal Haar space generator for K* if G is an N-dimensional Haar space generator for K for all $N \in \mathbb{N}$.

Example 1.5 Let

$$G(z) := \frac{\mathrm{e}^{Az+B}}{z}, \quad A, B \in \mathbb{C}. \tag{1}$$

Then, $G \in H(\mathbb{C} \backslash \{0\})$ and G is a universal Haar space generator for all compact subsets K of $\mathbb{C}$ which contain infinitely many points. Indeed, the function $h(z) := \sum_{k=1}^{N} \lambda_k G(z - s_k)$ with arbitrary $\lambda_k \in \mathbb{C}$ can have only $N - 1$ zeros in $\mathbb{C}$ for all $N \in \mathbb{N}$.

Let us collect some elementary properties of Haar space generators.

Lemma 1.6 1. A function G is an N-dimensional Haar space generator for any compact set $K \subset \mathbb{C}$ with at least N points if and only if G_1 is an N-dimensional Haar space generator for K, where G_1 is defined by

$$G_1(z) := \mathrm{e}^{Az+B} G(z), \quad A, B \in \mathbb{C}.$$

In particular, $G(z) := z^{n-1}\mathrm{e}^{Az+B}$, $n \in \mathbb{N}$, is an n-dimensional Haar space generator for any nonempty, compact $K \subset \mathbb{C}$, but it is never a $(j+1)$-dimensional Haar space generator for K if $j \geq n$. However, it is always a one-dimensional Haar space generator.

2. Let $K \subset \mathbb{C}$ be compact, and define $K_1 := aK + b$ where $a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C}$. Then, G is an N-dimensional Haar space generator for K if and only if G_1, defined by $G_1(z) := G(z/a)$ is an N-dimensional Haar space generator for K_1.

3. There is no inclusion property for Haar space generating mappings with respect to the dimension N: If G is an N-dimensional Haar space generator for K, then it is not necessarily an $(N-1)$- or an $(N+1)$-dimensional Haar space generator for K.

4. There is no inclusion property for Haar space generating mappings with respect to the inclusions of compact sets: If G is an N-dimensional Haar space generator for K, then it is not necessarily an N-dimensional Haar space generator for $\tilde{K}$ or for $\tilde{\tilde{K}}$ where $\tilde{K} \subset K \subset \tilde{\tilde{K}}$.

5. The property of being an N-dimensional Haar space generator is not continuous with respect to the monotone convergence of compact sets: If K_n is a sequence of compact sets, $K_{n+1} \subset K_n$ converging to a compact set K, and if G is an N-dimensional Haar space generator for all K_n then G is not necessarily an N-dimensional Haar space generator for K.

Proof:

1. Indeed, the linear combinations $\sum_{k=1}^{n} \mu_k G(z-t_k)$ and

$$\sum_{k=1}^{n} \lambda_k G_1(z-t_k) = \mathrm{e}^{Az} \sum_{k=1}^{n} \lambda_k \mathrm{e}^{-At_k+B} G(z-t_k) = \mathrm{e}^{Az} \sum_{k=1}^{n} \mu_k G(z-t_k)$$

vanish simultaneously. It is obvious, that z^{n-1} is an n-dimensional Haar space generator.

2. Let $z \in K$ and $t \in \mathbb{C}\backslash K$. Then, $\zeta := az+b \in K_1, \tau := at+b \in \mathbb{C}\backslash K_1$ and we have $G_1(\zeta - \tau) = G_1(az - at) = G(z-t)$.

3. Consider the compact set $K := \{z = x + \mathrm{i}y : x^2 + 4y^2 \leq 1\}$ and $G(z) := z^2$. Then, we have

   (a) G is a 1-dimensional Haar space generator for K since G does not vanish on $\mathbb{C}\backslash\{0\}$.

(b) G is not a 2-dimensional Haar space generator for K. Indeed, G is a 2-dimensional Haar space generator for K if and only if $G_1 := 1/z^2$ is a 2-dimensional Haar space generator for K. Apply Corollary 4.11, p. 13.

(c) G is a 3-dimensional Haar space generator by statement 1 of this lemma.

(d) G is not a 4-dimensional Haar space generator by statement 1 of this lemma.

4. We will see in Corollary 4.11 that $G(z) = 1/z^2$ is not a 2-dimensional Haar space generator for closed ellipses provided that the lengths of the two axes are different. On the other hand we have shown in HENGARTNER & OPFER, [2002] that G is a 2-dimensional Haar space generator for the closed unit disk. The inclusions

$$\{z : x^2 + 4y^2 \leq 1\} \subset \{z : x^2 + y^2 \leq 1\} \subset \{z : x^2 + \frac{y^2}{4} \leq 1\},$$

imply this part of the lemma, where $z := x + \mathrm{i}y$ was used.

5. The last statement of the lemma is established by considering the function $G(z) := 1/z^2$ and the sequence $K_n := \{z : x^2 + \frac{n}{n+1}y^2 \leq 1\}$, $n \in \mathbb{N}$ which converges monotonically decreasing to the closed unit disk $\overline{\mathbb{D}} := \{z : |z| \leq 1\}$ where G is not a 2-dimensional Haar space generator for K_n for all n but a 2-dimensional Haar space generator for $\overline{\mathbb{D}}$. □

In the paper cited above by HENGARTNER & OPFER [2002], it was shown that a function $G \in H(\mathbb{C}\backslash\{0\})$ is an analytic, universal Haar space generator for $K = \overline{\mathbb{D}}$ if and only if it is of the form (1). Our aim is to show here that this result holds for almost arbitrary compact sets in $\mathbb{C}$ containing infinitely many points. Actually, we conjecture, that this result holds true for all compact sets K with infinitely many points. In Section 2 we collect some preliminary results concerning Haar space generators for general compact sets $K \subset \mathbb{C}$. In Section 3, we study the case where $K\backslash\overline{K^\circ}$ is nonempty. We first prove the above result in Theorem 3.3. As an interesting consequence, we conclude in Theorem 3.4 that a function $G \in H(\mathbb{C}\backslash\{0\})$ which is a 1- and 2-dimensional Haar space generator for K, is either a universal Haar space generator for K and hence an N-dimensional Haar space generator for K for all $N \in \mathbb{N}$ or $G(z) = z\mathrm{e}^{Az+B}$ for $A, B \in \mathbb{C}$ which is not an N-dimensional Haar space generator whenever $N \geq 3$. Such a result does not hold for $K = \overline{\mathbb{D}}$. Indeed, $G(z) = z^2$ is an N-dimensional Haar space generator for $\overline{\mathbb{D}}$ if $N = 1, 2$ and 3, but not if $N = 4$.
In Section 4, we consider the case $K = \overline{K^\circ}$. Since K° is necessarily nonempty, K contains automatically infinitely many points.
Again, we can show the conclusions of Theorem 3.3 and Theorem 3.4 provided that K° is not a convex domain. The proofs given in these two sections are different. They even differ from our earlier proofs concerning the closed unit disk. It is interesting to note that the "nicest" compact sets are the most difficult ones. The remaining problem is to show that $G(z) = 1/z^2$ is not a universal Haar space generator for all convex compact sets. We can show this for the following cases:

1. K is an ellipse or a disk,

2. ∂K has a corner i.e., ∂K contains a point q where the opening angle at q seen from the inside is smaller than π, where ∂K denotes the boundary of K.

2 Results for general compact subsets of the complex plane

Let $K \subset \mathbb{C}$ be a given, nonempty, compact set. In this section we study properties of functions $G \in H(\mathbb{C}\backslash\{0\})$ which are 1- and 2-dimensional Haar space generators for K. Clearly, G is a 1-dimensional Haar space generator for K if and only if all G_s have no zeros in K, where

$$G_s(z) := G(z - s),\ z \in K,\ s \in \mathbb{C}\backslash K. \tag{2}$$

In this article we shall use the following notation.

Definition 2.1 Let $f(x_1, x_2, ..., x_n)$ be any function of $n > 1$ variables. If we consider f to be a function of the first $m < n$ variables $x_1, x_2, ..., x_m$ keeping the other variables $x_{m+1}, ..., x_n$ fixed, then we shall write $f(x_1, x_2, ..., x_m | x_{m+1}, ..., x_n)$.

Lemma 2.2 *Let K be an arbitrary, nonempty, compact subset of $\mathbb{C}$. Then, G is a 1-dimensional Haar space generator for K if and only if G does not vanish on $\mathbb{C}\backslash\{0\}$.*

Proof: (a) Suppose that $G(z^*) = 0$ for some $z^* \neq 0$. Without loss of generality, we may assume, by Lemma 1.6(2.), that $0 \in K$. Define $\lambda_0 := \sup\{\lambda \geq 0 : -\lambda z^* \in K\}$, $z_0 := -\lambda_0 z^*$ and $t_0 := -(1 + \lambda_0)z^*$. Then, we have $z_0 \in K$, $t_0 \in \mathbb{C}\backslash K$ and $G(z_0 - t_0) = G(z^*) = 0$. Thus, G is not a 1-dimensional Haar space generator.
(b) Let $G(z^*) \neq 0$ for all $z^* \neq 0$. Then, $G(z - t) \neq 0$ for all $z \in K$ and all $t \in \mathbb{C}\backslash K$. Thus, G is a 1-dimensional Haar space generator. □

Lemma 2.3 *Let G be 1- and 2-dimensional Haar space generators for K. Define G_s, G_t for $s \neq t$ according to (2). Then*

$$\mu(z|s,t) := \frac{G_t(z)}{G_s(z)} = \frac{G(z-t)}{G(z-s)}, \quad z \in K, \quad s, t \in \mathbb{C}\backslash K,\ s \neq t \tag{3}$$

is injective on K, which means that $\mu(z_1; s, t) = \mu(z_2; s, t)$ implies $z_1 = z_2$.

Proof: By the fact that G is a 2-dimensional Haar space generator, all non trivial functions in the space

$$V := \langle G_s, G_t \rangle,\ s \neq t,\ s, t \in \mathbb{C}\backslash K$$

have at most one zero in K. That means, that $v \in V\backslash\{0\}$, i. e. $v := \alpha G_s + \beta G_t$, $|\alpha| + |\beta| > 0$ has at most one zero in K. Since G is also a 1-dimensional Haar space generator, G_s has no zero in K. Suppose that there is a $\lambda \neq 0$ such that

$$\mu(z|s,t) := \frac{G_t(z)}{G_s(z)} = \frac{G(z-t)}{G(z-s)} = \lambda \neq 0$$

has several solutions in K for some s and t in $\mathbb{C}\backslash K$ then $v(z) = G_t(z) - \lambda G_s(z)$ has several zeros in K which leads us to a contradiction. □

Lemma 2.4 *Let K be an arbitrary, compact subset of $\mathbb{C}$ containing the two distinct points z_1 and z_2. Let $t \in \mathbb{C}\backslash K$ and let $G \in H(\mathbb{C}\backslash\{0\})$ be a 1- and 2-dimensional analytic Haar space generator for K. Let us define*

$$F(t, z_1, z_2) := \frac{G(z_1 - t)}{G(z_2 - t)}, \quad z_1, z_2 \in K, \quad t \in \mathbb{C}\backslash K. \tag{4}$$

Then, (a) $F(t|z_1, z_2)$ is univalent in $\mathbb{C}\backslash K$, and (b)

$$F(t, z_1, z_2) = (t - z_1)^m H(t, z_1, z_2), \tag{5}$$

for some $m \in \{-1, 0, 1\}$ and where $H(t|z_1, z_2)$ is a nonvanishing analytic function on $\overline{\mathbb{C}}\backslash(\{z_1\} \cup \{z_2\})$.

Proof: Let s and t be in $\mathbb{C}\backslash K, s \neq t$. Define the function μ as in (3) and

$$\nu(t, s, z_1, z_2) := \frac{\mu(z_1; t, s)}{\mu(z_2; t, s)} = \frac{F(t, z_1, z_2)}{F(s, z_1, z_2)} \tag{6}$$

and fix $z_1, z_2 \in K$ and $s, t \in \mathbb{C}\backslash K$ as required.
(a) Suppose $F(t|z_1, z_2) = F(s|z_1, z_2), t \neq s, s \in \mathbb{C}\backslash K$. This is equivalent to $\nu(t, s|z_1, z_2) = 1$ or to

$$\frac{G(z_1 - t)}{G(z_1 - s)} = \frac{G(z_2 - t)}{G(z_2 - s)} =: \lambda, \quad z_1, z_2 \in K, z_1 \neq z_2, \quad s, t \in \mathbb{C}\backslash K, s \neq t$$

for some $\lambda \in \mathbb{C}\backslash\{0\}$. By Lemma 2.3 this is impossible.
(b) By Lemma 2.2, $G(z) \neq 0$ for $z \neq 0$ and since G is a 2-dimensional analytic Haar space generator for K, $\nu(t|s, z_1, z_2) \in H(\mathbb{C}\backslash(\{z_1\} \cup \{z_2\})$ and is different from 1 outside $K, t \neq s$. Hence, infinity is not an essential singularity for $\nu(t|s, z_1, z_2)$. The same holds for the function $F(t|z_1, z_2)$, i.e., $F(t|z_1, z_2)/t^m$ is a nonvanishing and analytic function in a neighborhood of infinity for some integer m. Next, the property $\nu(t|s, z_1, z_2) \neq 1$ on $\mathbb{C}\backslash K$ for all $s \in \mathbb{C}\backslash K, s \neq t$, is equivalent to the fact that $F(t|z_1, z_2)$ is univalent on $\mathbb{C}\backslash K$ which implies that $m \in \{-1, 0, 1\}$. The analyticity of $F(t|z_1, z_2)$ on $\mathbb{C}\backslash(\{z_1\} \cup \{z_2\})$ implies that $F(t|z_1, z_2)$ has to be of the form stated in relation (5) with $m \in \{0, 1, -1\}$ and the lemma is established. ☐

In the next lemma we show that in Lemma 2.4, part (b) we must have $m = 0$.

Lemma 2.5 *Under the conditions of Lemma 2.4 we conclude that $F(t|z_1, z_2)$ is analytic and nonvanishing on $\overline{\mathbb{C}}\backslash(\{z_1\} \cup \{z_2\})$.*

Proof: We want to show that $m = 0$ in relation (5). Choose $0 < r < |z_1 - z_2|$ and put

$$m_1 := \frac{1}{2\pi} \int_{|t - z_1| = r} \mathrm{d} \log F(t|z_1, z_2),$$

$$m_2 := \frac{1}{2\pi} \int_{|t - z_2| = r} \mathrm{d} \log F(t|z_1, z_2).$$

Since $F(t|z_1, z_2) = (t - z_1)^m H(t|z_1, z_2) \in H(\mathbb{C}\backslash(\{z_1\} \cup \{z_2\})$ and does not vanish there, we conclude, by the argument principle, that for sufficiently large R:

$$\begin{aligned} m &= \frac{1}{2\pi}\int_{|t|=R} \mathrm{d}\log F(t|z_1,z_2) \\ &= \frac{1}{2\pi}\int_{|t-z_1|=r} \mathrm{d}\log F(t|z_1,z_2) + \frac{1}{2\pi}\int_{|t-z_2|=r} \mathrm{d}\log F(t|z_1,z_2) \\ &= m_1 + m_2. \end{aligned}$$

Next, let $S := \{t : |t - z_1| = r\}$. Then, for fixed z_1, z_2 the range $F(S, z_1, z_2)$ is a closed analytic curve. We conclude that m_1 is a finite and constant integer on the interval $0 < r < |z_1 - z_2|$. The same holds true for m_2. Finally, the special form $F(t|z_1, z_2) = \frac{G(z_1-t)}{G(z_2-t)}$ gives $m_1 = -m_2$ and hence, $m = 0$ which implies that $F(t|z_1, z_2)$ is analytic at infinity. □

We have seen that $\nu(t|s, z_1, z_2)$ defined in relation (6) satisfies $\nu(t|s, z_1, z_2) \neq 1$ on $\mathbb{C}\backslash K, t \neq s$, which is equivalent to saying that $F(t|z_1, z_2)$ is univalent on $\mathbb{C}\backslash K$. Our next result shall specify the form of F for the special case where there is a punctured neighborhood $V(z_2) := \{t : 0 < |t-z_2| < r_2\}$ of z_2 on which $\nu(t|s, z_1, z_2) \neq 1$, $t \neq s$, holds. For example, this is the case when $\{z_2\}$ is an isolated point of K.

3 The case where $K\backslash\overline{K^\circ}$ is nonempty

The next two lemmata sharpen Lemma 2.4.

Lemma 3.1 *Let K be an infinite compact subset of $\mathbb{C}$ containing the two distinct points z_1 and z_2. Let $t \in \mathbb{C}\backslash K$ and let $G \in H(\mathbb{C}\backslash\{0\})$ be a 1- and 2-dimensional analytic Haar space generator for K. Then, $F(t|z_1, z_2)$ defined in (4) is univalent in $\mathbb{C}\backslash[\overline{K^\circ}\cup\{z_1\}\cup\{z_2\}]$.*

Proof: If $K = \overline{K^\circ}$, then, Lemma 3.1 is a part of Lemma 2.4. Suppose now that $K\backslash\overline{K^\circ}$ is nonempty. Fix z_1 and $z_2 \in K, z_1 \neq z_2$. We denote by $\mathbb{D}_r(c) := \{z \in \mathbb{C} : |z-c| < r\}, r > 0$, the open disk of center c and radius r. Let s_0 and t_0 be two distinct points in $\mathbb{C}\backslash[\overline{K^\circ}\cup\{z_1\}\cup\{z_2\}]$, and suppose that $F(t_0|z_1, z_2) = F(s_0|z_1, z_2) = \lambda_0$. Define $\mathbb{D}_r(s_0)$ and $\mathbb{D}_r(t_0)$ such that the three sets $\mathbb{D}_r(s_0)\cap[\overline{K^\circ}\cup\{z_1\}\cup\{z_2\}]$, $\mathbb{D}_r(t_0)\cap[\overline{K^\circ}\cup\{z_1\}\cup\{z_2\}]$ and $\mathbb{D}_r(s_0)\cap\mathbb{D}_r(t_0)$ are empty. Define $D_1 := [\mathbb{D}_r(s_0)\cup\mathbb{D}_r(t_0)]\cap K$. Since $F(t|z_1, z_2)$ is an open mapping, there is a nonempty open disk $\Delta := \mathbb{D}(\lambda_0)$ in $F(\mathbb{D}_r(s_0)|z_1, z_2) \cap F(\mathbb{D}_r(t_0)|z_1, z_2)$. Furthermore, $\Delta\backslash F(D_1|z_1, z_2)$ is nonempty since $F(D_1|z_1, z_2)$ does not contain any nonempty open set. Choose $\lambda \in \Delta\backslash F(D_1|z_1, z_2)$. Then, there is an $s \in \mathbb{D}_r(s_0)\backslash D_1$ and a $t \in \mathbb{D}_r(t_0)\backslash D_1$ with $F(s|z_1, z_2) = F(t|z_1, z_2) = \lambda$. In other words, s and t are two different points in $\mathbb{C}\backslash K$ which by Lemma 2.4, contradicts the hypothesis that G is a 2-dimensional Haar space generator for F. □

Lemma 3.2 *Let K be an arbitrary compact subset of $\mathbb{C}$ containing infinitely many points and suppose that $K \neq K\backslash\overline{K^\circ}$. Let $G \in H(\mathbb{C}\backslash\{0\})$ be a 1- and 2-dimensional Haar space generator for K. Choose $z_1 \in K\backslash\overline{K^\circ}$ and $z_2 \in K$, $z_2 \neq z_1$. Then we have*

$$F(t|z_1, z_2) = c(z_1, z_2)\left(\frac{z_1 - t}{z_2 - t}\right)^m \tag{7}$$

for some $m \in \{-1, 1\}$ and where $c(z_1, z_2) \neq 0$ does not depend on t.

Proof: Since $z_1 \in K\backslash\overline{K^\circ}$ and $z_2 \neq z_1$, $F(t|z_1,z_2)$ is univalent in a punctured neighborhood of z_1 and hence we have for some $m \in \{-1,0,1\}$

$$F(t|z_1,z_2) = (z_1-t)^m c(t|z_1,z_2); \quad c_1(t=z_1|z_1,z_2) \neq 0, \tag{8}$$

where by Lemma 2.5, $c_1(t|z_1,z_2) \in H(\mathbb{C}\backslash\{z_2\}$. On the other hand, $F(t|z_1,z_2) = 1/F(t|z_2,z_1)$ which implies that

$$F(t|z_1,z_2) = c(t|z_1,z_2)\left(\frac{z_1-t}{z_2-t}\right)^m; \quad c(t=z_2|z_1,z_2) \neq 0, \tag{9}$$

for some $m \in \{0,1,-1\}$ and where $c(t|z_1,z_2) \in H(\overline{\mathbb{C}})$. Hence, by Liouville's theorem, $c(t|z_1,z_2)$ is independent of t. Finally, if $m=0$, then, $F(t|z_1,z_2)$ is independent of t and hence $F(t|z_1,z_2)$ is not univalent in $\mathbb{C}\backslash K$ which contradicts the hypothesis that G is a 2-dimensional Haar space generator. □

Theorem 3.3 *Let K be a compact subset of $\mathbb{C}$ containing infinitely many points such that $K\backslash\overline{K^\circ}$ is nonempty. Then, $G \in H(\mathbb{C}\backslash\{0\})$ is a universal Haar space generator for K if and only if G is of the form (1); in other words,*

$$G(z) := \frac{\mathrm{e}^{Az+B}}{z}, \quad A,B \in \mathbb{C}.$$

Proof: Fix $z_1 \in K\backslash\overline{K^\circ}$. Then, relation (7) holds for all $z_2 \in K$, $z_2 \neq z_1$ and all $t \in \mathbb{C}\backslash K$. Since K is a compact set containing infinitely many points we conclude by the identity principle that

$$c(z_2|z_1) := \frac{G(z_2-t)}{G(z_1-t)}\left(\frac{z_1-t}{z_2-t}\right)^m, \; m=1 \text{ or } m=-1 \tag{10}$$

admits as a function of z_2 an analytic continuation onto $\mathbb{C}$ which is independent of t. Hence, using (10), differentiation of $\log c(z_2,z_1)$ with respect to t yields

$$\frac{G'(z_2-t)}{G(z_2-t)} \pm \left[\frac{1}{z_2-t}\right] = \frac{G'(z_1-t)}{G(z_1-t)} \pm \left[\frac{1}{z_1-t}\right] = A \tag{11}$$

for all t where A is independent of z_2. Fix $t \in \mathbb{C}\backslash K$ and substitute $\zeta = z_2 - t$. The integration of (11) with respect to ζ implies either

$$G(\zeta) := \zeta\mathrm{e}^{A\zeta+B}$$

or

$$G(\zeta) := \frac{\mathrm{e}^{A\zeta+B}}{\zeta}, \quad A,B \in \mathbb{C}.$$

The first case is excluded by Lemma 1.6(1.) with $n=4$ and the theorem is established. □

The authors have shown (Hengartner & Opfer [2002]) the following result: Suppose that $G \in H(\mathbb{C}\backslash\{0\})$ is a 1- to 4-dimensional Haar space generator for the closed unit disk $\overline{\mathbb{D}}$. Then, G is a universal Haar space generator for $\overline{\mathbb{D}}$. As an immediate corollary of our theory here, we obtain the following result:

Theorem 3.4 *Let K be a compact subset of $\mathbb{C}$ containing infinitely many points such that $K\backslash\overline{K^\circ}$ is nonempty. If a function $G \in H(\mathbb{C}\backslash\{0\})$ is a 1- and 2-dimensional Haar space generator for K, then, either G is a universal Haar space generator for K or $G(z) = z\mathrm{e}^{Az+B}$ for $A,B \in \mathbb{C}$.*

4 The case where $K = \overline{K^\circ}$

Our first lemma shows a geometric property.

Lemma 4.1 Let $K = \overline{K^\circ}$ be a nonempty, compact subset of $\mathbb{C}$. Then, there are points $p_k \in \partial K$, numbers $\beta_k \in \mathbb{R}, \varepsilon \in (0, \frac{\pi}{2})$ and $\rho_k > 0,\ k = 1, 2, \ldots, n$ with $n \geq 3$ such that

1. the open sectors $Z_k := \{z : |\arg[(z - p_k)\mathrm{e}^{-\mathrm{i}\beta_k}]| < \pi/2 - \varepsilon\} \bigcap \{z : 0 < |z - p_k| < \rho_k\}$ satisfy
$$Z_k \subset K^\circ, \quad 1 \leq k \leq n,$$
2. and the sets $E_k := \bigcup_{z \in Z_k} \frac{z - p_k}{|z - p_k|}$ satisfy
$$\bigcup_{k=1}^{n} E_k \supset \partial\, \mathbb{D}.$$

Proof: Let $\Delta(q, \hat{r})$ be an open disk of center q and radius $\hat{r}$ and suppose that in $\Delta(q_0, 2\hat{r}) \subset K^\circ$ for some $q_0 \in K^\circ$. We now move $\Delta(q, \hat{r})$ in K° until it hits ∂K in a point p. We call such a point p a touching point on ∂K. It can be represented by $p = q + \hat{r}\mathrm{e}^{\mathrm{i}\beta}$ where q is the center of a disk which touches ∂K in p. Therefore, $\mathrm{e}^{\mathrm{i}\beta} = \frac{q-p}{|q-p|}$ which is not always unique. Denote by T the set of all touching points on ∂K. Since K is a compact set, T contains several points. Next, choose an $\varepsilon \in (0, \pi/2)$. Then, there is a $\rho > 0$ depending on p and on ε such that the open sector

$$Z(p, \varepsilon) := \{z : |\arg[(z - p)\mathrm{e}^{-\mathrm{i}\beta}]| < \pi/2 - \varepsilon\} \bigcap \{z : 0 < |z - p| < \rho(p, \varepsilon)\}$$

satisfies

$$Z(p, \varepsilon) \subset K^\circ, \quad p \in T. \tag{12}$$

Next, we claim that we may choose ε so small that the sets

$$E(p, \varepsilon) := \bigcup_{z \in Z(p,\varepsilon)} \frac{z - p}{|z - p|}, \quad p \in T$$

satisfy

$$\bigcup_{p \in T} E(p, \varepsilon) \supset \partial\, \mathbb{D}. \tag{13}$$

Indeed, if not, there is a point

$$\eta \in \partial\mathbb{D} \backslash \bigcup_{0<\varepsilon<\pi/2} \left[\bigcup_{p \in T} E(p, \varepsilon) \right]$$

from which it follows that K is not bounded in the direction of $-\eta$. Therefore, there is an $\varepsilon \in (0, \pi/2)$ such that the relations (12) and (13) hold. It remains to show that these relations hold for a finite subset of T. Let $T = p_j, j \in J$ and $E_j = E(p_j, \varepsilon), j \in J$. Then, each E_j is an open interval on $\partial\mathbb{D}$ and therefore, the set $E_j = E(p_j, \varepsilon), j \in J$ is a covering of the compact unit circle by open sets. By Heine-Borel Theorem, $\partial\mathbb{D}$ is covered by a finite number of the E_j. This proves Lemma 4.1. $\square$

It should be noted that the above lemma is in general not true if K is not compact. E.g. if K is a parallel strip, then K is closed but not compact, $K = \overline{K^\circ}$, but the lemma is not valid.

Let $K = \overline{K^\circ}$ be a nonempty, compact subset of $\mathbb{C}$ and let $G \in H(\mathbb{C}\backslash\{0\})$ be a 1- and 2-dimensional Haar space generator for K. Note that K contains a nonempty open set and therefore, the condition that K contains infinitely many points is satisfied. Let z_1 and $z_2 \neq z_1$ be fixed in K. Since the Lemmata 2.2, 2.4, 2.5 hold also for the case $K = \overline{K^\circ}$, we conclude that $F(t|z_1, z_2) := \frac{G(z_1-t)}{G(z_2-t)}$, $z_1, z_2 \in K$, $t \in \mathbb{C}\backslash K$ is analytic in the variable t on $\overline{\mathbb{C}}\backslash(\{z_1\} \cup \{z_2\})$. We derive our result by proving the following lemmata:

Lemma 4.2 *Let $K = \overline{K^\circ}$ be a nonempty, compact subset of $\mathbb{C}$ and let $G \in H(\mathbb{C}\backslash\{0\})$ be a 1- and 2-dimensional Haar space generator for K. Then, G has the form*

$$G(z) = z^m \mathrm{e}^{\phi(z)} \text{ where } \phi \in H(\mathbb{C}) \text{ and } m \in \mathbb{Z}. \tag{14}$$

Furthermore, we have the representation

$$F(t, z, z_1) := \frac{G(z-t)}{G(z_1-t)} = c(z, z_1) \left[\frac{z-t}{z_1-t}\right]^m. \tag{15}$$

Proof: Fix s far from K, say, for example, $s = 10+10\max_{z\in K}|z|$ and let t be an arbitrary point in K. Choose $n \in \mathbb{N}, \varepsilon \in (0, \pi/2), \beta_1 \ldots, \beta_n \in \mathbb{R}, p_1, \ldots, p_n$ in ∂K and $\rho_1, \ldots, \rho_n$ positive such that the conclusions of Lemma 4.1 hold. Consider again $\mu(z|t, s) := \frac{G(z-t)}{G(z-s)}$. Suppose that the origin is an essential, isolated singularity for G. Then $\mu(z|t, s)$ has an essential singularity at $z = t$ and is an analytic, nonvanishing function on $\mathbb{C}\backslash[\{t\} \cup \{s\}]$. We want to show that there is a $k, 1 \leq k \leq n$, such that the sector Z_k (defined in the previous lemma) contains infinitely many roots of the function $\mu(z|p_k, s - a_k) - 1$ where $a_k := t - p_k$. Note that $s - a_k$ is in $\mathbb{C} \setminus K$. Put $\rho = \min\{\rho_k : 1 \leq k \leq n\}$. Then we may choose $N \in \mathbb{N}, N > n + 1$, such that each sector

$$S_j =: \{z : |\arg(z-t) - 2j\pi/N| < 3\pi/N\} \cap \{z : |z-t| < \rho\}$$

lies in one of the sectors $Z_k + a_k$. This follows from a classical theorem due to Cantor which can be stated as follows: Let $\{O_\alpha\}_{\alpha\in J}$ be a covering of a compact set C by open sets. Then, there is a $\delta > 0$ such that two points w_1, w_2 in C belong to the same open set O_α whenever $|w_1 - w_2| < \delta$. Apply this result to the orthogonal projection of the $S_j - t$ and the $Z_k - p_k$ onto $\partial\mathbb{D}$. Since, by hypothesis, the origin is an essential, isolated singularity for G, we conclude that there is an S_{j_0} which contains infinitely many roots of $\mu(z|t, s) - 1$ which implies that there is a sector Z_{k_0} containing infinitely many roots $\{\zeta_i\}_{i=1}^\infty$ of $\mu(z|p_k, s - a_k) - 1$. Next, we move the new $t_1 = p_{k_0}$ a very little to the outside of K, say to $\tau = t_1 + b$. Then, several points $\eta_i = \zeta_i + b$ are still in K and $\sigma = s - a_{k_0} + b$ stays in $\mathbb{C}\backslash K$. In other words, we have $\mu(\eta_i|\tau, \sigma) = 1$ for several $\eta_i \in K$. This contradicts the univalence of $\mu(z|t, s)$ on K provided that t and s are not in K. Therefore, G cannot have an essential singularity and hence, it is of the form (14) and G has either a pole or a zero of order $m, m \in \mathbb{N}$, at the origin. By Lemma 2.4, $F(t|z, z_1)$ is analytic at infinity. Fix z and z_1 in K and vary t. Define

$$c(t|z, z_1) := \left[\frac{z_1-t}{z-t}\right]^m \frac{G(z-t)}{G(z_1-t)}.$$

Then, $c(t|z, z_1)$ is an analytic function without zeros on $\overline{\mathbb{C}}$. Therefore, by Liouville's Theorem, $c(t|z, z_1)$ is independent of t. Finally (15) holds for all $z \in K\backslash\{z_1\}$. Applying the identity principle, we conclude that (15) holds everywhere $\mathbb{C}$. □

In the next lemma, we show that ϕ of (14) is a linear function.

Lemma 4.3 *Suppose that $K = \overline{K^\circ}$ is a nonempty, compact subset of $\mathbb{C}$ and let $G \in H(\mathbb{C}\backslash\{0\})$ be a 1- and 2-dimensional Haar space generator for K. Then, G is of the form*

$$G(z) = \frac{\mathrm{e}^{Az+B}}{z^m}, \quad m \in \mathbb{N}. \tag{16}$$

Proof: Differentiation of

$$\log\ c(z, z_1) = \log\left[\frac{G(z-t)}{G(z_1-t)}\right] + m\ \log\left[\frac{z_1-t}{z-t}\right] \tag{17}$$

$$= \phi(z-t) - \phi(z_1-t) \tag{18}$$

with respect to the variable t implies

$$\phi'(z-t) - \phi'(z_1-t) = 0 \text{ for all } t. \tag{19}$$

If we substitute $\zeta = z - t$ for fixed $t \in \mathbb{C}\backslash K$, (19) becomes

$$\phi'(\zeta) = A \text{ for all } \zeta, \tag{20}$$

thus, A is independent of ζ. Therefore, we obtain $\phi(\zeta) = A\zeta + B$ and G is of the form $G(z) = z^m \mathrm{e}^{Az+B}$, $m \in \mathbb{Z}$. Finally, Lemma 1.6(1.) yields the relation (16). □

Next, we show that (16) does not define a 2-dimensional Haar space generator when $|m| \geq 3$.

Lemma 4.4 *Let $K = \overline{K^\circ}$ be a compact, nonempty subset of $\mathbb{C}$. Then G defined by $G(z) := \frac{\mathrm{e}^{Az+B}}{z^m}$, $m \in \mathbb{Z}$ does not define a 2-dimensional Haar space generator whenever $|m| \geq 3$.*

Proof: First, observe that for negative m, Lemma 4.4 is a stronger statement than Lemma 1.6(1.). Moreover, by the same Lemma 1.6(1.), it is enough to prove Lemma 4.4 for the function $G(z) := \frac{1}{z^m}$. We want to show that

$$\mu(z|t,s) := \frac{G(z-t)}{G(z-s)} = \left(\frac{z-s}{z-t}\right)^m, \quad s, t \in \mathbb{C}\backslash K, s \neq t, \tag{21}$$

is not univalent in K. In other words, we want to find t and s in $\mathbb{C}\backslash K$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that the equation

$$w = E(z|s,t) := \frac{z-s}{z-t} = \lambda\ \mathrm{e}^{2k\pi\mathrm{i}/m}, \quad 0 \leq k < m, \tag{22}$$

has a solution z in K for at least two different k . Let $\Delta : \{z : |z - z^*| \leq r\}$ be a closed disk in K with the largest possible radius r (in a rectangle there are many such disks). Then, Δ touches ∂K in at least two different points, say p and q. There is a $t \in \mathbb{C}\backslash K$ and an

$s \in \mathbb{C}\backslash K, s \neq t$, such that $|t-p| < \varepsilon$ and $|s-q| < \varepsilon$, where $\varepsilon := 0.01 \min(|p-s|, |q-t|)$. The image $E(\Delta|t,s)$ is a disk passing through the points $P := \frac{p-s}{p-t}$; $|P| > 100$ and $Q := \frac{q-s}{q-t}$; $|Q| < 0.01$. Furthermore, $E(\Delta|t,s)$ does not contain the points $w = 0, w = 1$ and $w = \infty$. Denote by w^* the center of the disk $E(\Delta|t,s)$ and choose $\lambda = -w^*/|w^*|$. Since $|m| \geq 3$, we conclude that $E(\Delta|t,s)$ contains at least 2 points w_1 and w_2 of the set $\{\lambda\, \mathrm{e}^{2k\pi \mathrm{i}/m}, 0 \leq k < m\}$. Finally, the two solutions we are looking for are $z_1 = E^{-1}(w_1|t,s)$ and $z_2 = E^{-1}(w_2|t,s)$. □

It remains to eliminate the case $m = 2$. Referring to Lemma 1.6(1.), it suffices to show that $G(z) := 1/z^2$ is not a universal Haar space generator. If $K = \overline{K^\circ}$ is not a convex set, we prove, that G is neither a 2- nor a 3- dimensional Haar space generator. We start with the following lemma.

Lemma 4.5 *Let K be a nonempty, compact subset of $\mathbb{C}$ and let $K_1 := aK + b, a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C}$, Then, $G(z) := 1/z^2$ is an N-dimensional Haar space generator for K_1 if and only if it is an N-dimensional Haar space generator for K.*

Proof: Suppose $G(z) := 1/z^2$ is an N-dimensional Haar space generator for K. Then

$$G_1(z) := \left(\frac{a}{z}\right)^2 = a^2 \frac{1}{z^2} = \mathrm{e}^{2\ln a} G(z)$$

is an N-dimensional Haar space generator for K_1. Finally, Lemma 1.6(2.) shows that $G(z) := 1/z^2$ is an N-dimensional Haar space generator for K_1. The reverse follows in the same way. □

In what follows, we are first interested in finding compact sets K for which $G(z) = 1/z^2$ is not a 2-dimensional Haar space generator. Suppose that there exist $s \in \mathbb{C}\backslash K$ and $t \in \mathbb{C}\backslash K, s \neq t$, such that $\frac{G(z-t)}{G(z-s)} = \lambda$ or, equivalently, $\frac{z-s}{z-t} = \pm\sqrt{\lambda}$ has two roots z_1 and z_2 in K. Then, we obtain from $\frac{z_2-s}{z_2-t} = -\frac{z_1-s}{z_1-t}$ the equation

$$z_2 = \frac{-2st + z_1(s+t)}{2z_1 - (s+t)} \tag{23}$$

or, equivalently,

$$t = \frac{2z_1z_2 - s(z_1+z_2)}{(z_1+z_2) - 2s}. \tag{24}$$

Summarizing, we obtain the following lemma.

Lemma 4.6 *Let K be a compact subset of $\mathbb{C}$ containing the two different points z_1 and z_2 and let t and s be outside K. If relation (23) or relation (24) holds, $G(z) := 1/z^2$ is not a 2-dimensional Haar space generator for K.*

As a first application, we obtain:

Lemma 4.7 *Let $K = \overline{K^\circ} \subset \mathbb{C}$. If K° is not a convex domain, then, $G(z) := \dfrac{\mathrm{e}^{Az+B}}{z^2}$ does not define a 2-dimensional Haar space generator for K.*

Proof: Since K is not a convex set, there are constants $a \neq 0$ and b such that $K_1 = aK + b$ contains the point $z_1 = 0$ and $z_2 = 1$ and that the open interval $(0, 1)$ on the real axis is outside K. Hence, there is a disk $D(\frac{1}{2}, r), r > 0$, which lies outside K. The relation (24) gives $t = \frac{s}{2s-1}$ which converges to $1/2$ as s tends to infinity. Therefore, there is an s outside K close to infinity, such that $t \in D(\frac{1}{2}, r)$ and hence not in K. By Lemma 4.6, we conclude that $G(z) := 1/z^2$ is not a 2-dimensional Haar space generator for K. □

So far we have shown the following two results:

Theorem 4.8 *Let $K = \overline{K^\circ}$ be a compact subset of $\mathbb{C}$ which is not the closure of a nonempty, convex set. Then, $G \in H(\mathbb{C}\backslash\{0\})$ is a universal Haar space generator for K if and only if G is of the form (1); in other words,*

$$G(z) := \frac{\mathrm{e}^{Az+B}}{z}, \quad A, B \in \mathbb{C}.$$

The result corresponding to Theorem 3.4 is the following.

Theorem 4.9 *Let K be as in Theorem 4.8. If a function $G \in H(\mathbb{C}\backslash\{0\})$ is a 1- and 2-dimensional Haar space generator for K, then, either G is a universal Haar space generator for K or $G(z) = z\mathrm{e}^{Az+B}$ for $A, B \in \mathbb{C}$ which is not an N-dimensional Haar space generator for K whenever $N \geq 3$.*

We now concentrate our attention on the case where K is the closure of a bounded convex domain. So far we have shown that a holomorphic universal Haar space generator G has to be either of the form (1) or of the form $G(z) := \frac{\mathrm{e}^{Az+B}}{z^2}, \quad A, B \in \mathbb{C}$. We shall exclude the second form for ellipses and compact, convex sets whose boundary contains a corner. First, we consider again the case $N = 2$. An immediate consequence of Lemma 4.6 is the following lemma.

Lemma 4.10 *Let K be a compact subset of $\mathbb{C}$ containing the two different points z_1 and z_2. If both points $\frac{z_1+z_2}{2} \pm \mathrm{i}\frac{z_1-z_2}{2}$ are not in K, then $G(z) := 1/z^2$ is not a 2-dimensional Haar space generator for K.*

Corollary 4.11 *The function $G(z) := 1/z^2$ is not a 2-dimensional Haar space generator for ellipses with different lengths of the axes.*

Corollary 4.11 does not apply for disks. Indeed, we have the following lemma.

Lemma 4.12 *The function $G(z) := 1/z^2$ is a 2-dimensional Haar space generator for all disks with positive radii.*

Proof: Indeed, suppose that $G(z) := 1/z^2$ is not a 2-dimensional Haar space generator for the closed unit disk $\overline{\mathbb{D}}$. Then, there must be a t and an $s, s \neq t$, outside $\overline{\mathbb{D}}$ and a $\lambda \in \mathbb{C}$ such that there are at least two points z_1 and z_2 in $\overline{\mathbb{D}}$ with

$$w := \frac{z-s}{z-t} = \pm\sqrt{\lambda}. \tag{25}$$

Since $w = 0$ and $w = \infty$ are not in the closure of the image $w(\mathbb{D})$ we conclude that $w(\overline{\mathbb{D}})$ is a disk which cannot contain $\sqrt{\lambda}$ and $-\sqrt{\lambda}$. Therefore, G is a 2-dimensional Haar space generator for $\overline{\mathbb{D}}$. Applying Lemma 4.5 yields the desired result. □

We now consider the case $N = 3$ i.e., the 3-dimensional Haar space generators. We have shown in HENGARTNER & OPFER, [2002] that $G(z) := \frac{e^{Az+B}}{z^2}$, $A, B \in \mathbb{C}$ is not a 3-dimensional Haar space generator for $\overline{\mathbb{D}}$ and hence not for any closed disk with positive radius.

In the next application we show that $G(z) := 1/z^2$ is not a 3-dimensional Haar space generator for any nonconvex compact set.

Lemma 4.13 *Let $K \subset \mathbb{C}$ be a nonempty, compact set which satisfies $K = \overline{K^\circ}$. If K° is not a convex domain, then $G(z) := \frac{e^{Az+B}}{z^2}$ does not define a 3-dimensional Haar space generator for K.*

Proof: The assumption on the compact set K implies that it contains infinitely many points. Suppose that $K = \overline{K^0}$ is not a convex set. Then, there are constants $a \neq 0$ and b such that $K_1 = aK + b$ contains the point $z_1 = 0$ and that the ray $\{z : \Re z > 0, \Im z = 0\} \cap (\mathbb{C} \backslash K_1)$ contains a nonempty bounded component $I := (z_2, z_3)$, $z_2, z_3 \in K_1$. Observe, that the three different points z_1, z_2, z_3 belong to K_1. By Lemma 4.5 it is sufficient to show that the function $G(z) := 1/z^2$ is not a 3-dimensional Haar space generator for K_1. Define for $\mu \in \mathbb{C}$ the function

$$L(t, z_1, z_2, z_3, \mu) := \Big(\frac{z_1 - t}{z_3 - t}\Big)^2 - \mu\Big(\frac{z_1 - t}{z_2 - t}\Big)^2, \quad t \notin K_1.$$

We want to prove that the function

$$L(t|z_1 = 0, z_2, z_3, \mu = z_3/z_2) = t^2\Big\{\Big(\frac{1}{z_3 - t}\Big)^2 - \frac{z_3}{z_2}\Big(\frac{1}{z_2 - t}\Big)^2\Big\}$$

is at least three-valent in $\mathbb{C}\backslash K_1$. In other words, we shall find three different points t_1, t_2, t_3 in $\mathbb{C}\backslash K_1$ and $\mu := z_3/z_2$ such that $L(t_1|z_1 = 0, z_2, z_3, \mu = z_3/z_2) = L(t_2|z_1 = 0, z_2, z_3, \mu = z_3/z_2) = L(t_3|z_1 = 0, z_2, z_3, \mu = z_3/z_2)$. Since $\mu = z_3/z_2$, it follows that $L(t|z_1 = 0, z_2, z_3, \mu)$ is at least two-valent in any neighborhood of infinity. Moreover, $L(t = \infty, z_1 = 0, z_2, z_3, \mu = z_3/z_2) = 1 - z_3/z_2$. Next, we solve the equation

$$L(t|z_1 = 0, z_2, z_3, \mu = z_3/z_2) = 1 - z_3/z_2, \tag{26}$$

which yields (by the help of `maple`) the two solutions $t_\pm := \Big(z_2 + z_3 \pm \sqrt{z_2^2 - z_2 z_3 + z_3^2}\,\Big)/3$. Consider the solution with the positive root,

$$t_+ := \Big(z_2 + z_3 + \sqrt{z_2^2 - z_2 z_3 + z_3^2}\,\Big)/3.$$

Then, we obtain for $0 = z_1 < z_2 < z_3$:

$$t_+ > (2z_2 + \sqrt{z_2^2})/3 = z_2 \quad \text{and} \quad t_+ < (2z_3 + \sqrt{z_3^2})/3 = z_3$$

which shows that $t_+ \in \mathbb{C}\backslash K_1$. Define $\mathbb{D}(t_+) = \{z : |z - t_+| \leq \rho\}$ and $\mathbb{D}(\infty) = \{z : |z| \geq R\}$, such that $\mathbb{D}(\infty) \cap \mathbb{D}(t_+) = \emptyset$ and $(\mathbb{D}(\infty) \cup \mathbb{D}(t_+)) \cap K_1 = \emptyset$. Then, there is a point $p \neq 1 - z_3/z_2$ in the interior of $L(\mathbb{D}(t_+)|z = 0, z_2, z_3, \mu = z_3/z_2) \cap L(\mathbb{D}(\infty)|z_1 = 0, z_2, z_3, \mu = z_3/z_2)$ such that there are at least two different points t_1 and t_2 in $\mathbb{D}(\infty)$ and at least one point $t_3 \in \mathbb{D}(t_+)$ which are mapped onto p. This proves this lemma. □

Theorem 4.14 *Let K be any compact convex subset of $\mathbb{C}$ such that ∂K contains a point q at which the opening angle α as seen from the inside of K° satisfies $0 \le \alpha < \pi$. Then, $G \in H(\mathbb{C}\backslash\{0\})$ is a universal Haar space generator for K if and only if G is of the form (1); in other words,*

$$G(z) := \frac{\mathrm{e}^{Az+B}}{z}, \quad A, B \in \mathbb{C}.$$

Proof: If $\alpha = 0$, then K is either a point or an interval and the conclusion follows from Theorem 3.3. Hence, assume that $0 < \alpha < \pi$. In order to show that $G(z) = 1/z^2$ is not a 3-dimensional Haar space generator, we want to find an appropriate choice of $\mu, \nu \in \mathbb{C}$ and $t, s, u \in \mathbb{C}\backslash K$ mutually different such that the polynomial

$$P(z) = (z-u)^2(z-s)^2 + \mu(z-u)^2(z-t)^2 + \nu(z-s)^2(z-t)^2 \tag{27}$$

has at least three zeros in K. First, put

$$\mu = \frac{s^3(t-u)}{t^3(u-s)}, \quad \nu = \frac{u^3(s-t)}{t^3(u-s)}.$$

This choice of μ and ν assures us that $z = 0$ is a double root of $P(z)$. Next, define for $r > 0$, $t = 1$, $s = -1 + \mathrm{i}\sqrt{r}$ and $u = -1 - \mathrm{i}\sqrt{r}$. Then, we obtain

$$P(z) = -z^4(r+4) - 2z^3(r+4)(r-1) + z^2(3r^2 + 15r + 12)$$

whose roots are

$$\begin{aligned} z_1 &= z_2 = 0, \\ z_3 &= 1 - r - \frac{\sqrt{r^4 + 9r^3 + 28r^2 + 48r + 64}}{r+4}, \\ z_4 &= 1 - r + \frac{\sqrt{r^4 + 9r^3 + 28r^2 + 48r + 64}}{r+4}. \end{aligned}$$

Observe, that z_3 lies for all $r > 0$ on the negative real axis. Suppose now that ∂K has at q an opening angle $\alpha \in (0, \pi)$ seen from the inside of K°. Choose $\sqrt{r} > \tan(\alpha/2)$ and place K in such a way that

1. $q = 0$,
2. K lies in the cone $\pi - \alpha/2 \le \arg\, z \le \pi + \alpha/2$.

Next, apply a dilatation $K_1 := aK, a > 0$, such that $z_3 \in K_1^\circ$ and then, a very small horizontal translation $K_2 := K_1 + b, b > 0$, such that

1. $z = 0$ and $z = z_3$ are in K_2°,
2. t, s and u are not in K_2.

Finally, apply a small perturbation of m or n such that the double root $z = 0$ splits into two different roots lying in K_2 and the perturbed third root (corresponding to z_3) is still in K_2. This shows that $G(z) = 1/z^2$ is not a 3-dimensional Haar space generator for K_2 and hence, by Lemma 4.5, not for K. □

All our results can be summarized in the following theorem.

Theorem 4.15 *Let $K = \overline{K^\circ}$ be a nonempty, compact subset of $\mathbb{C}$ and let $G \in H(\mathbb{C}\backslash\{0\})$. Then, the function G is a universal Haar space generator for K if and only if it is of one of the forms*

$$\text{(a)} \quad G(z) := \frac{\mathrm{e}^{Az+b}}{z} \qquad \textit{or} \qquad \text{(b)} \quad G(z) := \frac{\mathrm{e}^{Az+b}}{z^2}, \qquad A, B \in \mathbb{C}, \tag{28}$$

where the latter case can possibly happen only if K is convex, $K = \overline{K^\circ}$, K is not an ellipse (including the disk) and the boundary ∂K has no corner.

Actually, we conjecture that case (b) will never happen.

Conjecture 4.16 *Let $K \subset \mathbb{C}$ be a compact and convex set with the additional property that $K = \overline{K^\circ}$. Then, $G \in H(\mathbb{C}\backslash\{0\})$ is a universal Haar space generator for K if and only if it is of the form* (a) *of the previous theorem.*

5 A numerical example

Let us treat the problem of interpolating given data $(z_j, f_j) \in \mathbb{C}^2, j = 1, 2, \ldots, n$ in the space

$$V_n := \langle v_1, v_2, \ldots, v_n \rangle, \text{ where } v_j(z) := \exp(a(z - s_j)) \frac{1}{z - s_j}$$

with given constant $a \in \mathbb{C}$ and given shifts $s_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$. That means, we want to find the unique element $v \in V_n$ which satisfies

$$v(z_j) = f_j, \; j = 1, 2, \ldots, n. \tag{29}$$

A typical element $v \in V_n$ has the form

$$v(z) = \frac{\exp(az)}{q(z)} p(z), \text{ where } q(z) := \prod_{j=1}^{n} (z - s_j), \; p \in \Pi_{n-1}. \tag{30}$$

Thus, we need to determine p of (30). Because of (29), this is equivalent to

$$p(z_j) = f_j q(z_j) \exp(-az_j) =: \eta_j, \; j = 1, 2, \ldots, n. \tag{31}$$

Now, this is a simple polynomial interpolation problem $p(z_j) = \eta_j$, $j = 1, 2, \ldots, n$. Having found p, the desired solution is v according to (30).

Let us now choose $z_j := \exp(\frac{2\pi \mathrm{i} j}{n})$, $j = 0, 1, \ldots, n-1$ and $f_j = f(z_j)$ where $f := \frac{1}{\Gamma}$ is the inverse of the complex gamma function. The chosen function f is an entire function, thus, analytic in all of $\mathbb{C}$. We have now some freedom to choose the constant a and the shifts $s_1, s_2, \ldots, s_n$, as well as n. Not so long ago, Trefethen [2002] posed the question (among ten) of finding the error of the best cubic polynomial approximation p_3 to the given f on the unit disk $\overline{\mathbb{D}}$, which is equivalent to finding the best approximation on the unit circle $|z| = 1$. It turned out, that the desired error, i. e. $e := \max_{|z|=1} |f(z) - p_3(z)|$ is $e = 0.214...$ (actually, 10 digits were required), meaning that the best cubic polynomial approximation p_3 is not a good approximation to f. We will see here, also with the case $n = 4$, that even the interpolating function v gives a much smaller error on the unit circle.

We selected the shifts according to $s_1 = -3.67, s_2 = -3.57, s_3 = 3.57, s_4 = 3.67$ and chose $a = -1.009$ (some computer trials were involved) and solved the above interpolation problem for $n = 4$. Let v be the solution, then $\max_{|z|=1} |(f - v)(z)| = 0.033721$. The mapping $|(f - v)(z)|$ on $|z| = 1$ can be viewed in Figure 5.1, left side. Interestingly, the maximal error $\max_{|z|=1} |(f - v)(z)|$ depends sharply on the constant a, to be viewed in Figure 5.1, right side.

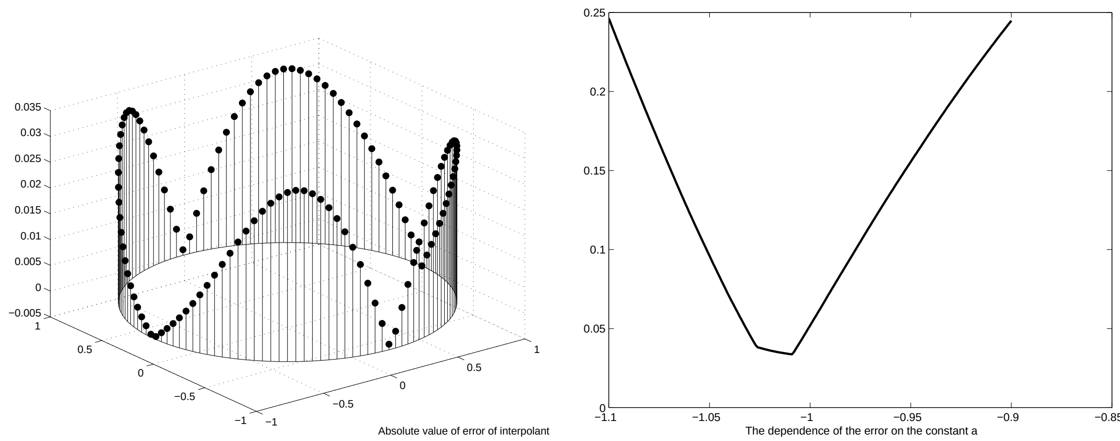

Figure 5.1 Absolute error $|f - v|$ of interpolant v (left) and dependence of the maximal absolute error on the constant a occurring in (30) (right)

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