Dimensional reduction for the generalized Seiberg-Witten equations and the Chern-Simons-Dirac functional

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Chapter 1

Introduction

The idea of Witten [Wit94] to replace the anti-selfduality equation by the Seiberg-Witten equations simplified many applications of gauge theory to the geometry of four-dimensional manifolds, in particular to the study of smooth structures. Similar to the Seiberg-Witten equations in dimension four, there are also Seiberg-Witten equations in dimension three. Replacing the Dirac operator by a nonlinear Dirac operator acting on sections in a fibre bundle, one obtains the generalized Seiberg-Witten equations. To construct such a Dirac operator, one has to require some properties of the typical fibre of this bundle. The spinor representation is replaced by a hyperkähler manifold, also called target manifold, with additional symmetries. Instead of sections in a spinor bundle, or equivalently equivariant maps from a principal $Spin^c$-bundle into the spinor representation, the spinors are now equivariant maps from a principal bundle into the target manifold, or equivalently sections in the associated fibre bundle. The generalized Seiberg-Witten equations in dimension three were introduced by Taubes in [Tau99]. In dimension four, these were studied by Pidstrygach in [Pid04].

In this diploma thesis we study the generalized Seiberg-Witten equations in dimensions three and four and, in particular, the generalized Seiberg-Witten equations on the cylinder over a three-dimensional manifold. Assuming temporal gauge, these equations reduce to the flow equations for a vector field on the configuration space of the three-dimensional manifold. Moreover, we prove that there is a functional on the configuration space whose gradient is this vector field. Such a functional is called Chern-Simons-Dirac functional. We study the properties of this functional and give explicit examples under certain assumptions on the target manifold.

One motivation to study the dimensional reduction and the Chern-Simons-Dirac functional is that it is essential in the constructions of the Seiberg-Witten Floer homology group for the usual Seiberg-Witten equations. The important invariants of smooth structures on four-manifolds can be encoded in Floer homology groups. In the case of Donaldson theory, this is an observation of Floer [Flo88]. A detailed account is given in [Don02]. The idea of this theory is to apply Morse theoretic constructions to the Chern-Simons functional on the infinite dimensional configuration space. There are also Seiberg-Witten Floer homology groups. In this case, the Chern-Simons-Dirac functional plays the role of the

Morse function on the infinite dimensional configuration space. A detailed account of the construction and properties of the Seiberg-Witten-Floer homology is given in [KM07]. Our Chern-Simons-Dirac functional generalized the one used to construct the Seiberg-Witten Floer homology groups.

We will first review some notions and constructions from differential geometry and gauge theory. In Chapter 3, we construct the nonlinear Dirac operator in dimensions three and four, and then formulate the generalized Seiberg-Witten equations in Chapter 4. The dimensional reduction of the Seiberg-Witten equations in four dimensions is studied in Chapter 5 and relates the Seiberg-Witten equations in dimensions three and four. Finally, in Chapter 6, we prove the existence of a Chern-Simons-Dirac functional for the generalized Seiberg-Witten equations and provide an example for such a functional for certain hyperkähler manifolds which permit a hyperkähler potential.

At this point, I would like to thank all those people who supported me during the time of my studies and the period of work on this diploma thesis. First of all, I am grateful to my advisor Prof. Pidstrygach for introducing me to many areas of mathematics related to differential geometry and gauge theory and for his comments and helpful suggestions. I am also grateful to Prof. Schick for his commitment as co-supervisor. I also want to thank all participants of the “tea seminar”, where many interesting topics were discussed. In particular, I am grateful to Vadim Alekseev, Henrik Schumacher, Kirstin Strokorb and Dr. Ulrich Pennig for a pleasant time and many fruitful discussions. Moreover, I am grateful to Carsten Thiel proofreading, many LaTeX-related hints and numerous interesting conversations. Last but not least, I would like to thank my parents for their never ending support.

Chapter 2

Preliminaries and notation

In this chapter we review some basis definitions and notions from differential geometry and gauge theory which we need later on, in particular fibre bundles, connections, hyperkähler manifolds, Clifford algebras and *Spin* groups.

2.1 Fibre bundles

Throughout this text all manifolds are smooth, paracompact and, if not stated otherwise, finite-dimensional.

2.1.1 Definition (fibre bundle). Let F be a manifold. A smooth map $\pi\colon E \to M$ between two manifolds is said to be a *smooth fibre bundle* with *typical fibre* F if for every $x \in M$ there is an open neighborhood $U \subset M$ of x (i.e. $x \in U$) and a diffeomorphism $\Phi_U\colon \pi^{-1}(U) \to U \times F$ satisfying $\mathrm{pr}_U \circ \Phi_U = \pi$. Such a pair (U, Φ_U) is called *bundle chart*. A *bundle atlas* is an open cover $\{U_i\}_{i\in I}$ of M with bundle charts $\{(U_i, \Phi_i)\}_{i\in I}$. We denote by $E_x := \pi^{-1}(\{x\})$ the *fibre* over $x \in M$. In particular, for a bundle chart (U_i, Φ_i) the restriction $\Phi_{i,x} := \mathrm{pr}_F \circ \Phi_i|_{F_x}\colon F_x \to F$ is a diffeomorphism. For a bundle atlas $\{(U_i, \Phi_i)\}_{i\in I}$, we have *transition functions*

$$\Phi_i \circ \Phi_j^{-1}\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F.$$

These define smooth maps Φ_{ij} to the group of diffeomorphisms of the fibre

$$\begin{aligned} \Phi_{ij}\colon U_i \cap U_j &\to \mathrm{Diff}\,(F)\,, \\ x &\mapsto \Phi_{i,x} \circ \Phi_{j,x}^{-1}, \end{aligned}$$

which satisfy the *cocycle conditions* $\Phi_{ij} \circ \Phi_{jk} = \Phi_{ik}$ and $\Phi_{ii} = \mathrm{id}_F$ for all $i, j, k \in I$. The family $\{\Phi_{ij}\}_{i,j\in I}$ is the *cocycle* for the bundle atlas $\{(U_i, \Phi_i)\}_{i\in I}$. A smooth map $s\colon M \to E$ satisfying $\pi \circ s = \mathrm{id}_M$ is said to be a *section* of $E \to M$. The space of all smooth sections is denoted by $\Gamma(M, E)$. Let $\pi\colon E \to M$ and $\pi'\colon E' \to M$ be two smooth fibre bundles over M. A *smooth bundle map* is a smooth map $f\colon E \to E'$ such that $\pi' \circ f = \pi$.

2.1.2 Definition (general connection). Let $\pi\colon E \to M$ be a smooth fibre bundle and $T\pi : TE \to TM$ the differential of π. The *vertical bundle* $\mathscr{V}_E$ is the subbundle $\ker(T\pi) \subset TE$. A *general connection* on $E \to M$ is a smooth subbundle $\mathscr{H} \subset TE$ such that $TE = \mathscr{V}_E \oplus \mathscr{H}$. We denote the projections to $\mathscr{V}_E$ and $\mathscr{H}$ by $\mathrm{pr}_{\mathscr{V}}\colon TE \to \mathscr{V}_E$ and $\mathrm{pr}_{\mathscr{H}}\colon TE \to \mathscr{H}$, respectively. These are homomorphisms of vector bundles over E.

2.1.1 Vector bundles

2.1.3 Definition. A smooth fibre bundle $\pi\colon E \to M$ is said to be a *(real/complex) vector bundle* if the typical fibre F is a $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and the transition maps are linear (i.e. $\Phi_{ij}(x) \in \mathrm{Aut}_{\mathbb{K}}(V)$ for all $i, j \in I$, $x \in M$).

Covariant derivative and connections

2.1.4 Remark. Let $E \to M$ be a vector bundle and consider the pullback $E \times_M E$. Note that $vl_E\colon E \times_M E \to \mathscr{V}_E$, $(v, w) \mapsto \frac{d}{dt}(v + tw)|_{t=0}$ is an isomorphism of vector bundles over E. Is is called *vertical lift*.

2.1.5 Definition. A *covariant derivative* on a vector bundle $E \to M$ a linear map

$$\nabla\colon \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f \otimes \nabla s \text{ for all } f \in C^\infty(M, \mathbb{R}), s \in \Gamma(M, E).$$

2.1.6 Remark. Given a vector bundle $\pi\colon E \to M$, there are two vector bundle structures on the total space TE. One the one hand, $\pi_E\colon TE \to E$ is the tangent bundle of E, on the other hand, $T\pi\colon TE \to TM$ is also a vector bundle.

2.1.7 Definition. Let $\pi\colon E \to M$ be a vector bundle. A general connection on E is said to be a *linear connection*, if the composition $TE \xrightarrow{\mathrm{pr}_{\mathscr{V}}} \mathscr{V} \subset TE$ is linear with respect to the vector bundle structure $T\pi\colon TE \to TM$. A *connector* on E is a smooth map $\mathcal{K}\colon TE \to E$ which satisfies $\mathcal{K} \circ vl_E = \mathrm{pr}_2\colon E \times_M E \to E$ and is a vector bundle homomorphism for both vector bundle structures on TE, i.e. the following two diagrams are vector bundle homomorphisms:

$$\begin{array}{ccc} TE & \xrightarrow{\mathcal{K}} & E \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi} & M \end{array} \qquad\qquad \begin{array}{ccc} TE & \xrightarrow{\mathcal{K}} & E \\ \downarrow{\scriptstyle T\pi} & & \downarrow{\scriptstyle \pi} \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

2.1.8 Remark ([KM97, 37.27]). Given a linear connection on a vector bundle $E \to M$, the composition

$$TE \xrightarrow{\mathrm{pr}_{\mathscr{V}}} \mathscr{V} \xrightarrow{vl_E^{-1}} E \times_M E \xrightarrow{\mathrm{pr}_2} E$$

is a connector. Conversely, given a connector $\mathcal{K}$, we can reconstruct the vertical projection $\mathrm{pr}_{\mathscr{V}} = vl_E \circ (\mathrm{pr}_E, \mathcal{K})\colon TE \to E \times_M E \to \mathscr{V}_E$ and the horizontal subbundle is $\mathscr{H} = \ker(\mathrm{pr}_{\mathscr{V}}) \subset TE$. Therefore, instead of specifying a linear connection, we can equivalently specify a connector on a vector bundle.

2.1.9 Remark. A connector $\mathcal{K}\colon TE \to E$ on a vector bundle $E \to M$ induces a covariant derivative on $\nabla\colon \Gamma(M,E) \to \Gamma(M, T^*M \otimes E)$:

$$\nabla_v s := \mathcal{K}(Ts(v)) \text{ for } v \in \Gamma(M,TM), s \in \Gamma(M,E).$$

Moreover, given a smooth map $f\colon N \to M$, the same formula defines a covariant derivative

$$\Gamma(N, f^*E) \to \Gamma(N, T^*N \otimes f^*E).$$

2.1.2 Group actions

2.1.10 Definition. Let M be a manifold and G a Lie group. A *smooth left action* of G on M is a smooth map

$$\begin{aligned} G \times M &\to M \\ (g,x) &\mapsto g \cdot x \end{aligned}$$

such that

1. for all $g \in G$ the map $L_g\colon M \to M$, $L_g(x) := g \cdot x$ is a diffeomorphism,
2. $1 \cdot x = x$ for all $x \in M$, where $1 \in G$ is the unit element in G, and
3. $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G, x \in M$.

Similarly, for a right action, one has a smooth map $M \times G \to M$, $(x,g) \mapsto x \cdot g$ and the maps $R_g : P \to P$, $R_g(x) = x \cdot g$ are diffeomorphisms. In many situations, we will also write gx for $g \cdot x$ in the case of a left action and xg for $x \cdot g$ in the case of a right action.

2.1.11 Remark. If a Lie group G acts smoothly on a manifold M, then we have an induced action of G on TM denoted by $G \times TM \ni (g,v) \mapsto g_* v \in TM$, where $g_* v := T_x L_g(v)$ for $v \in T_x M$.

2.1.12 Definition. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. For $\xi \in \mathfrak{g}$, the *fundamental vector field* $K_\xi^{M,G} \in \Gamma(M,TM)$ for a smooth left action of a Lie group G on a manifold M is

$$(K_\xi^{M,G})_x := \tfrac{d}{dt} \exp(t\xi) \cdot x|_{t=0} \in T_x M \text{ for } \xi \in \mathfrak{g}, x \in M.$$

Note that for $x \in M$, $\xi \in \mathfrak{g}$ and $g \in G$ we have

$$T_x L_g (K_\xi^{M,G})_x = \tfrac{d}{dt} g \exp(t\xi) \cdot x|_{t=0} = \tfrac{d}{dt} \exp(t Ad_g(\xi)) g \cdot x|_{t=0} = (K_{Ad_g(\xi)}^{M,G})_{gx}.$$

Here $Ad\colon G \to \mathrm{Aut}\,(\mathfrak{g})$ is the *adjoint representation* of G on its Lie algebra $\mathfrak{g}$. More precisely, we have $Ad_g := T_1 c_g$, where $c\colon G \to \mathrm{Aut}\,(G)$ is the *conjugation action*, i.e. $g \mapsto c_g,\ c_g(h) := ghg^{-1}$.

Similarly, for a smooth right action of G on M, the *fundamental vector field* $K_\xi^{M,G} \in \varGamma(M, TM)$ is

$$(K_\xi^{M,G})_x := \tfrac{d}{dt} x \cdot \exp(t\xi)|_{t=0} \in T_xM \text{ for } \xi \in \mathfrak{g}, x \in M.$$

Again, note that for $x \in M$, $\xi \in \mathfrak{g}$ and $g \in G$ we have

$$T_xR_g(K_\xi^{M,G})_x = \tfrac{d}{dt} x \cdot \exp(t\xi)g|_{t=0} = \tfrac{d}{dt} x \cdot g \exp(tAd_{g^{-1}}(\xi))|_{t=0} = (K^{M,G}_{Ad_{g^{-1}}(\xi)})_{xg}.$$

The fundamental vector field defines a G-equivariant linear map $\mathfrak{g} \to \varGamma(M, TM)$, $\xi \mapsto K_\xi^{M,G}$ from the Lie algebra $\mathfrak{g}$ with the adjoint action of G to $\varGamma(M, TM)$ with the induced action.

2.1.13 Definition. Let V be a vector space (over $\mathbb{R}$). Using the fundamental vector fields, we have a homomorphism

$$\begin{aligned} \iota_\mathfrak{g}\colon \varOmega^k(M, V) &\to \varOmega^{k-1}(M, \mathfrak{g}^* \otimes V), \\ \alpha &\mapsto \iota_\mathfrak{g}\alpha,\ \langle \iota_\mathfrak{g}\alpha, \xi\rangle = \iota_{K_\xi^{M,G}}\alpha \text{ for } \xi \in \mathfrak{g}. \end{aligned}$$

There is also a corresponding *Lie derivative*

$$\begin{aligned} \mathcal{L}_\mathfrak{g}\colon \varOmega^k(M, V) &\to \varOmega^k(M, \mathfrak{g}^* \otimes V), \\ \alpha &\mapsto \mathcal{L}_\mathfrak{g}\alpha, \langle \mathcal{L}_\mathfrak{g}\alpha, \xi\rangle := \mathcal{L}_{K_\xi^{M,G}}\alpha \text{ for } \xi \in \mathfrak{g}. \end{aligned}$$

Note that $\iota_\mathfrak{g}$ and $\mathcal{L}_\mathfrak{g}$ are related by $\mathcal{L}_\mathfrak{g} := d\iota_\mathfrak{g} + \iota_\mathfrak{g} d$.

2.1.14 Remark. Let $\rho\colon G \to \mathrm{Aut}\,(V)$ be a G-representation, M a manifold with a smooth (left) G-action. A k-form $\alpha \in \varOmega^k(M, V)$ is said to be G-*equivariant* if $L_g^*\alpha = \rho(g)\alpha$ for all $g \in G$. The space of equivariant k-forms is denoted by $\varOmega^k(M, V)^G$.

Let $\alpha \in \varOmega^k(M, V)^G$ be a G-equivariant k-form and $w_1, \cdots, w_{k-1} \in T_xM$ for some $x \in M$. Then

$$\begin{aligned} \langle L_g^*(\iota_\mathfrak{g}\alpha), \xi\rangle(w_1, \ldots, w_{k-1}) &= \alpha(K_\xi^{M,G}, TL_g(w_1), \ldots, TL_g(w_{k-1})) \\ &= \rho(g)\alpha(TL_{g^{-1}}K_\xi^{M,G}, w_1, \ldots, w_{k-1}) \\ &= \rho(g)\langle \iota_\mathfrak{g}\alpha, Ad_{g^{-1}}\xi\rangle(w_1, \ldots, w_{k-1}) \\ &= \langle (\rho(g) \otimes Ad_g^*)\iota_\mathfrak{g}\alpha, \xi\rangle(w_1, \ldots, w_{k-1}). \end{aligned}$$

Here $Ad^*\colon G \to \mathrm{Aut}\,(\mathfrak{g}^*)$ is the *coadjoint representation* $Ad_g^*(\nu)(\xi) = \nu(Ad_{g^{-1}}(\xi))$ for all $g \in G$, $\nu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. This proves that $\iota_\mathfrak{g}\alpha \in \varOmega^{k-1}(M, \mathfrak{g}^* \otimes V)^G$ and that $\iota_\mathfrak{g}$ maps G-equivariant forms into G-equivariant forms:

$$\iota_\mathfrak{g}\colon \varOmega^k(M, V)^G \to \varOmega^{k-1}(M, \mathfrak{g}^* \otimes V)^G.$$

2.1.3 Principal bundles

2.1.15 Definition (principal bundle). Let G be a Lie group and $\pi\colon P \to M$ a smooth fibre bundle with typical fibre G. We say that $\pi\colon P \to M$ is a *principal G-bundle* if the transition functions map to $G \subset \mathrm{Diff}\,(G)$, where G acts on itself by left multiplication (i.e. there are maps $g_{ij} : U_i \cap U_j \to G$ such that $\Phi_{ij}(x)(h) = g_{ij}(x)h$ for all $h \in G$). The maps $\{g_{ij}\}_{i,j\in I}$ again satisfy the *cocycle conditions* $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$ for all $x \in U_i \cap U_j \cap U_k$ and $g_{ii}(x) = 1$ for all $x \in U_i$, where $1 \in G$ is the unit element. Note that we have a right G-action on P, $(p, g) \mapsto pg := \Phi^{-1}_{i,\pi(p)}((\Phi_{i,\pi(p)}(p))g)$ for a bundle chart (U_i, Φ_i) with $\pi(p) \in U_i$, each fibre of P is a G-torsor and $P/G = M$.

Let $\lambda\colon G' \to G$ be a group homomorphism and $P \to M$ be a principal G-bundle. A *λ-reduction of P* is a principal G'-bundle $P' \to M$ and a smooth bundle map $f\colon P' \to P$ satisfying $f(pg) = f(p)\lambda(g)$ for all $p \in P'$ and $g \in G'$.

Two principal G-bundle $\pi\colon P \to M$, $\pi'\colon P' \to M$ are said to be *isomorphic* if there is a smooth G-equivariant diffeomorphism $f\colon P \to P'$.

2.1.16 Remark. Let $\pi\colon P \to M$ be a principal G-bundle with a bundle atlas $\{(U_i, \Phi_i)\}_{i\in I}$. The cocycle $\{g_{ij}\}_{ij\in I}$ determines an element in the first Čech cohomology set $\check{H}^1(M, G)$. We have a bijection between $\check{H}^1(M, G)$ and the isomorphism classes of principal G-bundles over M (cf. [Hir66, Thm 3.2.1] or [LM89, Appendix A]).

2.1.17 Example (bundle of frames). Let $E \to M$ be a real vector bundle of rank n. For $x \in M$ define

$$P_x := \{f\colon \mathbb{R}^n \to E_x | f \text{ linear isomorphism}\}.$$

This defines a bundle $P \to M$, called *bundle of linear frames in E*. Using the composition

$$GL_n(\mathbb{R}) \times P \ni (A, f) \mapsto A^*f = f \circ A \in P,$$

the bundle of linear frames is a principal $GL_n(\mathbb{R})$-bundle. Given a Riemannian metric g^E on E and an orientation, we can also study the *bundle of oriented orthonormal frames* $P_{SO(n)}$, where $(P_{SO(n)})_x = \{\, f \in P_x \mid f \text{ orientation preserving isometry}\,\}$. This principal $SO(n)$-bundle is a reduction of the bundle of linear frames P. In particular, we will be interested in the case when the vector bundle is the tangent bundle $TM \to M$.

Equivariant vector bundles and associated vector bundles

2.1.18 Definition. Let G be a Lie group. A vector bundle $\pi\colon E \to M$ with a smooth action of G on E is said to be an *equivariant vector bundle* if there is a smooth action of G on the base manifold M such that $\pi\colon E \to M$ is G-equivariant and $L_g\colon E_x \to E_{gx}$ is linear. Given a G-equivariant vector bundle $E \to M$, we denote $\pi_! E := E/G$.

2.1.19 Example. Let G be a Lie group.

1. Given a smooth action of G on a manifold M, there is an induced action of G on the tangent bundle TM. Equipped with these actions, $TM \to M$ is a G-equivariant vector bundle.

2. Given a smooth action of G on a manifold P and a G-representation V, the trivial vector bundle $P \times V \to P$ with the action $(h, (p, v)) \mapsto (ph^{-1}, hv)$ is a G-equivariant vector bundle. If $P \to M$ is a principal G-bundle, then $\pi_!(P \times V) \to M$ is the *associated vector bundle* which is denoted by $P \times_G V$.

Associated fibre bundles

The construction in the second part of Example 2.1.19 generalizes to arbitrary fibres:

2.1.20 Definition (associated fibre bundle). Let $P \to M$ be a principal G-bundle and let G act smoothly on a manifold F. Then G acts (from the left) on the product $P \times F$ by $(h, (p, f)) \mapsto (ph^{-1}, hf)$. The quotient by G is a fibre bundle over M with typical fibre F and is denoted by $P \times_G F := (P \times F)/G$.

2.1.21 Example. Let $P \to M$ be a bundle of orthonomal frames in a vector bundle $E \to M$ of rank k. Then $E = P \times_{O(k)} \mathbb{R}^k$. In particular, this holds for the tangent bundle $TM \to M$.

2.1.22 Proposition ([Bau09, Satz 2.9]). *Let $P \to M$ be a principal G-bundle and F a manifold with a smooth G-action. Then there is a bijection between the space of G-equivariant maps from P to F and the sections of the associated fibre bundle,*

$$\begin{aligned} C^\infty(P, F)^G &\to \Gamma(M, P \times_G F), \\ f &\mapsto s_f, \text{ where } s_f(x) = [x, f(x)] \text{ for } x \in M. \end{aligned}$$

2.1.23 Definition. Let $P \to M$ be a principal G-bundle and let V be a representation of G. A k-form $\alpha \in \Omega^k(P, V)$ on P with values in V is said to be *horizontal* if $\iota_{\mathfrak{g}}\alpha = 0$. The subspace of horizontal k-forms is denoted by $\Omega^k(P, V)_{hor} \subset \Omega^k(P, V)$.

The Proposition 2.1.22 generalizes to

2.1.24 Proposition ([Bau09, Satz 3.5]). *Let $P \to M$ be a principal G-bundle and V a G-representation. Then there is a bijection*

$$\Omega^k(P, V)^G_{hor} \to \Omega^k(M, P \times_G V).$$

2.1.4 Connections

2.1.25 Definition (connection 1-form). A *connection* 1*-form* on a principal G-bundle $P \to M$ is a G-equivariant 1-form $A \in \Omega^1(P, \mathfrak{g})^G$, satisfying $\iota_{\mathfrak{g}} A \equiv \mathrm{id}_{\mathfrak{g}}$. The subbundle $\mathscr{H}_A := \ker(A) \subset TP$ is the *horizontal bundle* or *horizontal distribution* for A.

The space of all connection 1-forms on P will be denoted by $\mathscr{A}(P)$.

2.1.26 Remark. The condition $\iota_{\mathfrak{g}} A \equiv \mathrm{id}_{\mathfrak{g}}$ for a connection 1-form $A \in \Omega^1(P, \mathfrak{g})^G$ can be written as

$$A(K_\xi^{P,G}) = \xi \text{ for all } \xi \in \mathfrak{g}.$$

Being equivariant means that for all $g \in G$ we have:

$$R_g^* A = Ad_{g^{-1}} A$$

where $Ad\colon G \to \mathrm{End}\,(\mathfrak{g})$ is the *adjoint representation* of G on its Lie algebra $\mathfrak{g}$. Here the inverse g^{-1} appears because we consider a left action of G on $\mathfrak{g}$ and a right action of G on P.

2.1.27 Proposition ([Bau09, Folgerung 3.1]). *The space of connection* 1*-forms* $\mathscr{A}(P)$ *is an affine space for the vector space* $\Omega^1(P, \mathfrak{g})^G_{hor}$.

2.1.28 Remark. A connection 1-form A on a principal G-bundle $P \to M$ induces a general connection $TP = \mathscr{H}_A \oplus \mathscr{V}_P$ and we denote the projections to $\mathscr{H}_A$ and $\mathscr{V}_P$ by $\mathrm{pr}_{\mathscr{H}_A}$ and $\mathrm{pr}_{\mathscr{V}_A}$ respectively. Since A is G-equivariant, this decomposition in also G-equivariant, i.e. $(\mathscr{H}_A)_{pg} = TR_g(\mathscr{H}_A)_p$ for all $p \in P$ and $g \in G$.

2.1.29 Remark. Let $P \to M$ be a principal G-bundle. Consider the adjoint representation of G on its Lie algebra $\mathfrak{g}$ and denote the associated vector bundle by $\mathfrak{g}_P := P \times_G \mathfrak{g}$. Using the isomorphism $\Omega^1(P, \mathfrak{g})^G_{hor} \cong \Omega^1(M, \mathfrak{g}_P)$ from Proposition 2.1.24, we can also think of $\mathscr{A}(P)$ as an affine space for the vector space $\Omega^1(M, \mathfrak{g}_P)$.

2.1.30 Remark. Note that there is a bijection between the set of covariant derivatives on a vector bundle and the connection 1-forms on its bundle of linear frames. The metric compatible covariant derivatives correspond to connection 1-forms on the bundle of orthonomal frames.

2.1.31 Definition. Let V be a G-representation and P a principal G-bundle with connection 1-form $A \in \mathscr{A}(P)$. The *covariant exterior derivative* for A is

$$d_A := \mathrm{pr}^*_{\mathscr{H}_A} d\colon \Omega^k(P, V)^G \to \Omega^{k+1}(P, V)^G_{hor}.$$

2.1.32 Definition (curvature). The *curvature* F_A of a connection 1-form $A \in \mathscr{A}(P)$ on a principal G-bundle $P \to M$ is

$$F_A := d_A A = dA + \tfrac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g})^G_{hor}.$$

Using the isomorphism $\Omega^2(P,\mathfrak{g})^G_{hor} \cong \Omega^2(M,\mathfrak{g}_P)$, we can also interpret the curvature as an element in $\Omega^2(M,\mathfrak{g}_P)$, which will also be denoted by F_A. If P is a bundle of frames in TM, we have $\mathfrak{g}_P \subset \mathrm{End}\,(TM)$ and the image of F_A under $\Omega^2(P,\mathfrak{g})^G_{hor} \to \Omega^2(M,\mathrm{End}\,(E))$ is the *curvature tensor* R^A.

2.1.33 Remark. For a horizontal 1-from $\alpha \in \Omega^1(P,\mathfrak{g})^G_{hor}$ on a principal G-bundle $P \to M$ and a connection 1-form $A \in \mathscr{A}(P)$, we have

$$\begin{aligned} F_{A+\alpha} &= d(A+\alpha) + \tfrac{1}{2}[A+\alpha, A+\alpha] = dA + d\alpha + \tfrac{1}{2}[A,A] + [A,\alpha] + \tfrac{1}{2}[\alpha,\alpha] \\ &= F_A + d_A\alpha + \tfrac{1}{2}[\alpha,\alpha]. \end{aligned}$$

2.1.34 Definition (canonical 1-form). Let M be an n-dimensional manifold and let $\pi\colon P \to M$ the bundle of linear frames in TM. The *canonical 1-form* $\theta \in \Omega^1(P,\mathbb{R}^n)^{GL_n(\mathbb{R})}_{hor}$ is

$$\theta_f(v) := f^{-1}(T_f\pi(v)) \text{ for } f \in P, v \in T_fP.$$

We will also denote the pullback of the canonical 1-form to any other bundle of frames by θ.

2.1.35 Definition (torsion form). Consider a bundle of frames $\pi\colon P \to M$ in the tangent bundle TM of an n-dimensional manifold M. Then for a connection 1-form $A \in \mathscr{A}(P)$ the *torsion form* $\Theta_A \in \Omega^2(P,\mathbb{R}^n)^{GL_n(\mathbb{R})}_{hor}$ is the covariant exterior derivative of the canonical 1-form θ:

$$\Theta_A := d_A\theta.$$

The image of Θ_A under the isomorphism $\Omega^2(P,\mathbb{R}^n)^{GL_n(\mathbb{R})}_{hor} \cong \Omega^2(M,TM)$ is the *torsion tensor* T^A.

Induced covariant derivative on associated vector bundles

Let $P \to M$ be a principal G-bundle and V a representation of G. Let $E = P \times_G V$ the associated vector bundle. A connection 1-form A induces a covariant derivative ∇^A on E: We define ∇^A to be the map which makes the following diagram commutative:

$$\begin{array}{ccc} C^\infty(P,V)^G & \xrightarrow{d_A} & \Omega^1(P,V)^G_{hor} \\ \wr\downarrow & & \downarrow\wr \\ \Gamma(M,E) & \xrightarrow{\nabla^A} & \Gamma(M,T^*M\otimes E) \end{array}$$

Here we use Proposition 2.1.24 for the vertical isomorphisms and on the right hand side additionally $\Omega^1(M,E) \cong \Gamma(M,T^*M\otimes E)$.

2.1.36 Remark. A section s in an associated bundle $E = P\times_G V$ is said to be *parallel* or *covariantly constant* with respect to A (or ∇^A) if $\nabla^A s = 0$. Usual examples are the tangent bundle TM, cotangent bundle T^*M, second symmetric power S^2T^*M of the cotangent bundle and the bundle of endomorphisms $\mathrm{End}\,(TM) = T^*M \otimes TM$ of TM. Examples of corresponding sections are a vector field $v \in \Gamma(M,TM)$, a 1-form $\alpha \in \Gamma(M,T^*M)$, a metric $g \in \Gamma(M,S^2T^*M)$ and a complex structure $I \in \mathrm{End}\,(TM)$.

2.1.37 Theorem ([Bau09, Satz 3.21, Aufgabe 3.7]). *Let $P \to M$ be a bundle of frames in TM. Let A be a connection 1-form on P. Then, in terms of covariant differentiation, the curvature and torsion tensors can be expressed as follows:*

$$R^A(v,w)u = \nabla^A_v \nabla^A_w u - \nabla^A_w \nabla^A_v u - \nabla^A_{[v,w]} u,$$
$$T^A(v,w) = \nabla^A_v w - \nabla^A_w v - [v,w],$$

where $u, v, w \in \Gamma(M, TM)$ are vector fields on M.

2.1.38 Proposition ([KMS93, §6.12]). *Let M be a manifold. There is the unique smooth map $\kappa_M \colon TTM \to TTM$ such that for all smooth $c \colon \mathbb{R}^2 \to M$:*

$$\tfrac{d}{dt}\tfrac{d}{ds} c(t,s)|_{s=0}|_{t=0} = \kappa_M \tfrac{d}{ds}\tfrac{d}{dt} c(t,s)|_{t=0}|_{s=0}.$$

This map $\kappa_M \colon TTM \to TTM$ is called the canonical flip *on M.*

2.1.39 Theorem ([KMS93, Thm 37.15]). *Let ∇ be a covariant derivative on TM with corresponding connector $\mathcal{K}$. Then the torsion tensor can be written as*

$$T^\nabla(v,w) = (\mathcal{K} \circ \kappa_M - \mathcal{K}) Tv \circ w \text{ for all } v, w \in \Gamma(M, TM).$$

In particular, a connection is torsion-free iff its connector satisfies $\mathcal{K} \circ \kappa_M = \mathcal{K}$.

2.1.40 Theorem ([KN96, Ch IV, Thm 2.2]). *Every Riemannian manifold M admits a unique covariant derivative ∇ on TM which is metric compatible (i.e. $\nabla g = 0$) and has vanishing torsion. This covariant derivative as well as the corresponding connection 1-form on the principal bundle of orthonomal frames are called* Levi-Civita connection*.*

2.1.5 Gauge group

2.1.41 Definition. The *gauge group* $\mathscr{G}(P)$ of a principal G-bundle $P \to M$ is the group of automorphism of P, i.e. G-equivariant diffeomorphisms $P \to P$:

$$\mathscr{G}(P) := \{\, \psi \colon P \to P \text{ diffeomorphism} \mid \psi(pg) = \psi(p)g \ \forall p \in P, g \in G \,\}.$$

Elements of the gauge group are called *gauge transformations.*

2.1.42 Note. Consider the G-action on itself by conjugation. A smooth G-equivariant map $g \colon P \to G$ induces a gauge transformation $\Psi \in \mathscr{G}(P)$, $\Psi(p) := pg(p)$. Conversely, for every gauge transformation $\Psi \in \mathscr{G}(P)$ there is a smooth G-equivariant map $g \colon P \to G$ such that $\Psi(p) = pg(p)$ for all $p \in P$. This implies that there is an isomorphism

$$\mathscr{G}(P) \cong C^\infty(P, G)^G.$$

In particular, we have an isomorphism $\mathrm{Lie}(\mathscr{G}(P)) \cong C^\infty(P, \mathfrak{g})^G$, where we consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. Using Proposition 2.1.22, we can also think of $\mathscr{G}(P) \cong C^\infty(P, G)^G$ as of section in the associated bundle $P \times_G G \to M$, where G acts on itself by conjugation. The Lie algebra $\mathrm{Lie}(\mathscr{G}(P))$ can also be identified with the sections in an associated bundle $\mathfrak{g}_P = P \times_G \mathfrak{g} \to M$ for the adjoint representation of G on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$:

$$\mathrm{Lie}(\mathscr{G}(P)) \cong \Gamma(M, \mathfrak{g}_P).$$

If G is abelian, then the bundle $\mathfrak{g}_P$ is a trivial bundle over M with fibre G. In this case the description as equivariant maps reduces to smooth maps from M to G and $\mathfrak{g}$, respectively:

$$\mathscr{G}(P) \cong C^\infty(M, G) \qquad \text{and} \qquad \mathrm{Lie}(\mathscr{G}(P) \cong C^\infty(M, \mathfrak{g}).$$

The group of connected components of the gauge group can be described in terms of principal bundles:

2.1.43 Lemma ([Don02, 2.5.2]). *Let $P \to M$ be a principal G-bundle. Then*

$$\pi_0(\mathscr{G}(P)) \cong \left\{ [Q] \in \check{H}^1(M \times S^1, G) \;\middle|\; Q|_{M \times \{0\}} \cong P \right\}.$$

Proof. First observe that $\pi_0(\mathscr{G}) = \mathscr{G}/\mathscr{G}_0$, where $\mathscr{G}_0$ is the connected component of the identity. For a gauge transformation $\psi \in \mathscr{G}(P)$ we construct a principal G-bundle on $M \times S^1$ as follows: Take the principal G-bundle $P \times [0,1]/\sim$ where $(p, 0) \sim (\psi(p), 1)$ for all $p \in P$. The map $\mathscr{G} \to \check{H}^1(M \times S^1, G)$ is invariant under the identity component $\mathscr{G}_0$ of the gauge group and induces the isomorphism. □

Action of the gauge group on connections

Let $P \to M$ be a principal G-bundle, $\psi \in \mathscr{G}(P)$ an element of the gauge group and $A \in \mathscr{A}(P)$ a connection 1-form. Pulling back the connection 1-form by the gauge transformation again produces a connection 1-form $\psi^* A \in \mathscr{A}(P)$ on P. We obtain a right action of the gauge group $\mathscr{G}(P)$ on the space of connections $\mathscr{A}(P)$.

2.1.44 Proposition ([Bau09, Satz 3.22]). *Let $A \in \mathscr{A}(P)$ be a connection 1-form, $\psi \in \mathscr{G}(P)$ a gauge transformation and $g : P \to G$ the corresponding G-equivariant map. Then*

$$\psi^* A = A^g, \text{ where } (A^g)_p := Ad_{g^{-1}(p)}(A_p) + (g^*\eta)_p \text{ for } p \in P.$$

Here $\eta \in \Omega^1(G, \mathfrak{g})^G$ denotes the left-invariant Maurer-Cartan form *on G, which is defined to be $\eta(v) := T_h L_{h^{-1}}(v) = h_*^{-1} v$ for $h \in G, v \in T_h G$. Furthermore, we have*

$$\psi^* F_A = F_{\psi^* A} = F_{A^g} = Ad_{g^{-1}}(F_A).$$

2.1.45 Lemma. *Given an G-equivariant smooth map $\xi \colon P \to \mathfrak{g}$ interpreted as an element of the Lie algebra $\mathrm{Lie}(\mathscr{G}(P))$ of the gauge group, the fundamental vector field for the action of the gauge group $\mathscr{G}(P)$ on the space of connections $\mathscr{A}(P)$ is*

$$(K_\xi^{\mathscr{A}(P), \mathscr{G}(P)})_A = d_A \xi \in \Omega^1(P, \mathfrak{g})^G_{hor} = T_A \mathscr{A}(P).$$

Proof. For $v \in T_pP$ we have:

$$\begin{aligned}\tfrac{d}{dt}\exp(t\xi)^*\eta(v)|_{t=0} &= \tfrac{d}{dt}T_{\exp(t\xi(p))}L_{\exp(-t\xi(p))}(T_p\exp(t\xi)(v))|_{t=0}\\ &= \tfrac{d}{dt}tT_{\exp(t\xi(p))}L_{\exp(-t\xi(p))}(T_{\xi(p)}\exp(T_p\xi(v)))|_{t=0}\\ &= \tfrac{d}{dt}tT_{\xi(p)}(\exp(-t\xi(p))\exp)(T_p\xi(v))|_{t=0}\\ &= T_p\xi(v).\end{aligned}$$

The last equality holds since

$$T(\exp(-t\xi(p))\exp) = \int_0^1 e^{-sad(t\xi(p))}ds = 1 - \tfrac{t}{2}ad(\xi(p)) + O(t^2).$$

For a proof of this formula, we refer the reader to [DK00, Thm 1.5.3]. Furthermore,

$$\tfrac{d}{dt}Ad_{\exp(t\xi)}|_{t=0} = T_1Ad(\tfrac{d}{dt}\exp(t\xi)|_{t=0}) = T_1Ad(\xi) = ad_\xi.$$

Finally, we conclude

$$\tfrac{d}{dt}Ad_{\exp(-t\xi)}(A) + \exp(t\xi)^*\eta|_{t=0} = ad_{-\xi}(A) + d\xi = d\xi + [A,\xi] = d_A\xi. \qquad \square$$

2.2 Hyperkähler manifolds

2.2.1 Definition (Kähler manifold). An *almost complex structure* on a manifold M is an endomorphism $I \in \mathrm{End}\,(TM)$ satisfying $I^2 = -\operatorname{id}_{TM}$. A *Kähler manifold* is a Riemannian manifold (M, g^M) with a parallel (with respect to the Levi-Civita connection) orthogonal almost complex structure $I \in \mathrm{End}\,(TM)$ such that the 2-form $\omega \in \Omega^2(M)$ is closed, where $\omega(v,w) = g^M(v, I(w))$ for all $v, w \in T_xM$. The symplectic form ω is called *Kähler form.*

2.2.2 Definition (hyperkähler manifold). A *hyperkähler manifold* is a Riemannian manifold (M, g^M) with three parallel (with respect to the Levi-Civita connection) orthogonal almost complex structures $I_1, I_2, I_3 \in \mathrm{End}\,(TM)$ such that $I_1I_2I_3 = -\operatorname{id}_{TM}$ and M is a Kähler manifold with respect to each of the three complex structures.

2.2.3 Remark ([Hit87, Lemma 6.8]). It is enough to require the existence of two anti-commuting orthogonal almost complex structures $I_1, I_2 \in \mathrm{End}\,(TM)$ (define $I_3 := I_1I_2$) such that the three 2-forms are closed: $d\omega_1 = d\omega_2 = d\omega_3 = 0$, where $\omega_\ell(v,w) = g^M(v, I_\ell(w))$ for all $v, w \in T_xM$ and $\ell \in \{1,2,3\}$.

2.2.4 Remark (dimensions and holonomy groups). The existence of the complex structure on a Kähler manifold M implies that the dimension of M is even. The existence of the three complex structures on a hyperkähler manifold M implies that the dimension of M is a multiple of 4. We also allow $\dim(M) = 0$. In this case, the identity is the only endomorphism of TM. However, it is a complex structure and we can take $I_1 = I_2 = I_3 = \operatorname{id}_{TM}$.

The holonomy group of a $2n$-dimensional Kähler manifold is contained in $U(n) \subset SO(2n)$. Conversely, every $2n$-dimensional manifold with holonomy group contained in $U(n) \subset SO(2n)$ is a Kähler manifold.

Let $\mathbb{H}$ be the skew field of *quaternions*. As a vector space we identify $\mathbb{H} \cong \mathbb{R}^4$. The holonomy group of a $4n$-dimensional hyperkähler manifold M is contained in $Sp(n) \subset SO(4n)$, where $Sp(n)$ is the group of $\mathbb{H}$-linear metric perserving automorphisms of $\mathbb{H}^n$. Conversely, every $4n$-dimensional manifold with holonomy group contained in $Sp(n) \subset SO(4n)$ is a hyperkähler manifold.

The group $Sp(1)$ can be identified with the sphere S^3 in the quaternions. We have an isomorphism $\mathbb{H} \supset S^3 \to Sp(1)$, $q \mapsto R_{\bar{q}}, R_{\bar{q}}(h) := h\bar{q}$ for $h \in \mathbb{H}$. We will from now on use this isomorphism to identify $Sp(1)$ with the sphere in the quaternions and its Lie algebra $\mathfrak{sp}(1)$ with the space of imaginary quaternions $\mathrm{Im}(\mathbb{H}) := \left\{ h \in \mathbb{H} \,\middle|\, \bar{h} = -h \right\}$.

2.2.5 Note (scalar multiplication). The tangent bundle of a hyperkähler manifold M is a bundle of $\mathbb{H}$-modules, i.e. we have a ring homomorphism called *scalar multiplication*

$$\begin{aligned} \mathcal{I}\colon \mathbb{H} &\to \mathrm{End}\,(TM)\,, \\ h &\mapsto \mathcal{I}_h, \end{aligned}$$

where $\mathcal{I}_h := h_0 \,\mathrm{id}_{TM} + h_1 I_1 + h_2 I_2 + h_3 I_3$ for $h = h_0 + h_1 i + h_2 j + h_3 k$. In particular, for all $\zeta \in \mathrm{Im}(\mathbb{H})$ with $\|\zeta\|^2 = 1$ we have

$$\mathcal{I}_\zeta^2 = \mathcal{I}_{\zeta^2} = -\mathcal{I}_{\zeta\bar{\zeta}} = -\mathcal{I}_1 = -\,\mathrm{id}_M$$

This implies that $\mathcal{I}$ maps the sphere $S^2 \subset \mathrm{Im}(\mathbb{H}) \subset \mathbb{H}$ into the space of complex structures on M. If $\dim(M) > 0$, then $\mathcal{I}$ is injective and we have a sphere of complex structures $\left\{ \sum_{\ell=1}^{3} \zeta_\ell I_\ell \,\middle|\, \sum_{\ell=1}^{3} \zeta_\ell^2 = 1 \right\}$.

We define a 2-form $\omega \in \Omega^2(M, \mathfrak{sp}(1)^*)$ as follows:

$$\langle \omega, \zeta \rangle := \omega_\zeta \text{ for all } \zeta \in \mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H}),$$

where $\omega_\zeta(v, w) = g^M(v, \mathcal{I}_\zeta w)$ for all $x \in M$ and $v, w \in T_x M$. If $\zeta \in \mathrm{Im}(\mathbb{H}) = \mathfrak{sp}(1)$ is of norm one, $\|\zeta\|^2 = 1$, then $\mathcal{I}_\zeta$ is an (almost) complex structure and ω_ζ the corresponding symplectic form.

2.2.6 Example. Consider $M = \mathbb{H}$. The tangent bundle is trivial, $T\mathbb{H} = \mathbb{H} \times \mathbb{H} \xrightarrow{\mathrm{pr}_1} \mathbb{H} = M$. For $(h, v) \in T\mathbb{H} = \mathbb{H} \times \mathbb{H}$ let

$$I_1((h, v)) := (h, iv) \qquad I_2((h, v)) := (h, jv) \qquad I_3((h, v)) := (h, kv).$$

This defines three complex structures $I_1, I_2, I_3 \in \mathrm{End}\,(T\mathbb{H})$ which are compatible with the standard metric $g_h^M(v, v') = \mathrm{Re}(v\bar{v}')$ for all $v, v' \in \mathbb{H} = T_h\mathbb{H}$. The scalar multiplication is given by $\mathcal{I}_{h'}((h, v)) = (h, h'v)$ for all $h' \in \mathbb{H}, (h, v) \in T\mathbb{H}$.

The three symplectic forms $\omega_\ell = g^M(\cdot, I_\ell(\cdot))$ for $\ell \in \{1,2,3\}$ are

$$\begin{aligned}\omega_1 &= -dh_0 \wedge dh_1 - dh_2 \wedge dh_3,\\ \omega_2 &= dh_1 \wedge dh_3 - dh_0 \wedge dh_2,\\ \omega_3 &= -dh_0 \wedge dh_3 - dh_1 \wedge dh_2,\end{aligned}$$

where $h = h_0 + ih_1 + jh_2 + kh_3$. Note that $i\omega_1 + j\omega_2 + k\omega_3 = \frac{1}{2} dh \wedge d\bar{h}$.

2.2.1 Group actions and moment maps

Consider the coadjoint representation $\mathfrak{g}^* = \mathrm{Lie}(G)^*$ of a Lie group G.

2.2.7 Definition (moment map). A smooth action of a Lie group G on a symplectic manifold (M, ω) is said to be a *symplectic action* if it fixes the symplectic form ω (i.e. $L_h^*\omega = \omega$ for all $h \in G$). A smooth map $\mu\colon M \to \mathfrak{g}^*$ is said to be a *moment map* for the symplectic G-action on M if

1. $d\mu = \iota_{\mathfrak{g}}\omega$ (moment map condition),
2. $\mu(gx) = Ad_g^*(\mu(x))$ for all $g \in G, x \in M$ (equivariance).

2.2.8 Proposition (existence/uniqueness of moment maps, [CdS01]).

1. *Let G be a compact connected Lie group. If a moment map $\mu\colon M \to \mathfrak{g}^*$ for a symplectic G-action on a symplectic manifold M exists, then the set of moment maps is a $[\mathfrak{g},\mathfrak{g}]^0$-torsor, where $[\mathfrak{g},\mathfrak{g}]^0$ is the annihilator of the commutator ideal $[\mathfrak{g},\mathfrak{g}]$ in $\mathfrak{g}^*$. In particular, if G is abelian, then the set of moment maps is $\{\, \mu + \nu\colon M \to \mathfrak{g}^* \mid \nu \in \mathfrak{g}^* \,\}$.*
2. *If G is a compact semisimple Lie group, then for any symplectic G-action there is a unique moment map.*

2.2.9 Definition (hyperkähler action). A smooth action of a Lie group G on a hyperkähler manifold (M, g^M, I_1, I_2, I_3) is said to be a *hyperkähler action*, if

1. G acts isometrically, i.e. for all $h \in G : L_h^* g^M = g^M$,
2. G respects the three complex structures, i.e. for all $h \in G : h_*I_1 = I_1h_*, h_*I_2 = I_2h_*$ and $h_*I_3 = I_3h_*$.

The definition of a moment map for a hyperkähler action is analoguous to the definition for symplectic actions, but now we have to take care of three symplectic structures.

2.2.10 Definition. Let (M, g^M, I_1, I_2, I_3) be a hyperkähler manifold with a hyperkähler action of a Lie group G. Consider the form $\omega \in \Omega^2(M, \mathfrak{sp}(1)^*)$. A smooth map $\mu\colon M \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$ is said to be a *hyperkähler moment map* for the G-action on M if

1. $d\mu = \iota_{\mathfrak{g}}\omega$ (moment map condition),

2. $\mu(gx) = Ad_g^*(\mu(x))$ for all $g \in G, x \in M$ (equivariance).

2.2.11 Remark. If $\mu\colon M \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$ is a hyperkähler moment map, then $d\langle \mu, \zeta\rangle = \iota_{\mathfrak{g}}\omega_\zeta$, and therefore $\langle \mu, \zeta\rangle\colon M \to \mathfrak{g}$ is a moment map for ω_ζ. In particular, let

$$\mu_1 := \langle \mu, i\rangle, \qquad \mu_2 := \langle \mu, j\rangle, \qquad \mu_3 := \langle \mu, k\rangle.$$

Then $\mu\colon M \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$ is a hyperkähler moment map iff μ_1, μ_2, μ_3 are moment maps for $\omega_1, \omega_2, \omega_3$, respectively.

2.2.12 Example. Consider the hyperkähler manifold $\mathbb{H}$ from Example 2.2.6 and for fixed $\ell \in \mathbb{Z}$ the action $S^1 \curvearrowright \mathbb{H},\ (z, h) \mapsto hz^\ell$. The fundamental vector field for this action is

$$(K_\xi^{\mathbb{H},S^1})_h = \tfrac{d}{dt} h \exp(t\ell\xi)|_{t=0} = \ell h\xi \in \mathbb{H} = T_h\mathbb{H}.$$

Consider the map $\tilde{\mu}\colon \mathbb{H} \to \mathrm{Im}(\mathbb{H}),\ \tilde{\mu}(h) = \frac{\ell}{2} hi\bar{h}$ and let $\mu_1, \mu_2, \mu_3 \in C^\infty(M, \mathbb{R})^{S^1}$ be defined by $\tilde{\mu} = i\mu_1 + j\mu_2 + k\mu_3$. Then

$$\begin{aligned} id\mu_1 + jd\mu_2 + kd\mu_3 = d\tilde{\mu} = (\tfrac{\ell}{2} dhi\bar{h} + \tfrac{\ell}{2} hid\bar{h}) &= \tfrac{1}{2}\iota_{\ell hi} dh \wedge d\bar{h} \\ &= \iota_{K_i^{\mathbb{H},S^1}}(i\omega_1 + j\omega_2 + k\omega_3). \end{aligned}$$

If we use the Ad-invariant scalar product $\langle \cdot, \cdot\rangle\colon i\mathbb{R} \otimes i\mathbb{R} \to \mathbb{R}$ with $\langle i, i\rangle = 1$, and the standard metric on $\mathrm{Im}(\mathbb{H})$ to identify $\mathfrak{sp}(1)^* = \mathrm{Im}(\mathbb{H})^* \cong \mathrm{Im}(\mathbb{H}) = \mathfrak{sp}(1)$, then $\mu := i \otimes \tilde{\mu}$ is a hyperkähler moment map.

2.2.2 Hyperkähler potential

2.2.13 Definition (Kähler potential). Let (M, g^M, I) be a Kähler manifold with Kähler form ω. For a 1-form $\alpha \in \Omega^1(M)$ define $I\alpha \in \Omega^1(M)$ by $I\alpha(v) := \alpha(I(v))$ for all $v \in TM$. A smooth function $\rho\colon M \to \mathbb{R}$ is said to be a *Kähler potential* if $dId\rho = 2\omega$.

2.2.14 Remark. In terms of complex valued differential forms and Dolbeault operators, we have

$$i\partial\bar{\partial}\rho = i(\partial + \bar{\partial})\bar{\partial}\rho = id\bar{\partial}\rho = id\tfrac{1}{2}(1 + iI)d\rho = -\tfrac{1}{2}dId\rho$$

for all $\rho \in C^\infty(M, \mathbb{R})$. Therefore, a smooth function ρ is a Kähler potential iff $-i\partial\bar{\partial}\rho = \omega$.

2.2.15 Definition (hyperkähler potential). A smooth map $\rho\colon M \to \mathbb{R}$ on a hyperkähler manifold (M, g^M, I_1, I_2, I_3) is said to be a *hyperkähler potential* if ρ is a Kähler potential for each of the three complex structures:

$$dI_\ell d\rho = 2\omega_\ell \text{ for all } \ell \in \{1, 2, 3\}.$$

2.2.16 Example. Consider the hyperkähler manifold $M = \mathbb{H}$ (cf. Example 2.2.6) and the function $\rho : \mathbb{H} \to \mathbb{R}$, $\rho(h) = \frac{1}{2}\|h\|^2$. We have

$$d\rho = \sum_{\ell=0}^{3} h_\ell dh_\ell.$$

For a complex structure I_ζ we have

$$d\mathcal{I}_\zeta d\rho = \sum_{\ell=0}^{3} dh_\ell \mathcal{I}_\zeta dh_\ell.$$

More explicitly,

$$\begin{aligned}
I_1 dh_0 &= -dh_1, & I_1 dh_1 &= dh_0, & I_1 dh_2 &= -dh_3, & I_1 dh_3 &= dh_2,\\
I_2 dh_0 &= -dh_2, & I_2 dh_1 &= dh_3, & I_2 dh_2 &= dh_0, & I_2 dh_3 &= -dh_1,\\
I_3 dh_0 &= -dh_3, & I_3 dh_1 &= -dh_2, & I_3 dh_2 &= dh_1, & I_3 dh_3 &= dh_0,
\end{aligned}$$

and therefore

$$\begin{aligned}
dI_1 d\rho &= \sum_{\ell=0}^{3} dh_\ell I_1 dh_\ell = -dh_0 \wedge dh_1 + dh_1 \wedge dh_0 - dh_2 \wedge dh_3 + dh_3 \wedge dh_2 = 2\omega_1,\\
dI_2 d\rho &= \sum_{\ell=0}^{3} dh_\ell I_2 dh_\ell = -dh_0 \wedge dh_2 + dh_1 \wedge dh_3 + dh_2 \wedge dh_0 - dh_3 \wedge dh_1 = 2\omega_2,\\
dI_3 d\rho &= \sum_{\ell=0}^{3} dh_\ell I_3 dh_\ell = -dh_0 \wedge dh_3 - dh_1 \wedge dh_2 + dh_2 \wedge dh_1 + dh_3 \wedge dh_0 = 2\omega_3.
\end{aligned}$$

This implies that $\rho\colon \mathbb{H} \to \mathbb{R}, \rho(h) = \frac{1}{2}\|h\|^2$ is a hyperkähler potential.

2.3 Clifford algebras and Spin groups

2.3.1 The Clifford algebra

2.3.1 Definition. Let V be a vector space (over $\mathbb{R}$) equipped with a quadratic form q. The *Clifford algebra* $Cl(V,q)$ is the quotient of the tensor algebra $\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$ by the ideal $\mathcal{I}(V,q)$ which is generated by elements of the form $v \otimes v + q(v)$ for $v \in V$:

$$Cl(V,q) := \mathcal{T}(V)/\mathcal{I}(V,q).$$

The equivalence class of an element $v_1 \otimes v_2 \otimes \cdots \otimes v_k \in \mathcal{T}(V)$ is denoted by $v_1 v_2 \cdots v_k$. The Clifford algebra $Cl(V,q)$ has the universal property that every linear map $f\colon V \to A$ into an associative algebra A (over $\mathbb{R}$) with unit satisfying $f(v)^2 + q(v) = 0$ for all $v \in V$ extends uniquely to a homomorphism $Cl(V,q) \to A$.

Let $\alpha\colon Cl(V,q) \to Cl(V,q)$ be the automorphism which extends $-\operatorname{id}_V\colon V \to V$. The even part $Cl^0(V,q)$ and the odd part $Cl^1(V,q)$ are defined as

$$Cl^\ell(V,q) := \ker(\alpha - (-1)^\ell \operatorname{id}) \text{ for } \ell \in \{0,1\}.$$

This defines a $\mathbb{Z}/2\mathbb{Z}$-grading on the Clifford algebra. Let now $Cl^\times(V,q)$ be the multiplicative group of units in the Clifford algebra $Cl(V,q)$ and define the group $Pin(V,q)$ to be the subgroup of $Cl^\times(V,q)$ generated by elements $v \in V$ with $q(v) = \pm 1$. The *Spin group* for V and q is

$$Spin(V,q) := Pin(V,q) \cap Cl^0(V,q).$$

If $V = \mathbb{R}^m$ is the m-dimensional Euclidean space and $q(v) = \|v\|^2$, then we denote the corresponding Clifford algebra by $Cl_m := Cl(\mathbb{R}^n, \|\cdot\|^2)$. The corresponding *Spin group* is denoted by $Spin(m) := Spin(\mathbb{R}^m, \|\cdot\|^2)$ Furthermore, we define

$$Spin^c(m) := (Spin(m) \times S^1)/\{(\pm 1, \pm 1)\}.$$

2.3.2 Note. The map

$$\begin{aligned} \Lambda^\bullet \mathbb{R}^m &\to Cl_m, \\ v_1 \wedge \cdots \wedge v_k &\mapsto v_1 \cdots v_k \end{aligned}$$

is an isomorphism of vector spaces. Restricting to $\Lambda^2\mathbb{R}^m$, we obtain an isomorphism $\operatorname{Lie}(Spin(m)) \cong \Lambda^2\mathbb{R}^m$.

2.3.3 Definition (volume element). The *volume element* of the Clifford algebra Cl_m is $vol_m := e_1 \cdots e_m$. The *complex volume element* is $vol_m^{\mathbb{C}} := i^{\lfloor \frac{m+1}{2} \rfloor} e_1 \cdots e_m \in Cl_m \otimes \mathbb{C}$.

2.3.4 Examples (Clifford algebras).

1. $Cl_1 \cong \mathbb{C}$, where $1 \mapsto 1, e_1 \mapsto i$.

2. $Cl_2 \cong \mathbb{H}$, where $1 \mapsto 1, e_1 \mapsto i, e_2 \mapsto j$.

3. $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$, where $1 \mapsto (1,1), e_1 \mapsto (-i,i), e_2 \mapsto (-j,j), e_3 \mapsto (-k,k)$.

4. $Cl_4 \cong M_2(\mathbb{H})$, where

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_0 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$e_1 \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad e_2 \mapsto \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}, \qquad e_3 \mapsto \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}.$$

The images of the volume elements $vol_3 = e_1e_2e_3$ and $vol_4 = e_0e_1e_2e_3$ under the isomorphisms above are $(1,-1)$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. For $m \in \{3,4\}$ we have $vol_m^{\mathbb{C}} = -vol_m$.

2.3.5 Note. The decomposition of Cl_3 in the previous example as a sum of two copies of the quaternions $\mathbb{H}$ corresponds to the decomposition $Cl_3 = Cl_3^+ \oplus Cl_3^-$: In Cl_3, the volume element $vol_3 = e_1e_2e_3$ is central and $(vol_3)^2 = 1$. Define two projections $\pi^+ := \frac{1}{2}(1 + vol_3)$ and $\pi^- := \frac{1}{2}(1 - vol_3)$, and $Cl_3^+ := \pi^+ Cl_3$, $Cl_3^- := \pi^- Cl_3$. In terms of quaternions, we have $\pi^+ = (1,0)$ and $\pi^- = (0,1)$, so $Cl_3^+ \cong \mathbb{H} \oplus \{0\} \subset \mathbb{H} \oplus \mathbb{H} \cong Cl_3$ and $Cl_3^- = \{0\} \oplus \mathbb{H} \subset \mathbb{H} \oplus \mathbb{H} \cong Cl_3$. The decomposition $Cl_3 = Cl_3^0 \oplus Cl_3^1$ into even and odd elements is given in terms of quaternions as $Cl_3^0 \cong \{\, (h,h) \in \mathbb{H} \oplus \mathbb{H} \mid h \in \mathbb{H} \,\} \subset \mathbb{H} \oplus \mathbb{H} \cong Cl_3$ and $Cl_3^1 \cong \{\, (h,-h) \in \mathbb{H} \oplus \mathbb{H} \mid h \in \mathbb{H} \,\} \subset \mathbb{H} \oplus \mathbb{H} \cong Cl_3$. The automorphism $\alpha \colon Cl_3 \to Cl_3$ is given in this picture by $\mathbb{H} \oplus \mathbb{H} \to \mathbb{H} \oplus \mathbb{H}, (h,h') \mapsto (h',h)$.

2.3.6 Proposition ([LM89, Ch I Thm 3.7]). *The map $\mathbb{R}^m \to Cl_{m+1}^0, v \mapsto ve_{m+1}$ induces an isomorphism*

$$Cl_m \xrightarrow{\sim} Cl_{m+1}^0$$

2.3.7 Remark. We will mostly be interested in the case $m = 3$, where we use the convention that $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathbb{R}^4 = \operatorname{span}\{e_0, e_1, e_2, e_3\}$. In this case we use the map $\mathbb{R}^3 \ni v \mapsto ve_0 \in Cl_4^0$. Note that $vol_3 \mapsto -vol_4$ and $vol_3^{\mathbb{C}} \mapsto -vol_4^{\mathbb{C}}$. If we use the isomorphisms from Examples 2.3.4, the composition $\mathbb{H} \oplus \mathbb{H} \cong Cl_3 \hookrightarrow Cl_4 \cong M_2(\mathbb{H})$ reads

$$\mathbb{H} \oplus \mathbb{H} \ni (h,h') \mapsto \begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix} \in M_2(\mathbb{H}).$$

2.3.8 Note. For $m \geq 3$ the Spin group $Spin(m)$ is the universal covering of $SO(m)$. In particular, we have a short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Spin(m) \xrightarrow{\lambda} SO(m) \to 1,$$

where λ of is the restriction of

$$\begin{aligned} Ad \colon Cl_m^\times &\to \operatorname{Aut}(Cl_m) \\ \varphi &\mapsto Ad_\varphi, \ Ad_\varphi(y) := \alpha(\varphi) y \varphi^{-1} \end{aligned}$$

to $Spin(m) \subset Cl_m^\times$ and $\mathbb{R}^m \subset Cl_m$. The differential $T_1\lambda \colon \mathfrak{spin}(m) \to \mathfrak{so}(m)$ is an isomorphism of Lie algebras. Here $\mathfrak{so}(n) = \operatorname{Lie}(SO(n))$ and $\mathfrak{spin}(n) = \operatorname{Lie}(Spin(n))$ are the Lie algebras of $SO(n)$ and $Spin(n)$, respectively.

This map is compatible with the embedding $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m+1}$, i.e. we have a commuting diagram

$$\begin{array}{ccc} Spin(m) & \hookrightarrow & Spin(m+1) \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ SO(m) & \hookrightarrow & SO(m+1) \end{array}$$

Similarly, we have a short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Spin^c(m) \xrightarrow{\lambda^c} SO(m) \times S^1 \to 1,$$

where $\lambda^c([(\varphi, z)]) = (\lambda(\varphi), z^2)$ for $[(\varphi, z)] \in Spin^c(m)$ and the $\mathbb{Z}/2\mathbb{Z}$ is the subgroup of $Spin^c(m)$ generated by $[(1,-1)] = [(-1,1)]$. Here $[(\varphi, z)] \in Spin^c(m)$ denotes the image of $(\varphi, z) \in Spin(m) \times S^1$ under the projection $Spin(m) \times S^1 \to (Spin(m) \times S^1)/\pm 1 = Spin^c(m)$.

2.3.9 Example. We will also use the quaternions to construct the universal covering of $SO(3)$. Identify $\mathbb{R}^3$ with the imaginary quaternions $\mathrm{Im}(\mathbb{H})$ and consider the homomorphism

$$Sp(1) \to SO(3),$$

which is mapping $\mathbb{H}^\times \supset Sp(1) \ni q \mapsto c_q \in SO(3)$, where $c_q(v) := qvq^{-1}$ for $v \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. Since $Sp(1)$ is simply connected and the kernel of this map is $\{\pm 1\} \subset Sp(1)$, we obtain an isomorphism

$$Sp(1) \cong Spin(3)$$

from the universal property of $Spin(3)$. The induced isomorphism on the level of Lie algebras is

$$\begin{aligned} \mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H}) &\to \Lambda^2\mathbb{R}^3, \\ i &\mapsto e_2 \wedge e_3, \\ j &\mapsto -e_1 \wedge e_3, \\ k &\mapsto e_1 \wedge e_2. \end{aligned}$$

This is also an isomorphism of $Spin(3)$-representations. If we again identify $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$, this isomorphism is given by the *Hodge star operator* $*\colon \mathbb{R}^3 \xrightarrow{\sim} \Lambda^2\mathbb{R}^3$.

2.3.10 Remark. Consider the diagonal embedding $\mathbb{H} \hookrightarrow \mathbb{H} \oplus \mathbb{H}$, $h \mapsto (h, h)$. Using the isomorphism from Examples 2.3.4, the group $Sp(1) \cong Spin(3)$ can be interpreted as the unit sphere in $Cl_3^0 \cong \mathbb{H} \hookrightarrow \mathbb{H} \oplus \mathbb{H} \cong Cl_3$. Its Lie algebra is $\mathfrak{sp}(1) \cong \mathrm{Im}(\mathbb{H}) \subset \mathbb{H} \cong Cl_3^0 \subset Cl_3$.

2.3.11 Example. There is a similar construction for $Spin(4)$. Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and consider the homomorphism

$$Sp(1) \times Sp(1) \to SO(4),$$

which is mapping $\mathbb{H}^\times \times \mathbb{H}^\times \supset Sp(1) \times Sp(1) \ni (q_+, q_-) \mapsto c_{q_+,q_-} \in SO(4)$, where $c_{q_+,q_-}(v) := q_+ v q_-^{-1}$. Again, notice that $Sp(1) \times Sp(1)$ is simply connected and the kernel of this map is $\{(\pm 1, \pm 1)\}$, so we obtain an isomorphism

$$Spin(4) \cong Sp(1) \times Sp(1).$$

To distinguish the two copies of $Sp(1)$, we will denote the first one by $Sp(1)_+$ and the second one by $Sp(1)_-$. The induced isomorphism of Lie algebras is given by

$$\begin{aligned} \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- &\xrightarrow{\sim} \Lambda^2\mathbb{R}^4, \\ (i, 0) &\mapsto \tfrac{1}{2}(e_0 \wedge e_1 + e_2 \wedge e_3), \\ (j, 0) &\mapsto \tfrac{1}{2}(e_0 \wedge e_2 - e_1 \wedge e_3), \\ (k, 0) &\mapsto \tfrac{1}{2}(e_0 \wedge e_3 + e_1 \wedge e_2), \\ (0, i) &\mapsto \tfrac{1}{2}(e_2 \wedge e_3 - e_0 \wedge e_1), \\ (0, j) &\mapsto -\tfrac{1}{2}(e_0 \wedge e_2 + e_1 \wedge e_3), \\ (0, k) &\mapsto \tfrac{1}{2}(e_1 \wedge e_2 - e_0 \wedge e_3). \end{aligned}$$

This is also an isomorphism of $Spin(4)$-representations. The Hodge star operator $*\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$ induces a direct sum decomposition $\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4$ of the $Spin(4)$-representation $\Lambda^2\mathbb{R}^4$, where

$$\Lambda^2_\pm\mathbb{R}^4 := \ker(\mathrm{id} \mp *\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4).$$

Note that the isomorphism $\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- \xrightarrow{\sim} \Lambda^2\mathbb{R}^4$ maps $\mathfrak{sp}(1)_+$ isomorphically to $\Lambda^2_+\mathbb{R}^4$ and $\mathfrak{sp}(1)_-$ isomorphically to $\Lambda^2_-\mathbb{R}^4$. In particular, we obtain isomorphisms of $Spin(4)$-representations

$$\mathfrak{sp}(1)_+ \cong \Lambda^2_+\mathbb{R}^4 \qquad \text{and} \qquad \mathfrak{sp}(1)_- \cong \Lambda^2_-\mathbb{R}^4.$$

2.3.12 Remark. The product structure of $Spin(4) = Sp(1)_+ \times Sp(1)_-$ is induced by the decomposition $Cl_4^0 \cong Cl_3 = Cl_3^+ \oplus Cl_3^-$. The two components $Sp(1)_+$ and $Sp(1)_-$ are the images of $Spin(4) \subset Cl_4^0 \cong Cl_3 = Cl_3^+ \oplus Cl_3^- \xrightarrow{\mathrm{pr}_\pm} Cl_3^\pm$. Using the isomorphism $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$ from Note 2.3.5, we can interpret $Sp(1)_+$ and $Sp(1)_-$ as the unit spheres in $Cl_3^+ \cong \mathbb{H} \oplus \{0\} \subset \mathbb{H} \oplus \mathbb{H} \cong Cl_3$ and $Cl_3^- \cong \{0\} \oplus \mathbb{H} \subset \mathbb{H} \oplus \mathbb{H} \cong Cl_3$ and the corresponding Lie algebras as $\mathfrak{sp}(1)_+ \cong \mathrm{Im}(\mathbb{H}) \subset \mathbb{H} \cong Cl_3^+$ and $\mathfrak{sp}(1)_- \cong \mathrm{Im}(\mathbb{H}) \subset \mathbb{H} \cong Cl_3^-$.

2.3.13 Example. Using the isomorphisms from Example 2.3.9 and Example 2.3.11, we have a commuting diagram

$$\begin{array}{ccc} Spin(3) & \hookrightarrow & Spin(4) \\ \downarrow\wr & & \downarrow\wr \\ Sp(1) & \hookrightarrow & Sp(1)_+ \times Sp(1)_- \end{array}$$

where the map at the bottom is the diagonal $Sp(1) \ni q \mapsto (q,q) \in Sp(1)_+ \times Sp(1)_-$.

2.3.14 Note. Notice that the composition

$$Sp(1) \cong Spin(3) \hookrightarrow Spin(4) \cong Sp(1)_+ \times Sp(1)_- \xrightarrow{\mathrm{pr}_\pm} Sp(1)_\pm$$

is the identity. On the level of Lie algebras, the composition

$$\mathbb{R}^3 \xrightarrow{*_3} \Lambda^2\mathbb{R}^3 \hookrightarrow \Lambda^2\mathbb{R}^4 \xrightarrow{(\cdot)_+} \Lambda^2_+\mathbb{R}^4,$$

is an isomorphism mapping

$$\mathbb{R}^3 \ni v \mapsto (e_0 \wedge v)_+ := \tfrac{1}{2}(1 + *_4)(e_0 \wedge v) = \tfrac{1}{2}(e_0 \wedge v + *_4 e_0 \wedge v) \in \Lambda^2_+\mathbb{R}^4.$$

This is an isomorphism of $SO(3)$-representations. Dually, we also have an isomorphism $\tau_0\colon (\mathbb{R}^3)^* \xrightarrow{\sim} \Lambda^2_+(\mathbb{R}^4)^*$ of $SO(3)$-representations. Here we used $*_3$ and $*_4$ for the Hodge star operators in dimension three and four, respectively.

2.3.2 Representations of the Clifford algebras and Spin groups

We will now collect some representations of Cl_m, $Spin(m)$ and $Spin^c(m)$ for $m \in \{3,4\}$. To write these in terms of quaternions, we will use the isomorphisms $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$ and $Cl_4 \cong M_2(\mathbb{H})$ from Examples 2.3.4 and also $Spin(3) \cong Sp(1)$ and $Spin(4) \cong Sp(1)_+ \times Sp(1)_-$ from Example 2.3.9 and Example 2.3.11. Furthermore, we also use $Spin^c(3) \cong (Sp(1) \times S^1)/\pm 1$ and $Spin^c(4) \cong (Sp(1)_+ \times Sp(1)_- \times S^1)/\pm 1$.

Representations of Cl_3

Consider the two irreducible Cl_3-representations

$$Cl_3 = Cl_3^+ \oplus Cl_3^- \xrightarrow{\pi_\pm} Cl_3^\pm \cong \mathbb{H} \to \mathrm{Aut}\,(\mathbb{H})\,,$$
$$Cl_3 \cong \mathbb{H} \oplus \mathbb{H} \ni (h_+, h_-) \longmapsto h_\pm \longmapsto (v \mapsto h_\pm v \text{ for } v \in \mathbb{H})\,.$$

Here $\mathbb{H}$ acts on itself by left multiplication. Since the decomposition of Cl_3 as a direct sum of two copies of the quaternions is the decomposition into the eigenspaces Cl_3^+ and Cl_3^- of the volume element vol_3, these two representation can be distinguished by the action of the volume element. The restrictions of these two representations to the even part $Cl_3^0 = \{\, (h,h) \in \mathbb{H} \oplus \mathbb{H} \mid h \in \mathbb{H}\,\}$ are isomorphic. Restricting further to $Spin(3) \subset Cl_3^0 \subset Cl_3$, we obtain the *spinor representation* S. We will only use the Cl_3-representation in which the volume element acts as the identity, which we also denote by S. This is the one induced by the projection to the first component $Cl_3 \cong \mathbb{H} \oplus \mathbb{H} \xrightarrow{\mathrm{pr}_1} \mathbb{H}$.

Here is a list of useful representations of $Spin(3)$ and $Spin^c(3)$:

Representations of $Spin(3)$ and $Spin^c(3)$

name	vector space	homomorphism	
$\mathbb{R}^3$	$\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$	$Sp(1) \to SO(3)$	$q \cdot v = qv\bar{q}$ for $v \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$
S	$\mathbb{H}$	$Sp(1) \to \mathrm{Aut}\,(\mathbb{H})$	$q \cdot h = qh$ for $v \in \mathbb{H} = S$
W	$\mathbb{H}$	$Spin^c(3) \to \mathrm{Aut}\,(\mathbb{H})$	$[(q,z)] \cdot h = \bar{q}hz$ for $v \in \mathbb{H}$

Here $q \in Sp(1)$, $z \in S^1$ and $[(q,z)] \in (Sp(1) \times S^1)/\pm 1 \cong Spin^c(3)$.

Representation of Cl_4

Consider the tautological irreducible representation of $Cl_4 \cong M_2(\mathbb{H})$ on $\mathbb{H}^2$. Restricting this representation to

$$Cl_4^0 \cong \left\{ \begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix} \,\middle|\, h, h' \in \mathbb{H} \right\} \cong \mathbb{H} \oplus \mathbb{H} \cong Cl_3,$$

we obtain a direct sum of the two irreducible representations of $Cl_3 \cong Cl_4^0$. As representations $Spin(4) \subset Cl_4^0$, these are the *spinor representations* which are denoted by S^+ and S^-. Note that this notation comes from the direct sum decomposition $Cl_3 = Cl_3^0 \oplus Cl_3^1$ and not from Cl_4. The element vol_4 acts as $-\,\mathrm{id}_{S^+}$ on S^+ and as id_{S^-} on S^-.

Here is a list of useful representations of $Spin(4)$ and $Spin^c(4)$:

Representations of $Spin(4)$ and $Spin^c(4)$

name	vector space	homomorphism	
$\mathbb{R}^4$	$\mathbb{R}^4 \cong \mathbb{H}$	$Spin(4) \to SO(4)$	$(q_+, q_-)\cdot h = q_+ h \bar{q}_-$ for $h \in \mathbb{H} \cong \mathbb{R}^4$
S^+	$\mathbb{H}$	$Spin(4) \to \mathrm{Aut}\,(\mathbb{H})$	$(q_+, q_-)\cdot h = q_+ h$ for $h \in \mathbb{H}$
S^-	$\mathbb{H}$	$Spin(4) \to \mathrm{Aut}\,(\mathbb{H})$	$(q_+, q_-)\cdot h = q_- h$ for $h \in \mathbb{H}$
$\mathbb{R}^4$	$\mathbb{R}^4 \cong \mathbb{H}$	$Spin^c(4) \to SO(4)$	$[(q_+, q_-, z)]\cdot h = q_+ h \bar{q}_-$ for $h \in \mathbb{H} \cong \mathbb{R}^4$
W^+	$\mathbb{H}$	$Spin^c(4) \to \mathrm{Aut}\,(\mathbb{H})$	$[(q_+, q_-, z)]\cdot h = q_+ h z$ for $h \in W \cong \mathbb{H}$
W^-	$\mathbb{H}$	$Spin^c(4) \to \mathrm{Aut}\,(\mathbb{H})$	$[(q_+, q_-, z)]\cdot h = q_- h z$ for $h \in W \cong \mathbb{H}$

Here $q_+ \in Sp(1)_+$, $q_- \in Sp(1)_-$, $z \in S^1$, $[(q_+, q_-, z)] \in (Sp(1)_+ \times Sp(1)_- \times S^1)/\pm 1 \cong Spin^c(4)$.

Clifford multiplication

2.3.15 Definition. Let V be a Cl_m-representation. Restricting to $Spin(m) \subset Cl_m$, we interpret V as a $Spin(m)$-representation. The *Clifford multiplication* is the map of $Spin(m)$-representations

$$c_m \colon \mathbb{R}^m \otimes V \to V,$$

which is obtained by restricting the Cl_m-action on V to $\mathbb{R}^m \subset Cl_m$. Similarly, a representation V of $Cl_m \otimes \mathbb{C}$ can be interpreted as a $Spin^c(m)$-representation by restriction to $Spin^c(m) \subset Cl_m \otimes \mathbb{C}$. The *Clifford multiplication* is again the homomorphism of $Spin^c(m)$-representations

$$c_m \colon \mathbb{R}^m \otimes V \to V,$$

which is obtained by restricting the $Cl_m \otimes \mathbb{C}$-action on V to $\mathbb{R}^m \subset Cl_m \subset Cl_m \otimes \mathbb{C}$.

We will now give the Clifford multiplication for the above $Spin^{(c)}(m)$-representations for $m \in \{3, 4\}$, which are restrictions of irreducible (complex) Cl_m-representations, in terms of quaternions: Consider the representation S of $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$ (cf. Examples 2.3.4) which is induced by the projection to the first component and left multiplication. If we restrict to $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3 \subset Cl_3$, we obtain the Clifford multiplication

$$\begin{aligned} \mathbb{R}^3 \otimes S \cong \mathrm{Im}(\mathbb{H}) \otimes S &\to S, \\ h \otimes h' &\mapsto \overline{h} h'. \end{aligned}$$

For the four-dimensional case, we use the isomorphism $Cl_4^0 \cong Cl_3$ from Proposition 2.3.6 and the Cl_4-representation $Cl_4 \otimes_{Cl_3} S$, where the Cl_4-action is given by left multiplication. The tensor product means that $\beta v e_0 \otimes h = \beta \otimes \bar{v} h$ for $\beta \in Cl_4, v \in \mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$ and $h \in S$.

2.3.16 Lemma. *There is an isomorphism of $Cl_4 \cong M_2(\mathbb{H})$-representations*

$$\begin{aligned} \Psi \colon \mathbb{H}^2 &\to Cl_4 \otimes_{Cl_3} S, \\ (v, w) &\mapsto 1 \otimes v + e_0 \otimes w, \end{aligned}$$

where $\mathbb{H}^2$ *is the tautological representation of* $Cl_4 \cong M_2(\mathbb{H})$ *and* Cl_4 *acts on* $Cl_4 \otimes_{Cl_3} S$ *by left multiplication. Restricting to* $Spin(4) \subset Cl_4^0$*, this induces an isomorphism of* $Spin(4)$*-representations*

$$S^+ \oplus S^- \cong Cl_4 \otimes_{Cl_3} S.$$

Proof. The element e_0 and Cl_4^0 generate Cl_4. This implies that Ψ is surjective and using dimension counting, we conclude that Ψ is an isomorphism of vector spaces. For $(v, w) \in \mathbb{H}^2$, we have

$$\begin{aligned} e_1\Psi(v,w) &= e_1 \otimes v + e_1e_0 \otimes w = e_0e_1e_0 \otimes v + e_1e_0 \otimes w = -e_0 \otimes iv - 1 \otimes iw \\ &= \Psi(-iw, -iv), \end{aligned}$$

and note that

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -iw \\ -iv \end{pmatrix}.$$

The same holds if we replace e_1 by e_2 or e_3 and i by j or k, respectively. Finally,

$$e_0.\Psi(v,w) = e_0 \otimes v + e_0e_0 \otimes w = e_0 \otimes v - 1 \otimes w = \Psi(-w, v).$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -w \\ v \end{pmatrix}.$$

Therefore, $\Psi \colon \mathbb{H}^2 \to Cl_3 \otimes_{Cl_3} S$ is an isomorphism of Cl_4-representations. Since the restriction of the tautological representation to Cl_4^0 is a direct sum $S^+ \oplus S^-$, we obtain an isomorphism of $Spin(4)$-representations $S^+ \oplus S^- \cong Cl_4 \otimes_{Cl_3} S$. □

2.3.17 Remark. Note that the Clifford multiplication for the tautological representation $\mathbb{H}^2$ is given by

$$\begin{aligned} \mathbb{R}^4 \cong \mathbb{H} \mapsto M_2(\mathbb{H}) \subset \operatorname{End}\left(\mathbb{H}^2\right), \\ h \mapsto \begin{pmatrix} 0 & -h \\ \bar{h} & 0 \end{pmatrix}. \end{aligned}$$

In particular, if we identify $\mathbb{R}^4 \cong \mathbb{H}$ then the restriction of the Clifford multiplication to $\mathbb{R}^4 \otimes S^+ \to S^-$ is given by $h \otimes h' \mapsto \bar{h}h'$.

2.3.18 Remark (Clifford multiplication for *Spin*^{c}). Let R_i be the complex structure on the Cl_3-representation S which is given by multiplication with i from the right. This induces an action of $Cl_3 \otimes \mathbb{C}$. This representation is denoted by W. Its restriction to $Spin^c(3)$ is the one in the list above. Similarly, we have an action of $Cl_4 \otimes \mathbb{C}$ on $\mathbb{H}^2$ in the four-dimensional case. Since the isomorphism from Lemma 2.3.16 is compatible with the complex structures, we obtain an isomorphism of $Spin^c(4)$-representations $W^+ \oplus W^- \cong Cl_4 \otimes_{Cl_3} W$ with the same Clifford multiplication as above.

2.3.19 Conclusion (Clifford multiplication in terms of quaternions).
In all considered cases, the Clifford multiplication is given by

$$\mathbb{H} \otimes \mathbb{H} \to \mathbb{H},$$
$$h \otimes h' \mapsto \overline{h}h'.$$

This can be interpreted as a homomorphism of $Spin(m)$ or $Spin^c(m)$-representations

$$\mathbb{R}^3 \otimes S \to S \quad \text{and} \quad \mathbb{R}^3 \otimes W \to W \quad \text{for } m = 3,$$
$$\mathbb{R}^4 \otimes S^+ \to S^- \quad \text{and} \quad \mathbb{R}^4 \otimes W^+ \to W^- \quad \text{for } m = 4,$$

where in the three-dimensional case, we take the restriction of the above homomorphism to $\mathrm{Im}(\mathbb{H}) \otimes \mathbb{H}$. Note that this reflects our choice of the irreducible Cl_3-representation S.

2.3.3 *Spin*-structures and *Spin*^{c}-structures

2.3.20 Definition (*Spin*-structure). A *Spin-structure* on an oriented Riemannian vector bundle $E \to M$ of rank $m \geq 3$ is a λ-reduction $P_{Spin(m)} \to P_{SO(m)}$, where $P_{SO(m)} \to M$ is the bundle of oriented orthonormal frames in E and $\lambda\colon Spin(m) \to SO(m)$ is the universal cover from Note 2.3.8. An oriented Riemannian vector bundle $E \to M$ is said to be *Spin* if a *Spin*-structure on $E \to M$ exists. A *Spin-structure* on an oriented m-dimensional Riemannian manifold M is a *Spin*-structure on $TM \to M$. An oriented Riemannian manifold M is said to be a *Spin-manifold* if a *Spin*-structure on M exists.

2.3.21 Definition (*Spin*^{c}-structure). A *Spin*^{c}*-structure* on an oriented Riemannian vector bundle $E \to M$ of rank $m \geq 3$ is a principal S^1-bundle $P_{S^1} \to M$ together with a λ^c-reduction $P_{Spin^c(m)} \to P_{SO(m)} \times_M P_{S^1}$, where $P_{SO(m)} \to M$ is the bundle of oriented orthonormal frames in E and $\lambda^c\colon Spin^c(m) \to SO(m) \times S^1$ is the 2-fold covering from Note 2.3.8. An oriented Riemannian vector bundle $E \to M$ is said to be *Spin*^{c} if a *Spin*^{c}-structure on $E \to M$ exists. A *Spin*^{c}*-structure* on an oriented m-dimensional Riemannian manifold M is a *Spin*^{c}-structure on $TM \to M$. An oriented m-dimensional Riemannian manifold M is said to be a *Spin*^{c}*-manifold* if a *Spin*^{c}-structure exists.

The following theorem answers the question for existence and uniqueness of *Spin*-structures and *Spin*^{c}-structures.

2.3.22 Theorem ([LM89, Ch II Thm 1.7, App D Thm D.2]).
An oriented Riemannian vector bundle $E \to M$ is Spin iff its second Stiefel-Whitney class $w_2(E) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ vanishes. In this case, the collection of isomorphism classes of Spin-structures is a $H^2(M, \mathbb{Z}/2\mathbb{Z})$-torsor.

An oriented Riemannian vector bundle $E \to M$ is Spin^c iff its second Stiefel-Whitney class $w_2(E) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ is a mod2 *reduction of an integral class. In this case, the collection of isomorphism classes of Spin^c-structures is a $H^2(M, \mathbb{Z}/2\mathbb{Z}) \oplus 2H^1(M, \mathbb{Z})$-torsor.*

2.3.23 Corollary.

1. *An oriented Riemannian manifold M is a Spin-manifold if and only if its second Stiefel-Whitney class $w_2(TM) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ vanishes.*

2. *An oriented Riemannian manifold M is a $Spin^c$-manifold if and only if its second Stiefel-Whitney class $w_2(TM) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ is a mod 2 reduction of an integral class.*

In dimensions three and four, the following two theorems guarantee the existence of $Spin^{(c)}$-structures.

2.3.24 Theorem (Stiefel, [Sti35]). *Every compact orientable three-dimensional manifold is parallelizable, i.e. the tangent bundle is trivial.*

2.3.25 Corollary. *Every compact three-dimensional oriented manifold is a Spin-manifold.*

2.3.26 Theorem (Whitney, [HH58]). *Every compact oriented four-dimensional Riemannian manifold is a $Spin^c$-manifold.*

2.3.4 Spinor bundles

2.3.27 Definition. Consider a *Spin*-structure $P_{Spin(m)} \to P_{SO(m)}$ on an oriented m-dimensional Riemannian manifold M. A *spinor bundle* is an associated vector bundle $E = P_{Spin(m)} \times_{Spin(m)} V$ where V is a Cl_m-module. Here V is interpreted as a $Spin(m)$-representation, using the embedding $Spin(m) \subset Cl_m$.

Let $P_{Spin^c(m)} \to P_{SO(m)} \times_M P_{S^1}$ be $Spin^c$-structure on an oriented m-dimensional Riemannian manifold M. A *complex spinor bundle* is an associated vector bundle $E = P_{Spin^c(m)} \times_{Spin^c(m)} V$ where V is a complex Cl_m-module. Here V is interpreted as a $Spin^c(m)$-representation, using the embedding $Spin^c(m) \subset Cl_m \otimes \mathbb{C}$. Sections of a (complex) spinor bundle are called *spinors*.

2.3.28 Example. For an irreducible Cl_m-representation S, we denote the spinor bundle by $\mathcal{S}$. In the $Spin^c(m)$ case, if W is the irreducible complex Cl_m-representation, we denote the complex spinor bundle by $\mathcal{W}$. For $m = 4$ we have the direct sum decompositions $S = S^+ \oplus S^-$ and $W = W^+ \oplus W^-$ (cf. subsection 2.3.2). The associated vector bundles for these representations are denoted by $\mathcal{S}^+, \mathcal{S}^-, \mathcal{W}^+, \mathcal{W}^-$, respectively.

2.3.29 Example (Dirac operator). Let M be an m-dimensional manifold M with a $Spin(m)$-structure $P_{Spin(m)} \to P_{SO(m)}$ and let S be an irreducible Cl_m-representation. The *Dirac operator* $\mathcal{D}$ is defined to be the composition

$$\mathcal{D} \colon \Gamma(M, \mathcal{S}) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes \mathcal{S}) \xrightarrow{c_m} \Gamma(M, \mathcal{S}),$$

where ∇ is the Levi-Civita connection on M and c_m is the Clifford multiplication induced by $(\mathbb{R}^m)^* \otimes S \cong \mathbb{R}^m \otimes S \to S$.

For $m = 4$, any irreducible Cl_4-representation S splits into a direct sum of two irreducible $Spin(4)$-representations $S = S^+ \oplus S^-$ and the Clifford multiplication with an element $v \in \mathbb{R}^m$ interchanges S^+ and S^-. In particular, we are interested in the restriction $\mathcal{D}^+$ of the Dirac operator $\mathcal{D}$:

$$\mathcal{D}^+ \colon \Gamma(M, \mathcal{S}^+) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes \mathcal{S}^+) \xrightarrow{c} \Gamma(M, \mathcal{S}^-).$$

2.3.30 Example ($Spin^c(m)$ Dirac operator). Let $P_{Spin^c(m)} \xrightarrow{(\pi_{SO}, \pi_{S^1})} P_{SO(m)} \times_M P_{S^1}$ be a $Spin^c(m)$-structure on an m-dimensional manifold M and let W be an irreducible complex Cl_m representation. Let φ_M be the Levi-Civita connection, $A \in \mathscr{A}(P_{Spin^c(m)})$ a connection 1-form such that $\pi_{SO}^* \varphi_M = \mathrm{pr}_{\mathfrak{so}(m)} A$ and ∇^A the corresponding covariant derivative. The *$Spin^c(m)$ Dirac operator* $\mathcal{D}_A$ is defined to be the composition

$$\mathcal{D}_A \colon \Gamma(M, \mathcal{W}) \xrightarrow{\nabla^A} \Gamma(M, T^*M \otimes \mathcal{W}) \xrightarrow{c_m} \Gamma(M, \mathcal{W}),$$

where c_m is the Clifford multiplication induced by $(\mathbb{R}^m)^* \otimes W \cong \mathbb{R}^m \otimes W \to W$.

If m is even, then the irreducible complex Cl_m-representation W splits into a direct sum of two irreducible $Spin^c(m)$-representations $W = W^+ \oplus W^-$ and the Clifford multiplication with an element $v \in \mathbb{R}^m$ interchanges W^+ and W^- (cf. [LM89, App D]). In particular, we are intersted in the restriction $\mathcal{D}_A^+$ of the Dirac operator $\mathcal{D}_A$:

$$\mathcal{D}_A^+ \colon \Gamma(M, \mathcal{W}^+) \xrightarrow{\nabla^A} \Gamma(M, T^*M \otimes \mathcal{W}^+) \xrightarrow{c_m} \Gamma(M, \mathcal{W}^-).$$

Chapter 3

The nonlinear Dirac operator

In this chapter, we construct the nonlinear Dirac operator in dimensions three and four associated to a hyperkähler manifold with permuting action. This Dirac operator was introduced by Taubes [Tau99] for three-dimensional manifolds and by Pidstrygach [Pid04] for four-dimensional manifolds.

3.1 The group $Spin^G_\varepsilon(m)$

In order to define the nonlinear generalization of the Dirac operator, we need the Lie group $Spin^G_\varepsilon(m)$, which will be the replacement of $Spin(m)$ or $Spin^c(m)$ in the construction of the $Spin$ or $Spin^c$ Dirac operator.

3.1.1 Definition. Let G be a compact Lie group and $\varepsilon \in Z(G)$ a central element of G satisfying $\varepsilon^2 = 1$. The element $(-1, \varepsilon) \in Spin(m) \times G$ generates a normal subgroup of order 2, which we denote by ± 1. For $m \in \{3, 4\}$ we define the group $Spin^G_\varepsilon(m)$ as

$$Spin^G_\varepsilon(m) := (Spin(m) \times G)/\pm 1.$$

3.1.2 Examples.

1. If $G = \mathbb{Z}/2\mathbb{Z}$ and $\varepsilon = -1$, then

$$Spin^{\mathbb{Z}/2\mathbb{Z}}_{-1}(m) = (Spin(m) \times \mathbb{Z}/2\mathbb{Z})/\pm 1 = Spin(m).$$

2. If $G = S^1$ and $\varepsilon = -1$, then

$$Spin^{S^1}_{-1}(m) = (Spin(m) \times S^1)/\pm 1 = Spin^c(m).$$

3. If $\varepsilon = 1$ and G is an arbitrary compact Lie group, then

$$Spin^G_1(m) = SO(m) \times G.$$

4. In particular, for the trivial group $G = 1$ we obtain

$$Spin_1^1(m) = Spin(m)/\pm 1 = SO(m).$$

3.1.3 Note. Denote by $\langle(1,\varepsilon)\rangle$ the (normal) subgroup of $Spin_\varepsilon^G(m)$ generated by $[(1,\varepsilon)] = [(-1,1)] \in Spin_\varepsilon^G(m)$ and by G/ε the quotient of G by the subgroup generated by ε. Then

$$Spin_\varepsilon^G(m)/\langle(1,\varepsilon)\rangle = SO(m) \times G/\varepsilon.$$

We have a short exact sequence

$$1 \to \langle(1,\varepsilon)\rangle \to Spin_\varepsilon^G(m) \xrightarrow{\lambda^G} SO(m) \times G/\varepsilon \to 1, \tag{3.1}$$

where $\lambda^G \colon Spin_\varepsilon^G(m) \to SO(m) \times G/\varepsilon$ is the quotient map. In particular, the Lie algebra $\mathfrak{spin}_\varepsilon^G(m) = \mathrm{Lie}(Spin_\varepsilon^G(m))$ of $Spin_\varepsilon^G(m)$ is

$$\mathfrak{spin}_\varepsilon^G(m) \cong \mathfrak{so}(m) \oplus \mathfrak{g}.$$

3.1.4 Remark. There is a second short exact sequence, which will be useful. We have an embedding $G \hookrightarrow Spin_\varepsilon^G(m)$ as a normal subgroup. The quotient of $Spin_\varepsilon^G(m)$ by G is $SO(m)$. Therefore the following sequence is exact

$$1 \to G \to Spin_\varepsilon^G(m) \to SO(m) \to 1.$$

3.1.5 Remark. Using the injection $\iota\colon Spin(m) \to Spin(m+1)$ we also obtain an injection

$$Spin_\varepsilon^G(m) = (Spin(m) \times G)/\pm 1 \xrightarrow{[\iota, id]} (Spin(m+1) \times G)/\pm 1 = Spin_\varepsilon^G(m+1).$$

3.1.6 Note. For $m = 3$ and $m = 4$, the isomorphisms $Spin(3) \cong Sp(1)$ from Example 2.3.9 and $Spin(4) \cong Sp(1)_+ \times Sp(1)_-$ from Example 2.3.11 induce isomorphisms $Spin_\varepsilon^G(3) \cong (Sp(1) \times G)/\pm 1$ and $Spin_\varepsilon^G(4) \cong (Sp(1)_+ \times Sp(1)_- \times G)/\pm 1$.

3.1.1 $Spin_\varepsilon^G(m)$-structures

Having the group $Spin_\varepsilon^G(m)$ at hand, we can study principal $Spin_\varepsilon^G(m)$-bundles and $Spin_\varepsilon^G(m)$-structures on m-dimensional manifolds. These generalize *Spin*-structures and $Spin^c$-structures and replace these in the construction of the Dirac operator.

3.1.7 Definition ($Spin_\varepsilon^G(m)$-structures). A $Spin_\varepsilon^G(m)$-*structure* on an oriented m-dimensional Riemannian manifold Z ($m \geq 3$) is a principal G/ε-bundle $P_{G/\varepsilon} \to Z$ together with a λ^G-reduction $\pi\colon Q_m \to P_{SO(m)} \times_Z P_{G/\varepsilon}$, where $P_{SO(m)} \to Z$ is the bundle of orthonormal frames in TZ, and $\lambda^G\colon Spin_\varepsilon^G(m) \to SO(m) \times G/\varepsilon$ is the homomorphism from Note 3.1.3. We will denote the components of π by $\pi_{SO}\colon Q_m \to P_{SO(m)}$ and $\pi_{G/\varepsilon}\colon Q_m \to P_{G/\varepsilon}$.

3.1.8 Examples.

1. A $Spin^G_1(m)$-structure on Z is the same as a principal G-bundle $P_G \to Z$. In this case $Q_m \cong P_{SO(m)} \times_Z P_G$.

2. A $Spin^{\mathbb{Z}/2\mathbb{Z}}_{-1}(m)$-structure on Z is the same as a *Spin*-structure on Z.

3. A $Spin^{S^1}_{-1}(m)$-structure on Z is the same as a $Spin^c$-structure on Z.

3.1.9 Remark. The quotient map $Spin^G_\varepsilon(m) \to SO(m) = Spin^G_\varepsilon(m)/G$ induces an isomorphism $Q_m/G \cong P_{SO(m)}$. Similarly, $Spin^G_\varepsilon(m)/Spin(m) = G/\varepsilon$ implies that $Q_m/Spin(m) \cong P_{G/\varepsilon}$.

3.1.10 Remark. From the short exact sequence (3.1) we obtain an exact sequence in Čech-cohomology:

$$\check{H}^1(Z,\langle(1,\varepsilon)\rangle)\to\check{H}^1(Z,Spin^G_\varepsilon(m))\to\check{H}^1(Z,SO(m))\oplus\check{H}^1(Z,G/\varepsilon)\to\check{H}^2(Z,\langle(1,\varepsilon)\rangle)$$

If $\varepsilon = 1$, then the first and the last term vanish, and we obtain a bijection

$$\check{H}^1(Z,Spin^G_1(m)) \cong \check{H}^1(Z,SO(m)) \oplus \check{H}^1(Z,G).$$

In this case, the principal $Spin^G_1(m)$-bundle Q_m is isomorphic to the fibre product of the bundle of oriented orthonormal frames $P_{SO(m)}$ and a principal G-bundle P_G, i.e. we have $Q_m \cong P_{SO(m)} \times_Z P_G$.

If $\varepsilon \neq 1$, the quotient $Q_m/\langle(1,\varepsilon)\rangle$ is a principal $SO(m) \times G/\varepsilon$-bundle. There is not necessarily a lift of this principal G/ε-bundle to a principal G-bundle. Given a principal G/ε-bundle $P_{G/\varepsilon}$, we observe from the exact sequence in Čech cohomology that the obstruction for the existence of a lift of $P_{SO(m)} \times_Z P_{G/\varepsilon}$ to a principal $Spin^G_\varepsilon(m)$-bundle is an element in $\check{H}^2(Z,\langle(1,\varepsilon)\rangle) \cong H^2(Z,\mathbb{Z}/2\mathbb{Z})$. This element is

$$w_2(P_{SO(m)}) + \delta(P_{G/\varepsilon}) \in H^2(Z,\mathbb{Z}/2\mathbb{Z}),$$

where $\delta\colon \check{H}^1(Z,G/\varepsilon) \to H^2(Z,\mathbb{Z}/2\mathbb{Z})$ is the map from the exact sequence in Čech-cohomology, which is induced by the short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to G \to G/\varepsilon \to 1.$$

For $G = \mathbb{Z}/2\mathbb{Z}$, we have $Spin^{\mathbb{Z}/2\mathbb{Z}}_{-1}(m) = Spin(m)$. In this case, $G/\varepsilon = 1$ is trivial and therefore $H^1(Z,G/\varepsilon) = 0$. We obtain $\delta = 0$ and the obstruction is the second Stiefel-Whitney class $w_2(P_{SO(m)}) \in H^2(Z,\mathbb{Z}/2\mathbb{Z})$ (cf. Theorem 2.3.22).

For $G = S^1$ and $\varepsilon = -1$, we have $Spin^{S^1}_{-1}(m) = Spin^c(m)$ and $\delta(P_{S^1}) = \tilde{c}_1(P_{S^1})$ is the mod 2 reduction of the first Chern class $c_1(P_{S^1})$ and there is a lift of $P_{SO} \times_Z P_{S^1}$ iff $w_2(P_{SO(m)}) \equiv c_1(P_{S^1}) \mod 2$ (cf. Theorem 2.3.22). For details on *Spin*-structures and $Spin^c$-structures we refer the reader to [LM89, Ch 2 §1, App A]. For similar computations for other groups G see [Zen06, Appendix].

3.1.2 Gauge group

We can now study the automorphism group of a $Spin^G_\varepsilon(m)$-structure.

3.1.11 Definition. Let $Q_m \to P_{SO(m)} \times_Z P_{G/\varepsilon}$ be a $Spin^G_\varepsilon(m)$-structure on Z. The gauge group of the $Spin(m)$-equivariant principal G-bundle $Q_m \to P_{SO(m)}$ is denoted by $\mathscr{G}_m$, i.e.

$$\begin{aligned}\mathscr{G}_m :=& \mathscr{G}(Q_m \to P_{SO(m)})^{Spin^G_\varepsilon(m)} \\ =& \left\{ \psi \in \mathscr{G}(Q_m \to P_{SO(m)}) \;\middle|\; \psi \text{ is } Spin^G_\varepsilon(m)\text{-equivariant} \right\}.\end{aligned}$$

We will refer to $\mathscr{G}_m$ as the *gauge group*.

Consider the action of $Spin^G_\varepsilon(m)$ on G, which is induced by the conjugation action of G. We can describe the gauge group in terms of equivariant maps:

3.1.12 Lemma.

$$\mathscr{G}(Q_m \to P_{SO(m)})^{Spin^G_\varepsilon(m)} \cong C^\infty(Q_m, G)^{Spin^G_\varepsilon(m)}.$$

Proof. First, note that $Q_m \to P_{SO(m)}$ is a principal G-bundle, and Note 2.1.42 implies $\mathscr{G}(Q_m \to P_{SO(m)}) \cong C^\infty(Q_m, G)^G$. Let $g\colon Q_m \to G$ be $Spin^G_\varepsilon(m)$-equivariant and $\psi\colon Q_m \to Q_m$ the corresponding automorphims, i.e. $\psi(p) = pg(p)$ for all $p \in Q_m$. Then for all $h \in Spin(m)$ and $p \in Q_m$:

$$\psi(ph) = phg(ph) = phh^{-1}g(p)h = pg(p)h = \psi(p)h.$$

This proves that ψ is $Spin(m)$-equivariant and therefore also $Spin^G_\varepsilon(m)$-equivariant. Conversely, if $\psi\colon Q_m \to Q_m$ is $Spin^G_\varepsilon(m)$-equivariant and $g\colon Q_m \to G$ the corresponding G-equivariant map, then for all $p \in Q_m, h \in Spin(m)$:

$$phh^{-1}g(p)h = pg(p)h = \psi(p)h = \psi(ph) = phg(ph),$$

and this implies $h^{-1}g(p)h = g(ph)$, so $g\colon Q_m \to G$ is $Spin(m)$-equivariant and therefore also $Spin^G_\varepsilon(m)$-equivariant. □

3.1.13 Corollary. *The Lie algebra of $\mathscr{G}_m$ is*

$$\mathrm{Lie}(\mathscr{G}_m) \cong C^\infty(Q_m, \mathfrak{g})^{Spin^G_\varepsilon(m)} \cong \Gamma(Z, \mathfrak{g}_{Q_m}).$$

3.1.14 Example. If the group G is abelian, then the action of $Spin^G_\varepsilon(m)$ on G is trivial, and we obtain

$$\mathscr{G}_m = \mathscr{G}(Q_m \to P_{SO(m)})^{Spin^G_\varepsilon(m)} \cong C^\infty(Q_m, G)^{Spin^G_\varepsilon(m)} \cong C^\infty(Z, G).$$

3.2 The target manifold

The next step is to replace the fibre of the spinor bundle by a target manifold M. We will now restrict to the dimensions three and four. In these cases, we can use the isomorphisms $Spin^G_\varepsilon(3) \cong (Sp(1) \times G)/\pm 1$ and $Spin^G_\varepsilon(4) \cong (Sp(1)_+ \times Sp(1)_- \times G)/\pm 1$ from Note 3.1.6. To construct a Dirac operator, we have to impose some requirements on M.

3.2.1 Definition. An action of $Sp(1)$ on a hyperkähler manifold M is said to be *permuting* if $Sp(1)$ acts by isometries and the induced action on the sphere of complex structures is the standard action of $Sp(1)$ on S^2, i.e.

$$q_* \mathcal{I}_\zeta \bar{q}_* = \mathcal{I}_{q\zeta\bar{q}} \text{ for all } q \in Sp(1), \zeta \in \mathrm{Im}(\mathbb{H}), \|\zeta\|^2 = 1.$$

Consider a permuting action of $Sp(1)$ on M and let G be a compact Lie group with a hyperkähler action on M which commutes with the $Sp(1)$-action. Furthermore, assume that $(-1, \varepsilon) \in Sp(1) \times G$ acts trivially on M. Therefore the action of $Sp(1) \times G$ on M descends to an action of $(Sp(1) \times G)/\pm 1 \cong Spin^G_\varepsilon(3)$. Such an action of $Spin^G_\varepsilon(3)$ is said to be *permuting*. An action of $Spin^G_\varepsilon(4)$ is said to be *permuting* if it is induced by a permuting action of $Spin^G_\varepsilon(3)$ via the homomorphism $Spin^G_\varepsilon(4) \to Spin^G_\varepsilon(4)/Sp(1)_- \cong Spin^G_\varepsilon(3)$.

3.2.2 Example. The first example of a hyperkähler manifold with permuting $Sp(1)$-action is the quaternionic vector space $\mathbb{H}^n$ with the standard metric. The tangent bundle is trivial and the complex structures are given by componentwise multiplication with i, j and k respectively, $I_1(v) = iv$, $I_2(v) = jv$ and $I_3(v) = kv$ for all $x \in \mathbb{H}^n$ and $v \in \mathbb{H}^n = T_x\mathbb{H}^n$. Consider the $Sp(1)$-action by multiplication $Sp(1) \times \mathbb{H}^n \ni (q, x) \mapsto qx \in \mathbb{H}^n$. The induced action on $T\mathbb{H}^n$ is again given by multiplication, and the action on the sphere of complex structures is $(q, \mathcal{I}_\zeta) \mapsto \bar{q}_* \mathcal{I}_\zeta q_*$, $\bar{q}_* \mathcal{I}_\zeta q_*(v) = \bar{q}\zeta q v = \mathcal{I}_{\bar{q}\zeta q}$ for all $q \in Sp(1), \zeta \in \mathfrak{sp}(1), \|\zeta\|^2 = 1, x \in \mathbb{H}^n$ and $v \in T_x\mathbb{H}^n$. This proves that the $Sp(1)$-action is permuting.

For the hyperkähler action, we can take any Lie subgroup G of $Sp(n)$ acting by $\mathbb{H}$-linear isometries on $\mathbb{H}^n$. In particular, this includes the following example: Let $n = 1$, $G = S^1$ and $\varepsilon = -1$. Define an action of $(Sp(1) \times G)/\pm 1$ on $M = \mathbb{H}$:

$$[(q, z)] \cdot h := qhz \text{ for } [(q, z)] \in (Sp(1) \times G)/\pm 1, h \in \mathbb{H}.$$

The hyperkähler structure on $\mathbb{H}$ is the same as in Example 2.2.6. The $G = S^1$ action on $M = \mathbb{H}$ is a hyperkähler action and that the $Sp(1)$ action is permuting. This is the representation W of $Spin^c(3) = (Sp(1) \times G)/\pm 1$ from subsection 2.3.2. If we interpret $M = \mathbb{H}$ as a hyperkähler manifold with permuting $Spin^c(4)$-action, we obtain the $Spin^c(4)$-representation W^+.

Properties of hypkähler manifolds with permuting action

The Hodge star operator $*\colon \mathbb{R}^3 \to \Lambda^2\mathbb{R}^3$ is an isomorphism of representations of $Spin(3) \cong Sp(1)$. Using the identification $\mathbb{R}^3 \cong \mathfrak{sp}(1)$, we obtain a homomorphism

$$\begin{aligned} \mathfrak{sp}(1) \xrightarrow{*} \Lambda^2\mathfrak{sp}(1) &\hookrightarrow \mathfrak{sp}(1) \otimes \mathfrak{sp}(1), \\ i \mapsto j \wedge k &\mapsto \tfrac{1}{2}(j \otimes k - k \otimes j), \\ j \mapsto k \wedge i &\mapsto \tfrac{1}{2}(k \otimes i - i \otimes k), \\ k \mapsto i \wedge j &\mapsto \tfrac{1}{2}(i \otimes j - j \otimes i). \end{aligned}$$

Dually, we have a homomorphism $\pi_{\mathfrak{sp}(1)^*}\colon \mathfrak{sp}(1)^* \otimes \mathfrak{sp}(1)^* \to \mathfrak{sp}(1)^*$. We will now recall some properties of hyperkähler manifolds with permuting actions. These were first studied by Swann [Swa91]. The third part of the following proposition is due to Boyer, Galicki, Mann [BGM93, Prop. 2.7] and the fourth part is due to Pidstrygach [Pid04, Section 2.2.1].

3.2.3 Proposition. *Let (M, g^M, I_1, I_2, I_3) be a hyperkähler manifold with permuting $Spin^G_\varepsilon(3)$-action. Then*

1. *The 2-form ω is $Spin^G_\varepsilon(3)$-equivariant, i.e. $\omega \in \Omega^2(M, \mathfrak{sp}(1)^*)^{Spin^G_\varepsilon(3)}$.*
2. *The $\mathfrak{sp}(1)$ Lie derivative of ω is $\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, \zeta \otimes \zeta' \rangle = -\langle \omega, [\zeta, \zeta'] \rangle$ for all $\zeta, \zeta' \in \mathfrak{sp}(1)$.*
3. *The 2-form $\omega \in \Omega^2(M, \mathfrak{sp}(1)^*)^{Spin^G_\varepsilon(3)}$ is exact, and in particular we have $\omega = d\gamma$ for $\gamma := -\frac{1}{2}\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}\omega \in \Omega^1(M, \mathfrak{sp}(1))^{Spin^G_\varepsilon(3)}$.*
4. *The map $\mu := -\iota_{\mathfrak{g}}\gamma \in C^\infty(M, \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*)^{Spin^G_\varepsilon(3)}$ is a hyperkähler moment map for the action of G on M.*

Proof.

1. Let $q \in Sp(1)$, $\zeta \in \mathfrak{sp}(1)$, $x \in M$ and $v, w \in T_xM$. We use that the action of $Sp(1)$ is permuting to obtain

$$\begin{aligned} \langle L_q^*\omega, \zeta \rangle(v, w) &= \omega_\zeta(q_*v, q_*w) = g^M(q_*v, \mathcal{I}_\zeta(q_*w)) = g^M(v, q_*^{-1}\mathcal{I}_\zeta(q_*w)) \\ &= g^M(v, \mathcal{I}_{Ad_{q^{-1}}(\zeta)}(w)) = \langle \omega, Ad_{q^{-1}}(\zeta) \rangle(v, w) \\ &= \langle Ad_q^* \circ \omega, \zeta \rangle(v, w). \end{aligned}$$

This proves that ω is $Sp(1)$-equivariant. Let $g \in G$, $\zeta \in \mathfrak{sp}(1)$, $x \in M$ and $v, w \in T_xM$. Since the action of G is hyperkähler,

$$\begin{aligned} \langle L_g^*\omega, \zeta \rangle(v, w) &= g^M(g_*v, \mathcal{I}_\zeta(g_*w)) = g^M(g_*v, g_*\mathcal{I}_\zeta(w)) = g^M(v, \mathcal{I}_\zeta(w)) \\ &= \langle \omega, \zeta \rangle(v, w). \end{aligned}$$

This proves that ω is G-invariant. Together with the $Sp(1)$-equivariance, this implies that $\omega \in \Omega^2(M, \mathfrak{sp}(1)^*)^{Spin^G_\varepsilon(3)}$.

2. Using the previous assertion, we obtain

$$\begin{aligned}\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, \zeta \otimes \zeta' \rangle &= \mathcal{L}_{K_\zeta^{M,Sp(1)}} \omega_{\zeta'} = \tfrac{d}{dt}(L_{\exp(t\zeta)})^* \omega_{\zeta'}|_{t=0} = \tfrac{d}{dt}\omega_{Ad_{\exp(-t\zeta)}(\zeta')}|_{t=0} \\ &= \langle \omega, \tfrac{d}{dt} Ad_{\exp(-t\zeta)}(\zeta')|_{t=0}\rangle = \langle \omega, -ad_\zeta(\zeta')\rangle \\ &= -\langle \omega, [\zeta, \zeta']\rangle\end{aligned}$$

for all $\zeta, \zeta' \in \mathfrak{sp}(1)$.

3. The definition of $\pi_{\mathfrak{sp}(1)^*}$ and the previous assertion imply

$$\begin{aligned}\langle \pi_{\mathfrak{sp}(1)^*} \mathcal{L}_{\mathfrak{sp}(1)}\omega, i\rangle &= \tfrac{1}{2}\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, j\otimes k - k \otimes j\rangle = -\tfrac{1}{2}\langle \omega, 2[j,k]\rangle = -2\langle \omega, i\rangle, \\ \langle \pi_{\mathfrak{sp}(1)^*} \mathcal{L}_{\mathfrak{sp}(1)}\omega, j\rangle &= \tfrac{1}{2}\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, k\otimes i - i \otimes k\rangle = -\tfrac{1}{2}\langle \omega, 2[k,i]\rangle = -2\langle \omega, j\rangle, \\ \langle \pi_{\mathfrak{sp}(1)^*} \mathcal{L}_{\mathfrak{sp}(1)}\omega, k\rangle &= \tfrac{1}{2}\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, i\otimes j - j \otimes i\rangle = -\tfrac{1}{2}\langle \omega, 2[i,j]\rangle = -2\langle \omega, k\rangle,\end{aligned}$$

and hence $\pi_{\mathfrak{sp}(1)^*}\mathcal{L}_{\mathfrak{sp}(1)}\omega = -2\omega$. Finally,

$$d\gamma = -\tfrac{1}{2} d\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}\omega = -\tfrac{1}{2}\pi_{\mathfrak{sp}(1)^*} d\iota_{\mathfrak{sp}(1)}\omega = -\tfrac{1}{2}\pi_{\mathfrak{sp}(1)^*}\mathcal{L}_{\mathfrak{sp}(1)}\omega = \omega.$$

4. The G-invariance of γ implies $\mathcal{L}_{\mathfrak{g}}\gamma = 0$. Since γ is equivariant and $\iota_{\mathfrak{g}}$ maps equivariant forms to equivariant forms, the map $\mu = -\iota_{\mathfrak{g}}\gamma \colon M \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$ is $Spin^G_\varepsilon(3)$-equivariant. We use the Cartan formula $\mathcal{L}_{\mathfrak{g}} = d\iota_{\mathfrak{g}} + \iota_{\mathfrak{g}} d$ to check the moment map condition

$$d\mu = -d\iota_{\mathfrak{g}}\gamma = -\mathcal{L}_{\mathfrak{g}}\gamma + \iota_{\mathfrak{g}} d\gamma = \iota_{\mathfrak{g}}\omega.$$

This proves that $\mu : M \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$ is a hyperkähler moment map. □

3.2.4 Remark. The second assertion of the previous proposition implies that a hyperkähler manifold M of dimension $\dim(M) > 0$ with permuting $Sp(1)$-action cannot be compact: For $\zeta \in \mathfrak{sp}(1)$, $\|\zeta\|^2 = 1$, the form ω_ζ is a Kähler form and exact. Therefore, the volume form is also exact, and hence M cannot be compact.

3.2.1 Target manifolds with hyperkähler potential

Among the hyperkähler manifolds with permuting $Spin^G_\varepsilon(3)$-action, there are those hyperkähler manifolds with permuting action, which admit a hyperkähler potential.

3.2.5 Example (Swann's construction). Let N be a compact quaternionic Kähler manifold with positive scalar curvature. Then Swann's construction [Swa91] produces a hyperkähler manifold $M = \mathcal{U}(N)$ with permuting $Sp(1)$-action. This is a fibre bundle $\mathcal{U}(N) \to N$ with typical fibre $\mathbb{H}^\times/\pm 1$. The fundamental vector fields for the permuting $Sp(1)$-action on $M = \mathcal{U}(N)$ satisfy $\mathcal{I}_\zeta K_\zeta^{M,Sp(1)} = -\chi$ for a vector field $\chi \in \Gamma(M, TM)$ and all $\zeta \in \mathfrak{sp}(1)$, $\|\zeta\|^2 = 1$. Moreover, $M = \mathcal{U}(N)$ has a hyperkähler potential $\rho = \frac{1}{2}\|\cdot\|^2$, where $\|\cdot\|$ is the norm on the fibres of $M = \mathcal{U}(N)$. Examples for compact quaternionic Kähler manifolds with positive scalar curvature are Wolf spaces. These are compact homogeneous quaternionic Kähler manifolds. There is a list of these manifolds, namely quaternionic projective spaces $\mathbb{HP}^n = \frac{Sp(n)}{Sp(n-1)\times Sp(1)}$, some complex Grassmannians $Gr_2(\mathbb{C}^n) = \frac{SU(n)}{S(U(n-2)\times U(2))}$, some oriented Grassmannians $\widetilde{Gr}_4(\mathbb{R}^n) = \frac{SO(n)}{SO(n-4)\times SO(4)}$ and five quotients of the exotic simply connected compact Lie groups G_2, F_3, E_6, E_7, E_8. The corresponding hyperkähler manifold $M = \mathcal{U}(N)$ for a Wolf space N is a certain coadjoint orbit of the simple complex Lie group (for details cf. [Swa91]).

Properties of hyperkähler manifolds with permuting action and potential

Consider the homomorphism of representations of $Spin(3) \cong Sp(1)$

$$\begin{aligned} \pi_{\mathbb{R}} \colon \mathfrak{sp}(1)^* \otimes \mathfrak{sp}(1)^* &\to \mathbb{R} \\ \alpha \otimes \beta &\mapsto \tfrac{1}{3}(\alpha(i)\beta(i) + \alpha(j)\beta(j) + \alpha(k)\beta(k)). \end{aligned}$$

In the following proposition, the third assertion is due to Henrik Schumacher and the last assertion first appeared in [Swa91].

3.2.6 Proposition (Target manifold with potential). *Let M be a hyperkähler manifold with permuting $Spin^G_\varepsilon(m)$-action such that $\mathcal{I}_\zeta K_\zeta^{M,Sp(1)} \in \Gamma(M, TM)$ is independent of $\zeta \in \mathfrak{sp}(1)$, $\|\zeta\|^2 = 1$. Denote $\chi := -\mathcal{I}_\zeta K_\zeta^{M,Sp(1)}$ and let ∇ be the Levi-Civita connection on M. Then*

1. $\gamma = \frac{1}{2}\iota_\chi \omega$ *and* $\mu = -\frac{1}{2}\iota_{\mathfrak{g}}\iota_\chi\omega$.
2. *The function* $\rho = -\pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\gamma$ *is a* $Spin^G_\varepsilon(m)$*-invariant hyperkähler potential* $\rho\colon M \to \mathbb{R}$ *satisfying* $\chi = \operatorname{grad}(\rho)$.
3. $\nabla\chi = \mathrm{id}_{\Gamma(M,TM)}$.
4. $d\mu(\chi) = 2\mu$.
5. $\rho = \frac{1}{2} g^M(\chi,\chi)$.

Proof. We denote the image of $i, j, k \in \operatorname{Im}(\mathbb{H})$ under the isomorphism $\operatorname{Im}(\mathbb{H}) \cong \mathfrak{sp}(1)$ by $\zeta_1, \zeta_2, \zeta_3$, respectively.

1. We have

$$\begin{aligned} \langle\gamma, i\rangle &= -\tfrac{1}{2}\langle\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}\omega, i\rangle = -\tfrac{1}{2}\langle\iota_{\mathfrak{sp}(1)}\omega, j\otimes k\rangle = -\tfrac{1}{2}\iota_{K_{\zeta_2}^{M,Sp(1)}}\omega_3 = \tfrac{1}{2}\iota_\chi\omega_1, \\ \langle\gamma, j\rangle &= -\tfrac{1}{2}\langle\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}\omega, j\rangle = -\tfrac{1}{2}\langle\iota_{\mathfrak{sp}(1)}\omega, k\otimes i\rangle = -\tfrac{1}{2}\iota_{K_{\zeta_3}^{M,Sp(1)}}\omega_1 = \tfrac{1}{2}\iota_\chi\omega_2, \\ \langle\gamma, k\rangle &= -\tfrac{1}{2}\langle\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}\omega, k\rangle = -\tfrac{1}{2}\langle\iota_{\mathfrak{sp}(1)}\omega, i\otimes j\rangle = -\tfrac{1}{2}\iota_{K_{\zeta_1}^{M,Sp(1)}}\omega_2 = \tfrac{1}{2}\iota_\chi\omega_3. \end{aligned}$$

   These can be combined into $\gamma = \frac{1}{2}\iota_\chi\omega$. Furthermore, this immediately implies that $\mu = -\iota_{\mathfrak{g}}\gamma = -\frac{1}{2}\iota_{\mathfrak{g}}\iota_\chi\omega$.

2. Consider the function $\rho := -\pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\gamma\colon M \to \mathbb{R}$. First, note that for each tangent vector $v \in TM$:

$$\begin{aligned}\pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\omega(v) &= \tfrac{1}{3}(\omega_1(K^{M,Sp(1)}_{\zeta_1}, v) + \omega_2(K^{M,Sp(1)}_{\zeta_2}, v) + \omega_3(K^{M,Sp(1)}_{\zeta_3}, v))\\ &= -\tfrac{1}{3}(g^M(I_1K^{M,Sp(1)}_{\zeta_1}, v) + g^M(I_2K^{M,Sp(1)}_{\zeta_2}, v) + g^M(I_3K^{M,Sp(1)}_{\zeta_3}, v))\\ &= g^M(\chi, v).\end{aligned}$$

Since $\gamma \in \Omega^1(M, \mathfrak{sp}(1)^*)^{Spin^G_\varepsilon(m)}$ is $Sp(1)$-equivariant, we have

$$\begin{aligned}\langle \mathcal{L}_{\mathfrak{sp}(1)}\gamma, \zeta \otimes \zeta'\rangle &= \tfrac{d}{dt}\langle (L_{\exp(t\zeta)})^*\gamma, \zeta'\rangle|_{t=0} = \tfrac{d}{dt}\langle Ad^*_{\exp(t\zeta)}\gamma, \zeta'\rangle|_{t=0}\\ &= \tfrac{d}{dt}\langle \gamma, Ad_{\exp(-t\zeta)}\zeta'\rangle|_{t=0} = -\langle \gamma, [\zeta, \zeta']\rangle\end{aligned}$$

for all $\zeta, \zeta' \in \mathfrak{sp}(1)$. In particular, $\pi_{\mathbb{R}}\mathcal{L}_{\mathfrak{sp}(1)}\gamma = 0$. We conlude

$$d\rho = -d\pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\gamma = -\pi_{\mathbb{R}}d\iota_{\mathfrak{sp}(1)}\gamma = -\pi_{\mathbb{R}}\mathcal{L}_{\mathfrak{sp}(1)}\gamma + \pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}d\gamma = \pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\omega = \iota_\chi g.$$

This implies that $\mathrm{grad}(\rho) = \chi$.

$$\mathcal{I}_\zeta d\rho(v) = d\rho(\mathcal{I}_\zeta(v)) = g^M(\chi, \mathcal{I}_\zeta(v)) = \iota_\chi\omega_\zeta(v) \text{ for all } v \in TM$$

and finally

$$d\mathcal{I}_\zeta d\rho = d\iota_\chi\omega_\zeta = 2d\langle\gamma, \zeta\rangle = 2\omega_\zeta,$$

so ρ is a hyperkähler potential.

3. Swann proves in [Swa91, Prop 5.6] that $f \in C^\infty(M, \mathbb{R})$ is a hyperkähler potential iff $\nabla(df) = g_M$. Therefore, $\nabla(d\rho) = g^M$. Using $\chi = \mathrm{grad}(\rho)$, we conclude that for all $x \in M$ and $v, w \in T_xM$

$$\begin{aligned}g^M(\nabla_v\chi, w) &= \nabla_v(g^M(\chi, w)) - g^M(\chi, \nabla_v w) = \nabla_v(d\rho(w)) - d\rho(\nabla_v w)\\ &= \nabla_v(d\rho)(w) = g^M(v, w),\end{aligned}$$

and therefore $\nabla_v\chi = v$ for all $v \in TM$.

4. The $Spin^G_\varepsilon(3)$-invariance of the potential $\rho\colon M \to \mathbb{R}$ implies the invariance of the 1-form $d\rho$. The group $Spin^G_\varepsilon(3)$ acts isometrically on M and hence $\chi = \mathrm{grad}(\rho)$ is $Spin^G_\varepsilon(3)$-equivariant, i.e. $T_xL_h(\chi_x) = \chi_{hx}$ for all $h \in Spin^G_\varepsilon(3), x \in M$. This implies that for all $x \in M, \nu \in \mathfrak{spin}^G_\varepsilon(3)$:

$$\langle (\mathcal{L}_{\mathfrak{spin}^G_\varepsilon(3)}\chi)_x, \nu\rangle = \tfrac{d}{dt}T_{\exp(t\nu)x}L_{\exp(-t\nu)}(\chi_{\exp(t\nu)x})|_{t=0} = \tfrac{d}{dt}\chi_x|_{t=0} = 0.$$

Therefore $\mathcal{L}_{\mathfrak{spin}^G_\varepsilon(3)}\chi = 0$ and in particular $\mathcal{L}_{\mathfrak{sp}(1)}\chi = 0$ and $\mathcal{L}_{\mathfrak{g}}\chi = 0$. The Lie derivatives of the symplectic forms are

$$\begin{aligned}\mathcal{L}_\chi\omega_1 &= -\mathcal{L}_{I_2K^{M,Sp(1)}_{\zeta_2}}\omega_1 = -d\iota_{I_2K^{M,Sp(1)}_{\zeta_2}}\omega_1 = -d\iota_{K^{M,Sp(1)}_{\zeta_2}}\omega_3 = -\mathcal{L}_{K^{M,Sp(1)}_{\zeta_2}}\omega_3,\\ \mathcal{L}_\chi\omega_2 &= -\mathcal{L}_{I_3K^{M,Sp(1)}_{\zeta_3}}\omega_2 = -d\iota_{I_3K^{M,Sp(1)}_{\zeta_3}}\omega_2 = -d\iota_{K^{M,Sp(1)}_{\zeta_3}}\omega_1 = -\mathcal{L}_{K^{M,Sp(1)}_{\zeta_3}}\omega_1,\\ \mathcal{L}_\chi\omega_3 &= -\mathcal{L}_{I_1K^{M,Sp(1)}_{\zeta_1}}\omega_3 = -d\iota_{I_1K^{M,Sp(1)}_{\zeta_1}}\omega_3 = -d\iota_{K^{M,Sp(1)}_{\zeta_1}}\omega_2 = -\mathcal{L}_{K^{M,Sp(1)}_{\zeta_1}}\omega_2.\end{aligned}$$

We use Proposition 3.2.3 to obtain

$$\begin{aligned}\mathcal{L}_\chi\omega_1 &= -\mathcal{L}_{K_{\zeta_2}^{M,Sp(1)}}\omega_3 = -\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, j\otimes k\rangle = 2\omega_1,\\ \mathcal{L}_\chi\omega_2 &= -\mathcal{L}_{K_{\zeta_3}^{M,Sp(1)}}\omega_1 = -\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, k\otimes i\rangle = 2\omega_2,\\ \mathcal{L}_\chi\omega_3 &= -\mathcal{L}_{K_{\zeta_1}^{M,Sp(1)}}\omega_2 = -\langle \mathcal{L}_{\mathfrak{sp}(1)}\omega, i\otimes j\rangle = 2\omega_3,\end{aligned}$$

and hence $\mathcal{L}_\chi\omega = 2\omega$.

For two vector fields $v, w \in \Gamma(M, TM)$ we have $[\mathcal{L}_v, \iota_w] = \iota_{[v,w]} = -\iota_{\mathcal{L}_w v}$ and therefore $[\mathcal{L}_\chi, \iota_{\mathfrak{spin}_\varepsilon^G(3)}] = -\iota_{\mathcal{L}_{\mathfrak{spin}_\varepsilon^G(3)}\chi} = 0$. In other words, $\mathcal{L}_\chi\iota_{\mathfrak{spin}_\varepsilon^G(3)} = \iota_{\mathfrak{spin}_\varepsilon^G(3)}\mathcal{L}_\chi$ and in particular, $\mathcal{L}_\chi\iota_{\mathfrak{g}} = \iota_{\mathfrak{g}}\mathcal{L}_\chi$ and $\mathcal{L}_\chi\iota_{\mathfrak{sp}(1)} = \iota_{\mathfrak{sp}(1)}\mathcal{L}_\chi$.

We use this to compute

$$\begin{aligned}\mathcal{L}_\chi\gamma &= \tfrac{1}{2}\mathcal{L}_\chi(\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}\omega) = \tfrac{1}{2}\pi_{\mathfrak{sp}(1)^*}\mathcal{L}_\chi\iota_{\mathfrak{sp}(1)}\omega\\ &= \tfrac{1}{2}\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}\mathcal{L}_\chi\omega = \tfrac{1}{2}\pi_{\mathfrak{sp}(1)^*}\iota_{\mathfrak{sp}(1)}2\omega\\ &= 2\gamma.\end{aligned}$$

Finally, we obtain

$$d\mu(\chi) = \mathcal{L}_\chi\mu = -\mathcal{L}_\chi\iota_{\mathfrak{g}}\gamma = -\iota_{\mathfrak{g}}\mathcal{L}_\chi\gamma = -2\iota_{\mathfrak{g}}\gamma = 2\mu.$$

5. Since $\mathcal{L}_\chi\iota_{\mathfrak{sp}(1)} = \iota_{\mathfrak{sp}(1)}\mathcal{L}_\chi$ and $\mathcal{L}_\chi\gamma = 2\gamma$, we get

$$\begin{aligned}g^M(\chi,\chi) &= d\rho(\chi) = \mathcal{L}_\chi\rho = -\mathcal{L}_\chi(\pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\gamma)\\ &= -\pi_{\mathbb{R}}\mathcal{L}_\chi(\iota_{\mathfrak{sp}(1)}\gamma) = -\pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\mathcal{L}_\chi\gamma = -2\pi_{\mathbb{R}}\iota_{\mathfrak{sp}(1)}\gamma\\ &= 2\rho.\end{aligned}$$

□

3.2.7 Corollary. *In the proof of the previous proposition, we also proved the following useful formulae:*

1. $\mathcal{L}_\chi\omega = 2\omega$,
2. $\mathcal{L}_\chi\gamma = 2\gamma$,
3. $\mathcal{L}_{\mathfrak{spin}_\varepsilon^G(3)}\chi = 0$, *and in particular* $\mathcal{L}_{\mathfrak{sp}(1)}\chi = 0$ *and* $\mathcal{L}_{\mathfrak{g}}\chi = 0$,
4. $\mathcal{L}_\chi\rho = 2\rho$.

3.2.8 Example. Consider the hyperkähler manifold $M = \mathbb{H}^n$ from Example 3.2.2 with the action of $Sp(1)$ on $\mathbb{H}^n$ given by left multiplication in each component. The fundamental vector field for this action is

$$(K_\zeta^{\mathbb{H}^n,Sp(1)})_x = \tfrac{d}{dt}\exp(t\zeta)x|_{t=0} = \zeta x \in \mathbb{H}^n = T_x\mathbb{H}^n \text{ for all } x \in \mathbb{H}^n, \zeta \in \mathfrak{sp}(1).$$

We obtain

$$\mathcal{I}_\zeta(K_\zeta^{\mathbb{H}^n,Sp(1)})_x = \zeta\zeta x = -x \in \mathbb{H}^n = T_x\mathbb{H}^n \text{ for all } \zeta \in \mathfrak{sp}(1), \|\zeta\|^2 = 1.$$

The vector field $\chi = -\mathcal{I}_\zeta K_\zeta^{\mathbb{H}^n,Sp(1)}$ is independent of $\zeta \in \mathfrak{sp}(1)$, $\|\zeta\|^2 = 1$. This is the *Euler vector field* $\chi_x = x \in \mathbb{H}^n = T_x\mathbb{H}^n$. The hyperkähler potential is

$$\rho(x) = \tfrac{1}{2}g^M(\chi_x,\chi_x) = \tfrac{1}{2}\|\chi_x\|^2 = \tfrac{1}{2}\|x\|^2.$$

3.3 Configuration space

We will now describe the configuration space for the generalized Seiberg-Witten equations in dimensions three and four, which is a product of an affine space of connections and the space of spinors. Therefore, we fix a compact Lie group G and a central element $\varepsilon \in Z(G)$ satisfying $\varepsilon^2 = 1$. We also fix a $Spin^G_\varepsilon(m)$-structure $Q_m \to Z$ on a oriented Riemannian manifold Z ($m = \dim(Z) \in \{3,4\}$) and a hyperkähler manifold M with permuting $Spin^G_\varepsilon(m)$-action. To simplify notation, we write $\hat{G}_m$ for $Spin^G_\varepsilon(m)$.

3.3.1 Connections

We have seen in Note 3.1.3 that the Lie algebra $\hat{\mathfrak{g}}_m$ of $\hat{G}_m = Spin^G_\varepsilon(m)$ splits as a direct sum $\hat{\mathfrak{g}}_m = \mathfrak{so}(m) \oplus \mathfrak{g}$. Let φ_Z be the Levi-Civita connection on $P_{SO(m)} \to Z$.

3.3.1 Definition. By $\mathscr{A}_m$ we denote the affine space of connections on $Q_m \to Z$ with $\mathfrak{so}(m)$-component given by the lift of the Levi-Civita connection φ_Z, i.e.

$$\mathscr{A}_m := \left\{ A \in \mathscr{A}(Q_m) \;\middle|\; \mathrm{pr}_{\mathfrak{so}(m)} \circ A = \pi^*_{SO(m)}\varphi_Z \right\}.$$

3.3.2 Lemma. *The space $\mathscr{A}_m$ is an affine space for the vector space $\Omega^1(Q_m, \mathfrak{g})^{\hat{G}_m}_{hor} \cong \Omega^1(Z, \mathfrak{g}_{Q_m})$.*

Proof. Let $A, A' \in \mathscr{A}_m$ be two connections. Then $A - A' \in \Omega^1(Q_m, \hat{\mathfrak{g}}_m)^{\hat{G}_m}_{hor}$. From $\mathrm{pr}_{\mathfrak{so}(m)} \circ A = \pi^*_{SO(m)}\varphi_Z = \mathrm{pr}_{\mathfrak{so}(m)} \circ A'$, we obtain that actually $A - A' \in \Omega^1(Q_m, \mathfrak{g})^{\hat{G}_m}_{hor}$. Conversely, let $A \in \mathscr{A}_m$ be a connection and $\alpha \in \Omega^1(Q_m, \mathfrak{g})^{\hat{G}_m}_{hor}$. Then $A + \alpha$ is again a connection 1-form on Q_m and

$$\mathrm{pr}_{\mathfrak{so}(m)} \circ (A + \alpha) = \mathrm{pr}_{\mathfrak{so}(m)} \circ A = \pi^*_{SO(m)}\varphi_Z. \qquad \square$$

3.3.3 Note. We obtain an isomorphism

$$\mathscr{A}_m \to \mathscr{A}(Q_m \to P_{SO(m)})^{Spin(m)}, \qquad A \mapsto \mathrm{pr}_{\mathfrak{g}} \circ A = A - \pi^*_{SO(m)}\varphi_Z,$$

with an inverse

$$\mathscr{A}(Q_m \to P_{SO(m)})^{Spin(m)} \to \mathscr{A}, \qquad a \mapsto \pi^*_{SO(m)}\varphi_Y + a.$$

Here $\mathscr{A}(Q_m \to P_{SO(m)})^{Spin(m)}$ is the space of $Spin(m)$-invariant connection 1-forms on the $Spin(m)$-equivariant principal G-bundle $Q_m \to Q_m/G = P_{SO(m)}$.

3.3.4 Notation. If Z is a compact oriented Riemannian manifold, then for a $Spin^G_\varepsilon(m)$-invariant smooth function $f\colon Q_m \to \mathbb{R}$, we denote by $\pi_! f$ the induced function $\pi_! f\colon Z \to \mathbb{R}$. To simplify notation, we define

$$\int_Z f := \int_Z \pi_! f * 1.$$

Here $*\colon \Omega^0(Z,\mathbb{R}) \to \Omega^m(Z,\mathbb{R})$ is the Hodge star operator and $1 \in C^\infty(Z,\mathbb{R})$ is the constant function with value 1. Therefore, $*1$ is the volume form on Z.

3.3.5 Remark. Let $Z = Q_m/\hat{G}_m$ be compact. Given an *Ad*-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$, the L^2-metric on $\Omega^1(Q_m,\mathfrak{g})^{\hat{G}_m}_{hor} \cong \Omega^1(Q_m/\hat{G}_m,\mathfrak{g}_{Q_m})$ defines a Riemannian metric on $\mathscr{A}_m$, considered as an (infinite dimensional) manifold:

$$g^{\mathscr{A}}: T_A\mathscr{A}_m \otimes T_A\mathscr{A}_m = \Omega^1(Q_m,\mathfrak{g})^{\hat{G}_m}_{hor} \otimes \Omega^1(Q_m,\mathfrak{g})^{\hat{G}_m}_{hor} \to \mathbb{R},$$
$$\alpha\otimes\beta \mapsto \int_Z \langle\alpha\wedge *\beta\rangle_{\mathfrak{g}}.$$

Here we implicitly used the isomorphism $\Omega^m(Q_m,\mathbb{R})^{\hat{G}_m}_{hor} \cong \Omega^m(Z,\mathbb{R})$.

3.3.2 Spinors

Let $Q_m \to P_{SO(m)} \times_Z P_{G/\varepsilon}$ be a $Spin^G_\varepsilon(m)$-structure on Z.

3.3.6 Definition (spinor). A smooth $\hat{G}_m$-equivariant map $u\colon Q_m \to M$ is said to be a *(generalized) spinor*. We will denote the space of spinors by

$$\mathscr{N}_m := C^\infty(Q_m, M)^{\hat{G}_m}.$$

3.3.7 Remark. Using Proposition 2.1.22, we can also interpret spinors as sections of the associated bundle $Q_m \times_{\hat{G}_m} M \to Z$.

3.3.8 Proposition. *The space of spinors $\mathscr{N}_m = C^\infty(Q_m,M)^{\hat{G}_m}$ is a smooth manifold. The tangent space at $u \in \mathscr{N}_m$ is $T_u\mathscr{N}_m = \Gamma_c(Q_m,u^*TM)^{\hat{G}_m} \cong C^\infty_c(Q_m,TM)^{\hat{G}_m}_u$, where $C^\infty_c(Q_m,TM)^{\hat{G}_m}_u := \left\{ v\in C^\infty_c(Q_m,TM)^{\hat{G}_m} \,\middle|\, \pi_M\circ v = u\right\} \subset C^\infty(Q_m,TM)^{\hat{G}_m}$. The projection of the tangent bundle is given by composition with π_M:*

$$T\mathscr{N}_m \subset C^\infty(Q_m,TM)^{\hat{G}_m} \xrightarrow{\pi_{\mathscr{N}}} \mathscr{N}_m,$$
$$v \mapsto \pi_M\circ v.$$

If Z is compact, then $T\mathscr{N}_m = C^\infty(Q_m,TM)^{\hat{G}_m}$ and there is a metric

$$T_u\mathscr{N}_m \times T_u\mathscr{N}_m \ni v,w \mapsto g^{\mathscr{N}}(v,w) := \int_Z g^M_u(v,w),$$

where g^M_u is the pullback metric on u^*TM. The connector $\mathcal{K}^M$ of the Levi-Civita connection on M induces a connector $TT\mathscr{N}_m \to T\mathscr{N}_m$:

$$\mathcal{K}^{\mathscr{N}}: C^\infty(Q_m,TTM)^{\hat{G}_m} \to C^\infty(Q_m,TM)^{\hat{G}_m},\ \xi\mapsto \mathcal{K}^M\circ\xi.$$

The corresponding covariant derivative $\nabla^{\mathscr{N}}$ is compatible with the metric $g^{\mathscr{N}}$ and torsion-free.

Proof. We discuss this in Appendix A. ☐

3.3.9 Remark. Proposition 2.1.22 shows that $C^\infty(Q_m, M)^{\hat{G}_m} \cong \Gamma(Z, Q_m \times_{\hat{G}_m} M)$, so we can think of spinors as sections in the associated fibre bundle $Q_m \times_{\hat{G}_m} M$ with typical fibre M. If Z is compact, then this isomorphism is smooth (cf. Note A.2.7). However, the description of spinors as equivariant maps will be more suitable for our purposes.

3.3.10 Definition (Configuration space). Let Z be an oriented Riemannian manifold of dimension $m = \dim(Z) \in \{3, 4\}$ and $Q_m \to P_{SO(m)} \times_Z P_{G/\varepsilon}$ a $Spin^G_\varepsilon(m)$-structure on a Z. The *configuration space for the Seiberg-Witten equations* is the product of the space of spinors $\mathscr{N}_m$ and the affine space of connections $\mathscr{A}_m$:

$$\mathscr{C}_m := \mathscr{N}_m \times \mathscr{A}_m.$$

Note that the spaces of spinors and connections and the configuration space as well as the gauge group depend on the $Spin^G_\varepsilon(m)$-structure. Since we always consider one fixed $Spin^G_\varepsilon(m)$-structure at a time, we use the short notations $\mathscr{N}_m, \mathscr{A}_m, \mathscr{C}_m$ and $\mathscr{G}_m$ althought they do not reflect these dependencies.

3.3.11 Proposition. *The configuration space $\mathscr{C}_m = \mathscr{N}_m \times \mathscr{A}_m$ is an (infinite dimensional) smooth manifold. If Z is compact, then $\mathscr{C}_m$ is a Riemannian manifold with a metric $g^{\mathscr{C}} = \mathrm{pr}^*_{\mathscr{N}} g^{\mathscr{N}} + \mathrm{pr}^*_{\mathscr{A}} g^{\mathscr{A}}$. Furthermore, the covariant derivative $\nabla^{\mathscr{N}}$ on $T\mathscr{N}$ and the tautological covariant derivative on the vector space $\mathscr{A}_m$ induce a metric compatible covariant derivative $\nabla^{\mathscr{C}}$ on $T\mathscr{C}_m$ with vanishing torsion.*

Proof. We discuss this in Lemma A.2.3, Lemma A.2.10 and Proposition A.2.11. ☐

3.3.3 The action of the gauge group on connections and spinors

3.3.12 Lemma. *The gauge group $\mathscr{G}_m$ acts by pullback (from the right) on the space of connections $\mathscr{A}_m$.*

Proof. Let $A \in \mathscr{A}_m$ be a connection and $\psi \in \mathscr{G}_m$ a gauge transformation. We have to prove that $\psi^* A \in \mathscr{A}_m$. Since ψ fixes the bundle $P_{SO(m)}$, we get

$$\mathrm{pr}_{\mathfrak{so}(m)} \circ \psi^* A = \psi^* \,\mathrm{pr}_{\mathfrak{so}(m)} \circ A = \psi^* \pi^*_{SO(m)} \varphi_Y = \pi^*_{SO(m)} \varphi_Z.$$ ☐

3.3.13 Lemma. *The gauge group $\mathscr{G}_m$ acts by pullback (from the right) on the space of spinors $\mathscr{N}_m$. This action can be written as $\mathscr{N}_m \times C^\infty(Q, G)^{\hat{G}_m} \ni (u, g) \mapsto g^{-1} u \in \mathscr{N}_m$.*

Proof. Let $\psi \in \mathscr{G}_m$ and $g \in C^\infty(Q_m, G)^{\hat{G}_m}$ be the equivariant map satisfying $\psi(p) = pg(p)$ for all $p \in Q_m$. Then

$$\psi^* u(p) = u(\psi(p)) = u(pg(p)) = (g(p))^{-1} u(p).$$ ☐

3.4 Covariant derivative

We will now define the first ingredient to our Dirac operator, the covariant derivative. Let $Q_m \to P_{SO(m)} \times_Z P_{G/\varepsilon}$ be a $Spin^G_\varepsilon(m)$-structure on Z and M a hyperkähler manifold with a permuting $Spin^G_\varepsilon(m)$-structure.

3.4.1 Definition. For a connection 1-form $A \in \mathscr{A}_m$ we define a *covariant derivative*

$$d_A^M \colon C^\infty(Q_m, M)^{\hat{G}_m} \to C^\infty(Q_m, (\mathbb{R}^m)^* \otimes TM)^{\hat{G}_m},$$
$$\langle (d_A^M u)(p), w\rangle := Tu(\tilde{w}) \text{ for } w \in \mathbb{R}^n.$$

Here $\tilde{w} \in T_pQ_m$ is the horizontal lift of $\pi_{SO}(p)(w) \in T_{\pi_Z(p)}Z$.

We will also use the following variation of the concept of covariant derivative: Consider a $\hat{G}_m$-equivariant vector bundle $E \to M$ with a fixed $\hat{G}_m$-equivariant connection on E and the corresponding connector $\mathcal{K}\colon TE \to E$. We define

$$d_{A,\mathcal{K}}^E \colon C^\infty(Q_m, E)^{\hat{G}_m} \to C^\infty(Q_m, (\mathbb{R}^m)^* \otimes E)^{\hat{G}_m},$$
$$d_{A,\mathcal{K}}^E v := (\mathrm{id}_{(\mathbb{R}^m)^*} \otimes \mathcal{K}) \circ d_A^E v,\ \ v \in C^\infty(Q_m, E)^{\hat{G}_m}.$$

Here $d_A^E \colon C^\infty(Q_m, E)^{\hat{G}_m} \to C^\infty(Q_m, (\mathbb{R}^m)^* \otimes TE)^{\hat{G}_m}$ is the covariant derivative defined above for the total space of the vector bundle $E \to M$.

3.4.2 Remark. For a representation $M = V$ of $\hat{G}_m$ the map d_A^M is the *covariant exterior derivative* from Definition 2.1.31 if we identify $C^\infty(Q_m, (\mathbb{R}^n)^* \otimes V)^{\hat{G}_m} \cong \Omega^1(Q_m, V)^{\hat{G}_m}_{hor}$.

3.4.3 Remark. Notice the difference between d_A^M and $d_{A,\mathcal{K}}^E$. While d_A^M generalizes the exterior covariant derivative, $d_{A,\mathcal{K}}^E$ is a combination of d_A^E and the connector $\mathcal{K}\colon TE \to E$. Consider a bundle of frames $F \to M$ in $E \to M$ with structure group $G \subset GL_k(\mathbb{R})$. Then $F \times_G \mathbb{R}^k = E$. Lifting $v \in C^\infty(Q_m, E)^{\hat{G}_m}$ to $\hat{v}\colon (\pi_M \circ v)^*F \to (\mathbb{R}^m)^* \otimes \mathbb{R}^k$, we can interpret the lift of $d_{A,\mathcal{K}}^E v$ to $(\pi_M \circ v)^*F$ as the exterior covariant derivative of $\hat{v}$ with respect to a connection 1-form on $(\pi_M \circ v)^*F$ induced by the connection 1-form A on Q_m and the connection 1-form on F corresponding to the connector $\mathcal{K}$. This approach is used in [Pid04].

The other extreme would be to consider the induced covariant derivative $\nabla^{A,\mathcal{K}}$ on the vector bundle $\pi_!(\pi_M \circ v)^*E \to Z$. From this perspective $d_{A,\mathcal{K}}^E v$ is the lift of $\nabla^{A,\mathcal{K}} s \in \Gamma(Z, T^*Z \otimes \pi_!(\pi_M \circ v)^*E)$ to Q_m, where $s \in \Gamma(Z, \pi_!(\pi_M \circ v)^*E)$ is the section corresponding to $v \in C^\infty(Q_m, E)^{\hat{G}_m}$.

However, for our purposes it is more convenient to work with $\hat{G}_m$-equivariant maps from Q_m to M or the $\hat{G}_m$-equivariant vector bundles. Moreover, this approach makes it easier to understand the generalized Dirac operator as a generalization of the usual Dirac operator, where the spinor representation is replaces by the hyperkähler manifold M.

3.4.4 Lemma. *Let $A \in \mathscr{A}_m$ and $\mathcal{K}$ the connector of a connection on $TM \to M$ with vanishing torsion. Then the covariant derivative*

$$d_A^M \colon C^\infty(Q_m, M)^{\hat{G}_m} \to C^\infty(Q_m, (\mathbb{R}^m)^* \otimes TM)^{\hat{G}_m}$$

is smooth and we have

1. $d_A^M u \in C^\infty(Q_m, (\mathbb{R}^n)^* \otimes TM)_u^{\hat{G}_m}$ *for* $u \in C^\infty(Q_m, M)^{\hat{G}_m}$ *and therefore defines an element* $\nabla^A u \in \varGamma(Z, T^*Z \otimes \pi_! u^* TM)$*,*
2. $T d_A^M = (\mathrm{id}_{(R^m)^*} \otimes \kappa_M) \circ d_A^{TM}$*,*
3. $(\mathrm{id}_{(R^m)^*} \otimes \mathcal{K}) \circ T d_A^M = d_{A,\mathcal{K}}^{TM}$*,*
4. $Tu(\mathrm{pr}_{\mathscr{H}_A}(v)) = Tu(v) + (K_{A(v)}^{M,\hat{G}_m})_{u(p)} \in T_{u(p)}M$ *for* $u \in \mathscr{N}_m, v \in T_p Q_m$*.*

Proof.

1. Let $w \in \mathbb{R}^m$, $p \in Q_m$ and $\tilde{w} \in T_p Q_m$ the horizontal lift of $\pi_{SO}(p)(w) \in TZ$. Then
$$\pi_M(\langle (d_A^M u)(p), w\rangle) = \pi_M(Tu(\tilde{w})) = u(\pi_{Q_m}(\tilde{w})) = u(p)$$
and therefore $d_A^M u \in C^\infty(Q_m, (\mathbb{R}^n)^* \otimes TM)_u^{\hat{G}_m}$. The image of $d_A^M u$ under the isomorphism $C^\infty(Q_m, (\mathbb{R}^n)^* \otimes TM)_u^{\hat{G}_m} \cong \varGamma(Q_m, (\mathbb{R}^n)^* \otimes u^*TM)^{\hat{G}_m} \cong \varGamma(Z, T^*Z \otimes \pi_! u^* TM)$ is denoted by $\nabla^A u$.

2. Let $v \in C^\infty(Q_m, TM)_u^{\hat{G}_m}$ and $\gamma \colon \mathbb{R} \to C^\infty(Q_m, M)^{\hat{G}_m}$ a smooth path representing $v = \frac{d}{dt}\gamma(t)|_{t=0}$. Let $w \in \mathbb{R}^m$ and $p \in Q_m$. Denote the horizontal lift of $\pi_{SO}(p)(w) \in T_{\pi_Z(p)}Z$ by $\tilde{w} \in T_p Q_m$. Let $\sigma \colon \mathbb{R} \to Q_m$ be a smooth path representing $\frac{d}{dt}\sigma(t)|_{t=0} = \tilde{w} \in T_p Q_m$. Then
$$\begin{aligned}\langle T_u d_A^M(v)(p), w\rangle &= \langle \tfrac{d}{dt}(d_A^M(\gamma(t)))(p)|_{t=0}, w\rangle = \tfrac{d}{dt}\langle (d_A^M(\gamma(t)))(p), w\rangle|_{t=0}\\ &= \tfrac{d}{dt} T\gamma(t)(\tilde{w})|_{t=0} = \tfrac{d}{dt}\tfrac{d}{ds}\gamma(t)(\sigma(s))|_{s=0}|_{t=0}\\ &= \kappa_M \tfrac{d}{ds}\tfrac{d}{dt}\gamma(t)(\sigma(s))|_{t=0}|_{s=0} = \kappa_M \tfrac{d}{ds} v(\sigma(s))|_{s=0} = \kappa_M Tv(\tilde{w})\\ &= \langle (\mathrm{id}_{(R^m)^*} \otimes \kappa_M) \circ d_A^{TM}(v)(p), w\rangle.\end{aligned}$$

3. Since the torsion of the connection on $TM \to M$ with connector $\mathcal{K}$ vanishes, we can use Theorem 2.1.39 to prove the third assertion:
$$(\mathrm{id}_{(R^m)^*} \otimes \mathcal{K}) \circ T(d_A^M) = (\mathrm{id}_{(R^m)^*} \otimes (\mathcal{K} \circ \kappa_M)) \circ d_A^{TM} = (\mathrm{id}_{(R^m)^*} \otimes \mathcal{K}) \circ d_A^{TM}.$$

4. For $\xi \in \hat{G}_m$, let $K_\xi^{Q_m,\hat{G}_m} \in \varGamma(Q_m, TQ_m)$ be the fundamental vector field for the right $\hat{G}_m$-action on Q_m. Then, for all $p \in Q_m$:
$$\begin{aligned}Tu((K_\xi^{Q_m,\hat{G}_m})_p) &= \tfrac{d}{dt} u(p\exp(t\xi))|_{t=0} = \tfrac{d}{dt}\exp(-t\xi)u(p)|_{t=0} = -(K_\xi^{M,\hat{G}_m})_{u(p)}\\ &= -\left(K^{M,\hat{G}_m}_{A(K_\xi^{M,\hat{G}_m})}\right)_{u(p)}.\end{aligned}$$

Since $A(v) = 0$ for horizontal $v \in TQ_m$, this equation can be written as

$$Tu(\mathrm{pr}_{\mathscr{V}_A}(v)) = -(K^{M,\hat{G}_m}_{A(v)})_{u(p)} \text{ for all } v \in TQ_m.$$

Finally, we obtain

$$Tu(\mathrm{pr}_{\mathscr{H}_A}(v)) = Tu(v) - Tu(\mathrm{pr}_{\mathscr{V}_A}(v)) = Tu(v) + (K^{M,\hat{G}_m}_{A(v)})_{u(p)} \text{ for all } v \in T_pQ_m.$$

□

3.4.5 Remark. Under the isomorphism $C^\infty(Q_m, (\mathbb{R}^m)^* \otimes TM)^{\hat{G}_m} \cong \Omega^1(Q_m, TM)^{\hat{G}_m}_{hor}$, the covariant exterior derivative $d^M_A u$ corresponds to $\mathrm{pr}^*_{\mathscr{H}_A} Tu$. The last assertion in Lemma 3.4.4 gives an explicit formula for $\mathrm{pr}^*_{\mathscr{H}_A} Tu$.

3.4.6 Example. For $G = S^1$ and $M = \mathbb{H}$ as in Example 3.2.2 we have

$$\mathscr{N}_3 = C^\infty(Q_3, \mathbb{H})^{Spin^c(3)} \cong \Gamma(Y, \mathcal{W}) \quad \text{and} \quad \mathscr{N}_4 = C^\infty(Q_4, \mathbb{H})^{Spin^c(4)} \cong \Gamma(X, \mathcal{W}^+).$$

In this case the generalized spinors are exactly the usual spinors. The covariant derivative is the usual covariant derivative.

3.5 Clifford multiplication and hyperkähler manifolds

We will now study the Clifford multiplication, which is the second ingedient for a Dirac operator. Let (M, g^M, I_1, I_2, I_3) be a hyperkähler manifold with a permuting $Spin^G_\varepsilon(3)$-action. We also have an induced action of $Spin^G_\varepsilon(3)$ on TM.

3.5.1 Clifford multiplication in three dimensions

The first observation is that we can use the scalar multiplication to construct an action of Cl_3 on TM.

3.5.1 Lemma. *The tangent bundle $TM \to M$ is a bundle of Cl_3-modules. The corresponding homomorphism $c_3\colon \mathbb{R}^3 \otimes TM \to TM$ is $Spin^G_\varepsilon(3)$-equivariant.*

Proof. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Since $(-I_\ell)^2 = -\mathrm{id}_{TM}$, the map

$$\begin{aligned} \mathbb{R}^3 &\to \mathrm{End}\,(TM), \\ e_\ell &\mapsto -I_\ell \text{ for } \ell \in \{1,2,3\} \end{aligned}$$

induces a homomorphism $c_3\colon Cl_3 \to \mathrm{End}\,(TM)$. Identifying $\mathbb{R}^3$ with $\mathrm{Im}(\mathbb{H})$, we have

$$c_3(h) = \mathcal{I}_{\overline{h}}\colon TM \to TM \text{ for all } h \in \mathrm{Im}(\mathbb{H}).$$

We will also denote the restriction $\mathbb{R}^3 \otimes TM \to TM$, $h \otimes v \mapsto c_3(h)(v) = \mathcal{I}_{\bar{h}}(v)$ by c_3. Let $[(q,g)] \in Spin^G_\varepsilon(3)$. Since the $Spin^G_\varepsilon(3)$-action is permuting, we have

$$g_* q_* c_3(h \otimes v) = g_* q_* \mathcal{I}_{\bar{h}}(v) = q_* \mathcal{I}_{\bar{h}}(g_* v) = q_* \mathcal{I}_{\bar{h}}(\bar{q}_* q_* g_* v) = \mathcal{I}_{q\bar{h}\bar{q}}(q_* g_* v) = c_3(qh\bar{q} \otimes q_* g_* v)$$

for all $h \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ and $v \in TM$. Therefore, $c_3 \colon \mathbb{R}^3 \otimes TM \to TM$ is $Spin^G_\varepsilon(3)$-equivariant. □

3.5.2 Note. We use the action of Cl_3 on TM induced by $e_\ell \mapsto -I_\ell$ and not $e_\ell \mapsto I_\ell$. Therefore,

$$c_3(vol_3) = c_3(e_1 e_2 e_3) = c_3(e_1) c_3(e_2) c_3(e_3) = (-I_1)(-I_2)(-I_3) = \mathrm{id}_{TM}\,.$$

This choice is the analogue of the choise of the Cl_3-representation S in Section 2.3.2, where the volume element also acts as the identity. It is also possible to use the other Cl_3-module structure to define a Dirac operator. However this choice will be useful in Chapter 5, where we study the Seiberg-Witten equations on the cylinder.

3.5.3 Remark. With the help of the isomorphism $g^\sharp \colon (\mathbb{R}^3)^* \cong \mathbb{R}^3$ induced by the standard metric on $\mathbb{R}^3$ we can also interpret the Clifford multiplication as a $Spin^G_\varepsilon(3)$-equivariant homomorphism

$$\begin{aligned} (\mathbb{R}^3)^* \otimes TM &\to TM, \\ x \otimes v &\mapsto c_3(g^\sharp(x))(v). \end{aligned}$$

We also have the corresponding map $c_3 \colon \Gamma(M, (\mathbb{R}^3)^* \otimes TM) \to \Gamma(M, TM)$ and for a $Spin^G_\varepsilon(3)$-structure $Q_3 \to P_{SO(3)} \times_Y P_{G/\varepsilon}$, this induces a smooth map

$$C^\infty(Q_3, (R^3)^* \otimes TM)^{Spin^G_\varepsilon(3)} \to C^\infty(Q_3, TM)^{Spin^G_\varepsilon(3)},$$

which we also denote by c_3. This will be the Clifford multiplication used in the definition of the Dirac operator on three-dimensional manifolds.

3.5.4 Lemma. *Let $\mathcal{K} \colon TTM \to TM$ be the connector for the Levi-Civita connection on M.*

1. *The Clifford multiplication $c_3 \colon \Gamma(M, (\mathbb{R}^3)^* \otimes TM) \to \Gamma(M, TM)$ is parallel with respect to the Levi-Civita connection, i.e. $\nabla(c_3) = 0$.*
2. *$\mathcal{K}T(c_3) = c_3 \circ (\mathrm{id}_{(R^3)^*} \otimes \mathcal{K})$.*

Proof. Let $s \in \Gamma(M, (\mathbb{R}^3)^* \otimes TM)$ and $v \in TM$. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and $\{e_1^*, e_2^*, e_3^*\}$ the dual basis of $(\mathbb{R}^3)^*$. Then $s = \sum_{\ell=1}^3 e_\ell^* \otimes s_\ell$ for $s_\ell = \langle s, e_\ell \rangle$. The complex structures are parallel ($\nabla I_\ell = 0$) and hence

$$\begin{aligned} \nabla_v(c_3(s))) &= \sum_{\ell=1}^3 \nabla_v(c_3(e_\ell^* \otimes s_\ell)) = -\sum_{\ell=1}^3 \nabla_v(I_\ell(s_\ell)) \\ &= -\sum_{\ell=1}^3 I_\ell(\nabla_v s_\ell) = \sum_{\ell=1}^3 c_3(e_\ell^* \otimes \nabla_v(s_\ell)) \\ &= c_3(\nabla_v(s)). \end{aligned}$$

This implies that the Clifford multiplication $c_3\colon \Gamma(M,(\mathbb{R}^3)^*\otimes TM)\to\Gamma(M,TM)$ is parallel. Let us now consider the connector $\mathcal{K}\colon TTM\to TM$ for the Levi-Civita connection. For a vertical $v\in(\mathbb{R}^3)^*\otimes TTM$, i.e. $v=(\mathrm{id}_{(\mathbb{R}^3)^*}\otimes vl_{TM})(v_1,v_2)$ for $v_1,v_2\in(\mathbb{R}^3)^*\otimes T_xM$, we have $(\mathrm{id}_{(\mathbb{R}^3)^*}\otimes\mathcal{K})(v)=(\mathrm{id}_{(\mathbb{R}^3)^*}\otimes\mathcal{K})((\mathrm{id}_{(\mathbb{R}^3)^*}\otimes vl_{TM})(v_1,v_2))=\mathrm{pr}_2(v_1,v_2)=v_2$ and therefore

$$\begin{aligned}\mathcal{K}(Tc_3(v)) &= \mathcal{K}(\tfrac{d}{dt}c_3(v_1+tv_2)|_{t=0}) = \mathcal{K}(\tfrac{d}{dt}c_3(v_1)+tc(v_2)|_{t=0})\\ &= \mathcal{K}(vl_{TM}(c_3(v_1),c_3(v_2))) = c_3(v_2)\\ &= c_3((\mathrm{id}_{(\mathbb{R}^3)^*}\otimes\mathcal{K})((\mathrm{id}_{(\mathbb{R}^3)^*}\otimes vl_{TM})(v_1,v_2)))\\ &= c_3((\mathrm{id}_{(\mathbb{R}^3)^*}\mathcal{K}((v))\end{aligned}$$

On the other hand, if $v\in(\mathbb{R}^3)^*\otimes TTM$ is not vertical, then $0\neq(\mathrm{id}_{(\mathbb{R}^3)^*}\otimes T\pi_M)(v)\in TM$. Since $w:=T\pi_M(v)\neq 0$, we can find a section $s\in\Gamma(M,(\mathbb{R}^3)^*\otimes TM)$ such that $v=Ts(w)\in(\mathbb{R}^3)^*\otimes TTM$. Then

$$\begin{aligned}\mathcal{K}Tc_3(v) &= \mathcal{K}T(c_3\circ s)(w) = \nabla_w(c_3(s)) = \nabla_w(c_3)(s)+c_3(\nabla_w(s))\\ &= c_3((\mathrm{id}_{(\mathbb{R}^3)^*}\otimes\mathcal{K})(Ts(w))) = c_3\circ(\mathrm{id}_{(\mathbb{R}^3)^*}\otimes\mathcal{K})(v)\end{aligned}$$

for all $v\in(\mathbb{R}^3)^*\otimes TTM$ with $(\mathrm{id}_{(\mathbb{R}^3)^*}\otimes T\pi_M)(v)\neq 0$. Combining the results for vertical and non-vertical vectors, we can conclude that

$$\mathcal{K}\circ Tc_3 = c_3\circ(\mathrm{id}_{(\mathbb{R}^3)^*}\otimes\mathcal{K}).$$

□

We will now give a different description of the spinor bundles. Note that the permuting action implies that

$$\mathcal{I}_{\bar q}q_*\mathcal{I}_\zeta = \mathcal{I}_{\bar q}q_*\mathcal{I}_\zeta\bar q_*q_* = \mathcal{I}_{\bar q}\mathcal{I}_{q\zeta\bar q}q_* = \mathcal{I}_\zeta\mathcal{I}_{\bar q}q_* \text{ for all } q\in Sp(1) \text{ and } \zeta\in\mathfrak{sp}(1), \|\zeta\|^2=1.$$

Therefore the diffeomorphism $\mathcal{I}_{\bar q}q_*\colon TM\to TM$ commutes with the complex structures and thus also with the scalar multiplication. We can define an action of $Sp(1)\times G$ on TM by

$$(Sp(1)\times G)\times TM\ni((q,g),v)\mapsto g_*q_*\mathcal{I}_{\bar q}v\in TM. \tag{3.2}$$

The element $(-1,\varepsilon)$ acts as $-\mathrm{id}_{TM}$. The bundle TM with this action is denoted by E to distinguish the action from the induced one of $\hat G_3$. Let $S=\mathbb{H}$ be the standard Cl_3-representation from Section 2.3.2. The element $-1\in Sp(1)\cong Spin(3)$ also acts as $-\mathrm{id}$ on S. Therefore the action of group $\hat G_3$ on $S\otimes_{\mathbb{C}}E$ is well-defined. Here we think of S as the trivial vector bundle with fibre S and use the complex structures $R_i\in\mathrm{End}\,(S)$ and $I_1\in\mathrm{End}\,(E)$ to form the tensor product (i.e. $hi\otimes v=h\otimes I_1(v)$ for all $h\in S=\mathbb{H}, v\in E$).

3.5.5 Lemma ([Hay06, Prop 3.1.1]). *For the two complex vector bundles $TM\otimes\mathbb{C}=(TM\otimes\mathbb{C},\mathrm{id}\otimes i)$ and $E=(TM,I_1)$ and the complex vector space $S=(\mathbb{H},R_i)$ we have an isomorphism of $\hat G_3$-equivariant vector bundles over M:*

$$\begin{aligned}\Psi\colon TM\otimes\mathbb{C} &\xrightarrow{\sim} S\otimes_{\mathbb{C}}E,\\ v\otimes z &\mapsto z\otimes v - jz\otimes I_2(v).\end{aligned}$$

Furthermore, $\Psi\circ(\mathcal{I}_h\otimes\mathrm{id}_{\mathbb{C}})=(\mathcal{I}_h\otimes\mathrm{id}_E)\circ\Psi$ for all $h\in\mathbb{H}$.

Proof. From

$$\begin{aligned}
\Psi(I_1(v) \otimes z) &= z \otimes I_1(v) + jz \otimes I_1 I_2(v) = iz \otimes v - ijz \otimes I_2(v) = (L_i \otimes \mathrm{id}_E)\Psi(v \otimes z),\\
\Psi(I_2(v) \otimes z) &= z \otimes I_2(v) + jz \otimes v = -jjz \otimes I_2(v) + jz \otimes v = (L_j \otimes \mathrm{id}_E)\Psi(v \otimes z),\\
\Psi(I_3(v) \otimes z) &= z \otimes I_3(v) - jz \otimes I_1(v) = -kjz \otimes I_2(v) + kz \otimes v = (L_k \otimes \mathrm{id}_E)\Psi(v \otimes z),
\end{aligned}$$

we can conclude that $\Psi \circ (\mathcal{I}_h \otimes \mathrm{id}_{\mathbb{C}}) = (\mathcal{I}_h \otimes \mathrm{id}_E) \circ \Psi$ for all $h \in \mathbb{H}$.

For all $g \in G$, $v \in TM$, $z \in \mathbb{C}$ we have

$$\Psi(g_* v \otimes z) = z \otimes g_* v - jz \otimes I_2(g_* v) = z \otimes g_* v - jz \otimes g_* I_2(v) = (\mathrm{id}_S \otimes g_*)(\Psi(v \otimes z)).$$

This proves that Ψ is G-equivariant. For $q \in Sp(1)$, $v \in TM$ and $z \in \mathbb{C}$ we have

$$\begin{aligned}
\Psi(q_* v \otimes z) &= \Psi(q_* \mathcal{I}_{\bar q} \mathcal{I}_q v \otimes z) = z \otimes q_* \mathcal{I}_{\bar q} \mathcal{I}_q v - jz \otimes I_2(q_* \mathcal{I}_{\bar q} \mathcal{I}_q v)\\
&= z \otimes q_* \mathcal{I}_{\bar q} \mathcal{I}_q v - jz \otimes q_* \mathcal{I}_{\bar q} I_2(\mathcal{I}_q v) = (\mathrm{id}_S \otimes q_* \mathcal{I}_{\bar q})(\Psi(\mathcal{I}_q v \otimes z))\\
&= (L_q \otimes q_* \mathcal{I}_{\bar q})(\Psi(v \otimes z)),
\end{aligned}$$

and therefore, Ψ is $Sp(1)$-equivariant and thus $Spin^G_\varepsilon(3)$-equivariant. □

3.5.6 Corollary. *For the Clifford multiplication we have*

$$\Psi \circ (c_3(v) \otimes \mathrm{id}_{\mathbb{C}}) = (c_3(v) \otimes \mathrm{id}_E) \circ \Psi \text{ for all } v \in \mathbb{R}^3.$$

3.5.7 Remark. The real structure on $TM \otimes \mathbb{C}$ given by complex conjugation corresponds to the real structure on $S \otimes_{\mathbb{C}} E$ given by $r := -R_j \otimes I_2$:

$$\Psi(v \otimes \bar z) = \bar z \otimes v - j\bar z \otimes I_2(v) = jzj \otimes I_2^2(v) - zj \otimes I_2(v) = (-R_j \otimes I_2)(\Psi(v \otimes z))$$

for all $v \in TM, z \in \mathbb{C}$. The restriction of $\Psi\colon TM \otimes \mathbb{C} \to S \otimes_{\mathbb{C}} E$ to the real parts is a $Spin^G_\varepsilon(3)$-equivariant isomorphism $\Psi\colon TM \to [S \otimes_{\mathbb{C}} E]_r$, and we have a commuting diagram

$$\begin{array}{ccc}
(\mathbb{R}^3)^* \otimes TM & \xrightarrow{c_3} & TM\\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi}\\
(R^3)^* \otimes [S \otimes_{\mathbb{C}} E]_r & \xrightarrow{c_3 \otimes \mathrm{id}_E} & [S \otimes_{\mathbb{C}} E]_r
\end{array}$$

where the map at the bottom is induced by the usual Clifford multiplication $(\mathbb{R}^3)^* \otimes S \to S$.

3.5.2 Clifford multiplication in four dimensions

To define the nonlinear Dirac operator in four dimensions, we need to replace the Clifford multiplication $\mathbb{R}^4 \otimes S^+ \to S^-$. In particular, we need a replacement for S^+ and S^- and the Cl_4-module $S^+ \oplus S^-$. In Lemma 2.3.16, we have seen that $S^+ \oplus S^- \cong Cl_4 \otimes_{Cl_4^0} S$. At this point, the following proposition is useful.

3.5.8 Proposition ([LM89, Ch I Prop 5.20]). *There is a natural equivalence between the category of (ungraded) Cl_4^0-modules and the category of $\mathbb{Z}/2\mathbb{Z}$-graded Cl_4-modules. The functors are given as follows: A Cl_4^0-module V is mapped to $Cl_4 \otimes_{Cl_4^0} V$ with the left multiplication as Cl_4-module structure and the grading induced by the grading of Cl_4. A $\mathbb{Z}/2\mathbb{Z}$-graded module $W = W^0 \oplus W^1$ is mapped to its even part W^0. Since the even elements preserve the grading, this is a Cl_4^0-module.*

We can apply the same construction to TM, which replaces the Cl_3-module S. Since TM is a bundle of left Cl_4^0-modules, we obtain a bundle of $\mathbb{Z}/2\mathbb{Z}$-graded Cl_4-modules $\widehat{TM}$. Since we also have to take care of the action of $Spin_\varepsilon^G(4)$ on M and TM, we again consider the bundle E. This is the bundle TM with the $Sp(1)_+ \times G$-action from (3.2). We define

$$\widehat{TM} := Cl_4 \otimes_{Cl_4^0} E,$$

with the grading induced by the $\mathbb{Z}/2\mathbb{Z}$-grading of Cl_4, i.e. $\widehat{TM} = \widehat{TM}^0 \oplus \widehat{TM}^1$, $\widehat{TM}^0 = Cl_4^0 \otimes_{Cl_4^0} E$ and $\widehat{TM}^1 = Cl_4^1 \otimes_{Cl_4^0} E$. We also consider the action of $Spin_\varepsilon^G(3)$ on $\widehat{TM}$, which is induced by the action of $Spin(4)$ on Cl_4 by left multiplication and the action of $Sp(1)_+ \times G$ on E:

$$Spin_\varepsilon^G(4) \times \widehat{TM} \ni ([(z,g)], \beta \otimes v) \mapsto z\beta \otimes \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* v \in \widehat{TM}$$

Here $\pi_+ z$ is the image of $z \in Spin(4) \to Sp(1)_+$. This is a well-defined action since $(-1,-1,\varepsilon)$ acts as $-\operatorname{id}_E$ on E and as $-\operatorname{id}_{Cl_4}$ on Cl_4, and

$$\begin{aligned} z\beta e_\ell e_0 \otimes \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* v &= -z\beta \otimes I_\ell \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* v = -z\beta \otimes \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* I_\ell g_* v \\ &= -z\beta \otimes \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* I_\ell v \end{aligned}$$

for all $z \in Spin(4)$, $\beta \in Cl_4$, $g \in G$ and $v \in E$.

Restricting the Cl_4-action by multiplication from the left to $\mathbb{R}^4 \subset Cl_4$, we obtain a *Clifford multiplication* $c_4 \colon \mathbb{R}^4 \to \operatorname{End}\left(\widehat{TM}\right)$, which interchanges the even and odd part of $\widehat{TM}$.

To describe the action of $Spin_\varepsilon^G(4)$ on $\widehat{TM}^1$, we have to consider yet another action of $Spin_\varepsilon^G(4)$ on TM:

$$Spin_\varepsilon^G(4) \times TM \ni ([(z,g)], w) \mapsto \mathcal{I}_{\pi_- z} \mathcal{I}_{\overline{\pi_+ z}}(\overline{\pi_+ z})_* g_* w \in TM,$$

where $z \in Spin(4)$ and $(\pi_+ z, \pi_- z) \in Sp(1)_+ \times Sp(1)_-$ its image under the isomorphism $Spin(4) \cong Sp(1)_+ \times Sp(1)_-$. We denote TM with this action by $\underset{\sim}{TM}$. The following lemma is the analogue of Lemma 2.3.16:

3.5.9 Lemma. *There is an equivariant isomorphism of $Spin_\varepsilon^G(4)$-equivariant vector bundles*

$$\begin{aligned} \Psi \colon TM \oplus \underset{\sim}{TM} &\xrightarrow{\sim} \widehat{TM} \\ (v,w) &\mapsto (1 \otimes v + e_0 \otimes w) \end{aligned}$$

In particular, $\widehat{TM}^0 \cong TM$ and $\widehat{TM}^1 \cong \underset{\sim}{TM}$ as $Spin_\varepsilon^G(4)$-equivariant vector bundles.

Under the isomorphism $\mathrm{End}\left(\widehat{TM}\right) \cong \mathrm{End}\left(TM \oplus \underset{\sim}{TM}\right)$*, the Clifford multiplication on* $\widehat{TM}$ *corresponds to the map*

$$e_0 \mapsto \begin{pmatrix} 0 & -\mathrm{id}_{TM} \\ \mathrm{id}_{TM} & 0 \end{pmatrix} \qquad \textit{and} \qquad e_\ell \mapsto \begin{pmatrix} 0 & c_3(e_\ell) \\ c_3(e_\ell) & 0 \end{pmatrix} \; \textit{for } \ell \in \{1,2,3\}.$$

Proof. The same argument as in Lemma 2.3.16 applied fibrewise shows that Ψ is an isomorphism of vector bundles. Furthermore,

$$c_4(e_0)\Psi(v,w) = c_4(e_0)(1 \otimes v + e_0 \otimes w) = -1 \otimes w + e_0 \otimes v = \Psi(-w,v)$$

and

$$\begin{aligned} c_4(e_\ell)(1 \otimes v + e_0 \otimes w) &= e_\ell \otimes v + e_\ell e_0 \otimes w = 1 \otimes c_3(e_\ell \otimes w) + e_0 \otimes c_3(e_\ell \otimes v) \\ &= \Psi(c_3(e_\ell \otimes w), c_3(e_\ell \otimes v)) \end{aligned}$$

for all $x \in M$, $v, w \in T_xM$ and $\ell \in \{1,2,3\}$. This proves the asserted formula for the Clifford multiplication.

Next, we prove that Ψ is G-equivariant. For $g \in G$ and $(v,w) \in TM \oplus TM$ we have

$$\Psi(g_*v, g_*w) = 1 \otimes g_*v + e_0 \otimes g_*w = (1 \otimes g_*)(\Psi(v,w)). \tag{3.3}$$

We now consider the $Spin(4)$-actions. Note that $Spin(4) \subset Cl_4^0 \cong Cl_3 = Cl_3^+ \oplus Cl_3^-$. The image of $e_1 \in Cl_3$ under the isomorphism $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$ from Examples 2.3.4 is $(-i,i) \in \mathbb{H} \oplus \mathbb{H}$ and $\pi_+(-i,i) = -i$, $\pi_-(-i,i) = i$. Therefore,

$$\begin{aligned} (L_{e_1e_0} \otimes \mathrm{id}_E)(\Psi(v,w)) &= e_1e_0 \otimes v + e_1e_0e_0 \otimes w = -1 \otimes I_1(v) + e_0 \otimes I_1(w) \\ &= 1 \otimes \mathcal{I}_{-i}(v) + e_0 \otimes \mathcal{I}_i(w) = 1 \otimes \mathcal{I}_{\pi_+(-i,i)}(v) + e_0 \otimes \mathcal{I}_{\pi_-(-i,i)}(w) \\ &= \Psi(\mathcal{I}_{\pi_+(-i,i)}(v), \mathcal{I}_{\pi_-(-i,i)}(w)) \end{aligned}$$

for all $(v,w) \in TM \oplus TM$. The same formula holds if we replace e_1 by e_2 or e_3 and i by j or k, respectively. Since the elements $e_\ell e_0$ ($\ell \in \{1,2,3\}$) generate Cl_4^0, we obtain

$$\begin{aligned} (L_z \otimes \mathrm{id}_E)(\Psi(v,w)) &= z(1 \otimes v + e_0 \otimes w) = 1 \otimes \mathcal{I}_{\pi_+z}v + e_0 \otimes \mathcal{I}_{\pi_-z}w \\ &= \Psi(\mathcal{I}_{\pi_+z}v, \mathcal{I}_{\pi_-z}w) \end{aligned} \tag{3.4}$$

for all $z \in Cl_4^0 \cong Cl_3$ and $(v,w) \in TM \oplus TM$. In particular, this holds for all elements of the group $Spin(4) \subset Cl_4^0$.

Furthermore,

$$\begin{aligned} (\mathrm{id}_{Cl_4} \otimes \mathcal{I}_{\overline{\pi_+z}}(\pi_+z)_*)(\Psi(v,w)) &= 1 \otimes \mathcal{I}_{\overline{\pi_+z}}(\pi_+z)_*v + e_0 \otimes \mathcal{I}_{\overline{\pi_+z}}(\pi_+z)_*w \\ &= \Psi(\mathcal{I}_{\overline{\pi_+z}}(\pi_+z)_*v, \mathcal{I}_{\overline{\pi_+z}}(\pi_+z)_*w) \end{aligned} \tag{3.5}$$

for all $z \in Spin(4)$ and $(v,w) \in TM \oplus TM$.

Finally, combining the equations (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned}(L_z \otimes \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_*)(\Psi(v,w)) &= (L_z \otimes \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_*)(\Psi(g_* v, g_* w)) \\ &= (L_z \otimes \mathrm{id}_E)(\Psi(\mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* v, \mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* w)) \\ &= \Psi(\mathcal{I}_{\pi_+ z}\mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* v, \mathcal{I}_{\pi_- z}\mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* w) \\ &= \Psi((\pi_+ z)_* g_* v, \mathcal{I}_{\pi_- z}\mathcal{I}_{\overline{\pi_+ z}}(\pi_+ z)_* g_* w)\end{aligned}$$

for all $z \in Spin(4)$, $g \in G$ and $(v,w) \in TM \oplus TM$. This proves that $\Psi\colon TM \oplus \underset{\sim}{TM} \to \widehat{TM}$ is $Spin^G_\varepsilon(4)$-equivariant. □

3.5.10 Corollary. $c_4(e_0)^{-1}c_4(e_\ell) = c_3(e_\ell) \in \mathrm{End}\,(TM) = \mathrm{End}\left(\widehat{TM}^0\right)$ *for* $\ell \in \{1,2,3\}$.

3.5.11 Corollary (4D Clifford multiplication). *We have*

$$\widehat{TM} \cong [(S^+ \oplus S^-) \otimes_{\mathbb{C}} E]_r$$

with $\widehat{TM}^0 \cong [S^+ \otimes_{\mathbb{C}} E]_r$ *and* $\widehat{TM}^1 \cong [S^- \otimes_{\mathbb{C}} E]_r$ *and the 4-dimensional Clifford multiplication can be interpreted as a* $Spin^G_\varepsilon(4)$*-equivariant homomorphism*

$$c_4\colon \mathbb{R}^4 \otimes TM \to [S^- \otimes_{\mathbb{C}} E]_r.$$

In particular, we have a commuting diagram

$$\begin{array}{ccccc} & & \mathbb{R}^4 \otimes \widehat{TM}^0 & \xrightarrow{\;c_3\;} & \widehat{TM}^1 \\ & \overset{\sim}{\nearrow} & \wr\downarrow & & \downarrow\wr \\ \mathbb{R}^4 \otimes TM & & & & \\ & \underset{\sim}{\searrow} & & & \\ & & \mathbb{R}^4 \otimes [S^+ \otimes_{\mathbb{C}} E]_r & \xrightarrow{\;c_3 \otimes \mathrm{id}_E\;} & [S^- \otimes_{\mathbb{C}} E]_r \end{array}$$

Proof. Using $TM \cong [S \otimes_{\mathbb{C}} E]_r$, we obtain an isomorphism of Cl_4-modules

$$\widehat{TM} = Cl_4 \otimes_{Cl_4^0} TM \cong Cl_4 \otimes_{Cl_4^0} [S \otimes_{\mathbb{C}} E]_r = [Cl_4 \otimes_{Cl_4^0} S \otimes_{\mathbb{C}} E]_r = [(S^+ \oplus S^-) \otimes_{\mathbb{C}} E]_r$$

where the even and odd parts are $\widehat{TM}^0 \cong [S^+ \otimes_{\mathbb{C}} E]_r$ and $\widehat{TM}^1 \cong [S^- \otimes_{\mathbb{C}} E]_r$. The real structure on $S^\pm \otimes_{\mathbb{C}} E$ is again $r = -R_j \otimes I_2$. □

3.5.12 Remark. Our convention from Note 3.5.2 implies that the restriction of $c_4(vol_4)$ to $TM = \widehat{TM}^0$ is

$$c_4(vol_4) = c_4(e_0e_1e_2e_3) = -c_4(e_1e_0e_2e_0e_3e_0) = -c_3(e_1e_2e_3) = -\mathrm{id}_{TM}\,.$$

3.5.13 Remark. Using the isomorphism $g^\sharp\colon (\mathbb{R}^4)^* \cong \mathbb{R}^4$ induced by the standard scalar product on $\mathbb{R}^4$, we can also understand the Clifford multiplication as a $Spin^G_\varepsilon(4)$-equivariant homomorphism

$$\begin{aligned}(\mathbb{R}^4)^* \otimes TM &\to \widehat{TM}^1 \cong [S^- \otimes E]_r \cong \underset{\sim}{TM}, \\ x \otimes v &\mapsto c_4(g^\sharp(x))(v).\end{aligned}$$

For a $Spin^G_\varepsilon(4)$-structure $Q_4 \to P_{SO(4)} \times_Y P_{G/\varepsilon}$, this induces a smooth map

$$C^\infty(Q_4, (R^4)^* \otimes TM)^{Spin^G_\varepsilon(4)} \to C^\infty(Q_4, \widehat{TM}^1)^{Spin^G_\varepsilon(4)},$$

which we also denote by c_4. This will be the Clifford multiplication used in the definition of the Dirac operator on four-dimensional manifolds.

3.6 Dirac operator

We define the Dirac operator as the composition of the covariant derivative and Clifford multiplication.

3.6.1 Definition (Dirac operator). The (three-dimensional) *Dirac operator* $\mathcal{D}_A$ for a connection $A \in \mathscr{A}_3$ is defined to be the composition

$$C^\infty(Q_3, M)^{\hat{G}_3} \xrightarrow{d_A^M} C^\infty(Q_3, (\mathbb{R}^3)^* \otimes TM)^{\hat{G}_3} \xrightarrow{c_3} C^\infty(Q_3, TM)^{\hat{G}_3},$$
$$\mathcal{D}_A u := c_3(d_A^M u).$$

The (four-dimensional) *Dirac operator* $\mathcal{D}_A^+$ for a connection $A \in \mathscr{A}_4$ is defined to be the composition

$$C^\infty(Q_4, M)^{\hat{G}_4} \xrightarrow{d_A^M} C^\infty(Q_4, (\mathbb{R}^4)^* \otimes TM)^{\hat{G}_4} \xrightarrow{c_4} C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4},$$
$$\mathcal{D}_A^+ u := c_4(d_A^M u).$$

3.6.2 Remark. Using the isomorphism $\widehat{TM}^1 \cong [S^- \otimes_{\mathbb{C}} E]_r$ from Corollary 3.5.11, we can also interpret the Dirac operator as a map $\mathcal{D}_A^+ \colon C^\infty(Q_4, M)^{\hat{G}_4} \to C^\infty(Q_4, [S^- \otimes_{\mathbb{C}} E]_r)^{\hat{G}_4}$.

3.6.3 Note. Notice that

$$\mathcal{F}_3 := C^\infty(Q_3, TM)^{\hat{G}_3} \to C^\infty(Q_3, M)^{\hat{G}_3} = \mathscr{N}_3$$
$$v \mapsto \pi_M \circ v$$

and

$$\mathcal{F}_4 := C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4} \to C^\infty(Q_4, M)^{\hat{G}_4} = \mathscr{N}_4$$
$$v \mapsto \pi_M \circ v$$

are vector bundles $\mathcal{F}_m \to \mathscr{N}_m$. The fibres of these bundles are $C^\infty(Q_3, TM)^{\hat{G}_3}_u \cong \Gamma(Y, \pi_! u^* TM)$ and $C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4}_u \cong \Gamma(X, \pi_! u^* \widehat{TM}^1) \cong \Gamma(X, \pi_! [S^- \otimes u^* E]_r)$, respectively. The first part of Lemma 3.4.4 implies that the Dirac operators $\mathcal{D}_A$ and $\mathcal{D}_A^+$ are sections of these bundles.

3.6.4 Example (*Spin*^{c} Dirac operator). For $M = \mathbb{H}$, $G = S^1$ as in Example 3.2.2, the tangent bundle $TM = \mathbb{H} \times \mathbb{H} \xrightarrow{\mathrm{pr}_1} \mathbb{H} = M$ is the trivial bundle with fibre $\mathbb{H}$.

Interpreting M as a hyperkähler manifold with permuting $Spin^{S^1}_{-1}(3) = Spin^c(3)$-action, this is the trivial bundle with fibre W. The equivariant map $d^M_A u \in C^\infty(Q_3, (\mathbb{R}^3)^* \otimes TM)^{\hat{G}_3}$ corresponds to the section $\nabla^A(u) \in \Gamma(Y, T^*Y \otimes \mathcal{S})$. Furthermore, $c_3(h \otimes v) = \mathcal{I}_{\bar{h}}(v)$ for $h \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$, $x \in M$ and $v \in T_xM = W$. This is the usual Clifford multiplication and therefore $\mathcal{D}_A$ is the usual $Spin^c(3)$ Dirac operator.

If we interpret the action as a permuting $Spin^{S^1}_{-1}(4)$-action, this is the trivial bundle with fibre W^+. The equivariant map $d^M_A u \in C^\infty(Q_4, (\mathbb{R}^4)^* \otimes TM)^{\hat{G}_4}$ corresponds to the section $\nabla^A(u) \in \Gamma(X, T^*X \otimes \mathcal{W}^+)$. Again, $c_4(h \otimes v) = \mathcal{I}_{\bar{h}}(v)$ for $h \in \mathbb{H} \cong \mathbb{R}^4$, $x \in M$ and $v \in T_xM = W^+$ is the usual Clifford multiplication and $\mathcal{D}^+_A$ is the usual $Spin^c(4)$-Dirac operator $\mathcal{D}^+_A \colon \Gamma(X, \mathcal{W}^+) \to \Gamma(X, \mathcal{W}^-)$.

3.6.5 Example (twisted Dirac operator). Let Y be an oriented 3-dimensional Riemannian manifold Y with a $Spin(3)$-structure $P_{Spin(3)} \to P_{SO(3)}$ and let $\xi \to Y$ be a Riemannian vector bundle of rank k with a metric compatible covariant derivative ∇^ξ. Consider a bundle P of orthonormal frames in ξ, so $\xi = P \times_{O(k)} \mathbb{R}^k$. The covariant derivative ∇^ξ corresponds to a connection a on P. Take $Q_3 = P_{Spin(3)} \times_Y P$, $G = \mathbb{Z}/2\mathbb{Z} \times O(k)$ and $\varepsilon = (-1, 1)$. Then $Spin^G_\varepsilon(3) = Spin(3) \times O(k)$. Let $M = S \otimes \mathbb{R}^k$ with the hyperkähler structure induced from S. Using the connection $A = a + \pi^*_{SO(m)}\varphi_Y \in \mathscr{A}_3$, we recover the twisted Dirac operator

$$\mathcal{D}^\xi \colon \Gamma(\mathcal{S} \otimes \xi) \xrightarrow{\nabla^{\mathcal{S}\otimes\xi}} \Gamma(T^*Z \otimes \mathcal{S} \otimes \xi) \xrightarrow{c_3 \otimes \mathrm{id}_\xi} \Gamma(\mathcal{S} \otimes \xi).$$

A similar construction can be done for $m = 4$, where we recover

$$\mathcal{D}^{\xi,+} \colon \Gamma(\mathcal{S}^+ \otimes \xi) \xrightarrow{\nabla^{\mathcal{S}^+\otimes\xi}} \Gamma(T^*Z \otimes \mathcal{S}^+ \otimes \xi) \xrightarrow{c_4 \otimes \mathrm{id}_\xi} \Gamma(\mathcal{S}^- \otimes \xi).$$

This construction can also be modified to work if only a $Spin^c(m)$-structure is given. One has to replace S by W (or $S^\pm$ by $W^\pm$) and take $G = S^1 \times O(k)$ with $\varepsilon = (-1, 1)$ and $Q_m = P_{Spin^c(m)} \times_Z P$. In this case, one has to choose an additional connection on P_{S^1} for the $Spin^c(m)$-structure.

3.6.1 The linearized Dirac operator

We will now linearize the Dirac operator in three dimensions. Let $Q_3 \to P_{SO(3)} \times_Y P_{G/\varepsilon}$ be a $Spin^G_\varepsilon(3)$-structure on a compact oriented Riemannian manifold Y.

3.6.6 Definition. Using the connector $\mathcal{K} \colon TTM \to TM$ for the Levi-Civita connection on M, we define the *linearized Dirac operator* $\mathcal{D}^{lin,u}_A$ (at $u \in C^\infty(Q_3, M)^{\hat{G}_3}$) to be

$$\begin{aligned} \mathcal{D}^{lin,u}_A \colon C^\infty(Q_3, TM)^{\hat{G}_3}_u &\to C^\infty(Q_3, TM)^{\hat{G}_3}_u, \\ v &\mapsto \mathcal{K} \circ T_u\mathcal{D}_A(v). \end{aligned}$$

3.6.7 Remark. Note that the linearized Dirac operator $\mathcal{D}_A^{lin,u}$ is the covariant derivative $\nabla^{\mathscr{N}}\mathcal{D}_A$ at $u \in \mathscr{N}_3$, where $\nabla^{\mathscr{N}}$ is the metric compatible covariant derivative corresponding to the connector $\mathcal{K}^{\mathscr{N}}$ in Proposition 3.3.8.

3.6.8 Lemma. *We have*

$$\mathcal{D}_A^{lin,u} = c_3 \circ d_{A,\mathcal{K}}^{TM},$$

where $\mathcal{K}\colon TTM \to TM$ is the connector for the Levi-Civita connection on M.

Furthermore, for each $v, w \in C^\infty(Q_3, TM)_u^{\hat{G}_3}$:

$$g^{\mathscr{N}}(\mathcal{D}_A^{lin,u} v, w) = g^{\mathscr{N}}(v, \mathcal{D}_A^{lin,u} w).$$

Proof. From Lemma 3.5.4 we obtain

$$\begin{aligned}\mathcal{D}_A^{lin,u}(v) &= \mathcal{K} \circ T_u \mathcal{D}_A(v) = \mathcal{K} \circ T(c_3) T(d_A^M)(v) = c_3 \circ (\mathrm{id}_{(R^3)^*} \otimes \mathcal{K}) \circ T(d_A^M)(v) \\ &= c_3 \circ d_{A,\mathcal{K}}^{TM}(v).\end{aligned}$$

Consider the covariant derivative ∇^{u^*TM} on $u^*TM \to Q_3$, which is the pullback of the Levi-Civita connection on M. For $Z \in TQ_3$ and $v \in C^\infty(Q_3, TM)_u^{\hat{G}_3} \cong \Gamma(Q_3, u^*TM)^{\hat{G}_3}$ we obtain

$$\nabla_Z^{u^*TM} v = \mathcal{K} T v(Z).$$

Since the Levi-Civita connection is compatible with the metric on M, the pullback ∇^{u^*TM} is compatible with the pullback metric on u^*TM:

$$g^M(\nabla^{u^*TM} v, w) + g^M(v, \nabla^{u^*TM} w) = d(g^M(v, w)) \text{ for all } v, w \in C^\infty(Q_3, TM)_u^{\hat{G}_3}.$$

Note that if we insert a horizontal lift $\tilde{X} \in TQ$ (with respect to A) of $X \in TY$, the right hand side is

$$d(g^M(v, w))(\tilde{X}) = d_A(g^M(v, w))(\tilde{X}) = d\pi_!(g^M(v, w))(X),$$

where $\pi_!(g^M(v, w)) \in C^\infty(Y, \mathbb{R})$ is induced by $g^M(v, w)\colon Q_3 \to \mathbb{R}$, and its exterior derivative on Y is $d\pi_!(g^M(v, w)) \in \Omega^1(Y, \mathbb{R})$.

Fix a point $p \in Q_3$, $y := \pi_Y(p)$ and let $X_\ell := \pi_{SO}(p)(e_\ell) \in T_yY$ for $\ell \in \{1, 2, 3\}$. Extend $X_\ell \in T_yY$ to (locally) parallel vector fields $X_\ell \in \Gamma(Y, TY)$. This means that $\nabla X_\ell = 0$ for the Levi-Civita connection ∇ on Y. Since TY is the associated bundle $TY = Q_3 \times_{\hat{G}_3} \mathbb{R}^3$, these correspond to $\hat{G}_3$-equivariant maps $f_\ell\colon Q_3 \to \mathbb{R}^3$. In particular, $X_\ell = \pi_{SO}(p)(e_\ell)$

implies that $f_\ell(p) = e_\ell$. With these choices, we obtain

$$
\begin{aligned}
g^M(\mathcal{D}_A^{lin,u}(v)(p), w(p)) =& g^M(c_3 d^{TM}_{A,\mathcal{K}} v(p), w(p)) \\
=& \sum_{\ell=1}^{3} g^M(c_3(e_\ell \otimes \nabla^{u^*TM}_{\tilde{X}_\ell} v)(p), w(p)) \\
=& -\sum_{\ell=1}^{3} g^M(\nabla^{u^*TM}_{\tilde{X}_\ell} v(p), c_3(e_\ell \otimes w)(p)) \\
=& -\sum_{\ell=1}^{3} g^M(\nabla^{u^*TM}_{\tilde{X}_\ell} v(p), c_3(f_\ell(p) \otimes w(p))) \\
=& -\sum_{\ell=1}^{3} d(g^M(v, c_3(f_\ell \otimes w)))(\tilde{X}_\ell) \\
&+ \sum_{\ell=1}^{3} g^M(v(p), \nabla^{u^*TM}_{\tilde{X}_\ell}(c_3(f_\ell \otimes w))).
\end{aligned}
\tag{3.6}
$$

Define a vector field $U_{v,w} \in \mathit{\Gamma}(Y, TY)$ by $g^Y(U_{v,w}, Z) = \pi_!(g^M(v, c_3(f_Z \otimes w)))$ for $Z \in \mathit{\Gamma}(Y, TY)$ and $f_Z \colon Q_3 \to \mathbb{R}^3$ the corresponding $\hat{G}_3$-equivariant map. Then the first summand on the right hand side of equation (3.6) is

$$
\begin{aligned}
-\sum_{\ell=1}^{3} d(g^M(v, c_3(f_\ell \otimes w)))(\tilde{X}_\ell) &= -\sum_{\ell=1}^{3} d\pi_!(g^M(v, c_3(f_\ell \otimes w)))(X_\ell) \\
&= -\sum_{\ell=1}^{3} d(g^Y(U_{v,w}, X_\ell))(X_\ell) \\
&= -\sum_{\ell=1}^{3} g^Y(\nabla_{X_\ell} U_{v,w}, X_\ell)) - \sum_{\ell=1}^{3} g^Y(U_{v,w}, \nabla_{X_\ell} X_\ell)) \\
&= -\sum_{\ell=1}^{3} g^Y(\nabla_{X_\ell} U_{v,w}, X_\ell)) \\
&= -\operatorname{div}(U_{v,w}).
\end{aligned}
$$

Since the Clifford multiplication c_3 as well as the vector fields X_ℓ ($\ell \in \{1,2,3\}$) are parallel (Lemma 3.5.4), the second summand on the right hand side of equation (3.6) is

$$
\begin{aligned}
\sum_{\ell=1}^{3} g^M(v(p), \nabla^{u^*TM}_{\tilde{X}_\ell}(c_3(f_\ell \otimes w))) &= \sum_{\ell=1}^{3} g^M(v(p), c_3(f_\ell \otimes \nabla^{u^*TM}_{\tilde{X}_\ell} w)(p)) \\
&= g^M(v(p), \mathcal{D}_A^{lin,u}(w)(p)).
\end{aligned}
$$

We obtain

$$
g^M(\mathcal{D}_A^{lin,u}(v)(p), w(p)) = g^M(v(p), \mathcal{D}_A^{lin,u}(w)(p)) - \operatorname{div}(U_{v,w})(y).
$$

In particular, integrating over the compact manifold Y, the integral of the divergence $\operatorname{div}(U_{v,w})$ vanishes and we obtain

$$
g^{\mathscr{N}}(\mathcal{D}_A^{lin,u} v, w) = g^{\mathscr{N}}(v, \mathcal{D}_A^{lin,u} w).
$$

□

Under the assumptions of Proposition 3.2.6, the nonlinear Dirac operator $\mathcal{D}_A$ is determined by its linearization:

3.6.9 Lemma. *Assume that the fundamental vector fields for the permuting action satisfy $\mathcal{I}_\zeta K_\zeta^{M,Sp(1)} = -\chi$ for a vector field $\chi \in \Gamma(M,TM)$ and all $\zeta \in \mathfrak{sp}(1), \|\zeta\|^2 = 1$. Then*

$$\mathcal{D}_A^{lin,u}(\chi \circ u) = \mathcal{D}_A(u).$$

Proof. Let $\rho\colon M \to \mathbb{R}$ be the hyperkähler potential from Proposition 3.2.6. Then $\chi = \operatorname{grad}(\rho)$ and $\mathcal{K}T\chi = \nabla^{\mathcal{K}}\chi = \operatorname{id}_{\Gamma(M,TM)}$, where $\mathcal{K}$ is the connector for the Levi-Civita connection on M. Using Lemma 3.4.4 we obtain

$$\langle (d_{A,\mathcal{K}}^{TM}(\chi \circ u))(p), w\rangle = \mathcal{K}T\chi T u(\tilde{w}) = Tu(\tilde{w}) = \langle (d_A^M u)(p), w\rangle$$

for all $w \in \mathbb{R}^3$, $p \in Q_3$ and where $\tilde{w} \in T_pQ_3$ is the horizontal lift of $\pi_{SO}(p)(w) \in T_{\pi_Y(p)}Y$. Therefore, $d_{A,\mathcal{K}}^{TM}(\chi \circ u) = d_A^M u$ and with the help of Lemma 3.6.8, we conclude that

$$\mathcal{D}_A^{lin,u}(\chi \circ u) = c_3(d_{A,\mathcal{K}}^{TM}(\chi \circ u)) = c_3(d_A^M u) = \mathcal{D}_A u. \qquad \square$$

3.6.2 The Dirac operator and the gauge group

3.6.10 Lemma. *Let $u \in \mathscr{N}_m$ be a spinor, $A \in \mathscr{A}_m$ a connection and $\psi \in \mathscr{G}_m$ a gauge transformation. Let $g\colon Q_m \to G$ be the map satisfying $\psi(p) = pg(p)$ for all $p \in Q_3$. Then*

$$\mathcal{D}_{\psi^*A}(\psi^* u) = g_*^{-1}\mathcal{D}_A(u) = \psi^*(\mathcal{D}_A(u)) \text{ for } m = 3$$

and

$$\mathcal{D}^+_{\psi^*A}(\psi^* u) = g_*^{-1}\mathcal{D}^+_A(u) = \psi^*(\mathcal{D}^+_A(u)) \text{ for } m = 4.$$

Proof. For a connection 1-form A and a gauge transformation $\psi \in \mathscr{G}_m$, we have

$$\mathscr{H}_{\psi^*A} = \ker(\psi^*A) = T\psi^{-1}\mathscr{H}_A.$$

The horizontal projection $\operatorname{pr}_{\mathscr{H}_{\psi^*A}}$ can also be expressed in terms of $\operatorname{pr}_{\mathscr{H}_A}$ and ψ:

$$\operatorname{pr}_{\mathscr{H}_{\psi^*A}} = T\psi^{-1} \operatorname{pr}_{\mathscr{H}_A} T\psi.$$

In particular, the horizontal lift $\tilde{w}^A \in T_{\psi(p)}Q_m$ of $\pi_{SO}(\psi(p))(w) \in T_{\pi_Y(p)}Y$ with respect to A is given by $\tilde{w}^A = T_p\psi(\tilde{w}^{\psi^*A})$, where $\tilde{w}^{\psi^*A} \in T_pQ_m$ is the horizontal lift of $\pi_{SO}(p)(w) \in T_{\pi_Y(p)}Y$ with respect to ψ^*A. We obtain

$$\langle (d_{\psi^*A}(\psi^*u))(p), w\rangle = T_{\psi(p)}u(T_p\psi(\tilde{w}^{\psi^*A})) = T_{\psi(p)}u(\tilde{w}^A) = \langle (d_A u)(\psi(p)), w\rangle,$$

and thus $d_{\psi^*A}(\psi^*u) = \psi^*(d_A u)$. Finally, for $m = 3$:

$$\begin{aligned}\mathcal{D}_{\psi^*A}(\psi^*u)(p) &= c_3(d_{\psi^*A}(\psi^*u)(p)) = c_3(d_A^M u(\psi(p))) \\ &= \psi^*(\mathcal{D}_A u)(p).\end{aligned}$$

The map $\mathcal{D}_A(u)$ is G-equivariant by construction, hence for $p \in Q_3$:

$$\mathcal{D}_{\psi^*A}(\psi^*u)(p) = \psi^*(\mathcal{D}_A(u)) = \mathcal{D}_A(u)(\psi(p)) = \mathcal{D}_A(u)(pg(p)) = g^{-1}(p)_*\mathcal{D}_A(u)(p).$$

The same arguments holds for $m = 4$ if we substitute $\mathcal{D}_A^+$ for $\mathcal{D}_A$ and c_4 for c_3. $\square$

Chapter 4

The Seiberg-Witten equations

In this chapter, we will study the Seiberg-Witten equations associated to a hyperkähler manifold with permuting $Spin^G_\varepsilon(m)$-action for $m \in \{3,4\}$. For this purpose, we fix a compact Lie group G, an central element $\varepsilon \in Z(M)$ satisfying $\varepsilon^2 = 1$, a $Spin^G_\varepsilon(3)$-structure $Q_3 \to P_{SO(3)} \times_Y P_{G/\varepsilon}$ on a 3-dimensional compact oriented Riemannian manifold Y and a $Spin^G_\varepsilon(4)$-structure $Q_4 \to P_{SO(4)} \times_X P_{G/\varepsilon}$ on a 4-dimensional compact oriented Riemannian manifold X. To write the Seiberg-Witten equations, we also fix an Ad-invariant scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$. We use this to identify $\mathfrak{g} \cong \mathfrak{g}^*$. For a compact Lie group G with semisimple Lie algebra $\mathfrak{g}$ we can take $\langle x, y \rangle_{\mathfrak{g}} = -B(x,y)$, where B is the *Killing form* $B(x,y) := \mathrm{tr}(ad(x)ad(y))$ for $x, y \in \mathfrak{g}$.

4.1 The moment map

Let M be a hyperkähler manifold with permuting $Spin^G_\varepsilon(m)$-action and let $\mu\colon M \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$ be the $Spin^G_\varepsilon(m)$-equivariant hyperkähler moment map for the G-action from Proposition 3.2.3.

4.1.1 Definition.

1. Composing a spinor $u \in \mathscr{N}_3$ with the moment map μ we obtain a smooth $\hat{G}_3$-equivariant map
$$Q_3 \xrightarrow{u} M \xrightarrow{\mu} \mathfrak{g}^* \otimes \mathfrak{sp}(1)^* \cong \mathfrak{g} \otimes (\mathbb{R}^3)^*.$$
This composition is a map in $C^\infty(Q_3, \mathfrak{g} \otimes (\mathbb{R}^3)^*)^{\hat{G}_3}$ and defines an element $\Phi_3(u) \in \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor} \cong \Omega^1(Y, \mathfrak{g}_{Q_3})$. Here we used the isomorphism of $\hat{G}_3$-representations $\mathfrak{sp}(1)^* \cong (\mathbb{R}^3)^*$, which is induced by the isomorphism $Sp(1) \cong Spin(3)$ from Example 2.3.9.

2. Composing a spinor $u \in \mathscr{N}_4$ with the moment map μ we obtain a smooth $\hat{G}_4$-equivariant map
$$Q_4 \xrightarrow{u} M \xrightarrow{\mu} \mathfrak{g}^* \otimes \mathfrak{sp}(1)^* \cong \mathfrak{g} \otimes \Lambda^2_+(\mathbb{R}^4)^*.$$

This composition is a map in $C^\infty(Q_4, \mathfrak{g} \otimes \Lambda^2_+(\mathbb{R}^4)^*)^{\hat{G}}$ and therefore defines an element $\Phi_4(u) \in \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}}_{hor} \cong \Omega^2_+(X, \mathfrak{g}_{Q_4})$. Here $\Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}}_{hor}$ denotes the image of $\Omega^2_+(X, \mathfrak{g}_{Q_4})$ under the isomorphism $\Omega^2(X, \mathfrak{g}_{Q_4}) \cong \Omega^2(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor}$. For the composition, we use the isomorphism of $\hat{G}_4$-representations $\mathfrak{sp}(1)^* \cong \Lambda^2_+(\mathbb{R}^4)^*$ from Example 2.3.11.

4.1.2 Lemma. *Let Z be compact. Then the maps $\Phi_3\colon \mathscr{N}_3 \to \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor}$ and $\Phi_4\colon \mathscr{N}_4 \to \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}}_{hor}$ are smooth and*

$$T\Phi_m = d\mu.$$

Proof. The map $u \mapsto \mu \circ u$ is smooth, as it is defined by composing with the moment map μ (cf. [KM97, Ch IX Thm 42.13]). We now compute the derivative: Let $u \in \mathscr{N}_m$ be a spinor, $v \in T_u\mathscr{N}_m$ a tangent vector represented by the smooth curve $\sigma\colon \mathbb{R} \to \mathscr{N}_m$ and $p \in Q_m$. Then

$$T_u\Phi_m(v)(p) = \tfrac{d}{dt}\mu(\sigma(t)(p))|_{t=0} = d\mu\left(\tfrac{d}{dt}\sigma(t)(p)|_{t=0}\right) = d\mu(v(p)).$$ □

4.2 Seiberg-Witten section and equations

We have now collected all the necessary ingrediants to write the generalized Seiberg-Witten equations in dimensions three and four.

4.2.1 Definition. Consider the map

$$\begin{aligned} \mathfrak{F}_3\colon \mathscr{C}_3 = \mathscr{N}_3 \times \mathscr{A}_3 &\to C^\infty(Q_3, TM)^{\hat{G}_3} \times \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor}, \\ (u, A) &\mapsto (\mathcal{D}_A u, *F_a + \Phi_3(u)), \end{aligned}$$

where $a = A - \pi^*_{SO(3)}\varphi_Y$ is the $\mathfrak{g}$-component of $A \in \mathscr{A}_3$ and $*\colon \Omega^2(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor} \to \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor}$ is the Hodge star operator induced by $*\colon \Lambda^2(\mathbb{R}^3)^* \to (\mathbb{R}^3)^*$. This map is called *Seiberg-Witten section in three dimensions.* The system of equations $\mathfrak{F}_3(u, A) = 0$ was introduced by Taubes in [Tau99]:

$$\begin{cases} \mathcal{D}_A(u) = 0 \\ *F_a + \Phi_3(u) = 0 \end{cases}$$

These are the *generalized Seiberg-Witten equations in three dimensions.*

In the first equation, the zero on the right hand side is the composition of the spinor u and the zero section $0 \in \Gamma(M, TM)$.

4.2.2 Remark. Using the isomorphism $\Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor} \cong \Omega^1(Y, \mathfrak{g}_{Q_3})$, we can also think of the Seiberg-Witten section as a map $\mathscr{N}_3 \times \mathscr{A}_3 \to C^\infty(Q_3, TM)^{\hat{G}_3} \times \Omega^1(Y, \mathfrak{g}_{Q_3})$ and of the second equation as an equation in $\Omega^1(Y, \mathfrak{g}_{Q_3})$.

4.2.3 Definition. As in the three-dimensional case, consider the map

$$\begin{aligned}\mathfrak{F}_4\colon \mathscr{C}_4 = \mathscr{N}_4 \times \mathscr{A}_4 &\to C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4} \times \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor},\\ (u, A) &\mapsto (\mathcal{D}_A u, F_a^+ + \Phi_4(u)).\end{aligned}$$

where $a = A - \pi^*_{SO(4)}\varphi_X$ is the $\mathfrak{g}$-component of $A \in \mathscr{A}_4$ and $F_a^+ \in \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_3}_{hor}$ is the selfdual part of the curvature F_a of a. This map is called *Seiberg-Witten section in four dimensions.* The system of equations $\mathfrak{F}_4(u, A) = 0$ was introduced by Pidstrygach in [Pid04]:

$$\begin{cases} \mathcal{D}_A(u) = 0\\ F_a^+ + \Phi_4(u) = 0\end{cases}$$

These are the *generalized Seiberg-Witten equations in four dimensions.*

In the first equation, the zero on the right hand side is the composition of the spinor u and the zero section $0 \in \Gamma(M, \widehat{TM}^1)$.

4.2.4 Remark. Using the isomorphism $\Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor} \cong \Omega^2_+(X, \mathfrak{g}_{Q_4})$, we can also think of the Seiberg-Witten section as a map $\mathscr{N}_4 \times \mathscr{A}_4 \to C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4} \times \Omega^2_+(X, \mathfrak{g}_{Q_4})$ and of the second equation as an equation in $\Omega^2_+(X, \mathfrak{g}_{Q_4})$.

4.2.5 Note. We will now explain why the maps $\mathfrak{F}_m$ are called Seiberg-Witten *sections.* In Note 3.6.3, we have seen that the Dirac operator is a section in the $\mathcal{F}_m \to \mathscr{N}_m$. Interpreting the second component of the Seiberg-Witten section as a section in a trivial vector bundle, we can think of the map $\mathfrak{F}_m$ as a section in a vector bundle $\mathcal{E}_m \to \mathscr{C}_m$ ($m \in \{3, 4\}$). These vector bundles are

$$\begin{aligned}\mathcal{E}_3 &:= C^\infty(Q_3, TM)^{\hat{G}_3} \times \pi^*_{\mathscr{A}} T\mathscr{A}_3 \to \mathscr{C}_3 = C^\infty(Q_3, M)^{\hat{G}_3} \times \mathscr{A}_3,\\ \mathcal{E}_4 &:= C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4} \times (\mathscr{A}_4 \times \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor}) \to \mathscr{C}_4 = C^\infty(Q_4, M)^{\hat{G}_4} \times \mathscr{A}_4.\end{aligned}$$

The fibres of these bundles are

$$\begin{aligned}(\mathcal{E}_3)_{(u,A)} &= C^\infty(Q_3, TM)^{\hat{G}_3}_u \oplus \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor}\\ &\cong \Gamma(Y, \pi_! u^* TM) \oplus \Omega^1(Y, \mathfrak{g}_{Q_3}) \qquad \text{for } (u, A) \in \mathscr{C}_3,\end{aligned}$$

$$\begin{aligned}(\mathcal{E}_4)_{(u,A)} &= C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4}_u \oplus \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor}\\ &\cong \Gamma(X, \pi_! u^* \widehat{TM}^1) \oplus \Omega^1(X, \mathfrak{g}_{Q_4}) \qquad \text{for } (u, A) \in \mathscr{C}_4.\end{aligned}$$

Furthermore, note that the vector bundles $\mathcal{E}_m$ are $\mathscr{G}_m$-equivariant vector bundles. If the three-dimensional base manifold Y is compact, then $\mathcal{E}_3 = T\mathscr{C}_3$.

4.2.6 Example (usual Seiberg-Witten equations). Consider $M = \mathbb{H}$ as in Example 3.2.2, $G = S^1$ and $\varepsilon = -1$. In this case, a $Spin^{S^1}_{-1}(m)$-structure is the same as a

$Spin^c(m)$-structure and the Dirac operator is the usual $Spin^c(m)$ Dirac operator (cf. Example 3.6.4). Consider the following isomorphism of complex vector spaces

$$\Psi\colon (\mathbb{C}^2, i) \to (\mathbb{H}, R_i),$$
$$(u_1, u_2) \mapsto u_1 + ju_2.$$

The moment map $\mu\colon \mathbb{H} \to i\mathbb{R} \otimes \mathrm{Im}(\mathbb{H})$ from Example 2.2.12 ($\ell = 1$), $\mu = i \otimes \tilde{\mu}$, where $\tilde{\mu}\colon \mathbb{H} \to \mathrm{Im}(\mathbb{H}), \tilde{\mu}(h) = \frac{1}{2}hi\bar{h}$ can be written as

$$\tilde{\mu}(u_1 + ju_2) = \tfrac{1}{2}(u_1 + ju_2)i(\bar{u}_1 - \bar{u}_2 j) = \tfrac{i}{2}((|u_1|^2 - |u_2|^2) + 2jiu_2\bar{u}_1).$$

We obtain

$$\Psi^{-1}\mathcal{I}_{\tilde{\mu}(u_1+ju_2)}\Psi = \begin{pmatrix} \frac{i}{2}(|u_1|^2 - |u_2|^2) & i\bar{u}_2 u_1 \\ iu_2\bar{u}_1 & \frac{i}{2}(|u_2|^2 - |u_1|^2) \end{pmatrix} \in \mathfrak{su}(2).$$

Note that this is $i(u \otimes u^*)_0$, where $u = (u_1, u_2) \in \mathbb{C}^2$ and $(u \otimes u^*)_0$ is the endomorphism $(u \otimes u^*)_0 \in \mathrm{End}\,(\mathbb{C}^2)\,, w \mapsto \langle w, u\rangle u - \frac{1}{2}\|u\|^2 w$ and $\langle \cdot, \cdot \rangle$ is the standard hermitian product on $\mathbb{C}^2$. We use the convention, that $\langle \cdot, \cdot \rangle$ is linear in the first component and antilinear in the second. Therefore, also writing $c_3\colon \mathbb{R}^3 \to \mathrm{End}\,(\mathbb{C}^2)$ for the Clifford multiplication induced by $\Psi\colon \mathbb{C}^2 \xrightarrow{\sim} \mathbb{H}$,

$$c_3(\tilde{\mu}(u_1 + ju_2)) = -\Psi^{-1}\mathcal{I}_{\tilde{\mu}(u_1+ju_2)}\Psi = -i(u \otimes u^*)_0.$$

Extending c_3 complex linearly and using $c_3(*F_a) = -c_3(F_a)$, we obtain

$$(u \otimes u^*)_0 - c_3(F_a) = ic_3(\tilde{\mu}(u_1 + ju_2)) - c(F_a) = c_3(\mu(u_1 + ju_2)) + c_3(*F_a).$$

Therefore, the second Seiberg-Witten equation in three dimensions can be reformulated as $(v \otimes v^*)_0 = c_3(F_a)$ and the Seiberg-Witten equations read

$$\begin{cases} \mathcal{D}_A u = 0 \\ c_3(F_a) = (u \otimes u^*)_0 \end{cases}$$

In the literature, this is the most common form of the Seiberg-Witten equations in three dimensions (cf. [KM07]).

Similarly, in four dimensions, we interpret $\mu\colon \mathbb{H} \to i\mathbb{R} \otimes \mathrm{Im}(\mathbb{H}) \cong i\mathbb{R} \otimes \Lambda^2_+\mathbb{R}^4$ and obtain

$$\Psi^{-1}c_4(\tilde{\mu}(u_1 + ju_2))\Psi = i(u \otimes u^*)_0.$$

Therefore, extending $c_4\colon \Lambda^2_+\mathbb{R}^4 \to \mathrm{End}\,(\mathbb{C}^2)$ complex linearly, we obtain

$$c_4(F_a^+) - (u \otimes u^*)_0 = c_4(F_a^+) + ic_4(\tilde{\mu}(u_1 + ju_2)) = c_4(F_a^+) + c_4(\mu(u_1 + ju_2)).$$

Again, the second Seiberg-Witten equation in four dimensions can be reformulated as $(u \otimes u^*)_0 = c_4(F_a^+)$ and the Seiberg-Witten equations are

$$\begin{cases} \mathcal{D}_A^+ u = 0 \\ c_4(F_a^+) = (u \otimes u)_0. \end{cases}$$

In the literature, this is the most common form of the Seiberg-Witten equations in four dimensions (cf. [KM07]). These equations were first considered in [Wit94].

4.2.7 Example. If the hyperkähler manifold is just one point $M = \{pt\}$, then the equations reduce to $F_a^+ = 0$ in four dimensions and the equation $F_a = 0$ in the three-dimensional case. The solutions are the anti-selfdual connection in four dimensions and flat connections in three dimensions.

4.2.1 The Seiberg-Witten equations and the gauge group

Note that the gauge group $\mathscr{G}_3$ also acts (from the right) on $C^\infty(Q_3, TM)^{\hat{G}_3} \times \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}$ by pullback and $\mathscr{G}_4$ acts on $C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4} \times \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor}$ by pullback.

4.2.8 Proposition. *The Seiberg-Witten sections*

$$\mathfrak{F}_3 \colon \mathscr{C}_3 \to C^\infty(Q_3, TM)^{\hat{G}_3} \times \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor}$$

and

$$\mathfrak{F}_4 \colon \mathscr{C}_4 \to C^\infty(Q_4, \widehat{TM}^1)^{\hat{G}_4} \times \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor}$$

are gauge equivariant.

Proof. We have proven in Lemma 3.6.10 that the first component of the Seiberg-Witten section is equivariant. Let $\psi \in \mathscr{G}_m$ and $g \colon Q_m \to G$ the corresponding equivariant map. For $(u, A) \in \mathscr{A}_m$ let $a \in \Omega^1(Q_m, \mathfrak{g})^{\hat{G}_m}$ be the $\mathfrak{g}$-component of A. Applying Proposition 2.1.44, we obtain

$$*F_{\psi^* a} = *F_{a^g} = *Ad_{g^{-1}} F_a = Ad_{g^{-1}} * F_a = \psi^*(*F_a)$$

for $m = 3$ and

$$F^+_{\psi^* a} = F^+_{a^g} = Ad_{g^{-1}} F^+_a = \psi^* F^+_a$$

for $m = 4$. Furthermore,

$$\Phi_m(\psi^* u) = \Phi_m(g^{-1} u) = Ad_{g^{-1}} \Phi_m(u) = \psi^*(\Phi_m(u)).$$

This proves that the second component is equivariant. □

4.2.9 Corollary. *The Seiberg-Witten sections $\mathfrak{F}_m$ ($m \in \{3, 4\}$) are $\mathscr{G}_m$-equivariant sections in the $\mathscr{G}_m$-equivariant vector bundles $\mathcal{E}_m$ from Note 4.2.5.*

4.2.10 Definition. The *moduli space* of solutions of the Seiberg-Witten equations is the quotient of the space of solutions of the Seiberg-Witten equations by the action of the gauge group. In the three-dimensional case we have:

$$\mathcal{M}^{3D}_{SW}(Q_3) := \mathfrak{F}_3^{-1}(0)/\mathscr{G}_3 = \{\, (u, A) \in \mathscr{N}_3 \times \mathscr{A}_3 \mid \mathcal{D}_A u = 0, *F_a + \Phi_3(u) = 0 \,\}/\mathscr{G}_3,$$

and in the four-dimensional case:

$$\mathcal{M}^{4D}_{SW}(Q_4) := \mathfrak{F}_4^{-1}(0)/\mathscr{G}_4 = \left\{\, (u, A) \in \mathscr{N}_4 \times \mathscr{A}_4 \;\middle|\; \mathcal{D}^+_A u = 0, F^+_a + \Phi_4(u) = 0 \,\right\}/\mathscr{G}_4.$$

Note that the moduli spaces depend on the $Spin^G_\varepsilon(m)$-structure, although we dropped the dependence in the notation of the configuration spaces.

One special property of the Seiberg-Witten equations is the interplay between the two equations. An example of this is the following lemma:

4.2.11 Lemma. *Let* $Q_3 \to P_{SO(3)} \times_Y P_{G/\varepsilon}$ *be a* $Spin^G_\varepsilon(3)$*-structure on a compact oriented 3-dimensional Riemannian manifold* Y. *Let* $w \in T_u\mathscr{N}_3 = C^\infty(Q_3, TM)^{\hat{G}_3}_u$ *and* $\alpha \in C^\infty(Q_3, (\mathbb{R}^3)^* \otimes \mathfrak{g})^{\hat{G}_3} \cong \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor}$. *Then*

$$g^M(\tfrac{d}{dt}\mathcal{D}_{A+t\alpha}u|_{t=0}(p), w(p)) = \langle T_u\Phi_3(w)(q), \alpha(q)\rangle_{(\mathbb{R}^3)^*\otimes\mathfrak{g}}$$

For the right hand side, we identify $\mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H})$ *with* $\mathbb{R}^3$ *and use the standard scalar product on* $(\mathbb{R}^3)^*$.

Proof. First, note that $d_{A+t\alpha}u(p) = d_Au(p) + t(K^{M,G}_{\alpha(p)})_{u(p)}$ and therefore,

$$\begin{aligned}\tfrac{d}{dt}\mathcal{D}_{A+t\alpha}u|_{t=0}(p) &= \tfrac{d}{dt}c_3(d_{A+t\alpha}u)(p)|_{t=0} = c_3\left(\tfrac{d}{dt}d_Au(p) + t(K^{M,G}_{\alpha(p)})_{u(p)}|_{t=0}\right)\\ &= c_3((K^{M,G}_{\alpha(p)})_{u(p)}).\end{aligned}$$

Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and $\{e_1^*, e_2^*, e_3^*\}$ the dual basis of $(\mathbb{R}^3)^*$. Decompose $\alpha = \sum_{\ell=1}^{3} e_\ell^* \otimes \alpha_\ell$ with $\alpha_\ell = \langle \alpha, e_\ell\rangle \in C^\infty(Q_3, \mathfrak{g})^{\hat{G}_3}$. Then

$$\begin{aligned}g^M(\tfrac{d}{dt}\mathcal{D}_{A+t\alpha}u|_{t=0}(p), w(p)) &= g^M(c_3(K^{M,\hat{G}}_{\alpha(p)})_{u(p)}, w(p))\\ &= g^M(\sum_{\ell=1}^{3} c_3(e_\ell^* \otimes (K^{M,G}_{\alpha_\ell(p)})_{u(p)}), w(p))\\ &= -\sum_{\ell=1}^{3} g^M(I_\ell((K^{M,G}_{\alpha_\ell(p)})_{u(p)}), w(p))\\ &= \sum_{\ell=1}^{3} g^M((K^{M,G}_{\alpha_\ell(p)})_{u(p)}, I_\ell(w(p)))\\ &= \sum_{\ell=1}^{3}\langle \iota_{\mathfrak{g}}\omega_\ell(w(p)), \alpha_\ell(p)\rangle_{\mathfrak{g}}\\ &= \sum_{\ell=1}^{3}\langle d\mu_\ell(w(p)), \alpha_\ell(p)\rangle_{\mathfrak{g}}.\end{aligned}$$

Using the identification $\mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ and the standard scalar product on $\mathbb{R}^3$ we obtain

$$\sum_{\ell=1}^{3}\langle d\mu_\ell(w(p)), \alpha_\ell(p)\rangle_{\mathfrak{g}} = \sum_{\ell=1}^{3}\langle d\mu(w(p)), e_\ell^* \otimes \alpha_\ell(p)\rangle_{(\mathbb{R}^3)^*\otimes\mathfrak{g}} = \langle d\mu(w(p)), \alpha(p)\rangle_{(\mathbb{R}^3)^*\otimes\mathfrak{g}}.$$

Finally, using Lemma 4.1.2 we conclude that

$$g^M(w(p), \tfrac{d}{dt}\mathcal{D}_{A+t\alpha}u|_{t=0}(p)) = \langle d\mu(w(p)), \alpha(p)\rangle_{(\mathbb{R}^3)^*\otimes\mathfrak{g}} = \langle T_u\Phi(w)(p), \alpha(p)\rangle_{(\mathbb{R}^3)^*\otimes\mathfrak{g}}. \qquad \square$$

Chapter 5

Seiberg-Witten equations on the cylinder

Consider a 3-dimensional compact oriented Riemannian manifold Y and let $X = \mathbb{R} \times Y$ be the cylinder over Y with the product metric. Denote $\pi_Y : X \to Y$ and $\pi_{\mathbb{R}} : X \to \mathbb{R}$ the projections to Y and $\mathbb{R}$. Let $P_{SO(3)} \to Y$ be the bundle of oriented orthonormal frames. Note that the bundle of oriented orthonormal frames $P_{SO(4)} \to X$ reduces to $SO(3)$. In particular, $P_{SO(4)} \cong \pi_Y^* P_{SO(3)} \times_{SO(3)} SO(4)$. Consider a $Spin_\varepsilon^G(3)$-structure $Q_3 \to P_{SO(3)} \times_Y P_{G/\varepsilon}$ on Y. Then

$$Q_4 := \pi_Y^* Q_3 \times_{\hat{G}_3} \hat{G}_4 \to P_{SO(4)} \times_X \pi_Y^* P_{G/\varepsilon}$$

is a $Spin_\varepsilon^G(4)$-structure on X. We have the following commuting diagram of principal bundles over the cylinder X

$$\begin{array}{ccc} \pi_Y^* Q_3 & \xrightarrow{\ i\ } & Q_4 \\ {\scriptstyle \pi_{SO(3)}}\downarrow & & \downarrow{\scriptstyle \pi_{SO(4)}} \\ \pi_Y^* P_{SO(3)} & \xrightarrow{\ i'\ } & P_{SO(4)} \end{array}$$

where the horizontal maps are the inclusions induced by $\hat{G}_3 \hookrightarrow \hat{G}_4$ and $SO(3) \hookrightarrow SO(4)$, respectively. The vertical maps are quotient maps for the G-actions. There is also a projection

$$\pi_3 : Q_4 = \mathbb{R} \times (Q_3 \times_{\hat{G}_3} \hat{G}_4) \to Q_3 \times_{\hat{G}_3} \hat{G}_4 \to Q_3 \times_{\hat{G}_3} \hat{G}_4 / Sp(1)_- \cong Q_3.$$

Furthermore, $\pi_3 \circ i = \pi_{Q_3} : \pi_Y^* Q_3 = \mathbb{R} \times Q_3 \to Q_3$ is the projection to Q_3.

5.1 Spinors on the cylinder

We will now reinterpret spinors on the cylinder $X = \mathbb{R} \times Y$ as smooth paths of spinors on Y.

5.1.1 Lemma. *There is a bijection*

$$C^\infty(Q_4, M)^{\hat{G}_4} \xrightarrow{\sim} C^\infty(\mathbb{R}, C^\infty(Q_3, M)^{\hat{G}_3}),$$
$$u \mapsto \check{u},\ \textit{where}\ (\check{u}(t))(\pi_3(q)) = u(t, q).$$

Proof. First observe that $Q_4 = \mathbb{R} \times Q_3 \times_{\hat{G}_3} \hat{G}_4$. We use the exponential law (cf. Proposition A.1.7) to get

$$\begin{aligned} C^\infty(Q_4, M)^{\hat{G}_4} &= C^\infty(\mathbb{R} \times Q_3 \times_{\hat{G}_3} \hat{G}_4, M)^{\hat{G}_4} \\ &\cong C^\infty(\mathbb{R}, C^\infty(Q_3 \times_{\hat{G}_3} \hat{G}_4, M)^{\hat{G}_4}) \\ &= C^\infty(\mathbb{R}, C^\infty(Q_3 \times_{\hat{G}_3} \hat{G}_4/Sp(1)_-, M)^{\hat{G}_4/Sp(1)_-}) \\ &= C^\infty(\mathbb{R}, C^\infty(Q_3, M)^{\hat{G}_3}). \end{aligned}$$

We also used that $Sp(1)_- \hookrightarrow \hat{G}_4$ acts trivially on M and that $\hat{G}_4/Sp(1)_- \cong \hat{G}_3$. □

5.2 Connections on the cylinder

5.2.1 Definition. Let $P \to Y$ be a principal H-bundle and $\pi_Y^* P \to X = \mathbb{R} \times Y$ its pullback to the cylinder. Since $\pi_Y^* P = \mathbb{R} \times P$, we have a vector field $\frac{\partial}{\partial t} \in \Gamma(\pi_Y^* P, T\pi_Y^* P)$. A connection 1-form A on $\pi^* P$ is said to be in *temporal gauge* if $\frac{\partial}{\partial t}$ is horizontal, i.e.

$$A\left(\frac{\partial}{\partial t}_{(t,p)}\right) = 0 \text{ for all } t \in \mathbb{R}, p \in P.$$

The subspace of connection 1-forms in temporal gauge is denoted by $\mathscr{A}^{tg}(\pi_Y^* P) \subset \mathscr{A}(\pi_Y^* P)$. For the principal $\hat{G}_4$-bundle $Q_4 = \pi_Y^*(Q_3 \times_{\hat{G}_3} \hat{G}_4)$, we denote the space of connection 1-forms in temporal gauge with $\mathfrak{so}(4)$-component equal to the pullback of the Levi-Civita connection by $\mathscr{A}_4^{tg} := \mathscr{A}_4 \cap \mathscr{A}^{tg}(Q_4)$.

5.2.2 Lemma. *Let $P \to Y$ be a principal H-bundle and $A \in \mathscr{A}^{tg}(\pi_Y^* P)$ a connection* 1*-form in temporal gauge. Consider a group homomorphism $\lambda\colon H \to H'$. Then the induced connection* 1*-form on $\pi_Y^* P \times_H H'$ is again in temporal gauge.*

Proof. The induced connection $A' \in \mathscr{A}(\pi_Y^* P \times_H H')$ satisfies $f^* A' = \lambda_* A$, where $f\colon \pi_Y^* P \to \pi_Y^* P \times_H H'$. Then

$$A'\left(\frac{\partial}{\partial t}_{(t,f(p))}\right) = A'\left(Tf\left(\frac{\partial}{\partial t}_{(t,p)}\right)\right) = f^* A'\left(\frac{\partial}{\partial t}_{(t,p)}\right) = \lambda_* A\left(\frac{\partial}{\partial t}_{(t,p)}\right) = 0.$$

For an arbitary element $p' \in \pi_Y^* P \times_H H'$, there is an element $h \in H'$ such that $p' = f(p)h$ for some $p \in \pi_Y^* P$. The H'-equivariance of the connection 1-form A' implies that

$$A'\left(\frac{\partial}{\partial t}_{(t,p')}\right) = A'\left(\frac{\partial}{\partial t}_{(t,f(p)h)}\right) = A'\left(T_{f(p)} R_h\left(\frac{\partial}{\partial t}_{(t,f(p))}\right)\right) = Ad_{h^{-1}} A'\left(\frac{\partial}{\partial t}_{(t,f(p))}\right) = 0.$$ □

For a connection 1-form on $\pi_Y^* P$ in temporal gauge we obtain a smooth path of connection 1-forms on P.

5.2.3 Lemma. *Let $P \to Y$ be a principal H-bundle and $\pi_Y^* P \to \mathbb{R} \times Y$ the pullback to the cylinder. Then*

$$\mathscr{A}^{tg}(\pi_Y^* P) \cong C^\infty(\mathbb{R}, \mathscr{A}(P)).$$

Proof. Let $A \in \mathscr{A}^{tg}(\pi_Y^* P)$ be a connection 1-form in temporal gauge. For each $(t,p) \in \pi_Y^* P = \mathbb{R} \times P$ we have a linear map $A_{(t,p)} \colon T_{(t,p)}\pi^* P \to \mathfrak{h}$. Consider the induced linear map $\check{A}(t)_p : T_p P \to \mathfrak{h}$ for each $t \in \mathbb{R}$. This can be given explicitly as $\check{A}(t)_p(v) = A_{(t,p)}(0, v)$ for $v \in T_p P, t \in \mathbb{R}$. For each $t \in \mathbb{R}$, we have a 1-form $\check{A}(t) \in \Omega^1(P, \mathfrak{h})^H$. Combining these, we obtain a smooth path of H-equivariant 1-forms $\check{A}$. Furthermore,

$$\check{A}(t)((K_\xi^{P,H})_p) = A((K_\xi^{\pi_Y^* P,H})_{(t,p)}) = \xi.$$

Hence, every connection 1-form $A \in \mathscr{A}^{tg}(\pi_Y^* P)$ in temporal gauge on $\pi_Y^* P$ induces a smooth path of connection 1-forms $\check{A} : \mathbb{R} \to \mathscr{A}(P)$.

Conversely, given a smooth path $\check{A} : \mathbb{R} \to \mathscr{A}(P)$, we can define an equivariant 1-form $A \in \Omega^1(\pi_Y^* P, \mathfrak{h})^H$ as $A := \pi_P^* \check{A}$, where $\pi_P \colon \pi_Y^* P \to P$ is the projection. More precisely,

$$A_{(t,p)}(v) := \check{A}(t)_p(T_{(t,p)}\pi_P(v)) \text{ for } v \in T_{(t,p)}\pi_Y^* P.$$

Since

$$A((K_\xi^{\pi_Y^* P,H})_{(t,p)}) = \check{A}(t)((K_\xi^{P,H})_p) = \xi \text{ for all } \xi \in \mathfrak{h},$$

this is indeed a connection 1-form. By definition, we have

$$A\left(\tfrac{\partial}{\partial t}_{(t,p)}\right) = \check{A}(t)\left(T_{(t,p)}\pi_P\left(\tfrac{\partial}{\partial t}_{(t,p)}\right)\right) = \check{A}(t)(0) = 0.$$

Therefore, we obtain a connection 1-form $A \in \mathscr{A}^{tg}(\pi_Y^* P)$ in temporal gauge.

These two constructions are inverses of each other since A is uniquely detemined by $A(\frac{\partial}{\partial t})$ and the induced linear map $\check{A}(t)_p : T_p P \to \mathfrak{h}$ for each $t \in \mathbb{R}$. ☐

5.2.4 Remark. Every connection $A \in \mathscr{A}(\pi_Y^* P)$ is gauge equivalent to a connection in temporal gauge. The reason for this is that there are solutions of the first order ordinary differential equation

$$A(\tfrac{\partial}{\partial t}) = -TR_{g^{-1}}(\tfrac{\partial g}{\partial t}).$$

If we add the initial conditions $g(0,p) = 1 \in G$ for all $p \in P$ we obtain a unique solution g. Then the pullback $\psi^* A$ of A with respect to the gauge transformation $\psi \in \mathscr{G}(\pi_Y^* P), \psi(p) = pg(p)$ is in temporal gauge (cf. [Fre95, Lemma 1.21]). This induces a bijection

$$\mathscr{A}(\pi_Y^* P)/\mathscr{G}' \xrightarrow{\sim} \mathscr{A}^{tg}(\pi_Y^* P) \cong C^\infty(\mathbb{R}, \mathscr{A}(P)).$$

Here $\mathscr{G}' := \{\, \psi \in \mathscr{G}(\pi_Y^* P) \mid \psi(0,p) = (0,p)\ \forall p \in P \,\}$. Note that $\mathscr{G}'$ is the kernel of the homomorphism $\mathscr{G}(\pi_Y^* P) \to \mathscr{G}(P)$ which sends a gauge transformation to its restriction to $\pi_{\mathbb{R}}^{-1}(\{0\})$. We have a splitting short exact sequence

$$1 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G}(\pi_Y^* P) \underset{s}{\overset{\longrightarrow}{\longleftarrow}} \mathscr{G}(P) \longrightarrow 1.$$

The split $s\colon \mathscr{G}(P) \to \mathscr{G}(\pi_Y^* P)$ is the homomorphism given by $s(\psi) := \mathrm{id}_{\mathbb{R}} \times \psi$. In particular, $\mathscr{G}(\pi_Y^* P)$ is isomorphic to the semidirect product $\mathscr{G}' \rtimes \mathscr{G}(P)$ with respect to $\gamma\colon \mathscr{G}(P) \to \mathrm{Aut}\,(\mathscr{G}')$, $\gamma(\psi)(\varphi) := (\mathrm{id}_{\mathbb{R}} \times \psi) \circ \varphi \circ (\mathrm{id}_{\mathbb{R}} \times \psi^{-1})$.

Moreover, note that the bijection $\mathscr{A}(\pi_Y^* P)/\mathscr{G}' \xrightarrow{\sim} C^\infty(\mathbb{R}, \mathscr{A}(P))$ is $\mathscr{G}(P)$-equivariant, where the action of $\mathscr{G}(P) = \mathscr{G}(\pi_Y^* P)/\mathscr{G}'$ on $\mathscr{A}(\pi_Y^* P)/\mathscr{G}'$ is the induced action and the action of $\mathscr{G}(P)$ on $C^\infty(\mathbb{R}, \mathscr{A}(P))$ is induced by the action of $\mathscr{G}(P)$ on $\mathscr{A}(P)$. We obtain a commutative diagram

$$\begin{array}{ccccc}
\mathscr{A}(\pi_Y^* P)/\mathscr{G}' & \xrightarrow{\sim} & \mathscr{A}^{tg}(\pi_Y^* P) & \xrightarrow{\sim} & C^\infty(\mathbb{R}, \mathscr{A}(P)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{A}(\pi_Y^* P)/\mathscr{G} & \xrightarrow{\sim} & \mathscr{A}^{tg}(\pi_Y^* P)/\mathscr{G}(P) & \xrightarrow{\sim} & C^\infty(\mathbb{R}, \mathscr{A}(P))/\mathscr{G}(P),
\end{array}$$

where the vertical maps are the quotients by the action of $\mathscr{G}(P) = \mathscr{G}(\pi_Y^* P)/\mathscr{G}'$.

Let us now return to the $Spin_\varepsilon^G(m)$-structures.

5.2.5 Note (Levi-Civita connection on the cylinder). The bundle $P_{SO(4)} \to X = \mathbb{R} \times Y$ of oriented orthonormal frames in TX reduces to a principal $SO(3)$-bundle: $P_{SO(4)} = \pi_Y^* P_{SO(3)} \times_{SO(3)} SO(4)$. The Levi-Civita connections φ_Y and φ_X on $P_{SO(3)}$ and $P_{SO(4)}$ are related by $j_* \pi_Y^* \varphi_Y = i'^* \varphi_X$, where $j_* : \mathfrak{so}(3) \to \mathfrak{so}(4)$ is the differential of the inclusion $j : SO(3) \to SO(4)$ and $i'\colon \pi_Y^* P_{SO(3)} \to P_{SO(4)}$ is the reduction. In particular, this implies that the Levi-Civita connection φ_X on the cylinder with product metric is in temporal gauge.

5.2.6 Lemma. *There is a bijection*

$$\mathscr{A}_4^{tg} \cong C^\infty(\mathbb{R}, \mathscr{A}_3).$$

Proof. First, note that $Q_4 = \pi_Y^* Q_3 \times_{\hat{G}_3} \hat{G}_4$. Using Lemma 5.2.3, a path $\check{A} \in C^\infty(\mathbb{R}, \mathscr{A}_3)$ defines a connection 1-form $\tilde{A} \in \mathscr{A}^{tg}(\pi_Y^* Q_3)$. Consider the induced connection $A \in \mathscr{A}(Q_4)$, which satisfies $i^* A = \iota_* \tilde{A}$, where $i\colon \pi_Y^* Q_3 \to Q_4$ and $\iota_*\colon \hat{\mathfrak{g}}_3 \to \hat{\mathfrak{g}}_4$ is the differential of the homomorphism $\iota\colon \hat{G}_3 \to \hat{G}_4$. This connection 1-form is again in temporal gauge by Lemma 5.2.2, i.e. $A \in \mathscr{A}^{tg}(Q_4)$. This defines a smooth map

$$\Phi : C^\infty(\mathbb{R}, \mathscr{A}_3) \hookrightarrow C^\infty(\mathbb{R}, \mathscr{A}(Q_3)) \cong \mathscr{A}^{tg}(\pi_Y^* Q_3) \to \mathscr{A}^{tg}(Q_4),\ \check{A} \mapsto A.$$

Let $\check{A} \in C^\infty(\mathbb{R}, \mathscr{A}_3)$. Its image $\tilde{A}$ in $\mathscr{A}^{tg}(\pi_Y^* Q_3)$ satisfies $\tilde{A} := \pi_{Q_3}^* \check{A}$. We use the isomorphisms $\mathscr{A}_m \to \mathscr{A}(Q_m \to P_{SO(m)})^{Spin(m)}$ from Note 3.3.3 and $\hat{\mathfrak{g}}_m = \mathfrak{so}(m) \oplus \mathfrak{g}$ to decompose the connection 1-forms into a part with values in $\mathfrak{so}(m)$ and one with values in $\mathfrak{g}$. The $\mathfrak{so}(3)$-component of $\check{A}(t)$ is given by the lift of the Levi-Civita connection $\varphi_Y \in \mathscr{A}(P_{SO(3)})$, i.e. $\mathrm{pr}_{\mathfrak{so}(3)} \circ \check{A}(t) = \pi_{SO(3)}^* \varphi_Y$ for all $t \in \mathbb{R}$. Hence, $\mathrm{pr}_{\mathfrak{so}(3)} \circ \check{A} = \pi_{SO(3)}^* \pi_Y^* \varphi_Y$ and also $\mathrm{pr}_{\mathfrak{so}(3)} \circ \tilde{A} = \pi_{SO(3)}^* \pi_Y^* \varphi_Y$ Consider the induced connection 1-form $A = \Phi(\check{A})$ on Q_4. Since

$$i^* \pi_{SO(4)}^* \varphi_X = \pi_{SO(3)}^* i'^* \varphi_X = \pi_{SO(3)}^* j_* \pi_Y^* \varphi_Y = j_* \pi_{SO(3)}^* \pi_Y^* \varphi_Y,$$

the uniqueness of the induced connection implies $\mathrm{pr}_{\mathfrak{so}(4)} \circ A = \pi^*_{SO(4)}\varphi_X$. This proves that the image of $C^\infty(\mathbb{R}, \mathscr{A}_3)$ is in $\mathscr{A}_4$. Furthermore, we know from Note 5.2.5 that $\pi^*_{SO(4)}\varphi_X$ is in temporal gauge. Therefore, Φ is a well defined map $\Phi\colon C^\infty(\mathbb{R}, \mathscr{A}_4) \to \mathscr{A}_4^{tg}$. To prove that this is a bijection, we only have to check that Φ induces a bijection on the $\mathfrak{g}$-components. Using the isomorphism from Note 3.3.3, we have to consider $C^\infty(\mathbb{R}, \mathscr{A}(Q_3 \to P_{SO(3)})^{Spin(3)})$. From Lemma 5.2.6 we know that $C^\infty(\mathbb{R}, \mathscr{A}(Q_3 \to P_{SO(3)})^{Spin(3)}) \cong \mathscr{A}^{tg}(\pi_Y^* Q_3 \to \pi_Y^* P_{SO(3)})^{Spin(3)}$. The last step is to map a connection 1-form $\tilde{A}_\mathfrak{g} \in \mathscr{A}^{tg}(\pi_Y^* Q_3 \to \pi_Y^* P_{SO(3)})^{Spin(3)}$ to the unique connection 1-form $A_\mathfrak{g} \in \mathscr{A}^{tg}(Q_4 \to P_{SO(4)})^{Spin(4)}$ satisfying $i^* A_\mathfrak{g} = \tilde{A}_\mathfrak{g}$, which is again an isomorphism. Thus, combining the observations about the two components, we obtain an isomorphism $C^\infty(\mathbb{R}, \mathscr{A}_3) \cong \mathscr{A}_4^{tg}$.□

5.2.7 Remark. Combining Lemma 5.1.1 and Lemma 5.2.6, we obtain a map

$$C^\infty(\mathbb{R}, \mathscr{C}_3) \to \mathscr{C}_4,$$

which is a bijection onto its image $\mathscr{N}_4 \times \mathscr{A}_4^{tg}$, the space of spinors and connections in temporal gauge on the cylinder.

5.3 The Seiberg-Witten equations on the cylinder

We will now study the Seiberg-Witten section and the Seiberg-Witten equations on the cylinder. One component of the target of the Seiberg-Witten sections are spaces of differential forms. The following lemma describes these on the cylinder.

5.3.1 Lemma. *There is an isomorphism*

$$\begin{aligned} \tau\colon C^\infty(\mathbb{R}, \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_m}_{hor}) &\xrightarrow{\sim} \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_m}_{hor}, \\ \alpha &\mapsto (dt \wedge \pi_3^* \alpha)_+. \end{aligned}$$

Proof. We use the isomorphism $\tau_0 : (\mathbb{R}^3)^* \to \Lambda^2_+(\mathbb{R}^4)^*$ of $SO(3)$-representations from Note 2.3.14. This induces an isomorphism

$$C^\infty(\pi_Y^* Q_3, (\mathbb{R}^3)^* \otimes \mathfrak{g})^{\hat{G}_3} \xrightarrow{\sim} C^\infty(\pi_Y^* Q_3, \Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g})^{\hat{G}_3}.$$

Since

$$Q_4 \times_{\hat{G}_4} (\Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g}) = \pi_Y^* Q_3 \times_{\hat{G}_3} \hat{G}_4 \times_{\hat{G}_4} (\Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g}) = \pi_Y^* Q_3 \times_{\hat{G}_3} (\Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g}),$$

we obtain

$$C^\infty(\pi_Y^* Q_3, \Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g})^{\hat{G}_3} \cong C^\infty(Q_4, \Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g})^{\hat{G}_4},$$

and finally

$$\begin{aligned} C^\infty(\mathbb{R}, \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor}) \cong C^\infty(\pi_Y^* Q_3, (\mathbb{R}^3)^* \otimes \mathfrak{g})^{\hat{G}_3} &\xrightarrow{\sim} C^\infty(Q_4, \Lambda^2_+(\mathbb{R}^4)^* \otimes \mathfrak{g})^{\hat{G}_4} \\ &\cong \Omega^2_+(Q_4, \mathfrak{g})^{\hat{G}_4}_{hor}. \end{aligned}$$

We will denote this isomorphism by τ. Note 2.3.14 implies that this can be written explicitly as $\alpha \mapsto (dt \wedge \pi_3^* \alpha)_+$, where $(dt \wedge \pi_3^* \alpha)_{(t,p)} := dt \wedge \pi_3^* \alpha(t)_p$. □

5.3.2 Lemma. *Let $A \in \mathscr{A}_4^{tg}$ be a connection in temporal gauge and $u \in \mathscr{N}_4$ a spinor. Then*

$$\tau(\Phi_3(\check{u})) = \Phi_4(u) \text{ and } \tau(\tfrac{d\check{a}}{dt} + *_3 F_{\check{a}}) = F_a^+$$

where $\Phi_3(\check{u}) \in C^\infty(\mathbb{R}, \Omega^1(Q_3, \mathfrak{g})_{hor}^{\hat{G}_3})$ is the map $t \mapsto \Phi_3(\check{u}(t))$.

Proof. The equivariant maps corresponding to $\Phi_4(u)$ and $\Phi_3(\check{u}(t))$ are

$$\mu \circ u \colon Q_4 \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^* \cong \mathfrak{g} \otimes \Lambda_+^2(\mathbb{R}^4)^*$$

and

$$\mu \circ \check{u}(t) \colon Q_3 \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^* \cong \mathfrak{g} \otimes (\mathbb{R}^3)^*.$$

The composition of the isomorphism $(\mathbb{R}^3)^* \cong (\mathfrak{sp}(1))^* \cong \Lambda_+^2(\mathbb{R}^4)^*$ is τ_0, so using Lemma 5.3.1 we obtain

$$\tau(\Phi_3(\check{u})) = \Phi_4(u).$$

Let $\pi_{Q_3} \colon \pi_Y^* Q_3 = \mathbb{R} \times Q_3 \to Q_3$ be the projection to Q_3. Since A is in temporal gauge, we have $a(\frac{\partial}{\partial t}) = 0$. The corresponding smooth path $\check{a} \colon \mathbb{R} \to \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}$ satisfies $i^* a = \pi_{Q_3}^* \check{a}$, and $\check{a}(t) \in \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}$ is the restriction of $i^* a$ to the fibre over $t \in \mathbb{R}$. More precisely,

$$i^* a_{(t,p)}(v) = \check{a}(t)_p (T_{(t,p)} \pi_{Q_3}(v)) \text{ for } (t,p) \in \pi_Y^* Q_3, v \in T_{(t,p)} \pi_Y^* Q_3.$$

Observe that $[i^*a, i^*a] = \pi_{Q_3}^* [\check{a}, \check{a}]$ and therefore,

$$i^* F_a = d i^* a + \tfrac{1}{2}[i^*a, i^*a] = dt \wedge \pi_{Q_3}^*(\tfrac{d\check{a}}{dt}) + \pi_{Q_3}^*(d\check{a} + \tfrac{1}{2}[\check{a}, \check{a}]) = dt \wedge \pi_{Q_3}^*(\tfrac{d\check{a}}{dt}) + \pi_{Q_3}^* F_{\check{a}}.$$

Using $*_4 \pi_{Q_3}^* F_{\check{a}} = dt \wedge \pi_{Q_3}^*(*_3 F_{\check{a}})$, we obtain

$$i^* F_a^+ = \tfrac{1}{2}(i^* F_a + *_4 i^* F_a) = \tfrac{1}{2}(dt \wedge \pi_{Q_3}^* \tfrac{d\check{a}}{dt} + \pi_{Q_3}^* F_{\check{a}} + \pi_{Q_3}^*(*_3 \tfrac{d\check{a}}{dt}) + dt \wedge \pi_{Q_3}^*(*_3 F_{\check{a}})).$$

Consider $\breve{\pi}_3 \colon Q_4 \to Q_4/Sp(1)_- = \pi_Y^* Q_3$. Note that $\breve{\pi}_3^* i^* a = a$, and therefore, $\breve{\pi}_3^* i^* F_a = F_a$. Finally, we use $\pi_{Q_3} \circ \breve{\pi}_3 = \pi_3 \colon Q_4 \to Q_3$ to compute

$$\begin{aligned}
\tau(\tfrac{d\check{a}}{dt} + *_3 F_{\check{a}}) &= (dt \wedge \pi_3^*(\tfrac{d\check{a}}{dt}) + dt \wedge \pi_3^*(*_3 F_{\check{a}}))_+ \\
&= \tfrac{1}{2}(dt \wedge \pi_3^*(\tfrac{d\check{a}}{dt}) + dt \wedge \pi_3^*(*_3 F_{\check{a}}) + *_4(dt \wedge \pi_3^*(\tfrac{d\check{a}}{dt})) + *_4(dt \wedge \pi_3^*(*_3 F_{\check{a}}))) \\
&= \tfrac{1}{2}(dt \wedge \pi_3^*(\tfrac{d\check{a}}{dt}) + dt \wedge \pi_3^*(*_3 F_{\check{a}}) + \pi_3^*(*_3 \tfrac{d\check{a}}{dt}) + \pi_3^* F_{\check{a}}) \\
&= \tfrac{1}{2}\breve{\pi}_3^*(dt \wedge \pi_3^*(\tfrac{d\check{a}}{dt}) + dt \wedge \pi_{Q_3}^*(*_3 F_{\check{a}}) + \pi_{Q_3}^*(*_3 \tfrac{d\check{a}}{dt}) + \pi_{Q_3}^* F_{\check{a}}) \\
&= \breve{\pi}_3^* i^* F_a^+ = F_a^+.
\end{aligned}$$

□

5.3.3 Theorem. *Let $Q_3 \to P_{SO(3)} \times_Y P_{G/\varepsilon}$ be $Spin_\varepsilon^G(3)$-structure on a 3-dimensional compact oriented Riemannian manifold Y and $X = \mathbb{R} \times Y$ the cylinder over Y. Furthermore, let $Q_4 := \pi_Y^* Q_3 \times_{\hat{G}_3} \hat{G}_4 \to P_{SO(4)} \times_X P_{G/\varepsilon}$ the associated $Spin_\varepsilon^G(4)$-structure, $A \in \mathscr{A}_4^{tg}$ a connection in temporal gauge and $u \in \mathscr{N}_4$ a spinor on the cylinder. Then*

$$c_4(e_0)^{-1} \mathcal{D}_A^+(u)(i(t,p)) = \tfrac{d\check{u}}{dt}(t)(p) + \mathcal{D}_{\check{A}(t)}(\check{u}(t))(p) \text{ for all } (t,p) \in \pi_Y^* Q_3$$

and

$$\tau(\tfrac{d\check{a}}{dt} + *_3 F_{\check{a}} + \Phi_3(\check{u})) = F_a^+ + \Phi_4(u).$$

In particular, $(u, A) \in \mathscr{C}_4$ *with* A *in temporal gauge satisfy the Seiberg-Witten equations on the cylinder iff* $\gamma \in C^\infty(\mathbb{R}, \mathscr{C}_3), \gamma(t) := (\check{u}(t), \check{A}(t))$ *is a solution of the downward flow equations for the Seiberg-Witten section*

$$\tfrac{d\gamma}{dt}(t) = -\,\mathfrak{F}_3(\check{u}(t), \check{A}(t)).$$

These equations can also be written as

$$\begin{cases} \frac{d\check{u}}{dt} = -\mathcal{D}_{\check{A}}(\check{u}), \\ \frac{d\check{A}}{dt} = -\,*_3\, F_{\check{a}} - \Phi_3(\check{u}). \end{cases}$$

Proof. Let $u \in \mathscr{N}_4 = C^\infty(Q_4, M)^{\hat{G}_4}$ and $\check{u} \in C^\infty(\mathbb{R}, \mathscr{N}_3)$ the corresponding path of spinors satisfying $u(t,p) = \check{u}(t)(\pi_3(p))$ for all $(t,p) \in Q_4$ (cf. Lemma 5.1.1). For $(t,p) \in \pi_Y^* Q_3$ and $i(t,p) \in Q_4$, we obtain $\pi_{SO(4)}(i(t,p))(e_0) = \frac{\partial}{\partial t} \in T_{\pi_X(i(t,p))}X$. Since A is in temporal gauge, the horizontal lift of $\frac{\partial}{\partial t} \in T_{\pi_X(i(t,p))}X$ is $\frac{\partial}{\partial t} \in T_{i(t,p)}Q_4$.

Then

$$\langle d_A^M u(i(t,p)), e_0\rangle = T_{i(t,p)}u(\tfrac{\partial}{\partial t}) = \tfrac{d\check{u}}{dt}(t)(p).$$

For $v \in \mathbb{R}^3 \subset \mathbb{R}^4$ we have

$$\langle d_A^M u(i(t,p)), v\rangle = T_{i(t,p)}u(\tilde{v}^A) = T_p\check{u}(t)(\tilde{v}^{\check{A}(t)}) = \langle d_{\check{A}(t)}^M \check{u}(t)(p), v\rangle.$$

Here, $\tilde{v}^A \in T_{i(t,p)}\pi_Y^* Q_3$ and $\tilde{v}^{\check{A}(t)} \in T_p Q_3$ are the horizontal lifts of $\pi_{SO(3)}(p)(v) \in T_{\pi_Y(p)}Y \subset T_{(t,p)}X$ with respect to the connection 1-forms $i^*A \in \mathscr{A}(\pi_Y^* Q_3)^{tg}$ and $\check{A}(t) \in \mathscr{A}(Q_3)$, respectively. Finally,

$$\begin{aligned} c_4(e_0)^{-1}\mathcal{D}_A^+ u(i(t,p)) &= c_4(e_0)^{-1} c_4(d_A^M u)(i(t,p)) \\ &= \sum_{\ell=0}^{3} c_4(e_0)^{-1} c_4(e_\ell \otimes \langle d_A^M u(i(t,p)), e_\ell\rangle) \\ &= c_4(e_0)^{-1} c_4(e_0 \otimes \tfrac{d\check{u}}{dt}(t)(p)) + \sum_{\ell=0}^{3} c_4(e_0)^{-1} c_4(e_\ell \otimes \langle d_{\check{A}(t)}^M \check{u}(t)(p), e_\ell\rangle) \\ &= \tfrac{d\check{u}}{dt}(t)(p) + \sum_{\ell=0}^{3} c_3(e_\ell \otimes \langle d_{\check{A}(t)}^M \check{u}(t)(p), e_\ell\rangle) \\ &= \tfrac{d\check{u}}{dt}(t)(p) + \mathcal{D}_{\check{A}(t)}\check{u}(t)(p). \end{aligned}$$

Therefore

$$c_4(e_0)^{-1}\mathcal{D}_A^+(u)(i(t,p)) = \tfrac{d\check{u}}{dt}(t)(p) + \mathcal{D}_{\check{A}(t)}(\check{u}(t))(p).$$

The second statement is a direct consequence of Lemma 5.3.2. ☐

5.3.4 Corollary. *Interpreting equivariant maps as sections in the corresponding associated bundles and using* $c_4(dt)\colon \pi_! u^* TM \xrightarrow{\sim} \pi_! u^* \widehat{TM}^1$ *to identify the even and odd part of* $\pi_! u^* \widehat{TM}$*, we obtain*

$$\mathcal{D}_A^+ u = \tfrac{d\check{u}}{dt} + \mathcal{D}_{\check{A}} \check{u}$$

for a connection $A \in \mathscr{A}_4^{tg}$*,* $\check{A} \in C^\infty(\mathbb{R}, \mathscr{A}_3)$ *the corresponding path of connections,* $u \in \Gamma(X, Q_4 \times_{\hat{G}_4} M)$ *and* $\check{u} \in C^\infty(\mathbb{R}, \Gamma(Y, Q_3 \times_{\hat{G}_3} M))$ *the corresponding path of spinors.*

Chapter 6

The Chern-Simons-Dirac functional

In this chapter, we will prove the existence of a functional, whose critical points coincide with the solutions of the three-dimensional Seiberg-Witten equations. For this purpose, we need the manifold structure, Riemannian metric and covariant derivative on the configuration space from Proposition 3.3.11. As we focus on the three-dimensional case, we will drop the index 3 and write $Q, \mathscr{A}, \mathscr{N}, \mathscr{G}$ for $Q_3, \mathscr{A}_3, \mathscr{N}_3, \mathscr{G}_3$. Again, we fix a $Spin^G_\varepsilon(3)$-structure $Q \to P_{SO(3)} \times_Y P_{G/\varepsilon}$ on a 3-dimensional compact oriented Riemannian manifold Y and an Ad-invariant scalar product on $\mathfrak{g}$. We will also make extensive use of Notation 3.3.4.

6.1 Existence of the Chern-Simons-Dirac functional

We have seen in Note 4.2.5 that the Seiberg-Witten equations determine a vector field $\mathfrak{F}\colon \mathscr{C} \to T\mathscr{C}$ on the configuration space $\mathscr{C} = \mathscr{N} \times \mathscr{A}$:

$$\mathfrak{F}(u, A) = (\mathcal{D}_A u, *F_a + \Phi_3(u)) \in C^\infty(Q_3, TM)^{\hat{G}_3}_u \times \Omega^1(Q_3, \mathfrak{g})^{\hat{G}_3}_{hor} = T_{(u,A)}\mathscr{C}.$$

Using the metric on the configuration space (cf. Proposition 3.3.11), this induces the following 1-form on the configuration space $\mathscr{C}$:

$$\mathfrak{F}^\flat(v, \alpha) := g^{\mathscr{C}}((\mathcal{D}_A(u), *F_a + \Phi_3(u)), (v, \alpha)) = g^{\mathscr{N}}(\mathcal{D}_A(u), v) + g^{\mathscr{A}}(*F_a + \Phi_3(u), \alpha).$$

6.1.1 Lemma. *The Seiberg-Witten* 1*-form is closed, i.e.* $d\mathfrak{F}^\flat = 0$.

Proof. Let $V, W \in \Gamma(\mathscr{C}, T\mathscr{C})$ two vector fields on $\mathscr{C}$. Using the metric and the metric compatible, torsion-free covariant derivative ∇ (cf. Proposition A.2.11) on the configuration space, we get

$$
\begin{aligned}
d\,\mathfrak{F}^\flat(V,W) = {} & V(\mathfrak{F}^\flat(W)) - W(\mathfrak{F}^\flat(V)) - \mathfrak{F}^\flat([V,W]) \\
= {} & V(g^{\mathscr{C}}(\mathfrak{F},W)) - W(g^{\mathscr{C}}(\mathfrak{F},V)) - g^{\mathscr{C}}(\mathfrak{F},[V,W]) \\
= {} & g^{\mathscr{C}}(\nabla_V\,\mathfrak{F},W) + g^{\mathscr{C}}(\mathfrak{F},\nabla_V W) \\
& - g^{\mathscr{C}}(\nabla_W\,\mathfrak{F},V) - g^{\mathscr{C}}(\mathfrak{F},\nabla_W V) \\
& - g^{\mathscr{C}}(\mathfrak{F},[V,W]) \\
= {} & g^{\mathscr{C}}(\nabla_V\,\mathfrak{F},W) - g^{\mathscr{C}}(\nabla_W\,\mathfrak{F},V) + g^{\mathscr{C}}(\mathfrak{F},T^{\nabla}(V,W)) \\
= {} & g^{\mathscr{C}}(\nabla_V\,\mathfrak{F},W) - g^{\mathscr{C}}(\nabla_W\,\mathfrak{F},V).
\end{aligned}
$$

We have to compute $g^{\mathscr{C}}(\nabla_V\,\mathfrak{F},W)$. Let $\gamma = (\gamma_1,\gamma_2)\colon I \to \mathscr{C}$ be a smooth curve with $\gamma(0) = (u,A)$ und $\frac{d}{dt}\gamma(t)|_{t=0} = V$. We can choose $\gamma_2(t) = A + t\alpha$ for $V = (v,\alpha)$. We have $\frac{d}{dt}F_{A+t\alpha}|_{t=0} = d\alpha + [A,\alpha] = d_A\alpha$ and therefore

$$
\begin{aligned}
\mathrm{pr}_{T\mathscr{A}}\,\nabla_V\,\mathfrak{F} &= \tfrac{d}{dt}\,\mathrm{pr}_{\mathscr{A}}(\mathfrak{F}(\gamma(t)))|_{t=0} = \tfrac{d}{dt} * F_{a+t\alpha} + \varPhi_3(\gamma_1(t))|_{t=0}, \\
&= *d_A\alpha + T\varPhi_3(v).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathrm{pr}_{T\mathscr{N}}\,\nabla_V\,\mathfrak{F} &= \mathrm{pr}_{T\mathscr{N}}\,\nabla_{(0,v)}\,\mathfrak{F} + \mathrm{pr}_{T\mathscr{N}}\,\nabla_{(\alpha,0)}\,\mathfrak{F} \\
&= \mathcal{D}^{lin,u}_A(v) + \tfrac{d}{dt}\mathcal{D}_{A+t\alpha}u|_{t=0}
\end{aligned}
$$

Finally, let $V = (\alpha,v), W = (\beta,w) \in T_u\mathscr{C} = \varOmega^1(Q,\mathfrak{g})^{\hat{G}}_{hor} \times C^\infty(Q,TM)^{\hat{G}}_u$. Using Lemma 3.6.8 and Lemma 4.2.11, we obtain:

$$
\begin{aligned}
g^{\mathscr{C}}(\nabla_V\,\mathfrak{F},W) = {} & g^{\mathscr{C}}(\mathrm{pr}_{T\mathscr{N}}\,\nabla_V\,\mathfrak{F},W) + g^{\mathscr{C}}(\mathrm{pr}_{T\mathscr{A}}\,\nabla_V\,\mathfrak{F},W) \\
= {} & \int_Y g^M(\mathcal{D}^{lin,u}_A(v),w) + \int_Y g^M(\tfrac{d}{dt}\mathcal{D}_{A+t\alpha}u|_{t=0},w) \\
& + \int_Y \langle *d_A\alpha \wedge *\beta\rangle_{\mathfrak{g}} + \int_Y \langle T\varPhi_3(v),\beta\rangle_{\mathbb{R}^3\otimes\mathfrak{g}} \\
= {} & \int_Y g^M(v,\mathcal{D}_A u(w)) + \int_Y \langle T\varPhi_3(v),\alpha\rangle_{\mathbb{R}^3\otimes\mathfrak{g}} \\
& + \int_Y \langle *d_A\beta \wedge *\alpha\rangle_{\mathfrak{g}} + \int_Y g^M(v,\tfrac{d}{dt}\mathcal{D}_{A+t\beta}u|_{t=0}) \\
= {} & g^{\mathscr{C}}(V,\mathrm{pr}_{T\mathscr{N}}\,\nabla_W\,\mathfrak{F}) + g^{\mathscr{C}}(V,\mathrm{pr}_{T\mathscr{A}}\,\nabla_W\,\mathfrak{F}) \\
= {} & g^{\mathscr{C}}(\nabla_W\,\mathfrak{F},V),
\end{aligned}
$$

and thus

$$
d\,\mathfrak{F}^\flat(V,W) = g(\nabla_V\,\mathfrak{F},W) - g(\nabla_W\,\mathfrak{F},V) = 0. \qquad \square
$$

6.1.2 Theorem. *There is a functional L_{CSD} on the universal cover $\widetilde{\mathscr{C}}$ of the configuration space $\mathscr{C}$ such that images in $\mathscr{C}$ of the critical points of L_{CSD} are the solutions of the Seiberg-Witten equations. Such a functional is called* Chern-Simons-Dirac functional.

Proof. Let $\widetilde{\mathscr{C}} \xrightarrow{\pi} \mathscr{C}$ be the universal covering of the configuration space $\mathscr{C}$ (for existence cf. [KM97, 27.14]). Since $\pi_1(\widetilde{\mathscr{C}}) = 0$, we also have $H^1(\widetilde{\mathscr{C}}, \mathbb{R}) = 0$ and using Proposition A.2.5 also $H^1_{dR}(\widetilde{\mathscr{C}}, \mathbb{R}) = 0$. This implies that all closed 1-forms on $\widetilde{\mathscr{C}}$ are exact. In particular, there exists a functional $L_{CSD} \colon \widetilde{\mathscr{C}} \to \mathbb{R}$ satisfying $dL_{CSD} = \widetilde{\mathfrak{F}}^\flat$, where $\widetilde{\mathfrak{F}}^\flat$ is the pullback of $\mathfrak{F}^\flat$ to $\widetilde{\mathscr{C}}$. The gradient of L_{CSD} is the lift $\widetilde{\mathfrak{F}} \in \Gamma(\widetilde{\mathscr{C}}, T\widetilde{\mathscr{C}})$ of $\mathfrak{F} \in \Gamma(\mathscr{C}, T\mathscr{C})$, $\operatorname{grad}(L_{CSD}) = \widetilde{\mathfrak{F}}$. In particular, let $(u, A) \in \widetilde{\mathscr{C}}$. Then

$$\mathfrak{F}(\pi(u, A)) = 0 \Leftrightarrow \widetilde{\mathfrak{F}}(u, A) = 0 \Leftrightarrow \operatorname{grad}(L_{CSD})(u, A) = 0.$$

The solutions of the Seiberg-Witten equations are the images in $\mathscr{C}$ of the critical point of the Chern-Simons-Dirac functional L_{CSD}. □

6.1.3 Remark. We can construct the Chern-Simons-Dirac functional using the Poincaré lemma. For the part which is only dealing with the connection, we will do this explicitly. This functional is called *Chern-Simons functional*. Since the space of connections $\mathscr{A}$ is an affine space and hence contractible, there is a functional $L_{CS} \colon \mathscr{A} \to \mathbb{R}$ satisfying

$$\frac{d}{dt} L_{CS}(A + t\alpha)|_{t=0} = \int_Y \langle \alpha \wedge F_a \rangle_{\mathfrak{g}},$$

where $a = \operatorname{pr}_{\mathfrak{g}} \circ A = A - \pi^*_{SO(3)}\varphi_Y$ is the $\mathfrak{g}$-component of $A \in \mathscr{A}$. One can construct such a functional as follows. Fix a reference connection $A_0 \in \mathscr{A}$ with $\mathfrak{g}$-component $a_0 = A_0 - \pi^*_{SO(3)}\varphi_Y$ and define

$$L_{CS}(A) := \int_0^1 \int_Y \langle (a - a_0) \wedge F_{a_0 + t(a - a_0)} \rangle_{\mathfrak{g}} dt.$$

Note that

$$F_{a_0 + t(a-a_0)} = F_{a_0} + t d_{a_0}(a - a_0) + \tfrac{t^2}{2}[a - a_0, a - a_0],$$

and

$$\tfrac{1}{2} d_{a_0}(a - a_0) = \tfrac{1}{2}(F_a - F_{a_0}) - \tfrac{1}{4}[a - a_0, a - a_0].$$

Therefore,

$$\begin{aligned} L_{CS}(A) &= \int_0^1 \int_Y \langle (a - a_0) \wedge (F_{a_0} + t d_{a_0}(a - a_0) + \tfrac{t^2}{2}[a - a_0, a - a_0]) \rangle_{\mathfrak{g}} dt \\ &= \int_Y \langle (a - a_0) \wedge (F_{a_0} + \tfrac{1}{2} d_{a_0}(a - a_0) + \tfrac{1}{6}[a - a_0, a - a_0]) \rangle_{\mathfrak{g}} \\ &= \tfrac{1}{2} \int_Y \langle (a - a_0) \wedge (F_{a_0} + F_a - \tfrac{1}{6}[a - a_0, a - a_0]) \rangle_{\mathfrak{g}}. \end{aligned}$$

By construction, the functional L_{CS} satisfies the desired condition. However, we will also proof this explicitly in Theorem 6.2.4.

6.1.4 Remark. Another way to look at the Chern-Simons functional is to observe that the Ad-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is an invariant polynomial on the Lie algebra $\mathfrak{g}$. With the help of the Chern-Weil homomorphism, this defines a cohomology class in $H^4(Y,\mathbb{R})$. The manifold Y is three-dimensional, so $H^4(Y,\mathbb{R})=0$ and the cohomology class vanishes. In this situation, Chern and Simons [CS74] constructed secondary characteristic classes, which depend on a connection $A\in\mathscr{A}(P)$ on a principal G-bundle $P\to Y$. These are 3-forms on the total space P. However, there is also a Chern-Simons form depending on two connections $A_0, A\in\mathscr{A}(P)$. This is a closed 3-form on Y and represents a cohomology class in $H^3(Y,\mathbb{R})$. The pairing of this class with the fundamental class $[Y]\in H_3(Y)$ is the Chern-Simons functional (for details cf. [Fre95], [Fre02]).

6.2 Hyperkähler potential and Chern-Simons-Dirac functional

Let us now assume that the fundamental vector fields for the permuting $Sp(1)$-action satisfy $\mathcal{I}_\zeta K_\zeta^{M,Sp(1)}=-\chi$ for a vector field $\chi\in\Gamma(M,TM)$ and all $\zeta\in\mathfrak{sp}(1)$, $\|\zeta\|^2=1$. We use the hyperkähler potential ρ on M from Proposition 3.2.6 with $\operatorname{grad}(\rho)=\chi$. On the Lie algebra $\mathfrak{g}=\operatorname{Lie}(G)$ of the compact Lie group G we fix an Ad-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}\colon\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$. We also fix a connection $A_0\in\mathscr{A}$. The $\mathfrak{g}$-component of A_0 will be denoted $a_0=A_0-\pi_{SO(3)}^*\varphi_Y$.

6.2.1 Definition. The *Chern-Simons functional* $L_{CS}\colon\mathscr{A}\to\mathbb{R}$ is

$$L_{CS}(A):=\tfrac{1}{2}\int\limits_Y\langle(a-a_0)\wedge(F_{a_0}+F_a-\tfrac{1}{6}[a-a_0,a-a_0])\rangle_{\mathfrak{g}},$$

where $a=A-\pi_{SO(3)}^*\varphi_Y$ is the $\mathfrak{g}$-component of $A\in\mathscr{A}$.

The *Dirac functional* $L_D\colon\mathscr{A}\times\mathscr{N}\to\mathbb{R}$ is

$$L_D(u,A):=\tfrac{1}{2}\int\limits_Y d\rho(\mathcal{D}_A(u)).$$

The *Chern-Simons-Dirac functional* $L_{CSD}\colon\mathscr{A}\times\mathscr{N}\to\mathbb{R}$ is

$$L_{CSD}(u,A):=L_{CS}(A)+L_D(u,A).$$

Since $\operatorname{grad}(\rho)=\chi$, we can alternatively write

$$L_D(u,A)=\tfrac{1}{2}\int\limits_Y g^M(\chi\circ u,\mathcal{D}_A(u)).$$

Note that the Chern-Simons functional and the Chern-Simons-Dirac functional depend on the fixed connection A_0.

6.2.2 Example. If the group G is abelian, we obtain

$$L_{CS}(A) := \tfrac{1}{2}\int_Y \langle (a-a_0)\wedge(F_{a_0}+F_a)\rangle_{\mathfrak{g}}$$

For $G = S^1$, the Chern-Simons functional is

$$L_{CS}(A) = -\tfrac{1}{2}\int\limits_Y (a-a_0)\wedge(F_a+F_{a_0})$$

Here, we interpret the imaginary valued differential forms as complex valued forms and use the multiplication in $\mathbb{C}$. This Chern-Simons functional and the corresponding Chern-Simons-Dirac functional for $M = \mathbb{H}$ has been studied in detail in [KM07].

6.2.3 Example (Chern-Simons on trivial bundles). Consider the case when $Q \to Q/Spin(3) \cong P_{SO(3)}$ is a trivial G-bundle. Fix a trivialization $Q \cong P_{SO(3)} \times G$. Then the Maurer-Cartan form $\eta \in \Omega^1(G,\mathfrak{g})^G$ induces a $Spin(3)$-invariant connection 1-form $a_0 := \mathrm{pr}_G^*\,\eta$ on $Q \to P_{SO(3)}$. We can take $A_0 := \pi_{SO(3)}^*\varphi_Y + a_0$ as the fixed connection for the Chern-Simons functional. In particular, the Maurer-Cartan equation $\eta + \frac{1}{2}[\eta,\eta] = 0$ implies that a_0 is flat:

$$F_{a_0} = d\,\mathrm{pr}_G^*\,\eta + \tfrac{1}{2}[\mathrm{pr}_G^*\,\eta, \mathrm{pr}_G^*\,\eta] = \mathrm{pr}_G^*(d\eta + \tfrac{1}{2}[\eta,\eta]) = 0.$$

For a connection $A \in \mathscr{A}$:

$$F_a = F_{a_0} + d_{a_0}(a-a_0) + \tfrac{1}{2}[a-a_0, a-a_0] = d_{\mathrm{pr}_G^*\eta}(a-a_0) + \tfrac{1}{2}[a-a_0,a-a_0].$$

Denote the image of $a - a_0$ under the isomorphism $\Omega^1(Q,\mathfrak{g})^{\hat{G}}_{hor} \cong \Omega^1(Y,\mathfrak{g})$ by b. Then $d_{\mathrm{pr}_G^*\eta}(a-a_0)$ corresponds to $db \in \Omega^1(Y,\mathfrak{g})$. Therefore, we can write the Chern-Simons functional as

$$\begin{aligned}
L_{CS}(A) &= \tfrac{1}{2}\int\limits_Y \langle (a-a_0)\wedge(F_a + F_{a_0} - \tfrac{1}{6}[a-a_0,a-a_0]\rangle_{\mathfrak{g}} \\
&= \tfrac{1}{2}\int\limits_Y \langle (a-a_0)\wedge(d_{\mathrm{pr}_G^*\eta}(a-a_0) + \tfrac{1}{2}[a-a_0,a-a_0] - \tfrac{1}{6}[a-a_0,a-a_0])\rangle_{\mathfrak{g}} \\
&= \tfrac{1}{2}\int\limits_Y \langle b\wedge(db + \tfrac{1}{3}[b,b])\rangle_{\mathfrak{g}}.
\end{aligned}$$

For $G = SU(2)$ and $\langle x,y\rangle_{\mathfrak{su}(2)} = -B_{\mathfrak{su}(2)}(x,y) = -4\,\mathrm{tr}(xy)$, this is the form of the Chern-Simons functional, which is usually presented in the literature.

6.2.4 Theorem. *The gradient of the Chern-Simons-Dirac functional $L_{CSD}\colon \mathscr{C} \to \mathbb{R}$ from Definition 6.2.1 is the Seiberg-Witten vector field $\mathfrak{F}_3\colon \mathscr{C} \to T\mathscr{C}$, i.e.*

$$\mathrm{grad}(L_{CSD})(u,A) = (*F_a + \Phi_3(u), \mathcal{D}_A(u)) = \mathfrak{F}_3(u,A) \textit{ for all } (u,A) \in \mathscr{C}.$$

Proof. First, observe that for two connections $A, A_0 \in \mathscr{A}$ with $\mathfrak{g}$-components a, a_0, respectively:

$$\begin{aligned} F_a &= da + \tfrac{1}{2}[a,a] \\ &= \tfrac{1}{2}[a_0,a_0] + da + [a, a-a_0] - \tfrac{1}{2}[a,a] + [a,a_0] - \tfrac{1}{2}[a_0,a_0] \\ &= F_{a_0} + d_a(a-a_0) - \tfrac{1}{2}[a-a_0, a-a_0]. \end{aligned}$$

For $A \in \mathscr{A}$, $\alpha \in \Omega^1(Q,\mathfrak{g})^{\hat{G}}_{hor}$, we use Stokes' theorem and the Ad-invariance of the scalar product to obtain

$$\begin{aligned} \tfrac{d}{dt} L_{CS}(A+t\alpha)|_{t=0} =& \tfrac{d}{dt}\tfrac{1}{2}\int_Y \langle (a+t\alpha-a_0)\wedge(F_{a_0}+F_{a+t\alpha})\rangle_{\mathfrak{g}}|_{t=0} \\ &- \tfrac{d}{dt}\tfrac{1}{12}\int_Y \langle (a+t\alpha-a_0)\wedge[a+t\alpha-a_0, a+t\alpha-a_0]\rangle_{\mathfrak{g}}|_{t=0} \\ =& \tfrac{1}{2}\int_Y \langle \alpha\wedge(F_{a_0}+F_a - \tfrac{1}{6}[a-a_0,a-a_0]\rangle_{\mathfrak{g}} \\ &+ \tfrac{1}{2}\int_Y \langle (a-a_0)\wedge(d_a\alpha - \tfrac{1}{3}[\alpha, a-a_0])\rangle_{\mathfrak{g}} \\ =& \tfrac{1}{2}\int_Y \langle \alpha\wedge(F_{a_0}+F_a+d_a(a-a_0)) \\ &- \tfrac{1}{2}\int_Y \langle \alpha\wedge \tfrac{1}{3}[a-a_0,a-a_0] - \tfrac{1}{6}[a-a_0,a-a_0])\rangle_{\mathfrak{g}} \\ =& \int_Y \langle \alpha\wedge F_a\rangle_{\mathfrak{g}} = g^{\mathscr{A}}(\alpha, *F_a). \end{aligned} \tag{6.1}$$

Applying Lemma 4.2.11 and Proposition 3.2.6, we get

$$\begin{aligned} \tfrac{d}{dt} L_D(u, A+t\alpha)|_{t=0} &= \tfrac{d}{dt}\tfrac{1}{2}\int_Y g^M(\chi\circ u, \mathcal{D}_{A+t\alpha}(u))|_{t=0} \\ &= \tfrac{1}{2}\int_Y \langle \alpha, d\mu(\chi\circ u)\rangle_{(\mathbb{R}^3)^*\otimes\mathfrak{g}} \\ &= \int_Y \langle \alpha, \mu\circ u\rangle_{(\mathbb{R}^3)^*\otimes\mathfrak{g}} \\ &= g^{\mathscr{A}}(\alpha, \Phi_3(u)). \end{aligned} \tag{6.2}$$

We can now use the metric compatibility of the covariant derivate (cf. Proposition A.2.11), as well as Lemma 3.6.8 and Proposition 3.2.6 to get

$$\begin{aligned}
2\tfrac{\partial L_D}{\partial u}(v) &= \nabla_v^{\mathscr{N}}(g^{\mathscr{N}}(\chi\circ u, \mathcal{D}_A(u))) \\
&= \int_Y g^M(\chi\circ u, \nabla_v^M \mathcal{D}_A(u)) + \int_Y g^M(\nabla_v^M(\chi\circ u), \mathcal{D}_A(u)) \\
&= \int_Y g^M(\chi\circ u, \mathcal{D}_A^{lin,u}(v)) + \int_Y g^M(v, \mathcal{D}_A(u)) \\
&= \int_Y g^M(\mathcal{D}_A^{lin,u}(\chi\circ u), v) + \int_Y g^M(v, \mathcal{D}_A(u)) \\
&= 2\int_Y g^M(\mathcal{D}_A(u), v).
\end{aligned} \tag{6.3}$$

Combining equation (6.1), equation (6.2) and equation (6.3), we obtain

$$\operatorname{grad}(L_{CSD})(u,A) = (*F_a + \varPhi_3(u), \mathcal{D}_A u) = \mathfrak{F}(u,A) \text{ for all } (u,A)\in\mathscr{C}. \qquad \square$$

6.2.5 Corollary. *The critical points of the Chern-Simons-Dirac functional are the solutions of the* 3*-dimensional Seiberg-Witten equations.*

6.2.6 Corollary. *Let* $Q_3 \to P_{SO(3)} \times_Y P_{G/\varepsilon}$ *be a* $Spin_\varepsilon^G(3)$*-structure on a compact oriented* 3*-dimensional manifold* Y *and* $Q_4 \to P_{SO(4)} \times_X \pi_Y^* P_{G/\varepsilon}$ *the associated* $Spin_\varepsilon^G(4)$*-structure on the cylinder* $X = \mathbb{R}\times Y$*. Then* $(u,A)\in\mathscr{C}_4$ *with* A *in temporal gauge satisfy the Seiberg-Witten equations on the cylinder iff the path* $\gamma\in C^\infty(\mathbb{R},\mathscr{C}_3), \gamma(t) := (\check{u}(t), \check{A}(t))$ *is a solution of the downward gradient flow equation for the Chern-Simons-Dirac functional*

$$\tfrac{d\gamma}{dt}(t) = -\operatorname{grad}(L_{CSD})(\check{u}(t), \check{A}(t)).$$

Proof. This follows immediately from Theorem 5.3.3 and $\operatorname{grad}(L_{CSD}) = \mathfrak{F}_3$. □

6.2.7 Example. For $G = S^1$ and $M = \mathbb{H}$ as in Example 4.2.6, the Chern-Simons-Dirac functional is

$$\begin{aligned}
L_{CSD}(u,A) =& \tfrac12 \int_Y \langle (a-a_0)\wedge(F_{a_0} + F_a - \tfrac16[a-a_0, a-a_0])\rangle_{\mathfrak{g}} \\
&+ \tfrac12 \int_Y g^M(\chi\circ u, \mathcal{D}_A(u)) \\
=& -\tfrac12 \int_Y (a-a_0)\wedge(F_a + F_{a_0}) + \tfrac12\int_Y \langle u, \mathcal{D}_A(u)\rangle.
\end{aligned}$$

This is the Chern-Simons-Dirac functional used in [KM07] to define the Seiberg-Witten Floer homology groups.

6.2.1 The Chern-Simons-Dirac functional and the gauge group

6.2.8 Lemma. *The functional $L_D\colon \mathscr{C} \to \mathbb{R}$ in gauge invariant.*

Proof. Let $u \in \mathscr{N}$ a spinor, $A \in \mathscr{A}$ a connection 1-form and $\psi \in \mathscr{G}$ a gauge transformation. We know from Lemma 3.6.10 that $\mathcal{D}_{\psi^* A}(\psi^* u) = (g^{-1})_* \mathcal{D}_A(u)$, where $g\colon Q \to G$ is defined by the equation $\psi(p) = pg(p)$ for all $p \in Q$. Since $\rho\colon M \to \mathbb{R}$ is G-invariant by Proposition 3.2.6, the map $d\rho\colon TM \to \mathbb{R}$ is also G-invariant and we obtain

$$L_D(\psi^* u, \psi^* A) = \tfrac{1}{2} \int_Y d\rho(\mathcal{D}_{\psi^* A}(\psi^* u)) = \tfrac{1}{2} \int_Y d\rho(g_*^{-1} \mathcal{D}_A(u)) = \tfrac{1}{2} \int_Y d\rho(\mathcal{D}_A(u))$$
$$= L_D(u, A). \qquad \square$$

6.2.9 Lemma. *The functional $L_{CS}\colon \mathscr{A} \to \mathbb{R}$ is $\mathscr{G}_0$-invariant, where $\mathscr{G}_0$ is the identity component of the gauge group $\mathscr{G}$.*

Proof. Let $\xi \in C^\infty(Q, \mathfrak{g})^{\hat{G}} \cong \mathrm{Lie}(\mathscr{G})$. Using Stokes' theorem and the Bianci identity, we obtain

$$dL_{CS}((K_\xi^{\mathscr{G},\mathscr{A}})_A) = \int_Y \langle d_a \xi \wedge F_a \rangle_{\mathfrak{g}} = \int_Y \langle \xi, d_a F_a \rangle_{\mathfrak{g}} = 0.$$

For $\psi_t(p) := p \exp(t\xi(p)), t \in \mathbb{R}$ we have

$$\tfrac{d}{dt} L_{CS}(\psi_{s+t}^* A)|_{t=0} = dL_{CS}(\tfrac{d}{dt} \psi_{s+t}^* A|_{t=0}) = dL_{CS}\left(\tfrac{d}{dt} (A^{\exp(s\xi)})^{\exp(t\xi)}|_{t=0}\right)$$
$$= dL_{CS}\left((K_\xi^{\mathscr{A},\mathscr{G}})_{A^{\exp(t\xi)}}\right) = 0,$$

and therefore $L_{CS}(\psi_t^* A) = L_{CS}(\psi_0^* A) = L_{CS}(A)$ for all $t \in \mathbb{R}$. This proves that the functional L_{CS} is invariant under the image of the exponential map. The exponential map for infinite dimensional Lie groups is not necessarily a surjection onto the identity component. However, the gauge group is locally exponential (cf. [Woc06, Thm 3.1.11]). This means that image of the exponential map at least generates the identity component. Replacing A by $\varphi^* A$ in the equation above for a gauge transformation $\varphi \in \mathscr{G}$, we obtain

$$L_{CS}(\varphi^* A) = L_{CS}(\psi_t^* \varphi^* A) \text{ for all } t \in \mathbb{R}$$

Therefore, if L_{CS} is invariant under φ, then it is also invariant under $\varphi \circ \psi$. Using induction for $N \in \mathbb{N}$, this proves that L_{CS} is invariant under $\exp(\mathrm{Lie}(\mathscr{G}))^N = \left\{ \prod_{\ell=0}^N \exp(\xi_\ell) \,\middle|\, \xi_l \in \mathrm{Lie}(\mathscr{G}) \right\}$ for all $N \in \mathbb{N}$. Since the gauge group is locally exponential, we can now conclude that L_{CS} is $\mathscr{G}_0 = \bigcup_{N=0}^\infty \exp(\mathrm{Lie}(\mathscr{G}))^N$-invariance. $\square$

Combining Lemma 6.2.8 and Lemma 6.2.9, we obtain:

6.2.10 Theorem. *The Chern-Simons-Dirac functional $L_{CSD}\colon \mathscr{C} \to \mathbb{R}$ is invariant under the connected component $\mathscr{G}_0$ of the identity in the gauge group.*

6.2.11 Remark. The Chern-Simons functional depends on the fixed connection $A_0 \in \mathscr{A}$. Writing $L_{CS}^{A_0}\colon \mathscr{A} \to \mathbb{R}$ for the Chern-Simons functional for fixed connection A_0, we find that

$$L_{CS}^{\psi^* A_0}(\psi^* A) = L_{CS}^{A_0}(A) \text{ for } \psi \in \mathscr{G}.$$

In particular,

$$L_{CS}^{\psi^* A_0}(A) = L_{CS}^{A_0}((\psi^{-1})^* A) = L_{CS}^{A_0}(A) \text{ for all } \psi \in \mathscr{G}_0.$$

Therefore, the Chern-Simons functional only depends on the choice of the class of A_0 in $\mathscr{A}/\mathscr{G}_0$. Notice that the Chern-Simons functional is not in general $\mathscr{G}$-invariant. In other words, the Chern-Simons functional does in general depend on the class of A_0 in $\mathscr{A}/\mathscr{G}_0$, not only on the class of A_0 in $\mathscr{A}/\mathscr{G}$.

For $G = S^1$ and $G = SU(2)$, this dependence can be described in terms of topological data. For $G = S^1$ the gauge transformation determines a map $g\colon Y \to S^1$, which represents a class $[g] \in H^1(Y, \mathbb{Z}) = [Y, K(\mathbb{Z}, 1)] = [Y, S^1]$ and

$$L_{CS}(A) - L_{CS}(A^g) = 2\pi^2([g] \smile c_1(P_{S^1}))[Y].$$

Here $c_1(P_{S^1})$ is the first Chern class of the principal S^1-bundle $P_{S^1} = Q/Spin(3)$, $\smile\colon H^1(Y, \mathbb{Z}) \times H^2(Y, \mathbb{Z}) \to H^3(Y, \mathbb{Z})$ is the cup product and $[Y] \in H_3(Y, \mathbb{Z})$ denotes the fundamental class of Y (cf. [KM07, Lemma 4.1.3]).

In the case of $G = SU(2)$, note that every principal $SU(2)$-bundle over Y is trivial. Given a trivialization, a gauge transformation determines a map $g\colon Y \to SU(2)$, and $L_{CS}(A) - L_{CS}(A^g)$ is given (up to a constant factor) by the degree $\deg(g)$ of $g\colon Y \to SU(2)$ (cf. [Flo88]).

Chapter 7

Conclusion

The dimensional reduction of the generalized Seiberg-Witten equations is similar to the dimensional reduction of the usual Seiberg-Witten equations. As we have seen in Theorem 5.3.3, the generalized Seiberg-Witten equations on a cylinder over a three-dimensional manifold can be rewritten as downward flow equations for the vector field $\mathfrak{F}_3 \in \Gamma(\mathscr{C}_3, T\mathscr{C}_3)$ on the configuration space which is given by the generalized Seiberg-Witten equations. Moreover, there is also a Chern-Simons-Dirac functional for the generalized Seiberg-Witten equations (Theorem 6.1.2). The gradient of the Chern-Simons-Dirac functional is the vector field $\mathfrak{F}_3$. Therefore, the generalized Seiberg-Witten equations on the cylinder are equivalent to the downward gradient flow equations of the Chern-Simons-Dirac functional (Corollary 6.2.6).

In the case of a target manifold M with permuting action and a vector field $\chi \in \Gamma(M, TM)$ such that $\chi = -\mathcal{I}_\zeta K_\zeta^{M,Sp(1)}$ for all $\zeta \in \mathrm{Im}(\mathbb{H})$ with $\|\zeta\|^2 = 1$, we explicitly constructed such a functional (Theorem 6.2.4). In this case, the Chern-Simons-Dirac functional is also invariant under the identity component of the gauge group (Theorem 6.2.10). For the usual Seiberg-Witten equations, one can use this functional to construct the Seiberg-Witten Floer homology groups (cf. [KM07]). These constructions are infinite-dimensional analogues of the construction of the Morse homology groups, where the Chern-Simons-Dirac functional plays the role of the Morse function. In particular, the critical points and the gradient flow equations (Theorem 5.3.3) are important ingredients. It might be interesting to construct Floer homology groups for the Chern-Simons-Dirac functional for the generalized Seiberg-Witten equations. However, there are several obstacles to overcome. In particular, one has to carefully analyse the moduli spaces of generalized Seiberg-Witten equations in three and four dimensions. Again, the moduli spaces of solutions of the gradient flow equations are of particular interest since these are used to construct the boundary operator of the Floer complex. In particular, a suitable class of pertubations is needed to obtain non-degeneracy of the critical points of the Chern-Simons-Dirac functional and a smooth structure on the moduli spaces using Fredholm theory and the Sard-Smale theorem. Another challenge is to deal with reducible solutions.

When we do not assume the existence of a vector field χ as above, less is know about the Chern-Simons-Dirac functional and its properties. In particular, it might only exist on a cover $\widetilde{\mathscr{C}}$ of the configuration space $\mathscr{C}$. To understand this phenomenon, one has to study the space of periods $\left\{ \int_\gamma \mathfrak{F}_3^\flat(\dot\gamma) \,\middle|\, \gamma \in \pi_1(\mathscr{C}) \right\}$ of the Seiberg-Witten 1-form $\mathfrak{F}_3^\flat$.

However, the existence of the Chern-Simons-Dirac functional and its properties, in particular in the case when the target manifold admits a vector field χ as above, give rise to some hope that it might be possible to define Seiberg-Witten Floer homology groups for the generalized Seiberg-Witten equations.

Appendix A

Infinite dimensional manifolds

In this appendix, we collect some statements about infinite dimensional manifolds which have been used in the previous chapters. For a detailed and exhaustive treatment of the convenient calculus, which is used to describe these infinite-dimensional manifolds, we refer the reader to [KM97].

A.1 Manifolds of mappings

A.1.1 Proposition ([KM97, Thm 42.1]). *Let Q and M be finite dimensional smooth manifolds. Then the space $C^\infty(Q, M)$ of all smooth maps from Q to M is a smooth manifold modeled on the topological vector spaces*

$$\Gamma_c(Q, f^*TM) = \varinjlim_K \Gamma_K(Q, f^*TM)$$

of smooth compactly supported sections of the pullback bundles along $f\colon Q \to M$. Here $\Gamma_K(Q, f^*TM)$ is the space of smooth sections with support in a compact subset $K \subset Q$ and $\Gamma_c(Q, f^*TM)$ is the inductive limit of $\Gamma_K(Q, f^*TM)$, where K run through the compact subsets of Q.

A.1.2 Remark. If Q is compact, then $\Gamma_c(Q, f^*TM) = \Gamma(Q, f^*TM)$ is a Fréchet space with the usual compact-open C^∞-topology.

A.1.3 Remark. Note that $\Gamma_c(Q, f^*TM) \subset C^\infty(Q, TM)$ for all $f\colon Q \to M$ and therefore, we can interpret $TC^\infty(Q, M) \subset C^\infty(Q, TM)$ and the projection in the tangent bundle $TC^\infty(Q, M) \to C^\infty(Q, M)$ is the restriction of $C^\infty(Q, TM) \to C^\infty(Q, M), v \mapsto \pi_M \circ v$. If Q_m is compact, then $TC^\infty(Q, M) \cong C^\infty(Q, TM)$.

A.1.4 Remark ([KM97, Thm 42.3]). The manifold $C^\infty(Q, M)$ has separable connected components and is smoothly paracompact (i.e. it admits a smooth partition of unity) and Lindelöf. Furthermore, $C^\infty(Q, M)$ is metrizable if Q is compact.

A.1.5 Lemma. *Let H be a compact Lie group, P a principal H-bundle and M a Riemannian manifold with a smooth isometric H-action (all finite dimensional). Then the space $C^\infty(Q, M)^H$ of all smooth H-equivariant maps from Q to M is a closed submanifold of $C^\infty(Q, M)$, modeled on the vector spaces $\Gamma_c(Q, f^*TM)^H$ of smooth compactly supported sections of the pullback bundles along $f \in C^\infty(Q, M)^H$. Furthermore, $C^\infty(Q, M)^H$ is smoothly paracompact.*

Proof. For $f \in C^\infty(Q, M)^H$, consider the closed subspace of H-equivariant sections $\Gamma_c(Q, f^*TM)^H \subset \Gamma_c(Q, f^*TM)$. The charts in [KM97, Thm 42.1] use the exponential map for the Riemannian metric on M. Since the H-action is isometric, this is H-equivariant and we obtain the required submanifold charts.

Since $C^\infty(Q, M)^H \subset C^\infty(Q, M)$ is closed and $C^\infty(Q, M)$ is smoothly paracompact, $C^\infty(Q, M)^H$ is also smoothly paracompact (cf. [KM97, 27.11]). □

A.1.6 Remark. Another way to construct the smooth structure on $C^\infty(Q, M)^H$ is to use Proposition 2.1.22 and interpret it as the space of sections of the associated bundle $Q \times_H M$ (cf. [KM97, Thm 42.20]).

A.1.7 Proposition (exponential law, [KM97, Thm 42.14]). *Let M and N be two (finite dimensional) manifolds and X a compact (finite dimensional) manifold. Then there is a canonical bijection*

$$C^\infty(N, C^\infty(X, M)) \xrightarrow{\sim} C^\infty(N \times X, M).$$

A.2 The configuration space

A.2.1 Configuration space as an infinite dimensional manifold

We will now study the *configuration space* $\mathscr{C}_m = \mathscr{N}_m \times \mathscr{A}_m$ for the Seiberg-Witten equations. We denote by $\mathrm{pr}_{\mathscr{N}} \colon \mathscr{N}_m \times \mathscr{A} \to \mathscr{N}_m$ and $\mathrm{pr}_{\mathscr{A}} \colon \mathscr{N}_m \times \mathscr{A}_m \to \mathscr{A}_m$ the two projections from the configurations space $\mathscr{C}_m$ to its factors. The following lemma is a consequence of Lemma A.1.5:

A.2.1 Lemma. *The space of spinors $\mathscr{N}_m = C^\infty(Q_m, M)^{\hat{G}_m}$ is a smooth manifold with tangent spaces $T_u\mathscr{N}_m = \Gamma_c(Q_m, u^*TM)^{\hat{G}_m} \cong C^\infty_c(Q_m, TM)^{\hat{G}_m}_u$, where $\Gamma_c(Q_m, TM)^{\hat{G}_m}_u := \left\{ v \in C^\infty_c(Q_m, TM)^{\hat{G}_m} \;\middle|\; \pi_M \circ v = u \right\}$. The projection of the tangent bundle is given by composition with π_M:*

$$\begin{aligned} T\mathscr{N}_m \subset C^\infty(Q_m, TM)^{\hat{G}_m} &\xrightarrow{\pi_{\mathscr{N}_m}} \mathscr{N}_m, \\ v &\mapsto \pi_M \circ v. \end{aligned}$$

Furthermore, $\mathscr{N}_m$ is smoothly paracompact.

A.2.2 Remark. If Q is compact, then $\mathscr{N}_m$ is a Fréchet manifold and the total space of the tangent bundle is $T\mathscr{N}_m \cong C^\infty(Q, TM)^{\hat{G}_m}$.

A.2.3 Lemma. *The configuration space $\mathscr{C}_m$ is a smooth (infinite dimensional) manifold which is smoothly paracompact. If Z is compact, then $\mathscr{C}_m$ is a Fréchet manifold and*

$$T_{(u,A)}\mathscr{C}_m = C^\infty(Q_m, TM)_u^{\hat{G}_m} \oplus \Omega^1(Q_m, \mathfrak{g})_{hor}^{\hat{G}_m},$$

where $C^\infty(Q_m, TM)_u^{\hat{G}_m} := \left\{ v \in C^\infty(Q_m, TM)^{\hat{G}_m} \;\middle|\; \pi_M \circ v = u \right\}$.

Proof. We already know that the space of spinors is a smooth manifold. Since $\mathscr{A}_m$ is an affine space for the vector space $\Omega^1(Q_m, \mathfrak{g})_{hor}^{\hat{G}_m} \cong C^\infty(Q_m, \mathfrak{g} \otimes (\mathbb{R}^m)^*)^{\hat{G}_m}$, the space of connections $\mathscr{A}_m$ is a smooth manifold modeled on $C_c^\infty(Q_m, \mathfrak{g} \otimes (\mathbb{R}^m)^*)^{\hat{G}_m}$. If Z is compact, then $T_A\mathscr{A}_m = \Omega^1(Q_m, \mathfrak{g})_{hor}^{\hat{G}_m}$ for all $A \in \mathscr{A}_m$. □

A.2.4 Remark. We will not give any more details concerning the manifold structure here, as it is only important here, that we can use the usual calculus [KM97, Ch VII] and differential geometry [KM97, Ch VIII] for the configuration space $\mathscr{C}_m$. This is described in detail in [KM97, Ch VI-IX]. Note however, that one has to be quite careful generalizing from finite-dimensional to infinite-dimensional manifolds, even more if one considers manifolds modeled on topological vector spaces more general than Hilbert spaces or Banach spaces. Some notions which are equivalent in finite dimensions generalize to non-equivalent notions in the case of infinite-dimensional manifolds. For example, we understand tangent vectors as equivalence classes of smooth curves in the manifold and not as derivations, since this is the convenient notion in the infinite-dimensional setting. Another difference is that in many cases in infinite dimensions, the tangent bundle and the cotangent bundle are not isomorphic. In particular, a Riemannian metric will no longer identify tangent and cotangent bundle, but will only provide a homomorphism from the tangent to the cotangent bundle, which usually fails to be surjective.

A.2.5 Proposition ([KM97, Thm 34.7]). *Let M be a smooth, smoothly paracompact manifold. Then the de Rham cohomology of M and the singular cohomology with coefficients in $\mathbb{R}$ are canonically isomorphic.*

A.2.6 Corollary. *The de Rham cohomology and the singular cohomology with coefficients in $\mathbb{R}$ of the manifold $\mathscr{N}_m$ are canonically isomorphic. The same holds for the configuration space $\mathscr{C}_m$.*

Spinors and sections of associated bundles

A.2.7 Note. Using Proposition 2.1.22, we can also understand the space of spinors as the space of sections $\Gamma(Z, Q_m \times_{\hat{G}_m} M)$. This is again a submanifold of $C^\infty(Z, Q_m \times_{\hat{G}_m} M)$. When Q_m is compact, the bijection in Proposition 2.1.22 is even a diffeomorphism. This can be seen as follows: First notice that a map between two manifolds is smooth iff the composition with every smooth curve in the source manifold is a smooth curve in the target manifold. The space of smooth curves in $\varGamma(Z, Q_m \times_{\hat{G}_m} M)$ is $C^\infty(\mathbb{R}, \varGamma(Z, Q_m \times_{\hat{G}_m} M)) = \varGamma(\mathbb{R} \times Z, Q_m \times_{\hat{G}_m} M)$ and the space of smooth curves in $C^\infty(Q_m, M)^{\hat{G}_m}$ is $C^\infty(\mathbb{R}, C^\infty(Q_m, M)^{\hat{G}_m}) = C^\infty(\mathbb{R} \times Q_m, M)^{\hat{G}_m}$. The map between these spaces of curves is again the one in Proposition 2.1.22, in particular a bijection, and we conclude that the bijection from Proposition 2.1.22 as well as its inverse are smooth.

A.2.2 A metric on the configuration space

Let now $Q_m \to P_{SO(m)} \times_Z P_{G/\varepsilon}$ be a $Spin^G_\varepsilon(m)$-structure on a compact oriented Riemannian manifold Z ($\dim(Z) = m \in \{3, 4\}$).

A.2.8 Lemma. *For $u \in \mathscr{N}_m$ let $g^M_u : u^*TM \otimes u^*TM \to \mathbb{R}$ be the pullback metric defined by*

$$g^M_u((p, v), (p, w)) := g^M_{u(p)}(v, w) \text{ for } (p, v), (p, w) \in u^*TM \subset Q_m \times TM.$$

For $v, w \in C^\infty(Q_m, TM)^{\hat{G}_m}_u \cong \varGamma(Q_m, u^*TM)^{\hat{G}_m}$ define

$$g^{\mathscr{N}}(v, w) := \int_Z g^M_u(v, w),$$

where we use Notation 3.3.4 for the $\hat{G}_m$-invariant map $g^M_u(v, w) : Q_m \to \mathbb{R}$. This defines a Riemannian metric on the space of spinors $\mathscr{N}_m$.

Proof. Let $v, w \in T_u\mathscr{N}_m$. Then

$$g^{\mathscr{N}}_u(v, w) = \int_Z g^M_u(v, w) = \int_Z g^M_u(w, v) = g^{\mathscr{N}}_u(w, v)$$

and

$$g^{\mathscr{N}}_u(v, v) = \int_Z g^M_u(v, v) = \|v\|^2_{L^2} \geq 0$$

Furthermore, $g^{\mathscr{N}}_u(v, v) = 0$ iff $v = 0$. The linearity of $g^{\mathscr{N}}$ is a direct consequence of the linearity of g^M. □

A.2.9 Remark. The pullback metric is often denoted by u^*g^M. However, unlike the pullback of differential forms, the definition of the pullback metric does not involve the differential of u.

Next, we define a metric on the configuration space $\mathscr{C}_m$. For $(u, A) \in \mathscr{C}_m$ and $V = (v, \alpha), W = (w, \beta) \in T_{(u,A)}\mathscr{C}_m = C^\infty(Q_m, TM)^{\hat{G}_m}_u \oplus \varOmega^1(Q_m, \mathfrak{g})^{\hat{G}_m}_{hor}$ define

$$g^{\mathscr{C}}_{(u,A)}(V, W) := g^{\mathscr{N}}(v, w) + g^{\mathscr{A}}(\alpha, \beta).$$

Here

$$g^{\mathscr{A}}(\alpha, \beta) := \int_Z \langle \alpha \wedge *\beta \rangle_{\mathfrak{g}}$$

is the L^2-metric on $\varOmega^m(Q_m, \mathfrak{g})^{\hat{G}_m}_{hor}$ induced by the Ad-invariant scalar product. Note that we implicitly use the isomorphism $\varOmega^m(Q_m, \mathbb{R})^{\hat{G}_m}_{hor} \cong \varOmega^m(Z, \mathbb{R})$.

A.2.10 Lemma. *This defines a Riemannian metric $g^{\mathscr{C}}$ on the configuration space $\mathscr{C}_m$.*

A.2.3 The covariant derivative on the configuration space

The next step is to define a covariant derivative on the configuration space $\mathscr{C}_m$, which is both compatible with the metric $g^{\mathscr{C}}$ and torsion-free.

The bundle $\mathrm{pr}_{\mathscr{A}}^* T\mathscr{A} \to \mathscr{C}_m$ is trivial with fibre $\Omega^1(Q_m, \mathfrak{g})^{\hat{G}_m}_{hor}$. We can interpret a section s of this bundle as a map $\tilde{s}\colon \mathscr{C}_m \to \Omega^1(Q_m, \mathfrak{g})^{\hat{G}_m}_{hor}$. With this understood, we have a tautological covariant derivative:

$$\begin{aligned}\nabla\colon \Gamma(\mathscr{C}_m, \mathrm{pr}_{\mathscr{A}}^* T\mathscr{A} \otimes T\mathscr{C}_m) &\to \Gamma(\mathscr{C}_m, \mathrm{pr}_{\mathscr{A}}^* T\mathscr{A})\\ (\nabla_{(v,\alpha)} s) &= \Big(\gamma(0), \tfrac{d}{dt}\tilde{s}(\gamma(t))|_{t=0}\Big)\end{aligned}$$

for $(v, \alpha) = \frac{d}{dt}(\gamma(t))|_{t=0}$.

Recall that $TC^\infty(Q_m, M)^{\hat{G}_m} = C^\infty(Q_m, TM)^{\hat{G}_m}$ and

$$\pi_{\mathscr{N}}\colon C^\infty(Q_m, TM)^{\hat{G}_m} \to C^\infty(Q_m, M)^{\hat{G}_m}$$

is $v \mapsto \pi_M \circ v$, where $\pi_M\colon TM \to M$ is the tangent bundle of M. Similarly, $TT\mathscr{N}_m = C^\infty(Q_m, TTM)^{\hat{G}_m}$. We define a horizontal bundle $\mathscr{H}_{T\mathscr{N}_m} \subset TT\mathscr{N}_m$ and a vertical bundle $\mathscr{V}_{T\mathscr{N}_m} \subset TT\mathscr{N}_m$ as follows: Let $TTM = \mathscr{H}_{TM} \oplus \mathscr{V}_{TM}$ be the decomposition induced by the Levi-Civita on M. Since $\hat{G}_m$ acts isometrically on M, this decomposition is $\hat{G}_m$-equivariant. Define

$$\mathscr{H}_{T\mathscr{N}_m} := C^\infty(Q_m, \mathscr{H}_{TM})^{\hat{G}_m} \qquad \text{and} \qquad \mathscr{V}_{T\mathscr{N}_m} := C^\infty(Q_m, \mathscr{V}_{TM})^{\hat{G}_m}.$$

The corresponding connector $\mathcal{K}^{\mathscr{N}}, v \mapsto \mathcal{K}^m \circ v$ is given by composition with $\mathcal{K}^M$. This induces a covariant derivative

$$\begin{aligned}\nabla^{\mathscr{N}}\colon \Gamma(\mathscr{N}_m, T\mathscr{N}_m) \times \Gamma(\mathscr{N}_m, T\mathscr{N}_m) &\to \Gamma(\mathscr{N}_m, T\mathscr{N}_m),\\ (V, W) &\mapsto \nabla_V W,\ (\nabla_V W)_u := \mathcal{K}^{\mathscr{N}}(T_u W(V_u))\end{aligned}$$

We get

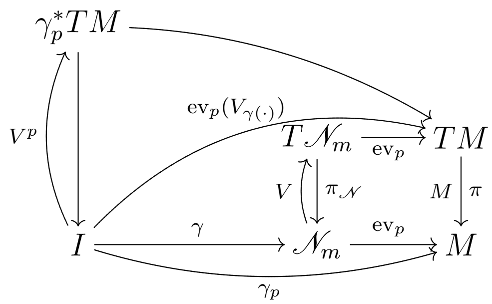
$$(\nabla_V W)_u = \mathcal{K}^{\mathscr{N}}(T_u W(V_u)) = \mathcal{K}^M \circ T_u W(V_u).$$

A.2.11 Proposition. *This covariant derivative $\nabla^{\mathscr{N}}$ is compatible with the metric $g^{\mathscr{N}}$ and torsion-free, i.e.*

1. $U(g^{\mathscr{N}}(V, W)) = g^{\mathscr{N}}(\nabla_U V, W) + g^{\mathscr{N}}(V, \nabla_U W)$ *for all* $U, V, W \in \Gamma(\mathscr{N}_m, T\mathscr{N}_m)$,
2. $\nabla_V W - \nabla_W V - [V, W] = 0$ *for all vector fields* $V, W \in \Gamma(\mathscr{N}_m, T\mathscr{N}_m)$.

Proof. For $p \in Q_m$, consider the evaluation map $\mathrm{ev}_p\colon C^\infty(Q_m, M)^{\hat{G}_m} \to M, \mathrm{ev}_p(u) := u(p)$. Similarly, we have an evaluation map $\mathrm{ev}_p\colon C^\infty(Q_m, TM)^{\hat{G}_m} \to TM$. Let $\gamma\colon I \to \mathscr{N}_m$ be a smooth curve of spinors satifying $\gamma(0) = u$ and $\frac{d}{dt}\gamma(t)|_{t=0} = U_u \in C^\infty(Q_m, TM)_u^{\hat{G}_m}$. We denote $\gamma^p := \mathrm{ev}_p \circ \gamma$.

For $V \in \Gamma(\mathscr{N}_m, T\mathscr{N}_m)$, the following diagram commutes:



Here V^p is the section of $\gamma_p^* TM$ given by $\mathrm{ev}_p \circ V \circ \gamma$. In particular,

$$(V^p)_0 = \mathrm{ev}_p(V_u)$$

The Levi-Civita connection on M and the corresponding covariant derivative ∇^M induce a covariant derivative $\gamma_p^* \nabla^M$ on $\gamma_p^* TM$ which is compatible with the metric $g^M_{\gamma_p}$ on I. Furthermore,

$$\begin{aligned}(\gamma_p^* \nabla_1 V^p)_0 &= \mathrm{pr}_{\mathscr{V}_{TM}}(\tfrac{d}{dt}(V^p)_t|_{t=0}) = \mathrm{pr}_{\mathscr{V}_{TM}}(\tfrac{d}{dt}\,\mathrm{ev}_p(V_{\gamma(t)})|_{t=0})\\ &= \mathrm{pr}_{\mathscr{V}_{TM}}(\mathrm{ev}_p(T_u V(U))) = \mathrm{ev}_p((\nabla_U V)_u),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{ev}_p\left(\tfrac{d}{dt} g^M_{\gamma(t)}(V_{\gamma(t)}, W_{\gamma(t)})|_{t=0}\right)\\ &= \tfrac{d}{dt}\,\mathrm{ev}_p\left(g^M_{\gamma(t)}(V_{\gamma(t)}, W_{\gamma(t)})\right)|_{t=0}\\ &= \tfrac{d}{dt}\left((g^M_\gamma)_t(\mathrm{ev}_p(V_{\gamma(t)}), \mathrm{ev}_p(W_{\gamma(t)}))\right)|_{t=0}\\ &= \tfrac{d}{dt}\left((g^M_{\gamma_p})_t((V^p)_t, (W^p)_t)\right)|_{t=0}\\ &= g^M_u((\gamma_p^*\nabla_1 V^p)_0, (W^p)_0) + g^M_u((V^p)_0, (\gamma_p^*\nabla_1 W^p)_0)\\ &= g^M_u(\mathrm{ev}_p((\nabla_U V)_u), \mathrm{ev}_p(W_u)) + g^M_u(\mathrm{ev}_p(V_u), \mathrm{ev}_p((\nabla_U W)_u)).\end{aligned}$$

We can now compute

$$\begin{aligned}U(g^{\mathscr{N}}(V,W)) &= \tfrac{d}{dt} g^{\mathscr{N}}_{\gamma(t)}(V_{\gamma(t)}, W_{\gamma(t)})|_{t=0} = \tfrac{d}{dt}\int_Z g^M_{\gamma(t)}(V_{\gamma(t)}, W_{\gamma(t)})|_{t=0}\\ &= \int_Z \tfrac{d}{dt} g^M_{\gamma(t)}(V_{\gamma(t)}, W_{\gamma(t)})|_{t=0} = \int_Z (g^M_u((\nabla_U V)_u, W_u) + g^M_u(V_u, (\nabla_U W)_u))\\ &= g^{\mathscr{N}}(\nabla_U V, W) + g^{\mathscr{N}}(V, \nabla_U W).\end{aligned}$$

This proves that the covariant derivative $\nabla^{\mathscr{N}}$ is compatible with the metric $g^{\mathscr{N}}$. For the second part of the statement, we use the formula for the torsion from Theorem 2.1.39.

Since $\mathrm{ev}_p \circ \kappa_{\mathscr{N}} = \kappa_M \circ \mathrm{ev}_p$ for $p \in Q_m$ and the torsion of the Levi-Civita connection on M vanishes, we have

$$T^{\nabla}(V, W) = (\mathcal{K}^{\mathscr{N}} \circ \kappa_{\mathscr{N}} - \mathcal{K}^{\mathscr{N}}) \circ TV \circ W = (\mathcal{K}^{M} \circ \kappa_{M} - \mathcal{K}^{M}) \circ T_u V \circ W = 0. \qquad \square$$

A.2.12 Corollary. *The tautological covariant derivative* $\nabla^{\mathscr{A}}$ *and the metric compatible, torsion-free covariant derivative* $\nabla^{\mathscr{N}}$ *determine a metric compatible, torsion-free covariant derivative* $\nabla^{\mathscr{C}}$ *on the tangent bundle* $T\mathscr{C} \to \mathscr{C}$*.*

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