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The Benefit Function Approach to Modeling Price-Dependent Demand Systems: An Application of Duality Theory

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Abstract

In this article we advocate more extensive use of the benefit function in specifying price-dependent or inverse demand models. In particular, we demonstrate how duality theory may be used to establish the inter-relationships between the Marshallian (or Hicksian) inverse demands and Luenberger's adjusted price functions, allowing estimable inverse demands to be derived directly from a benefit function. We also make use of a numerical inversion estimation method to rectify the "unobservability of utility problem" encountered in the empirical analysis of these inverse demands. To illustrate the usefulness of the proposed methods, we estimate two systems of inverse demands for Japanese quarterly fish consumption. Results generally indicate that the proposed methods are promising and operationally feasible so that we have opened up a wider range of empirical inverse demand specifications that can be subjected to tight theoretical restrictions.

***JEL** classification*: D11, D12

Keywords: Benefit Functions; Duality Theory; Numerical Inversion Estimation Method.

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Abstract

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Price-dependent or inverse demand systems, in which quantities are exogenous and prices are the dependent variables, have been studied extensively in consumption economics.[1] Most of the studies of these systems have made use of either the direct utility function or the distance function to generate inverse Marshallian or Hicksian demands by applying, respectively, the Hotelling-Wold identity or the Shephard-Hanoch lemma. Recently, additional attention has been given to the benefit function, which was first introduced and developed by Luenberger (1992). This function is now recognized to be of particular value in welfare analysis because its aggregation property makes it attractive to analyze welfare changes for heterogeneous consumers.[2] For instance, benefit functions of different individuals could be directly summed to obtain meaningful aggregate benefit, which could be used to measure the welfare implications of changes in the economy.

Despite its obvious potential for policy applications, there are few theoretical and empirical applications beyond those considered originally by Luenberger (1992) & (1995), Chambers, Chung and Färe (1996), and Baggio and Chavas (2006). A possible reason for the scarcity of applications is that the benefit function is not a convenient vehicle for generating empirical demand models, as the dual relationships between inverse Marshallian (or Hicksian) demands and the adjusted price functions derived from a benefit function are not well established. We provide here an attempt to establish such relationships.

The first aim of this paper is to advocate a more practical use of benefit functions in representing preferences and specifying inverse demand models. In particular, it proposes the exploitation of additional duality relationships to generate systems of price-dependent demand functions alternative to the more typical approaches to deriving inverse Marshallian and Hicksian systems. As will be clear from the following discussion, the price-dependent demand systems derived from direct utility functions, distance functions and benefit functions are intimately related by a series of relationships, which allow simple transformations from any one to the others. Combining one of these relationships with known results for expenditure-normalized inverse prices allows expenditure-normalized inverse prices to be derived directly from the benefit function (a result to be referred to as the Hotelling-Wold Analogue for the benefit function). In this way the theoretical and empirical analysis based on benefit functions is greatly facilitated, and it is such analysis that forms the main theme of this paper.

The second aim is to demonstrate the feasibility of using benefit functions to specify estimable yet general and regular price-dependent demand systems. Differentiation of a chosen benefit function with respect to quantities yields Luenberger's adjusted price functions, according to the envelope theorem. Since these functions are explicit in the unobservable utility level, in most cases they do not have a closed-form representation as their Marshallian counterparts i.e. in terms of the observable variables such as quantities.[3] This "operational complexity" however need not hamper the empirical exploitation of benefit functions.[4] A simple one-dimensional numerical inversion allows estimation of the parameters of a benefit function via the parameters of the implied inverse Marshallian demands. The formal theory for using a benefit function in this context will be developed and illustrated in the next section of this paper.

The Benefit Function and Its Additional Duality Properties

With Ω^N (or Ω^N_+) denoting the non-negative (or positive) orthant, let $\mathbf{x}\in\Omega^N$ represent an N-vector of commodities, $\mathbf{p}\in\Omega^N_+$ the corresponding price vector, and $\mathbf{g}\in\Omega^N$ ($\mathbf{g}\neq\mathbf{0}$) an arbitrary (fixed) N-vector of goods that serves as a reference bundle. Suppose that individual preferences are represented by a direct utility function u = U($\mathbf{x}$). Following Luenberger (1992) & (1995), the benefit function (B) for these preferences is defined as:

(1) $$\mathrm{B}(\mathbf{x}, u; \mathbf{g}) = \mathrm{Max}_{b}\ \{b \text{ s.t. } \mathrm{U}(\mathbf{x} - b\mathbf{g}) \geq u, \text{ and } \mathbf{x} \geq b\mathbf{g}\},$$

which measures how many units of $\mathbf{g}$ an individual is willing to give up to move from a utility u to the point $\mathbf{x}$.[5] Provided that the direct utility function is continuous, increasing and quasi-concave in $\mathbf{x}$, then the benefit function is continuous, increasing and concave in $\mathbf{x}$, decreasing in u, and satisfies a translation property: $\mathrm{B}(\mathbf{x}+\alpha\mathbf{g},\ u) = \alpha + \mathrm{B}(\mathbf{x},\ u)$. Luenberger has shown that u = U($\mathbf{x}$) implies that B($\mathbf{x}$, u) = 0 if $\mathbf{g}$ is a "good" (that is, $\mathrm{U}(\mathbf{x}+\alpha\mathbf{g}) > \mathrm{U}(\mathbf{x})$ for all $\mathbf{x}$ and $\alpha > 0$) and that B($\mathbf{x}$, u) = 0 implies $\mathrm{U}(\mathbf{x}) = u$ if $\mathbf{x} > 0$.

Luenberger (1992) and (1995) proves a duality between the benefit and cost functions corresponding to the utility function, and hence there is a duality between the benefit function and the corresponding direct utility function. Applying the envelope theorem, Luenberger introduces what he refers to as the "adjusted price functions", which can be derived from a benefit function via simple differentiation; i.e.

(2) $$\mathrm{P}_{\mathrm{i}}^{\mathrm{L}}(\mathbf{x}, u; \mathbf{g}) = \partial \mathrm{B}/\partial x_{\mathrm{i}} = \mathrm{B}_{x_{\mathrm{i}}}$$

where the superscript "L" is to remind us that (2) are the Luenberger functions derived from a benefit function. Since the translation property implies

(3) $$\sum_{\mathrm{j}} \frac{\partial \mathrm{B}}{\partial x_{\mathrm{j}}} g_{\mathrm{j}} = 1,$$

it follows that these price functions satisfy the normalization: $\sum_{\mathrm{j}} \mathrm{P}_{\mathrm{j}}^{\mathrm{L}} g_{\mathrm{j}} = 1$. Because of the dependence of the equation systems in (2) on quantities **x** and utility u, the functions $\mathrm{P}_{\mathrm{i}}^{\mathrm{L}}$ are clearly analogous to inverse Hicksian demands. An issue with this analogy is the potential dependence of the derivatives in (2) on the choice of reference vector **g**. To clarify this analogy and establish a useful notation, consider first the direct Marshallian ($\mathrm{X}_{\mathrm{i}}^{\mathrm{M}}$) and Hicksian ($\mathrm{X}_{\mathrm{i}}^{\mathrm{H}}$) demand functions which are the solutions to the following constrained optimization problems:

(4) $$\mathrm{X}_{\mathrm{i}}^{\mathrm{M}}(\mathbf{p}, c) \text{ — solution of} \rightarrow \text{Max}_{\mathbf{x}} \{\mathrm{U}(\mathbf{x}) \text{ s.t. } \mathbf{p}'\mathbf{x} \leq c\}, \text{ and}$$

(5) $$\mathrm{X}_{\mathrm{i}}^{\mathrm{H}}(\mathbf{p}, u) \text{ — solution of} \rightarrow \text{Min}_{\mathbf{x}} \{\mathbf{p}'\mathbf{x} \text{ s.t. } \mathrm{U}(\mathbf{x}) \geq u\},$$

where c is a level of total expenditure, and the superscript "M" for Marshallian (or "H" for Hicksian) helps to clarify ideas, and is motivated by the arguments (**p**, c) (or (**p**, u)) of the corresponding functions.

The budget constraint $\mathbf{p}'\mathbf{x} \le c$ (in fact $\mathbf{p}'\mathbf{x} = c$ by non-satiation) in (4) has two implications for the direct Marshallian demand systems: i) because the solutions satisfy the one-dimensional constraint, only N-1 of the N equations in (4) are functionally independent; and ii) because of the linearity of the budget constraint $\mathrm{X}_{\mathrm{i}}^{\mathrm{M}}$ is homogeneous of degree zero in ($\mathbf{p}$, c), and only N-1 of the N equations in (4) are then linearly independent. Unless one is willing to actually condition on c, then in order to define the concept of inverse demands corresponding to these direct demands some normalization is required. A standard normalization is to define expenditure-normalized prices $\mathbf{r} = \mathbf{p}/c$ and hence use homogeneity to rewrite $\mathrm{X}_{\mathrm{i}}^{\mathrm{M}}$ as

(6) $$x_{\mathrm{i}} = \mathrm{X}_{\mathrm{i}}^{\mathrm{M}}\left(\frac{\mathbf{p}}{c}, \frac{c}{c}\right) = \mathrm{X}_{\mathrm{i}}^{\mathrm{M}}(\mathbf{r},1),$$

which can, at least in principle, then be inverted to give the inverse Marshallian demands

(7) $$r_{\mathrm{i}} = \mathrm{R}_{\mathrm{i}}^{\mathrm{IM}}(\mathbf{x})$$

where the superscript "IM" (for inverse Marshallian) refers to the arguments ($\mathbf{x}$) of the corresponding functions. The inverse Marshallian demands (7) can most straightforwardly be defined by the standard dual approach as the solution to

(8) $$\mathrm{U}(\mathbf{x}) = \mathrm{Min}_{\mathbf{r}}\left\{\mathrm{U}^{\mathrm{M}}(\mathbf{r}) \text{ s.t. } \mathbf{r}'\mathbf{x} = 1\right\}$$

(where U^{M} is the indirect utility function), in which case the inverse Marshallian demands follow from the envelope theorem as

(9) $$r_{\mathrm{i}} = \mathrm{R}_{\mathrm{i}}^{\mathrm{IM}}(\mathbf{x}) = \frac{\mathrm{U}_{x_{\mathrm{i}}}(\mathbf{x})}{\sum_{\mathrm{j}=1} x_{\mathrm{j}}\mathrm{U}_{x_{\mathrm{j}}}(\mathbf{x})},$$

a result usually referred to as the Hotelling-Wold identity.

Similarly, the Hicksian direct demands (5) are homogeneous of degree zero in **p** so that again some normalization of prices is required. However, even with the standard normalization of prices derivation of inverse demands is not straightforward. As is well known, the Slutsky and Antonelli matrices are singular of rank N-1, and thus there is an essential singularity in the relation between direct and inverse Hicksian demands.[6]

Analogously to the case of inverse Marshallian demands, the inverse Hicksian demands can also be defined as the solution to a dual optimization problem: D(**x**, *u*) = Min $_{\mathbf{r}}$ {**r'x** s.t. C(**r**, *u*) = 1}, where D(**x**, *u*) is the distance function, and C(**r**, u) = Min $_{\mathbf{x}}$ {**r'x** s.t. u ≥ U(**x**)} is the normalized cost function, which uses the same normalization of prices as above. In this case, the envelope theorem gives $r_i = \mathrm{R}_i^{\mathrm{IH}}(\mathbf{x}, u) = \mathrm{D}_{x_i}(\mathbf{x}, u)$, a result often known as the Shephard-Hanoch lemma. Of course a further alternative approach to generating inverse Hicksian demands is to specify the constrained optimization: Min $_{\mathbf{x}}$ {**r'x** : U(**x**) ≥ *u*} and to manipulate the first order conditions to solve for the **r** as dependent variables, as functions of **x** and *u* as independent variables.

Another possible normalization of prices is to use the arbitrary reference vector **g** introduced in the definition of the benefit function, and apply the normalization **p'g** = 1. This amounts to the introduction of the alternative set of normalized prices $s_i = p_i / \mathbf{p'g}$ and allows the Hicksian demands to be written as

$$x_i = \mathrm{X}_i^{\mathrm{H}}(\mathbf{p}, u) = \mathrm{X}_i^{\mathrm{H}}\left(\frac{\mathbf{p}}{\mathbf{p'g}}, u\right) = \mathrm{X}_i^{\mathrm{H}}(\mathbf{s}, u). \tag{10}$$

Inverse demands might be written in the notation

$$s_i = \mathrm{S}_i^{\mathrm{IH}}(\mathbf{x}, u). \tag{11}$$

A similar singularity exists between the Hicksian demands and the Luenberger price functions as between the direct and inverse Hicksian demands, as shown by Luenberger (1996), and again the use of an appropriate dual result is the most straightforward way to define the "inverse" relationship. Luenberger (1992) shows that provided U(**x**) is quasi-concave and continuous, the following duality relationship holds:

$$\text{(12)} \qquad \mathrm{B}(\mathbf{x}, u; \mathbf{g}) = \mathrm{Inf}_{\,\mathrm{s}}\left\{\mathbf{s}'\mathbf{x} - \mathrm{C}(\mathbf{s}, u) \text{ s.t. } \mathbf{s}'\mathbf{g} = 1, \mathbf{s} \geq 0\right\}.$$

Luenberger (1996) then defines the "adjusted price function" as an envelope result:

$$\text{(13)} \qquad S(\mathbf{x}, u;\ \mathbf{g}) = \mathrm{Argmin}_{\,\mathrm{s}}\left\{\mathbf{s}'\mathbf{x} - \mathrm{C}(\mathbf{s},\ u) \text{ s.t. } \mathbf{s}'\mathbf{g} = 1,\ \mathbf{s} > 0\right\}.$$

With sufficient differentiability this envelope result is that

$$\text{(14)} \qquad \mathrm{S}_{\mathrm{i}}^{\mathrm{IH}} = \frac{\partial \mathrm{B}}{\partial \mathrm{x}_{\mathrm{i}}} = \mathrm{P}_{\mathrm{i}}^{\mathrm{L}}(\mathbf{x}, u)$$

where, for clarity, the notation $\mathrm{P}_{\mathrm{i}}^{\mathrm{L}}(\mathbf{x}, u)$ has been replaced by $\mathrm{S}_{\mathrm{i}}^{\mathrm{IH}}(\mathbf{x}, u)$ to reflect the specific normalization of prices. The matrix of derivatives of these functions with respect to quantities is singular because of the implication (3) of the translation property. System (14) is an inverse in the sense that:[7]

$$\text{(15)} \qquad \mathrm{P}_{\mathrm{i}}^{\mathrm{L}}\left[\mathbf{X}^{\mathrm{H}}(\mathbf{p}, u)\right] = \frac{p_{\mathrm{i}}}{\mathbf{p}'\mathbf{g}} = s_{\mathrm{i}}.$$

While inverse Marshallian or Hicksian demands cannot give absolute prices, the two alternative types of normalized prices can be simply related. Since

$$\text{(16)} \qquad \frac{\mathbf{r}}{\mathbf{r}'\mathbf{g}} = \frac{\mathbf{p}/c}{\mathbf{p}'\mathbf{g}/c} = \frac{\mathbf{p}}{\mathbf{p}'\mathbf{g}} = \mathbf{s}, \text{ and } \frac{\mathbf{s}}{\mathbf{s}'\mathbf{x}} = \frac{\mathbf{p}/\mathbf{p}'\mathbf{g}}{\mathbf{p}'\mathbf{x}/\mathbf{p}'\mathbf{g}} = \frac{\mathbf{p}}{\mathbf{p}'\mathbf{x}} = \mathbf{r} \quad \text{then}$$

$$\text{(17)} \qquad \mathrm{S}_{\mathrm{i}}(\mathbf{x}, u) = \frac{\mathrm{R}_{\mathrm{i}}(\mathbf{x}, u)}{\mathbf{R}'\mathbf{g}}; \quad \text{and } \mathrm{R}_{\mathrm{i}}(\mathbf{x}, u) = \frac{\mathrm{S}_{\mathrm{i}}(\mathbf{x}, u)}{\mathbf{S}'\mathbf{x}}.$$ [8]

These results suggest a two-step procedure in which the benefit function can be used to construct "standard" (i.e. functions defining expenditure-normalized prices) inverse Hicksian demand functions:

(18)
$$\mathrm{R}_{\mathrm{i}}^{\mathrm{IH}}(\mathbf{x},\ u)=\frac{\partial \mathrm{B}/\partial x_{\mathrm{i}}}{\sum_{\mathrm{j}=1}^{\mathrm{N}} x_{\mathrm{j}}\, \partial \mathrm{B}/\partial x_{\mathrm{j}}}.$$

With this background, it is illustrative to see how (18) can be derived more directly. Recall that the benefit function is an implicit representation of the direct utility function:

(19)
$$\mathrm{B}\left[\mathbf{x}, \mathrm{U}(\mathbf{x})\right] \equiv 0.$$

Differentiating once gives $\mathrm{B}_{x_{\mathrm{i}}}+\mathrm{B}_u \mathrm{U}_{x_{\mathrm{i}}} \equiv 0$, implying that $\mathrm{B}_{x_{\mathrm{i}}}=-\mathrm{B}_u \mathrm{U}_{x_{\mathrm{i}}}$. Weighting by quantities and summing, this implies that $\sum_{\mathrm{j}} x_{\mathrm{j}} \mathrm{B}_{x_{\mathrm{j}}} = \text{-}\mathrm{B}_u \sum_{\mathrm{j}} x_{\mathrm{j}} \mathrm{U}_{x_{\mathrm{j}}}$, and using these two results we find that $\mathrm{R}_{\mathrm{i}}^{\mathrm{IH}}(\mathbf{x},\ u) = \frac{\mathrm{B}_{x_{\mathrm{i}}}}{\sum_{\mathrm{j}} x_{\mathrm{j}} \mathrm{B}_{x_{\mathrm{j}}}} = \frac{\mathrm{U}_{x_{\mathrm{i}}}}{\sum_{\mathrm{j}} x_{\mathrm{j}} \mathrm{U}_{x_{\mathrm{j}}}} = \mathrm{R}_{\mathrm{i}}^{\mathrm{IM}}(\mathbf{x})$ where the last equality follows by the Hotelling-Wold Identity.[9]

Estimation of demand systems is usually carried out using budget shares. Thus we collect the foregoing results together in a form that is referred to as:

The Hotelling-Wold Analogue for the Benefit Function: Given a functional form for a benefit function satisfying the appropriate regularity conditions, the corresponding inverse Hicksian and Marshallian share equations can be derived as

(20)
$$\mathrm{W}_{\mathrm{i}}^{\mathrm{IH}}(\mathbf{x}, u) = x_{\mathrm{i}} \mathrm{R}_{\mathrm{i}}^{\mathrm{IH}}(\mathbf{x}, u) = \frac{x_{\mathrm{i}} \mathrm{B}_{x_{\mathrm{i}}}(\mathbf{x}, u)}{\sum_{\mathrm{j}=1}^{\mathrm{N}} x_{\mathrm{j}} \mathrm{B}_{x_{\mathrm{j}}}(\mathbf{x}, u)}, \text{ and}$$

(21)
$$\mathrm{W}_{\mathrm{i}}^{\mathrm{IM}}(\mathbf{x}) = \mathrm{W}_{\mathrm{i}}^{\mathrm{IH}}\left[\mathbf{x},\ \mathrm{U}(\mathbf{x})\right] = \frac{x_{\mathrm{i}} \mathrm{B}_{x_{\mathrm{i}}}[\mathbf{x},\ \mathrm{U}(\mathbf{x})]}{\sum_{\mathrm{j}=1}^{\mathrm{N}} x_{\mathrm{j}} \mathrm{B}_{x_{\mathrm{j}}}[\mathbf{x},\ \mathrm{U}(\mathbf{x})]}$$

where U(**x**) is obtained by inverting the identity function (19).

Considerations in the Choice of a Functional Form for a Benefit Function

Econometric analysis of demand systems essentially relates to empirical representations of underlying preference orderings within the framework of "rational" consumer behavior. A standard result on preference ordering representation is the existence of a direct utility function, but its application in the standard situation of endogenous quantities and exogenous prices and expenditure requires the analytical solution of a nonlinear optimization problem, which limits applicability to simple and empirically unacceptable functional forms.

Duality theory provides the way forward for empirical work. Provided the dual function satisfies appropriate regularity conditions (typically curvature, monotonicity and homogeneity conditions), then the problem of an analytic solution of an optimization problem is avoided, and replaced by the need to specify a regular dual function. So the standard results are the followings: to specify Marshallian demands, we represent preferences by an indirect utility function and apply Roy's Identity; to specify inverse Marshallian demands, we represent preferences by a direct utility function and apply the Hotelling-Wold Identity; to specify Hicksian demands, we represent preferences by a cost function and apply Shephard's Lemma; to specify inverse Hicksian demands, we represent preferences by a distance function and apply the Shephard-Hanoch lemma (or by a benefit function and apply Luenberger's result (14)).

Since utility is an unobservable variable, one may think that empirical work should be restricted to the first two of these. However, the imposition of regularity conditions is crucial, since the duality results only apply in regions of regularity. Because the indirect utility function and direct utility function are required to be quasi-convex or quasi-concave respectively, it is quite difficult to construct reasonably general functional forms, because, for example, linear combinations of quasi-convex functions are not necessarily quasi-convex, and decreasing quasi-convex functions of decreasing quasi-convex functions are not necessarily quasi-convex. Hence only simple regular representations of preferences are available in these cases. Note that this is not the case with concave or convex functions, for which various composition rules maintain regularity. For example, positive linear combinations of increasing concave functions are increasing concave, and increasing concave functions of increasing concave functions are also increasing and concave. One can construct arbitrary rank (in the sense of Lewbel (1992)) cost and distance functions that are regular over unbounded (and policy relevant) regions. These functions can then be used to represent Marshallian functions by the use of a simple one dimensional numerical inversion (a technique introduced in McLaren, Rossiter, and Powell (2000)), a small cost to pay for the enhanced regularity properties of the resulting representation of preferences. The purpose of this paper is to extend this analysis to the use of benefit functions, thus further extending the capacity to represent preferences by regular functional forms. Because of the translation property, regular benefit functions are not as easy to generalize as are regular cost and distance functions. However, the following result is straightforward to demonstrate. Given m regular benefit functions each with reference vector g, then a positive weighted average of these m functions is a regular benefit function with reference vector g.

Note that the use of “flexible” functional forms is not attractive. Flexible functional forms (such as Translog) have one attractive property: the ability to represent an arbitrary set of price and income elasticities at a point in price-income space. Usually they cannot be constrained to satisfy the required regularity properties (apart from homogeneity) even at this particular point, but far more damaging is that they cannot be constrained to satisfy the regularity properties required by a dual specification even over the sample space, let alone over points outside the sample space where we may wish to carry out policy analysis. (As an illustration, CGE models typically do not use flexible functional forms.) Thus well-known "flexible" functional forms are not necessarily useful for empirical specification of dual representations of preferences.

The Numerical Inversion Estimation Method

As shown in (20) and (21), the benefit function, together with its derivative property, provides a convenient vehicle for generating inverse Hicksian share systems. Specifically, for a chosen reference bundle **g** and a parametric specification of B satisfying certain conditions, one could obtain a share system by the above result. If we could invert the benefit function B explicitly to give the implied direct utility function U(**x**), then the inverse Hicksian shares could be "Marshallianized" by replacing the *u* by U(**x**) as shown in (21).[10] In practice, however, it is only in simple cases that it is possible to obtain a closed-form solution for U(**x**) for an arbitrary specification of B; it depends heavily on the particular parametric form of B. This paper focuses on the class of benefit functions for which such explicit inversion is not available; that is, solving B(**x**, *u*) = 0 for U(**x**) may not be accomplished analytically, and thus the benefit function cannot be equivalently represented by a closed form direct utility function.

For a given parametric form for the benefit function with parameters $\boldsymbol{\theta}$, the inverse Marshallian share system could be expressed implicitly by the set of functions:

$$(22) \qquad \frac{p_i x_i}{\mathbf{p}'\mathbf{x}} = \mathrm{W}_i^{\mathrm{IH}}(\mathbf{x}, u; \boldsymbol{\theta}) = \frac{x_i \mathrm{B}_{x_i}}{\sum_j x_j \mathrm{B}_{x_j}}, \quad \text{and}$$

$$(23) \qquad \mathrm{B}(\mathbf{x}, u; \boldsymbol{\theta}) = 0.$$

Provided that the benefit function is strictly decreasing in *u*, then it becomes feasible to numerically invert (23) to express *u* as a function of **x** and $\boldsymbol{\theta}$. Therefore, given a specific functional form for B and $\boldsymbol{\theta}$, the corresponding inverse share system can be written as:

$$(24)\quad W_i^{IH}(\mathbf{x}, u; \boldsymbol{\theta}) = \frac{x_i B_{x_i}(\mathbf{x}, u; \boldsymbol{\theta})}{\sum_j x_j B_{x_j}(\mathbf{x}, u; \boldsymbol{\theta})} = \frac{x_i B_{x_i}[\mathbf{x}, U(\mathbf{x}; \boldsymbol{\theta}); \boldsymbol{\theta}]}{\sum_j x_j B_{x_j}[\mathbf{x}, U(\mathbf{x}; \boldsymbol{\theta}); \boldsymbol{\theta}]} = W_i^{IM}(\mathbf{x}; \boldsymbol{\theta}),$$

where $u = U(\mathbf{x}; \boldsymbol{\theta})$ is the numerical solution of the identity function $B(\mathbf{x}, u; \boldsymbol{\theta}) = 0$ for u, solved at the given values of $\mathbf{x}$ and $\boldsymbol{\theta}$.

In a maximum likelihood search for the parameters of the inverse budget shares, explicit solution of the Marshallian inverse demands is not necessary; all that is required is that software capable of solving the identity function (23) be imbedded in the maximum likelihood computer routine. At each iterative step of the maximization of the likelihood function, there is a given set of parameter values. For these parameter values, (23) may be numerically inverted to recover the value of utility consistent with the given values of $\mathbf{x}$. Then, this value of utility can be used to eliminate the value of u from the inverse Hicksian share system.

Benefit Function Specification

In this section, we examine the two specifications on which our empirical analysis is based.[11] The first specification, the Simple Non-Additive Benefit (SNAB) function, serves to make the theoretical arguments developed in Section 2 less abstract and provide a bridge to our empirical analysis. The choice is motivated by a number of reasons, mainly the simplicity of the functional structure, the ease of imposing and maintaining regularity conditions, and the fact that the number of parameters will not increase rapidly with the number of inputs under consideration. More importantly, it is general enough to include “implicitly additive preference structure” as hypothesis to be tested rather than maintained. For purposes of comparison, we present and estimate the budget share equations corresponding to a second specification, the Baggio and Chavas (2006) model (to be referred to as the B&C model).

The Simple Non-Additive Benefit (SNAB) Function

Suppose that preferences are represented by the following form:

$$\text{(25)} \qquad \mathrm{B}(\mathbf{x}, u) = \sum_{\mathrm{j}} \delta_{\mathrm{j}} x_{\mathrm{j}} + \sum_{\mathrm{j}} \Phi_{\mathrm{j}} \log(x_{\mathrm{j}} - \gamma_{\mathrm{j}}) - \log(u) / \mathrm{X1}^{\eta},$$

where $\mathrm{X1} = \prod_{\mathrm{j}} x_{\mathrm{j}}^{\mu_{\mathrm{j}}}$ with $\Sigma_{\mathrm{j}}\mu_{\mathrm{j}} = 1$, and $\Phi_{\mathrm{i}} = (\alpha_{\mathrm{i}} + \beta_{\mathrm{i}} u) / (1 + u)$ are the utility varying coefficients with $\Sigma_{\mathrm{j}}\alpha_{\mathrm{j}} = \Sigma_{\mathrm{j}}\beta_{\mathrm{j}} = 1$. The structure (25) maintains all of the regularity properties in the quantities (increasing and concave in **x**) of the benefit function over the regions $\log(u) \geq 0$ and $x_{\mathrm{i}} > \gamma_{\mathrm{i}}$ $\forall$ i provided that the following conditions are satisfied:

$$\text{(26)} \qquad \gamma_{\mathrm{i}} \geq 0,\ \delta_{\mathrm{i}} \geq 0 \text{ and } 0 \leq \alpha_{\mathrm{i}},\ \beta_{\mathrm{i}},\ \mu_{\mathrm{i}},\ \eta \leq 1.$$

We have seen from (3) that $\sum_{\mathrm{j}} \frac{\partial \mathrm{B}}{\partial x_{\mathrm{j}}} g_{\mathrm{j}} = 1$ for all **x** and *u*, requiring that $\Sigma_{\mathrm{j}}\delta_{\mathrm{j}} g_{\mathrm{j}} = 1$, and $\sum_{\mathrm{j}} \frac{\Phi_{\mathrm{j}} \mathrm{g}_{\mathrm{j}}}{x_{\mathrm{j}} - \gamma_{\mathrm{j}}} = \sum_{\mathrm{j}} \frac{\mu_{\mathrm{j}}}{x_{\mathrm{j}}} \mathrm{g}_{\mathrm{j}} = 0$. Furthermore, imposition of the restriction $\eta = 0$ gives a benefit function (to be referred to as the restricted SNAB) which is consistent with Hanoch's (1975) implicitly additive preference structure.

As indicated by (22), differentiation of (25) after some manipulation gives the inverse Hicksian budget share system:

$$\text{(27)} \qquad \mathrm{W}_{\mathrm{i}}^{\mathrm{IH}}(\mathbf{x}, u) = \frac{\delta_{\mathrm{i}} x_{\mathrm{i}} + \Phi_{\mathrm{i}} x_{\mathrm{i}} / (x_{\mathrm{i}} - \gamma_{\mathrm{i}}) + \eta \cdot \mu_{\mathrm{i}} \log(u) / \mathrm{X1}^{\eta}}{\sum_{\mathrm{j}} \left[\delta_{\mathrm{j}} x_{\mathrm{j}} + \Phi_{\mathrm{j}} x_{\mathrm{j}} / (x_{\mathrm{j}} - \gamma_{\mathrm{j}}) + \eta \cdot \mu_{\mathrm{j}} \log(u) / \mathrm{X1}^{\eta} \right]}.$$

It is evident from (25) that it is impossible to solve explicitly for the value of *u* in terms of **x** and $\boldsymbol{\theta}$. In order to convert (27) to a Marshallian system, the unobservable *u* in (27) has to be replaced by the numerical inversion of (25) at $\mathrm{B} = 0$.

The B&C Model

The B&C model is obtained from the following specification of the benefit function:

(28) $$\mathrm{B}(\mathbf{x}, u) = \mathrm{X1} - \frac{u \cdot \mathrm{X2}}{1 - u \cdot \mathrm{X3}},$$

where Xk, k=1, 2 and 3, are three positive and continuous quantity functions. The B&C model results if Xk are specified as:

(29) $$\mathrm{X1} = \sum_{\mathrm{j}} \alpha_{\mathrm{j}} x_{\mathrm{j}} + 0.5 \sum_{\mathrm{j}} \sum_{\mathrm{i}} \gamma_{\mathrm{ij}} x_{\mathrm{i}} x_{\mathrm{j}}, \ \mathrm{X2} = \prod_{\mathrm{j}} x_{\mathrm{j}}^{\beta_{\mathrm{j}}}, \text{ and } \mathrm{X3} = \sum_{\mathrm{j}} \delta_{\mathrm{j}} x_{\mathrm{j}},$$

where α_j, β_j, δ_j, and γ_{ij} are the parameters. Symmetry of the Hessian matrix of the benefit function, implied by Young's theorem, requires that $\gamma_{ij} = \gamma_{ji}$. Furthermore, $\sum_{\mathrm{j}} \frac{\partial \mathrm{B}}{\partial x_{\mathrm{j}}} g_{\mathrm{j}} = 1$ must hold for all **x** and *u*, requiring that:

$$\sum_{\mathrm{j}} \frac{\partial \mathrm{X1}}{\partial x_{\mathrm{j}}} g_{\mathrm{j}} = 1, \text{ and } \sum_{\mathrm{j}} \frac{\partial \mathrm{X2}}{\partial x_{\mathrm{j}}} g_{\mathrm{j}} = \sum_{\mathrm{j}} \frac{\partial \mathrm{X3}}{\partial x_{\mathrm{j}}} g_{\mathrm{j}} = 0.$$

These generate the following restrictions:

(30) $$\sum_{\mathrm{j}} \alpha_{\mathrm{j}} g_{\mathrm{j}} = 1, \text{ and } \sum_{\mathrm{i}} \gamma_{\mathrm{ij}} g_{\mathrm{i}} = \sum_{\mathrm{j}} \gamma_{\mathrm{ij}} g_{\mathrm{j}} = \sum_{\mathrm{j}} \beta_{\mathrm{j}} g_{\mathrm{j}} = \sum_{\mathrm{j}} \delta_{\mathrm{j}} g_{\mathrm{j}} = 0.$$

Functions (28) and (29), on application of the Hotelling-Wold Analogue, generate the following system of inverse Hicksian budget share equations:

(31) $$\mathrm{W}_{\mathrm{i}}^{\mathrm{IH}}(\mathbf{x}, u) = \frac{\left(\alpha_{\mathrm{i}} + \sum_{\mathrm{j}} \gamma_{\mathrm{ij}} x_{\mathrm{j}}\right) \cdot x_{\mathrm{i}} - \frac{\beta_{\mathrm{i}} u \cdot \mathrm{X2}}{1 - u \cdot \mathrm{X3}} - \delta_{\mathrm{i}} x_{\mathrm{i}} \cdot \mathrm{X2} \cdot \left(\frac{u}{1 - u \cdot \mathrm{X3}}\right)^2}{\sum_{\mathrm{j}} \left[\left(\alpha_{\mathrm{j}} + \sum_{\mathrm{k}} \gamma_{\mathrm{jk}} x_{\mathrm{k}}\right) \cdot x_{\mathrm{j}} - \frac{\beta_{\mathrm{j}} u \cdot \mathrm{X2}}{1 - u \cdot \mathrm{X3}} - \delta_{\mathrm{j}} x_{\mathrm{j}} \cdot \mathrm{X2} \cdot \left(\frac{u}{1 - u \cdot \mathrm{X3}}\right)^2\right]}.$$

Elimination of *u* from (31) by the analytical inversion of (28) at the optimum (setting (28) equal to zero) leads immediately to the inverse Marshallian demand system, given by:

(32) $$W_i^{IH}(\mathbf{x}, u) = \frac{\left(\alpha_i + \sum_j \gamma_{ij} x_j\right) \cdot x_i - \beta_i \cdot X1 - \delta_i x_i \cdot \dfrac{X1^2}{X2}}{\sum_j \left[\left(\alpha_j + \sum_k \gamma_{jk} x_k\right) \cdot x_j - \beta_j \cdot X1 - \delta_j x_j \cdot \dfrac{X1^2}{X2}\right]}.$$

It is also transparent that, given the values of parameters and quantities of goods, the numerical inversion of (28) at the optimum to give u in terms of $\mathbf{x}$ and $\boldsymbol{\theta}$ and its substitution in (31) would give the same results as analytical inversion.

Brief Remarks on the Database, Estimation and Stochastic Specification

Price-dependent or inverse demand systems have been used recently to characterize short-run demand behavior for food, agricultural and fishery products. These systems seem especially useful in markets for agricultural and natural resource commodities where in the short run it is reasonable to argue that supplies are close to being perfectly inelastic. To illustrate the modeling and estimation strategies outlined in the preceding sections, the general SNAB function, its nested case (setting η to zero) and the B&C model were estimated using quarterly Japanese data on six categories of fish products – i) High Value Fish; ii) Medium Value Fish; iii) Low Value Fish; iv) Cuttlefish, Squid & Octopus; v) Lobster, Shrimp & Crab; and vi) Shellfish covering the period January 1985 through December 2005. The data used are based on those of Eales, Durham and Wessells (1997) for 1985 to 1992, and are extended for 1993 to 2005.[12] The data were further aggregated to quarterly frequency resulting in 84 usable observations, and were deseasonalized and mean centered prior to estimation.[13]

One important remaining issue is the choice of reference bundle **g** which, because of the requirement $\sum_j \frac{\partial B}{\partial x_j} g_j = 1$ implies restrictions on the parameters defining B. To simplify matters, we choose **g** to be an N-vector (0, 0,…, 1)' implying that all valuations are made relative to the value of the last commodity (shellfish). In other words, the Luenberger price function of shellfish (P_6^L) is normalized to unity. This choice of **g** then implies the following parameter restrictions:

i) the SNAB function: $\delta_6 = 1$, and $\alpha_6 = \beta_6 = \mu_6 = 0$; and

ii) the B&C model: $\alpha_6 = 1$, and $\beta_6 = \delta_6 = \gamma_{6j} = \gamma_{j6} = 0$ (j = 1 to 6).[14]

Since the GAUSS language is ideally suited for handling the implicit representation of functional relationships, the price-dependent demand systems may be estimated by using the GAUSS 3.6.27 computer package with the modules NLSYS and CML. The estimation method is non-linear Maximum Likelihood, and the inequality restrictions in (26) are imposed when estimating the systems.

To implement the empirical analysis, the model has to be imbedded within a stochastic framework. To do so, we assume that the budget share equations are stochastic due to errors of optimization. Let w_{it} denote the ith budget share at time t, $\mathbf{z}_t$ a vector of all exogenous variables, and $\mathbf{w}_t^n = (w_{1t},\ldots\ldots, w_{(N-1)t})'$ an (N-1) x 1 vector of w_{it}.[15] The budget share system to be estimated may then be expressed compactly as:

(33) $$\mathbf{w}_t^n = \mathbf{W}^n(\mathbf{x}_t; \theta) + \mathbf{e}_t^n, \; t = 1,\ldots\ldots, T,$$

where $\mathbf{W}^n(.)$ is the vector of deterministic components of the budget share equations, and $\mathbf{e}_t^n$ is a vector of the error terms e_{it}. To allow for serially correlated error terms, the following fourth-order autoregressive scheme is specified:[16]

(34) $$\mathbf{e}_t^n = \mathbf{R}_*^n \mathbf{e}_{t-4}^n + \boldsymbol{\varepsilon}_t^n, \; t = 2,\ldots\ldots, T,$$

where $\mathbf{R}_*^n$ is an (N-1) x (N-1) autocorrelation matrix, and $\boldsymbol{\varepsilon}_t^n$ is a vector of serially uncorrelated error terms characterized by a multivariate normal distribution with zero mean and a constant contemporaneous covariance matrix Ω. By using (34), (33) could be rewritten as:

(35) $$\mathbf{w}_t^n = \mathbf{W}^n(\mathbf{z}_t; \theta) + \mathbf{R}_*^n [\mathbf{w}_{t-4}^n - \mathbf{W}^n(\mathbf{z}_{t-4}; \theta)] + \boldsymbol{\varepsilon}_t^n, \; t = 2,\ldots\ldots, T,$$

which forms the basis for the empirical work.

While there are several ways in which the autocorrelation matrix $\mathbf{R}_*^n$ may be parameterized, preliminary analysis revealed that Moschini and Moro's parameterization gave reasonable results and resulted in significant parameter parsimony. Following Moschini and Moro's (1994) procedure, the N x N counterpart to matrix $\mathbf{R}_*^n$ (matrix $\mathbf{R}_*$) is specified by:

$$\mathbf{R}_* = \boldsymbol{\rho}^* - \frac{\boldsymbol{\rho}\boldsymbol{\rho}'}{\boldsymbol{\iota}'\boldsymbol{\rho}}$$

where $\boldsymbol{\rho} = (\rho_1,\ldots\ldots, \rho_N)'$, $\boldsymbol{\rho}^* = \text{diag}(\rho_1,\ldots\ldots, \rho_N)$, $\boldsymbol{\iota}$ is an N x 1 vector of ones, and ρ_1 to ρ_N are the autocorrelation coefficients. In estimation, the typical elements of $\mathbf{R}_*^n$ (R_{*ij}^n) are recovered by using the identity $R_{*ij}^n = R_{*ij} - R_{*iN}$ where R_{*ij} is the typical element of matrix $\mathbf{R}_*$. Accordingly, estimation of the equation systems with fourth order autoregressive error terms can be carried out based on the system (35), with N additional parameter ($\rho_1,\ldots\ldots, \rho_N$) to estimate in addition to parameters $\boldsymbol{\theta}$.

Because of the adding-up restriction of the budget shares, contemporaneous errors $\boldsymbol{\varepsilon}_t$ are correlated with a singular variance covariance matrix Ω. To cope with the singular error structure, the system (35) is estimated by deleting one of the budget share equations in the share systems (27) and (32). The coefficients of the deleted share equation can be recovered by using the theoretical restrictions in conjunction with the estimated coefficients of the other share equations. As usual, the estimation should be independent of which equation is excluded.

Empirical Results and Their Interpretation

Analysis of the Estimates: Comparative results for the three specifications are presented in table 1. The most important point to highlight from the results is that the general and restricted SNAB satisfy the required regularity conditions for all observations. Regarding the single equation fit and performance, all three specifications fit the data reasonably well given that estimation is in share form: the R^2 values range from 49.4% for high value fish (implied by the B&C) to 89.6% for lobster (implied by the restricted SNAB). The serial correlation properties of the error terms as shown in the Durbin-Watson statistics are no longer severely pathological although there is some evidence of remaining autocorrelation in the residuals. This should probably be considered to be the small cost paid for the simplicity of Moschini and Moro's (1994) method for specifying autoregressive errors.

For the general and restricted SNAB functions, the main point to make is that the restricted model is rejected in favor of its generalization on the basis of a χ^2 test. As can be seen, the computed chi-square value (χ^2) is 8.604 which far exceeds the critical value for χ^2 of 3.841 for the 5% significance level. This leads to the conclusion that the implicitly additive benefit function (or restricted SNAB) is overly restrictive. Of interest is that the B&C, while containing four (or three) more free parameters than the general (or restricted) SNAB function, has a substantially lower likelihood function value (1380.096 versus 1397.501 or 1380.096 versus 1393.199). This indicates a preference for general and restricted SNAB over B&C. Thus, the likelihood function value ranks the models (from most to least preferred) as follows: general SNAB, restricted SNAB and B&C.

To obtain further insights into the relative performance of the three specifications, Pollak and Wales' (1991) Likelihood Dominance Criterion (LDC) test is performed. The results of this test are shown in table 2. In all cases, LDC test statistics are less than the lower bound of the critical range, which means the models with fewer parameters (general and restricted SNAB) are preferred to the model with more parameters (B&C). Consequently, the LDC comparisons suggest B&C is not supported by the data, whereas the general and restricted SNAB are preferred. The preferred model is therefore based on the general SNAB function; its detailed parameter estimates are reported in table 3.

Analysis of the Elasticity Estimates:[17] The quantity and scale elasticity estimates for the general SNAB evaluated at the sample means of the exogenous variables are reported in table 4. Overall, these estimates offer no surprises. The Hicksian and Marshallian own quantity elasticities (h_{ii} and m_{ii}) are negative, although their magnitudes are fairly similar to those reported by, for example, Barten and Bettendorf (1989), Holt and Bishop (2002), and Wong and McLaren (2005). Additionally, most of these elasticities are generally greater than minus one, suggesting that all types of fish (except lobster) are own quantity inelastic, whereas the corresponding direct demands for fish are price elastic.

With respect to the derived Hisksian cross quantity elasticities (h_{ij}), they are generally small in magnitude – the largest Hicksian cross-quantity elasticity is for shellfish with respect to high value fish, illustrating weak gross substitutability among all types of fish. These findings are fairly similar to those obtained in Belgium by Barten and Bettendorf (1989), in U.S. by Holt and Bishop (2002), and in Japan by Wong and McLaren (2005). Regarding the Marshallian cross quantity elasticities, magnitudes for these estimates are smaller in absolute terms than their Hicksian counterparts; all cross quantity effects are, however, still very small. We also find that most fish pairs are gross q-substitutes as indicated by the negative signs.

Turning to the scale elasticities (y_i), the estimates are consistently negative whilst low value fish has the largest scale effect.[18] More importantly, the estimated y_i are fairly different from minus one, suggesting that preferences are non-homothetic. Of interest is that earlier estimates of the scale elasticities of Japanese fish consumption by Eales, Durham and Wessells (1997), and Wong and McLaren (2005) range from -0.16 to -1.95. Prima facie, our estimates of scale elasticities are somewhat comparable to those in the early studies, although they adopted different functional forms.

Analysis of Estimated Welfare Change: The main reason for imposing regularity restrictions such as (26) is to obtain consistent estimates for welfare losses caused by quantity restrictions. By applying the theory developed by Luenberger (1995) and (1996), the estimated inverse share system may be used to examine welfare changes associated with forced reduction in fish landings. Suppose that an individual's consumption bundle is changed from $\mathbf{x}^0$ with utility u^0 to $\mathbf{x}^1$ with utility u^1. Then the compensating benefit (CB) is defined by:

$$\text{(36)} \qquad CB = B(\mathbf{x}^1, u^0) - B(\mathbf{x}^0, u^0) = B(\mathbf{x}^1, u^0)$$

where the base utility u^0 is defined implicitly from $B(\mathbf{x}^0, u^0) = 0$. Intuitively, CB is the maximum amount of $\mathbf{g}$ that individuals are willing to give up in order to reach the utility level u^0 while facing the quantity $\mathbf{x}^1$. A positive (negative) value for CB indicates that consumers are better (worse) off while facing quantities $\mathbf{x}^1$.

In a similar manner, the equivalent benefit (EB) for a change in quantity from $\mathbf{x}^0$ to $\mathbf{x}^1$ is defined as:

$$\text{(37)} \qquad EB = B(\mathbf{x}^1, u^1) - B(\mathbf{x}^0, u^1) = - B(\mathbf{x}^0, u^1),$$

where u^1 is defined implicitly from $B(\mathbf{x}^1, u^1) = 0$. According to Luenberger (1995) and (1996), EB is the minimum amount of $\mathbf{g}$ needed to move individuals to the new utility level u^1 while facing the initial quantities $\mathbf{x}^0$. As for CB, a positive (negative) value for

EB suggests that consumers are better (worse) off under $\mathbf{x}^1$ than under $\mathbf{x}^0$.

By using equations (36) and (37) along with the preferred model (the general SNAB function), we may compute the welfare change associated with an arbitrary 10% catch reduction for a particular fish species. In the application, welfare change (CB and EB) estimates are obtained on an annualized basis for 1985-2005, as well as at the sample means. The CB and EB estimated for selected years are reported in table 5.

Note first that, as expected, CB and EB are negative in all instances, indicating that Japanese consumers are made worse off after the reduction in the harvest of an individual fish species. For example, the CB for a 10% reduction in the supply of high (or medium) value fish is -0.259 (or -0.511) unit of x_6. Furthermore, the largest (smallest) welfare loss associated with the supply reduction is for medium (or low) value fish. Interestingly, the numerical differences between the CB and EB estimates are rather small, amounting to no more than 0.117 unit of x_6 in all instances. In general, discrepancies between CB and EB are relatively small for medium and low value fish, lobster and cuttlefish, and are the largest for high value fish.

Narrow fluctuations over time in CB and EB estimates are observed for medium and low value fish, lobster and cuttlefish. On the other hand, we find that there are considerable variations in the magnitude of EB for high value fish across years. Particularly, in 1985 the EB estimate associated with a 10% reduction in high value fish catch was -0.371 unit of x_6, whereas the comparable estimate for 1995 was -0.186 unit of x_6, over a 50% decrease (in absolute value). Possibly, this result simply reflects the decreased value of high value fish in 1995 versus 1985.

Conclusion

The application of duality theory in consumer demand studies has allowed specification of a wide range of functional forms, which has helped considerably in the generation of empirical price-dependent demand systems. For the most part, specification has concentrated upon either the direct utility function or the distance function. Recently, more attention has been paid to the benefit function, but this has been mainly in the context of study of welfare issues. In this paper, we advocate a more extensive use of benefit functions in specifying inverse demand models by exploring the inter-relationships between the inverse Marshallian (or Hicksian) demands, and Luenberger's adjusted price-dependent demands. It has been demonstrated that for a chosen benefit function, application of an analogue to the Hotelling-Wold Identity yields expressions for inverse Hicksian normalized price functions. While these functions are explicit in the level of utility, in most cases they do not have a closed-form representation as corresponding Marshallian functions i.e. in terms of observable variables. This aspect, however, need not hinder estimation, and was solved by applying a numerical inversion estimation method, as illustrated in Section 2.

The implementation of the proposed methods relies on relatively simple functional forms to specify the benefit function, and the one used in this paper (referred to as the SNAB function), allows a simple generalization away from implicitly additive preferences. The application of the SNAB function was illustrated with an application to Japanese fish demand. Results in Section 4 generally indicate that this new specification is statistically preferred over the Baggio and Chavas (2006) model. Results also show that the modeling procedures and estimation methods employed here are promising and operationally feasible, and that the general and restricted SNAB functions satisfy their required regularity conditions for all observations in the sample period. This leads to the conclusion that specification of preferences in terms of the benefit function may open up a wider range of empirical price-dependent demand specifications that may be constrained to satisfy tight theoretical restrictions.

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Table 1: Single Equation and System Measures of Fit

	The General SNAB	The Restricted SNAB (η = 0)	The B&C
Log Likelihood Values	1397.501	1393.199	1380.096
No. of Parameters	27	26	31
R^2			
High Value Fish	0.550	0.514	0.494
Medium Value Fish	0.752	0.751	0.750
Low Value Fish	0.652	0.648	0.649
Lobster	0.895	0.896	0.893
Cuttlefish	0.706	0.708	0.728
Shellfish	0.683	0.683	0.695
Durbin-Watson Statistics			
High Value Fish	2.485	2.621	2.741
Medium Value Fish	2.474	2.492	2.497
Low Value Fish	2.155	2.136	2.424
Lobster	1.831	1.873	1.819
Cuttlefish	1.780	1.758	2.039
Shellfish	1.946	1.937	2.158
Likelihood Ratio Test			
H_0: $\eta = 0 \ \forall$ i (***rejected***)			
Test Statistic = 8.604 and $\chi^2_{1,\,5\%}$ = 3.841			

Table 2: Summary Statistics for Non-Nested Comparisons

Comparison (M1 versus M2)	Test Statistic	Critical Value Range (5% Significance Level)
General SNAB– Null Model v.s. B&C – Alternative Model	-8.703	(2.444, 3.615)
Restricted SNAB– Null Model v.s. B&C – Alternative Model	-6.552	(3.060, 4.375)

Note: M1 (or M2) is the model with fewer (or more) independent parameters in the model comparison.

In each cell, the first value is the lower bound of the LDC critical ranges, computed as: $0.5 \times$ [C(N2+1)-C(N1+1)], whilst the second value is the lower bound of the LDC critical ranges, computed as $0.5 \times$ [C(N2-N1+1)-C(1)], where N1 (or N2) denotes the number of parameters in M1 (or M2), and C(v) denotes the critical value of a χ^2 statistic with v degrees of freedom at the chosen significance level.

Table 3: Parameter Estimates for the Preferred Model (Asymptotic T Ratios in Parentheses)

α_1	0.302	(2.558)	δ_1	0.337	(5.078)	η	0.061	(5.290)
α_2	0.182	(1.960)	δ_2	0.608	(9.163)	γ_1	0.100	(0.362)
α_3	0.155	(5.853)	δ_3	0.055	(3.809)	γ_2	0.590	(5.977)
α_4	0.124	(2.808)	δ_4	0.000	(0.001)	γ_3	0.040	(1.128)
α_5	0.236	(6.114)	δ_5	0.000	—	γ_4	0.080	(1.342)
β_1	0.117	(0.896)	μ_1	0.189	(0.914)	γ_5	0.000	—
β_2	0.268	(2.280)	μ_2	0.257	(0.593)	γ_6	0.000	—
β_3	0.036	(0.399)	μ_3	0.194	(0.517)	ρ_1	0.999	(199.980)
β_4	0.228	(1.816)	μ_4	0.146	(0.260)	ρ_2	0.999	(142.843)
β_5	0.351	(1.675)	μ_5	0.215	(0.231)	ρ_3	0.999	(164.104)
						ρ_4	0.974	(145.612)
						ρ_5	0.998	(209.385)
						ρ_6	0.480	(26.661)

Note: The constraints $0 \leq \gamma_{5}, \gamma_6 \leq 1$ and $0 \leq \delta_5 \leq 1$ were binding, and hence no t-values are reported.

The estimated t-ratios must be interpreted with care since the standard asymptotic theory is unfortunately inapplicable when parameters are subject to inequality constraints.

Table 4: Elasticity Estimates for the General SNAB function (Asymptotic T Ratios in Parentheses)

Commodity i	**Hicksian Quantity Elasticities**					
	h_{i1}	h_{i2}	h_{i3}	h_{i4}	h_{i5}	h_{i6}
High Value	-0.286	0.267	0.053	0.118	0.132	-0.285
	(-7.018)	(6.874)	(6.116)	(7.489)	(6.799)	(-9.448)
Medium Value	0.211	-0.447	0.072	0.145	0.168	-0.150
	(6.278)	(-3.927)	(4.644)	(4.344)	(9.895)	(-4.994)
Low Value	0.244	0.366	-0.608	0.174	0.190	-0.366
	(9.965)	(8.870)	(-12.595)	(5.572)	(6.999)	(-7.726)
Lobster	0.250	0.483	0.083	-0.929	0.191	-0.078
	(12.413)	(12.700)	(9.194)	(-32.504)	(9.315)	(-4.801)
Cuttlefish	0.240	0.456	0.081	0.155	-0.814	-0.117
	(11.779)	(12.759)	(9.184)	(9.079)	(-37.543)	(-7.078)
Shellfish	-0.642	-0.491	-0.234	-0.085	-0.148	-0.255
	(-8.494)	(-5.005)	(-6.180)	(-2.721)	(-4.889)	(-13.751)
Commodity i	**Marshallian Quantity Elasticities**					
	m_{i1}	m_{i2}	m_{i3}	m_{i4}	m_{i5}	m_{i6}
High Value	-0.574	-0.153	-0.029	-0.014	-0.028	-0.414
	(-19.909)	(-2.091)	(-11.295)	(-9.491)	(-18.762)	(-10.318)
Medium Value	-0.073	-0.859	-0.009	0.012	0.011	-0.276
	(-13.465)	(-6.127)	(-3.676)	(7.923)	(8.203)	(-10.061)
Low Value	-0.229	-0.323	-0.742	-0.044	-0.073	-0.578
	(-15.441)	(-4.399)	(-16.387)	(-16.264)	(-21.822)	(-9.495)
Lobster	-0.037	0.065	0.002	-1.061	0.031	-0.207
	(-6.149)	(0.832)	(0.786)	(-63.350)	(16.447)	(-9.325)
Cuttlefish	-0.060	0.018	-0.005	0.017	-0.982	-0.252
	(-10.624)	(0.235)	(-2.167)	(10.611)	(-1033.101)	(-10.557)
Shellfish	-0.087	-0.035	-0.013	0.007	0.003	-0.300
	(-12.783)	(-0.484)	(-6.779)	(6.125)	(7.294)	(-11.633)
	Scale Elasticities					
	y_1	y_2	y_3	y_4	y_5	y_6
	-1.212	-1.194	-1.989	-1.207	-1.264	-0.423
	(-13.519)	(-10.193)	(-20.291)	(-12.215)	(-13.424)	(-4.888)

Table 5: Annualized Compensating and Equivalent Benefit for a 10% Increase in Supply of All Types of Fish

Fish Category	CB	EB
	1985	
High Value	-0.259	-0.376
Medium Value	-0.511	-0.510
Low Value	-0.074	-0.074
Lobster	-0.083	-0.083
Cuttlefish	-0.125	-0.125
Shellfish	-0.100	-0.100
	1995	
High Value	-0.256	-0.186
Medium Value	-0.526	-0.525
Low Value	-0.069	-0.069
Lobster	-0.099	-0.099
Cuttlefish	-0.147	-0.147
Shellfish	-0.100	-0.100
	2005	
High Value	-0.249	-0.262
Medium Value	-0.502	-0.501
Low Value	-0.074	-0.074
Lobster	-0.091	-0.090
Cuttlefish	-0.134	-0.133
Shellfish	-0.100	-0.100
	Sample Average	
High Value	-0.248	-0.224
Medium Value	-0.514	-0.513
Low Value	-0.071	-0.071
Lobster	-0.093	-0.093
Cuttlefish	-0.139	-0.138
Shellfish	-0.100	-0.100

[1] See, for example, Eales and Unnevehr (1994), Holt (2002), Holt and Bishop (2002), and Wong and McLaren (2005).

[2] See Luenberger (1992), pp. 468-469.

[3] In Baggio and Chavas (2006), since the specification of the benefit function can be analytically inverted to derive the implied direct utility function, the closed form inverse Marshallian demands can be derived. Thus the inverse Marshallian demands could have been derived just as easily by applying the Hotelling-Wold Identity to the corresponding direct utility

function, and thus this does not represent an advance in our ability to represent a wider class of underlying preference orderings.

[4] See McLaren et al (2000), and Wong and McLaren (2005).

[5] The definition of a benefit function is illustrated in Luenberger (1995), pp. 98-99.

[6] See Deaton (1979).

[7] See Luenberger (1996), p. 449.

[8] Superscripts have been suppressed for simplicity. The mapping from R to S is in fact equation 2.12 in Chambers, Chung and Fare (1996).

[9] The same derivation could apply to the distance function D(**x**, *u*), with the identity D(**x**, *u*)=1. Thus it should be true that the benefit and the distance functions are related by the functional identity $\frac{B_{x_i}}{\sum_j x_j B_{x_j}} = \frac{D_{x_i}}{\sum_j x_j D_{x_j}} = \frac{D_{x_i}}{D} = D_{x_i}$, by homogeneity of D, and D = 1, which provides further insight into the relation between these functions. See also Chambers, Chung and Fare (1996). This relationship also says something about the way in which the reference vector **g** may enter into the functional specification of B, since the corresponding inverse Hicksian demands are independent of **g**.

[10] This is essentially the procedure employed in Baggio and Chavas (2006).

[11] See Luenberger (1992), p. 469-472 for examples of using other functional forms to represent the benefit function.

[12] See Eales, Durham and Wessells (1997), p. 1157 for a complete description of the data.

[13] The seasonal adjustment of the data set was done with the help of SAMA procedure in TSP version 4.5 package. An alternative approach to accounting for seasonality's effects is to specify each δ_i (or α_i) parameter in (27) (or (31)) to be a function of three quarterly dummy variables and a constant, as suggested by Eales and Unnevehr (1994), and Holt and Bishop

(2002). Note however that this will significantly increase the number of parameters, which may create estimation convergence problems.

[14] The ith adjusted price function is derived by applying the envelope theorem to (25) and (28). Thus,

i) the SNAB function $\rightarrow$ $P_i^L(\mathbf{x}, u; \mathbf{g}) = \delta_i + \dfrac{\Phi_i}{x_i - \gamma_i} + \dfrac{\eta \cdot \mu_i}{x_i}\dfrac{\log(u)}{X1^{\eta}}$; and

ii) the B&C model $\rightarrow$ $P_i^L(\mathbf{x}, u; \mathbf{g}) = \alpha_i + \sum_j \gamma_{ij} x_j - \dfrac{\beta_i u \cdot X2}{(1 - u \cdot X3)x_i} - \delta_i \cdot X2 \cdot \left(\dfrac{u}{1 - u \cdot X3}\right)^2$.

Provided that $\mathbf{g}$ = (0, 0, 0, 0, 0, 1) and noting that $\sum_{i=1}^{N} P_i^L g_i = 1$ must hold for all $\mathbf{x}$ and u, these imply the following parameter restrictions:

i) the SNAB function: $\alpha_6 = \beta_6 = \mu_6 = 0$; and

ii) the B&C model: $\alpha_6 = 1$, and $\beta_6 = \delta_6 = \gamma_{6j} = 0$ (j = 1 to 6).

[15] The superscript “n” indicates the last row (row and column) of the respective vector (matrix) has been annihilated.

[16] Preliminary analysis revealed significant autocorrelation in the residuals of (33) at lag four.

[17] For reasons of brevity, the elasticity equations derived from the general SNAB are not presented below. The derivations of these equations are available separately.

[18] Scale elasticities (y_i), reported in the third part of table 3, measure the potential response of commodity price to a proportionate increase in all commodities. For example, the scale elasticity for high value fish is -1.212, which indicates that a 1% proportionate increase in all commodities will reduce the price (or the marginal value) of this fish category by about 1.212%.

Appendix: Elasticity Equations for the SNAB Function

Let M_{ij} denote the Marshallian quantity elasticities for commodity i with respect to x_j, S_i the scale elasticity of commodity i, and H_{ij} the Hicksian quantity elasticities for commodity i with respect to x_j. To facilitate thinking about preferences in terms of a benefit function, the quantity and scale elasticity functions can be written in terms of **x** and u:

$$M_{ij} = \frac{\partial \log(R_i^M)}{\partial \log(x_j)} = \frac{\partial \log\left\{B_{x_i}[\mathbf{x}, U(\mathbf{x})] / \sum_j B_{x_j}[\mathbf{x}, U(\mathbf{x})] x_j\right\}}{\partial \log(x_j)}, \; S_i = \sum_j M_{ij}, \text{ and}$$

$$H_{ij} = \frac{\partial \log(R_i^H)}{\partial \log(x_j)} = \frac{\partial \log\left[B_{x_i}(\mathbf{x}, u) / \sum_j B_{x_j}(\mathbf{x}, u) x_j\right]}{\partial \log(x_j)}.$$

Given the functional form of the SNAB function, it follows that the quantity and scale elasticity equations are expressed as:

$$M_{ij} = -\delta_{ij} + \frac{\delta_{ij} V_i + U_{ij} - \mu_i Z_j}{B_{x_i} x_i} - \left[\frac{V_j + \sum_j U_{ij} - Z_j}{\sum_j B_{x_j} x_j}\right],$$

$$S_i = -1 + \sum_i \left(\frac{\delta_{ij} V_i + U_{ij} - \mu_i Z_j}{B_{x_i} x_i}\right) - \sum_i \left[\frac{V_j + \sum_j U_{ij} - Z_j}{\sum_j B_{x_j} x_j}\right], \text{ and}$$

$$H_{ij} = -\delta_{ij} + \frac{\delta_{ij} V_i + U_{ij} - \mu_i \mu_j \eta^2 \log(u) / X1^\eta}{B_{x_i} x_i} - \left[\frac{V_j - \mu_j \eta^2 \log(u) / X1^\eta}{\sum_j B_{x_j} x_j}\right],$$

where $V_i = \delta_i x_i + \dfrac{\Phi_i x_i}{x_i - \gamma_i} - \dfrac{\Phi_i x_i^2}{(x_i - \gamma_i)^2}$,

$$U_{ij} = \frac{x_i}{x_i - \gamma_i} \frac{\partial \Phi_i}{\partial \log(u)} \frac{\partial \log(U)}{\partial \log(x_j)} \text{ in which } \frac{\partial \log(U)}{\partial \log(x_j)} = \frac{-B_{x_j} x_j}{\sum_{j'} \frac{\partial \Phi_{j'}}{\partial \log(u)} \log(x_{j'} - \gamma_{j'}) - \frac{1}{X1^\eta}},$$

$$\frac{\partial \Phi_i}{\partial \log(u)} = \frac{(\beta_i - \alpha_i) u}{(1+u)^2}, \; B_{x_i} = \partial B / \partial x_i = \delta_i + \Phi_i / (x_i - \gamma_i) + \frac{\mu_i \eta \log(u)}{x_i X1^\eta}, \text{ and}$$

$$Z_j = \frac{\eta}{X1^\eta}\left[\frac{-B_{x_j} x_j}{\sum_{j'} \frac{\partial \Phi_{j'}}{\partial \log(u)} \log(x_{j'} - \gamma_{j'}) - \frac{1}{X1^\eta}} - \mu_j \eta \log(u)\right].$$