On zeta functions associated to symmetric matrices (II): Functional equations and special values

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In the previous paper [5], we gave an explicit formula for zeta functions associated to symmetric matrices. In this paper, we shall give functional equations of these zeta functions. We also give special values of the zeta functions at non-positive integers , poles and residues. As an application, we give the contribution of "central unipotent" elements to the dimension of the space $S_k(\Gamma_n(N))$ of Siegel cusp forms belonging to the principal congruence subgroup $\Gamma_n(N)$. This leads us to an explicit conjecture on dimensions of $S_k(\Gamma_n(N))$ for $k > 2n$ and $N \geq 3$ for arbitrary n.

Historically, functional equations for zeta functions associated with a large class of prehomogeneous vector spaces were investigated by M. Sato and T. Shintani (see [12]), and the special cases treated in this paper were studied by Shintani [16]. These are functional equations between $n+1$ dimensional vectors each of whose components is the zeta function associated with each real orbit of the prehomogeneous vector spaces. Later, Satake and Faraut simplified these functional equations by giving a kind of diagonalization. (See [9], [8]) Since we have given the zeta functions explicitly in the previous paper, we can calculate everything explicitly and we can show that, in this diagonalization, only two, or three components are non-zero, if n is odd, or even, respectively. So, our functinal equations are much simpler than the previously known ones. Actually, we give a more direct proof of the functional equations independently of the arguments by Shintani or Satake-Faraut. In fact, when n is odd, the zeta functions are described by the Riemann zeta functions, so the functional equations are easily obtained. When n is even, our argument is based on the functional equations of the Mellin transform of real analytic Eisenstein series of half integral weights and we can get the results for $n \geq 4$ by some standard arguments. The case $n = 2$ is the pathological case, since the volumes $\mu(x)$ in the definition of our zeta functions are not finite for some x in this case. Hence we need a modification of the definition of the zeta functions to obtain a good functional equation. Such modification has been done first by T. Shintani [16] and later by F. Sato [11] in more intrinsic way. Here we also give the third alternative proof of this functional equation for $n = 2$ from our standpoint. The prototype of the proof of this style is found in Sturm [18] for an easier half of our cases where $n = 2$. For all $n \geq 1$, this direct approach allows us to evaluate special values, situation of poles, and their residues. This paper is organized as follows. In section 1, we state functional equations and poles and residues of our zeta functions for $n \geq 3$. In section 2, we review the Fourier coefficients of the real analytic Eisenstein series of half integral weight, which is more or less known by [15], [2], [21], [18], [3]. In section 3, we introduce a Cohen type linear combination of Eisenstein series ([2]) for a general parameter. For even $n \geq 4$, we use it to show the functional equation between Dirichlet series $D_n^*(s, \delta)$ and $D_n(s, \delta)$ which appear in some part of our zeta functions. In section 4, we prove the functional equations for $n \geq 3$. In section 5, we assume $n = 2$, and give the alternative proof described above. In section 3 and 5, we need some calculation on the Mellin transform of Eisenstein series with parameters. Goldfeld and Hoffstein [3] calculated this kind of Mellin transform for completely different purposes from ours. Our calculation is very similar to theirs in several points, but the details and aims are different. So, we write the proofs in detail here. The results in section 6 was announced before in [4], and we give here the proofs. We evaluate the values of our zeta functions at non positive integers. As a application, we give the contribution of central uniponent conjugacy classes to the dimension of Siegel cusp forms. This also leads to a conjecture for an explicit dimension formular for Siegel cusp forms, which will be stated there.

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1 Results on functional equations, poles and residues

In this section, we shall give results on functional equations between our zeta fnctions, their poles and residues. In this section, we mainly treat the case $n \geq 3$. The proof will be given in section 4. The case $n = 1$ is the trivial case and will be omitted in this paper. The case $n = 2$ is somewhat special but has been known by Shintani [16] and F. Sato [11]. This case will be treated separately in section 5 from our stand point with a new proof.

First, we recall the results from the previous paper part I [5]. Let L_n be the set of $n \times n$ symmetric matrices and L_n^* be the set of half integral symmetric matrices. For $L = L_n$ or L_n^*, our zeta function $\zeta_i(s, L)$ was originally defined for each i with $0 \leq i \leq n$ by a certain infinite sum over elements of L with signature $(i, n-1)$. We have shown in our previous paper that $\zeta_i(s, L)$ depends only on $\delta = (-1)^{n-i}$ and $\epsilon = (-1)^{(n-i)(n-i+1)/2}$ and denoted it by $\zeta(s, L, \delta, \epsilon)$. (As for the precise definition, see [5].) We review the explicit formula for this zeta function.

For the sake of simplicity, we put

$$b_n = \frac{|\prod_{i=1}^{[(n-1)/2]} B_{2i}|}{2^{n-1}([\frac{n-1}{2}])!}$$

where B_k is the k-th Bernoulli number defined by $te^t/(e^t-1)=\sum_{k=0}^{\infty} B_k t^k/k!$. When n is even, we also use the notation

$$B'_{n/2}=2(\frac{n}{2})!(2\pi)^{-n/2}\zeta(\frac{n}{2})(-1)^{1+n/2}.$$

For odd n, we put

$$\begin{aligned}
Q_n(s) &= \zeta(s-\frac{n-1}{2})\prod_{i=1}^{(n-1)/2}\zeta(2s-(2i-1)),\\
R_n(s) &= \zeta(s)\prod_{i=1}^{(n-1)/2}\zeta(2s-2i).
\end{aligned}$$

If n is odd with $n\geq 3$, by [5] Theorem 1.2, we have

$$\begin{aligned}
\zeta(s,L_n^*,\delta,\epsilon) &= b_n 2^{(n-1)s}(Q_n(s)+\epsilon\delta^{(n+1)/2}(-1)^{(n^2-1)/8}R_n(s)),\\
\zeta(s,L_n,\delta,\epsilon) &= b_n(2^{(n-1)/2}Q_n(s)+\epsilon\delta^{(n+1)/2}(-1)^{(n^2-1)/8}R_n(s)).
\end{aligned}$$

For even n, we put

$$A_n(s)=\prod_{i=1}^{n/2-1}\zeta(2s-2i),\qquad \text{and}\qquad B_n(s)=\prod_{i=1}^{n/2}\zeta(2s-(2i-1)).$$

If n is even with $n\geq 4$ or $(n,\delta)=(2,1)$, we also put

$$D_n^*(s,\delta)=(-1)^{[n/4]}2(2\pi)^{-n/2}\Gamma(n/2)\sum_{(-1)^{n/2}\delta d_K>0}|d_K|^{(n-1)/2-s}L(\frac{n}{2},\chi_K)\frac{\zeta(2s)\zeta(2s-n+1)}{L(2s-\frac{n}{2}+1,\chi_K)},$$

where d_K runs through all the discriminant of quadratic fields K or $\mathbf{Q}\oplus\mathbf{Q}$ such that $(-1)^{n/2}\delta D_K>0$ and χ_K is the Dirichlet character associated with K. Let $D_n^*(s,\delta)=\sum_{d=1}^{\infty}H(n/2,d,\delta)d^{-s}$. We define

$$D(s,\delta)=\sum_{i=1}^{\infty}H(\frac{n}{2},4d,\delta)d^{-s}.$$

By [5] Theorem 1.3, we have

$$\begin{aligned}
\zeta(s,L_n^*,\delta,\epsilon) &= b_n 2^{ns}((-1)^{[n/4]}D_n^*(s,\delta)A_n(s)+\epsilon\delta_n(-1)^{n(n+2)/8}\frac{2|B'_{n/2}|}{n}B_n(s))\\
\zeta(s,L_n,\delta,\epsilon) &= b_n((-1)^{[n/4]}D_n(s,\delta)A_n(s)+\epsilon\delta_n(-1)^{n(n+2)/8}\frac{2^{(n+2)/2}|B'_{n/2}|}{n}B_n(s)),
\end{aligned}$$

where $\delta_n=1$ if $(-1)^{n/2}\equiv\delta \bmod 4$, and $\delta_n=0$ otherwise.

Using the notation of Shintani [16], for any $s \in \mathbf{C}$ and a positive integer n, we put

$$\gamma_n(s) = \prod_{i=0}^{n-1} \Gamma(s+1+\frac{i}{2}).$$

When n is odd with $n \geq 3$, then $\zeta(s, L, 1, \epsilon) = \zeta(s, L, -1, (-1)^{(n+1)/2}\epsilon)$ for $L = L_n$, or L_n^*. So, it is enough to give a functional equation between $\zeta(s, L, 1, 1)$ and $\zeta(s, L, 1, -1)$.

Theorem 1 *When n is odd with $n \geq 3$, then*

$$\begin{aligned} \begin{pmatrix} \zeta(\frac{n+1}{2}-s, L_n^*, 1, 1) \\ \zeta(\frac{n+1}{2}-s, L_n^*, 1, -1) \end{pmatrix} &= 2^{-ns+(n^2-1)/2}\pi^{-ns+n(n-1)/4}\gamma_n(s-\frac{n+1}{2}) \\ &\quad \begin{pmatrix} a(s)+b(s) & -a(s)+b(s) \\ a(s)-b(s) & -a(s)-b(s) \end{pmatrix} \begin{pmatrix} \zeta(s, L_n, 1, 1) \\ \zeta(s, L_n, 1, -1) \end{pmatrix} \end{aligned}$$

where

$$a(s) = (\cos\frac{\pi s}{2})(\cos\pi s)^{(n-1)/2}$$

and

$$b(s) = (-1)^{(n^2-1)/8}(\sin\pi s)^{(n-1)/2} \times \begin{cases} \cos\frac{\pi s}{2}, & \textit{if } n \equiv 1 \bmod 4, \\ \sin\frac{\pi s}{2}, & \textit{if } n \equiv 3 \bmod 4. \end{cases}$$

Next, we give poles and residues of the above zeta functions. When $s_0 \in C$ is the possible simple pole of $\zeta(s, L_n^*, \delta, \epsilon)$, or $\zeta(s, L_n, \delta, \epsilon)$, respectively, we denote by $r_{n,\delta,\epsilon}(s_0)$, or $r^*_{n,\delta,\epsilon}(s_0)$ the residue of each zeta function at s_0, respectively. We understand that this value is 0, when the zeta function is holomorphic at s_0. When n is odd, we can determine the poles and residues exactly, as is shown in the following Theorem.

Theorem 2 *We assume that $n \geq 3$ is an odd integer. Then, the zeta function $\zeta(s, L_n^*, \delta, \epsilon)$ or $\zeta(s, L_n, \delta, \epsilon)$ has possible poles only at $s = l/2$, where l is an integer with $2 \leq l \leq n+1$. The residues are given as follows.*

(1) When $n = 3$, we get

$$\begin{aligned} r^*_{3,\delta,\epsilon}(1) &= (2\epsilon-1)b_3, \\ r_{3,\delta,\epsilon}(1) &= (\epsilon-1)b_3/2. \end{aligned}$$

In particular, $r_{3,\delta,1}(1) = 0$. When $n > 3$, we get $r^*_{3,\delta,\epsilon}(1) = r_{3,\delta,\epsilon}(1) = 0$, and the zeta functions are holomorphic at 1.

(2)For each natural integer j with $2 \leq j \leq (n-1)/2$, we get

$$\begin{aligned} r^*_{n,\delta,\epsilon}(j) &= 2^{(n-1)j-1}b_n\zeta(j-(n-1)/2)\prod_{i=1,i\neq j}^{(n-1)/2}\zeta(2j-2i+1), \\ r_{n,\delta,\epsilon}(j) &= 2^{(n-3)/2}b_n\zeta(j-(n-1)/2)\prod_{i=1,i\neq j}^{(n-1)/2}\zeta(2j-2i+1). \end{aligned}$$

In particular, if $j < (n-1)/2$ and $j \equiv (n-1)/2 \bmod 2$, then the zeta functions are holomorphic at $s = j$.

(3) We get

$$\begin{aligned} r^*_{n,\delta,\epsilon}((n+1)/2) &= 2^{(n^2-1)/2} b_n \prod_{i=1}^{(n-1)/2} \zeta(n-2i+2), \\ r_{n,\delta,\epsilon}((n+1)/2) &= 2^{(n-1)/2} b_n \prod_{i=1}^{(n-1)/2} \zeta(n-2i+2). \end{aligned}$$

(4)For each natural number j with $1 \le j \le (n-1)/2$, we get

$$\begin{aligned} r^*_{n,\delta,\epsilon}(j+1/2) &= 2^{(n-1)(j+1/2)} \delta^{(n+1)/2} \epsilon (-1)^{(n^2-1)/8} b_n \zeta(j+1/2) \prod_{i=1}^{(n-1)/2} \zeta(2j-2i+1), \\ r_{n,\delta,\epsilon}(j+1/2) &= \delta^{(n+1)/2} \epsilon (-1)^{(n^2-1)/8} b_n \zeta(j+1/2) \prod_{i=1}^{(n-1)/2} \zeta(2j-2i+1). \end{aligned}$$

When n is even with $n \ge 4$, then $\zeta(s, L, 1, 1) = \zeta(s, L, 1, -1)$ if $n \equiv 2 \bmod 4$, and $\zeta(s, L, -1, 1) = \zeta(s, L, -1, -1)$ if $n \equiv 0 \bmod 4$. We give the functional equation for each case.

Theorem 3 *We put $c(s) = (\cos \pi s)^{n/2-1} \sin \pi s$. When $n \equiv 0 \bmod 4$,*

$$\begin{pmatrix} \zeta(\frac{n+1}{2} - s, L^*_n, 1, 1) \\ \zeta(\frac{n+1}{2} - s, L^*_n, 1, -1) \\ \zeta(\frac{n+1}{2} - s, L^*_n, -1, 1) \end{pmatrix} =$$

$$2^{-ns+n^2/2-1} \pi^{-ns+n(n-1)/4} \gamma_n(s - \frac{n+1}{2})$$

$$\times \begin{pmatrix} (\cos \pi s)^{n/2} + (-1)^{n(n-2)/8} (\sin \pi s)^{n/2} & (\cos \pi s)^{n/2} - (-1)^{n(n-2)/8} (\sin \pi s)^{n/2} & 0 \\ (\cos \pi s)^{n/2} - (-1)^{n(n-2)/8} (\sin \pi s)^{n/2} & (\cos \pi s)^{n/2} + (-1)^{n(n-2)/8} (\sin \pi s)^{n/2} & 0 \\ (\cos \pi s)^{n/2-1} & (\cos \pi s)^{n/2-1} & -2c(s) \end{pmatrix}$$

$$\times \begin{pmatrix} \zeta(s, L_n, 1, 1) \\ \zeta(s, L_n, 1, -1) \\ \zeta(s, L_n, -1, 1) \end{pmatrix}.$$

When $n \equiv 2 \bmod 4$ with $n \ge 6$,

$$\begin{pmatrix} \zeta(\frac{n+1}{2} - s, L^*_n, 1, 1) \\ \zeta(\frac{n+1}{2} - s, L^*_n, -1, 1) \\ \zeta(\frac{n+1}{2} - s, L^*_n, -1, -1) \end{pmatrix} =$$

$$2^{-ns+n^2/2-1} \pi^{-ns+n(n-1)/4} \gamma_n(s - \frac{n+1}{2})$$

$$\times\begin{pmatrix} 2(\cos\pi s)^{n/2} & 0 & 0 \\ 2(\cos\pi s)^{n/2-1} & c(s)+(-1)^{n(n-2)/8}(\sin\pi s)^{n/2} & c(s)-(-1)^{n(n-2)/8}(\sin\pi s)^{n/2} \\ 2(\cos\pi s)^{n/2-1} & c(s)-(-1)^{n(n-2)/8}(\sin\pi s)^{n/2} & c(s)+(-1)^{n(n-2)/8}(\sin\pi s)^{n/2} \end{pmatrix}$$
$$\times\begin{pmatrix} \zeta(s,L_n,1,1) \\ \zeta(s,L_n,-1,1) \\ \zeta(s,L_n,-1,-1) \end{pmatrix}.$$

Now, we shall give the location of poles and their residues.

Theorem 4 *We assume that $n \geq 4$ is an even integer. Then, the zeta function $\zeta(s, L_n^*, \delta, \epsilon)$ or $\zeta(s, L_n, \delta, \epsilon)$ has possible poles only at $s = l/2$, where l is an integer with $1 \leq l \leq n+1$. The residues are given as follows.*

(1) Unless $(n,\delta) = (4,-1)$, we get

$$\begin{aligned} r^*_{n,\delta,\epsilon}(1) &= 2^n b_n \epsilon \delta_n (-1)^{(n(n+2)/8} \frac{|B_{n'/2}|}{n} \prod_{i=2}^{n/2} \zeta(3-2i), \\ r_{n,\delta,\epsilon}(1) &= 2^{-n/2} r^*_{n,\delta,\epsilon}(1), \end{aligned}$$

and if $(n,\delta) = (4,-1)$, we get

$$r^*_{4,-1,\epsilon}(1) = r_{4,-1,\epsilon}(1) = 12 \cdot \pi^{-4}\zeta(3).$$

(2)For each natural integer j with $2 \leq j \leq n/2$, we get

$$\begin{aligned} r^*_{n,\delta,\epsilon}(j) &= 2^{nj} b_n \epsilon \delta_n (-1)^{n(n+2)/8} \frac{|B_{n'/2}|}{n} \prod_{i=1, i\neq j}^{n/2} \zeta(2j-2i+1), \\ r_{n,\delta,\epsilon}(j) &= 2^{n(1/2-j)} r^*_{n,\delta,\epsilon}(j). \end{aligned}$$

In particular, if $(-1)^{n/2} \equiv -\delta \bmod 4$, then the zeta functions are holomorphic at $s = j$.

(3)We get

$$\begin{aligned} r^*_{n,\delta,\epsilon}((n+1)/2) &= 2^{n^2/2} b_n \pi^{-n/2} \Gamma(n/2)\zeta(n) \prod_{i=1}^{n/2-1} \zeta(n+1-2i), \\ r_{n,\delta,\epsilon}((n+1)/2) &= 2^{n^2/2} r^*_{n,\delta,\epsilon}((n+1)/2). \end{aligned}$$

(4)For each natural number j with $1 \leq j \leq n/2-1$, we get

$$\begin{aligned} r^*_{n,\delta,\epsilon}(j+1/2) &= 2^{n(j+1/2)-1} (-1)^{[n/4]} b_n D^*_n(j+1/2,\delta) \prod_{i=1, i\neq j}^{n/2-1} \zeta(2i+1-2j), \\ r_{n,\delta,\epsilon}(j+1/2) &= (-1)^{[n/4]} b_n D_n(j+1/2,\delta) \prod_{i=1, i\neq j}^{n/2-1} \zeta(2i+1-2j). \end{aligned}$$

Remark, We do not know whether $D^*_n(j+1/2,\delta)$ or $D_n(j+1/2,\delta)$ vanishes or not.

2 Fourier coefficients of Eisenstein series

In this section, we review several results on Fourier coefficients of some real analytic or holomorphic Eisenstein series of half integral weight of one variable. These are investigated by Shimura [15], Siegel [17], Cohen [2], Zagier [21], Sturm [18], Goldfeld and Hoffstein [3] and so on. Although we need some modification of their results, the way of the proof is essentially due to them. We shall sometimes sketch the proof for the reader's convenience. For any $\sigma, z \in \mathbf{C}$ with $z \neq 0$, we put $z^\sigma = e^{\sigma \log(z)}$, where $\arg(\log(z))$ is taken so that $-\pi < \arg(\log(z)) \leq \pi$. We denote the upper half plane by $H = \{z \in \mathbf{C}; \mathrm{Im}(z) > 0\}$. For the sake of simplicity, we write $e(z) = e^{2\pi i z}$. For each $s \in \mathbf{C}$, each odd (positive or negative) integer k, and $z \in H$, define the Eisenstein series $E(k, \sigma, z)$ by

$$E(k,\sigma,z) = y^{\frac{\sigma}{2}} \sum_{d=1,\ odd}^{\infty} \sum_{c=-\infty}^{\infty} \left(\frac{4c}{d}\right) \epsilon_d^{-k} (4cz+d)^{\frac{k}{2}} |4cz+d|^{-\sigma},$$

where $\left(\frac{*}{*}\right)$ is the quadratic residue symbol whose precise meaning is as was given in Shimura [13], p.442 and $\epsilon_d = 1$, or i, if $d \equiv 1$, or $3 \bmod 4$, respectively. When $-k + 2\sigma - 4 > 0$, this series converges absolutely and uniformly. We also define $E^*(k, \sigma, z)$ by

$$\begin{aligned} E^*(k,\sigma,z) &= E(-\frac{1}{4z})(-2iz)^{\frac{k}{2}} \\ &= y^{\frac{\sigma}{2}} 2^{\frac{k}{2}-\sigma} e(-\frac{k}{8}) \sum_{d=1,\ odd}^{\infty} \sum_{b=-\infty}^{\infty} \left(\frac{-b}{d}\right) \epsilon_d^{-k} (dz+b)^{\frac{k}{2}} |dz+b|^{-\sigma}. \end{aligned}$$

As usual, put

$$\Gamma_0(4) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z); c \equiv 0 \bmod 4\}.$$

For any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, put $j(\gamma, z) = \theta(\gamma z)/\theta(z)$, where $\theta(z) = \sum_{n \in Z} e^{2\pi i n^2 z}$. It is known that

$$j(\gamma, z) = \epsilon_d (\frac{c}{d})(cz+d)^{1/2}.$$

We get

$$E(k,\sigma,\gamma z) = j(\gamma,z)^{-k/2} E(k,\sigma,z) \text{ and } E^*(k,\sigma,\gamma z) = j(\gamma,z)^{-k/2} E^*(k,\sigma,z),$$

where we put $\gamma z = (az+b)(cz+d)^{-1}$. So, these functions are not holomorphic if $\sigma \neq 0$, but behave like modular forms of weight $-k/2$. When $k < -5$ and $\sigma = 0$, two holomorphic functions $E(-k, 0, z)$ and $E^*(-k, 0, z)$ gives a basis of the orthogonal complement of cusp forms of weight $-k/2$ in modular forms of weight $-k/2$ belonging to $\Gamma_0(4)$ (cf. Shimura loc. cit.). After Siegel, define a confluent hypergeometric function $W(y,\alpha,\beta)$ by

$$W(y,\alpha,\beta)=\int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}e^{-yu}du,$$

where $y>0$ and $\alpha,\beta\in\mathbf{C}$. This is convergent for $\mathrm{Re}(\beta)>0$ and meromorphically continued to the whole $(\alpha,\beta)\in\mathbf{C}^2$. For any integer d and α, $\beta\in\mathbf{C}$ with $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\alpha+\beta)>1$, we also define $\tau_d(y,\alpha,\beta)$ by

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)\tau_d(y,\alpha,\beta)=\begin{cases} d^{\alpha+\beta-1}e^{-2\pi dy}W(4\pi dy,\alpha,\beta) & (d>0)\\ |d|^{\alpha+\beta-1}e^{-2\pi|d|y}W(4\pi|d|y,\beta,\alpha) & (d<0)\\ \Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} & (d=0)\end{cases}$$

This is also meromorphically continued. The Fourier expansions of $E(k,\sigma,z)$ and $E^*(k,\sigma,z)$ are of the following form:

$$\begin{aligned} y^{-\sigma/2}E(k,\sigma,z) &= 1+\sum_{d=-\infty}^{\infty}\beta(d,\sigma,k)e(dx)\tau_d(y,\frac{\sigma-k}{2},\frac{\sigma}{2}),\\ y^{-\sigma/2}2^{\sigma-k/2}e(\frac{k}{8})E^*(k,\sigma,z) &= \sum_{d=-\infty}^{\infty}\alpha(d,\sigma,k)e(dx)\tau_d(y,\frac{\sigma-k}{2},\frac{\sigma}{2}),\end{aligned}$$

where

$$\begin{aligned} \alpha(d,\sigma,k) &= \sum_{c=1}^{\infty}(\frac{-4}{c})\epsilon_c^{-k}c^{k/2-\sigma}\sum_{m=1}^{c}(\frac{m}{c})e(dm/c),\\ \beta(d,\sigma,k) &= \sum_{c=1}^{\infty}(4c)^{k/2-\sigma}\sum_{m=1}^{4c}\epsilon_m^{-k}(\frac{4c}{m})e^{2\pi imd/4c}.\end{aligned}$$

To get Fourier coefficients suitable for our purpose, we put

$$F(k,\sigma,z)=E(k,\sigma,z)+2^{k/2-\sigma}(e(k/8)+e(-k/8))E^*(k,\sigma,z).$$

This kind of linear combination has been already considered by Cohen [2], Zagier [21] and Sturm [18] for $\sigma=0$. We need the coefficients also for $\sigma\neq 0$. which was calculated in Goldfeld and Hoffstein [3]. Since we need slightly different details here, we review this. We write the Fourier expansion of $F(k,\sigma,z)$ as

$$\begin{aligned} y^{-\sigma/2}F(k,\sigma,z) &= \sum_{d=-\infty}^{\infty}c_d(y)d(dx)\\ &= 1+\sum_{d=-\infty}^{\infty}C(d,\sigma,k)e(dx)\tau_d(y,\frac{\sigma-k}{2},\frac{\sigma}{2}),\end{aligned}$$

where each $c_d(y)$ is the Fourier coefficients depending on y and d and each $C(d,\sigma,k)$ is a constant which does not depend on z. Following the method of Shimura [15] and Sturm [18], we can calcultate $C(d,\sigma,k)$ explicitly. For the convenience of the readers, we sketch here the calculation. By definition,

$$C(d,\sigma,k)=\beta(d,\sigma,k)+2^{k-2\sigma}(1+i^{-k})\alpha(d,\sigma,k).$$

By Sturm [18] p.227 Lemma 2 and the arguments in p.228, we have

$$\begin{aligned} C(d,\sigma,k) &= \alpha(d,\sigma,k)\{2^{k-2\sigma}(1+i^{-k})+\sum_{r=2}^{\infty}\sum_{l=1}^{2^r}\epsilon_l^{-k}(\frac{2^r}{l})e(\frac{dl}{2^r})2^{(k/2-\sigma)r}\} \\ &= \alpha(d,\sigma,k)A(d,\sigma,k), \end{aligned}$$

where

$$\begin{aligned} A(d,\sigma,k) &= 2^{k-2\sigma}(1+i^{-k})+\sum_{r=2}^{\infty}\frac{1}{2}((1+i^{-k})U(r)+(1-i^{-k})V(r))2^{(k/2-\sigma)r} \\ U(r) &= \sum_{l=1}^{2^r}(\frac{2^r}{l})e(\frac{dl}{2^r}), \\ V(r) &= \sum_{l=1}^{2^r}(\frac{-2^r}{l})e(\frac{dl}{2^r}). \end{aligned}$$

Assume that $d\neq 0$ and put $d=2^t d_0$, where d_0 is an odd (positive or negative) integer and t is a non-negative integer. If t is even, then we get that $U(2i)=2^{2i-1}$ for $1\leq i\leq t/2$, $U(t+3)=(\frac{2}{d_0})2^{t+3/2}$, and $U(r)=0$ for the other r, and that $V(t+2)=2^{t+1}i(\frac{-1}{d_0})$, $V(t+3)=2^{t+3/2}(\frac{-2}{d_0})$ and $V(r)=0$ for the other r. If t is odd, then $V(r)=0$ for all r and $U(2i)=2^{2i-1}$ for $1\leq i\leq (t-1)/2$, $U(t+1)=-2^t$ and $U(r)=0$ for the other r. Hence, if t is even, then we get

$$\begin{aligned} &\sum_{r=2}^{\infty}\frac{1}{2}(1+i^{-k})U(r)2^{(k/2-\sigma)r}= \\ &\quad (1+i^{-k})(\frac{2^{k-2\sigma}(1-2^{(k/2-\sigma+1)t})}{1-2^{k-2\sigma+2}}+2^{3(k/2-\sigma)+1/2+(k/2-\sigma+1)t}(\frac{2}{d_0})), \\ &\sum_{r=2}^{\infty}\frac{1}{2}(1-i^{-k})V(r)2^{(k/2-\sigma)r}= \\ &\quad (-1)^{(k+1)/2}(1+i^{-k})(2^{3(k/2-\sigma)+1/2}\cdot 2^{(k/2-\sigma+1)t}(\frac{-2}{d_0})+2^{k-2\sigma+(k/2-\sigma+1)t}(\frac{-1}{d_0})), \end{aligned}$$

and if t is odd, then

$$\begin{aligned} &\sum_{r=2}^{\infty}\frac{1}{2}(1+i^{-k})U(r)2^{(k/2-\sigma)r}= \\ &\quad (1+i^{-k})(-2^{k/2-\sigma-1}\cdot 2^{(k/2-\sigma+1)t}+\frac{2^{k-2\sigma}(1-2^{(k-2\sigma+2)(t-1)/2})}{1-2^{k-2\sigma+2}}), \end{aligned}$$

$$\sum_{r=2}^{\infty} \frac{1}{2}(1-i^{-k})V(r)2^{(k/2-\sigma)r} = 0$$

We get $(-1)^{(k+1)/2}(\frac{-1}{d_0}) = 1$, or -1, if $d_0 \equiv (-1)^{(k+1)/2} \bmod 4$, or not, respectively.

Lemma 2.1 *(1) If $d_0 \equiv (-1)^{(k+1)/2} \bmod 4$, then*

$$\begin{aligned}\sum_{t=0}^{\infty} A(d_0 2^{2t}, \sigma, k) 2^{2t(\sigma-k/2-1-s)} &= (1+i^{-k})2^{k-2\sigma+1} \\ &\times \frac{(1+2^{k/2-\sigma+1/2}(\frac{2}{d_0}))(1-2^{\sigma-k/2-3/2-2s}(\frac{2}{d_0}))}{(1-2^{-2s})(1-2^{2\sigma-k-2-2s})}.\end{aligned}$$

(2) If $d_0 \not\equiv (-1)^{(k+1)/2} \bmod 4$, then

$$\sum_{t=0}^{\infty} A(d_0 2^{2t}, \sigma, k) 2^{2t(\sigma-k/2-1-s)} = (1+i^{-k})\frac{2^{-1-2s}(1-2^{k-2\sigma+1})}{(1-2^{-2s})(1-2^{2\sigma-k-2-2s})}.$$

(3) For any odd d_0, we get

$$\sum_{t=0}^{\infty} A(d_0 2^{2t+1}, \sigma, k) 2^{(2t+1)(\sigma-k/2-1-s)} = (1+i^{-k})\frac{2^{\sigma-k/2-2-3s}(1-2^{k-2\sigma+1})}{(1-2^{-2s})(1-2^{2\sigma-k-2-2s})}.$$

By Shimura [15] Proposition 2, for any square free integer m, we get

$$\sum_{f=1}^{\infty} \alpha(mf^2, \sigma, k) f^{2\sigma-k-2-2s} = \frac{L_2(\sigma-(k+1)/2, \omega_1)\zeta_2(2s)\zeta(2s-2\sigma+k+2)}{\zeta_2(2\sigma-k-1)L_2(2s-\sigma+(k+3)/2, \omega_1)},$$

where we denote by ζ_2 or L_2 the usual zeta or L functions such that their Euler 2-factors are omitted, and ω_1 is the primitive character defined by

$$\omega_1(a) = \left(\frac{-1}{a}\right)^{(k+1)/2} \left(\frac{4m}{a}\right)$$

for $(a, 4m) = 1$. Here $(*/*)$ is the quadratic residue symbol, the precise meaning of which is in Shimura [13]. Now, for any square free integer m, we calculate the Dirichlet series

$$\sum_{f=1}^{\infty} e(\frac{k}{8})C(mf^2, \sigma, k)(|m|f^2)^{\sigma-k/2-1-s}.$$

By the above formula (2) and (3) in Lemma 2.1, if f is odd, then we get $C(mf^2, \sigma, k) = 0$, unless m is odd and $m \equiv (-1)^{(k+1)/2} \bmod 4$. Hence, we can assume that $d_K f_1^2 = (-1)^{(k+1)/2} m f^2$ for some integer f_1 and the fundamental discriminant d_K of the quadratic field $K = Q((-1)^{(k+1)/2}m)$ or $d_K = 1$ when $m = (-1)^{(k+1)/2}$.

Proposition 2.2

$$\sum_{f=1}^{\infty} e(\frac{k}{8})C((-1)^{(k+1)/2}d_K f^2, \sigma, k)(|d_K|f^2)^{\sigma-k/2-1-s} =$$

$$(-1)^{(k^2-1)/8}2^{k-2\sigma+3/2}\frac{|d_K|^{\sigma-k/2-1}}{|d_K|^s} \times \frac{L(\sigma-(k+1)/2, \chi_K)\zeta(2s)\zeta(2s-2\sigma+k+2)}{\zeta(2\sigma-k-1)L(2s-\sigma+(k+3)/2, \chi_K)},$$

where χ_K is the Kroneker character attached to K, and regarded as the trivial character, if $d_K = 1$.

Proof. This can be shown by calculationg a kind of convolution product of the Dirichelte series in Lemma 2.1 and Shimura's results, noting that $e(\frac{k}{8})(1+i^{-k}) = e(\frac{k}{8}) + e(-\frac{k}{8}) = \sqrt{2}(-1)^{(k^2-1)/8}$. q.e.d.

Proposition 2.3 *If $\sigma > k/2+1$, then we have*

$$C(0, \sigma, k) = 2^{k-2\sigma+1}(1+i^{-k})\frac{\zeta(2\sigma-k-2)}{\zeta(2\sigma-k-1)}.$$

Proof. When $d = 0$, we get $V(r) = 0$ for all r, and $U(r) = 2^{r-1}$, or 0 for r even, or odd, respectively. Since

$$\sum_{r=1}^{\infty} U(2r)2^{(k/2-\sigma)2r} = \frac{2^{k-2\sigma+1}}{1-2^{k-2\sigma+2}},$$

we get

$$A(0, \sigma, k) = 2^{k-2\sigma+1}(1+i^{-k})\frac{1-2^{k-2\sigma+1}}{1-2^{k-2\sigma+2}}.$$

On the other hand, by Shimura [15] p.89 (3.9), we have

$$\alpha(0, \sigma, k) = \frac{\zeta_2(2\sigma-k-2)}{\zeta_2(2\sigma-k-1)}.$$

Hence, we get our results.

Corollary 2.4 *We get*

$$c_0(y) = 1 + 2^{3k/2-3\sigma+7/2}(-1)^{(k^2-1)/8}\pi\frac{\Gamma(\sigma-k/2-1)\zeta(2\sigma-k-2)}{\Gamma((\sigma-k)/2)\Gamma(\sigma/2)\zeta(2\sigma-k-1)}y^{1-\sigma+k/2}.$$

3 Dirichlet series associated to Eisenstein series

3.1

First, we prove functional equations between $D_n^*(\frac{n+1}{2}-s,\delta)$ and $D_n(s,\delta)$, where $\delta=1$, or -1. The case $(n,\delta)=(2,-1)$ is exceptional, and we need some special treatment in this case. So, we exclude this case for a moment, and treat it in section 5. Our proof is based on the usual argument by Hecke on functional equations of Mellin transforms of automorphic forms. This type of functional equation has been more or less calculated by Goldfeld and Hoffstein [3] for a different purpose, but the modular forms in question are slightly different and we need some new calculation here. For any function $f(z)$ on H, we define two operators $f|_kW_4$ and $f|U_4$ by

$$\begin{aligned}(f|_kW_4)(z) &= f(-1/4z)(-2iz)^{k/2},\\ (f|U_4)(z) &= \frac{1}{4}\sum_{\nu=0}^{3} f(\frac{z+\nu}{4}).\end{aligned}$$

We define $F(z)=F(k,\sigma,z)$ as in the last section.

Lemma 3.1 *We have the following relation.*

$$F|_kW_4 = 2^{(k+1)/2}(-1)^{(k^2-1)/8}F|U_4.$$

Proof. This can be proved in the same way as in Kohnen [6], and we sketch the outline. Put

$$g_1(z)=\frac{1}{4}(F(\frac{z+1}{4})+F(\frac{z+3}{4})),$$

and

$$g_2(z)=\frac{1}{4}(F(\frac{z}{4})+F(\frac{z+2}{4})).$$

Then, $F|U_4=g_1+g_2$. As was shown in the last section, the coefficients of $F(z)$ of $e(dx)$ are 0, unless $d\equiv 0$, or $(-1)^{(k+1)/2}$ mod 4. Since $e(d/2)+1 = e(d/4)+e(3d/4)$, if $d\equiv 0$ mod 4, or $d\equiv 1$ mod 2, we get $g_1=g_2$. Hence, using the same argument as in Kohnen [6] p.255, we get

$$\begin{aligned}F|U_4|_kW_4 &= 2g_1|_kW_4\\ &= 2^{-k/2-1}(F(z-\frac{1}{4})e(-k/8)+F(z+\frac{1}{4})e(k/8)).\end{aligned}$$

It is easy to see that $e(-d/4)e(-k/8)+e(d/4)e(k/8)=\sqrt{2}(-1)^{(k^2-1)/8}$, if $d\equiv 0$, or $(-1)^{(k+1)/2}$ mod 4. Hence, $F|U_4|_kW_4 = 2^{-(k+1)/2}(-1)^{(k^2-1)/8}F$, and since W_4^2 acts trivially on F by Shimura [14] p.448, we prove our lemma. q.e.d.

Now, we put $g(z) = y^{-\sigma/2}F(k,\sigma,z)$ and $h(z) = (y^{-\sigma/2}F(k,\sigma,z))|U_4$. We denote the Fourier expansion of $g(z)$ as

$$g(z) = \sum_{d=-\infty}^{\infty} c_d(y)e(dx).$$

Then, we get

$$h(z) = \sum_{d=-\infty}^{\infty} c_{4d}(y/4)e(dx).$$

Put $G(y) = g(iy)$ and $H(y) = h(iy)$. Then, by the above lemma, we get $G(1/4y) = (-1)^{(k^2-1)/8}\sqrt{2}y^{\sigma-k/2}H(y)$, and $G(1/2y) = C_k y^{\sigma-k/2}H(y/2)$, where $C_k = 2^{(k+1)/2-\sigma}(-1)^{(k^2-1)/8}$. Define two Dirichlet series by

$$\begin{aligned}
\Phi_{\sigma,k}(s) &= \int_0^\infty (G(y)-c_0(y))y^{s-1}dy,\\
\Psi_{\sigma,k}(s) &= \int_0^\infty (H(y)-c_0(y/4))y^{s-1}dy.
\end{aligned}$$

Then, by the usual argument of Hecke, we get

$$\begin{aligned}
2^s\Phi_{\sigma,k}(s) &= \int_0^\infty (G(y/2)-c_0(y/2))y^{s-1}dy\\
&= \int_1^\infty (G(y/2)-c_0(y/2))y^{s-1}dy + \int_0^1 (G(y/2)-c_0(y/2))y^{s-1}dy\\
&= \int_1^\infty (G(y/2)-c_0(y/2))y^{s-1}dy + \int_1^\infty (C_kH(y/2)y^{\sigma-k/2}-c_0(1/2y))y^{-s-1}dy\\
&= \int_1^\infty (G(y/2)-c_0(y/2))y^{s-1}dy + C_k\int_1^\infty (H(y/2)-c_0(y/8))y^{\sigma-k/2-s-1}dy\\
&\quad -\int_1^\infty c_0(1/2y)y^{-s-1}dy + C_k\int_1^\infty c_0(y/8)y^{\sigma-k/2-s-1}dy.
\end{aligned}$$

We have $c_0(y) = 1 + c(\sigma)y^{1-\sigma+k/2}$ for a certain function of $c(\sigma)$ of σ which does not depend on y. (The exact value is given in section 2, Corollary 2.4.) So, if $s > \sigma - k/2$ and $s > 1$, then

$$\begin{aligned}
&2^s\Phi_{\sigma,k}(s) + \int_1^\infty c_0(1/2y)y^{-s-1}dy - C_k\int_1^\infty c_0(y/8)y^{\sigma-k/2-s-1}dy\\
=\ &2^s\Phi_{\sigma,k}(s) + \frac{c(\sigma)2^{\sigma-k/2-1}}{s-\sigma+k/2+1} + C_k\frac{c(\sigma)8^{\sigma-k/2-1}}{1-s} + \frac{1}{s} + C_k\frac{1}{\sigma-k/2-s}\\
=\ &\int_1^\infty (G(y/2)-c_0(y/2))y^{s-1}dy + C_k\int_1^\infty (H(y/2)-c_0(y/8))y^{\sigma-k/2-s-1}dy.
\end{aligned}$$

Now, denote the last line above by $J_k(\sigma,s)$. Then, it can be proved that $(\sigma - (k+3)/2)(\sigma-(k+2)/2)\zeta(2\sigma-k-1)J_k(\sigma,s)$ is holomorphic at whole $\sigma, s \in \mathbf{C}$. Hence, $\Phi_{\sigma,k}(s)$ is meromorphically continued to the whole $(\sigma,s)\in\mathbf{C}^2$ and

$$\zeta(2\sigma-k-1)(\sigma-(k+2)/2)(\sigma-(k+3)/2)s(s-1)(s-\sigma+k/2)(s-\sigma+k/2+1)\Phi_{\sigma,k}(s)$$

is holomorphic on $(\sigma, s) \in \mathbf{C}^2$. In the same way, we get

$$\begin{aligned}
2^s \Psi_{\sigma,k}(s) &= \int_1^\infty (H(y/2) - c_0(y/8)) y^{s-1} dy + C_k^{-1} \int_1^\infty (G(y/2) - c_0(y/2)) y^{\sigma-k/2-s-1} dy \\
&\quad - \int_1^\infty c_0(1/8y) y^{-s-1} dy + C_k^{-1} \int_1^\infty c_0(y/2) y^{\sigma-k/2-s-1} dy. \\
&= C_k^{-1} J_k(\sigma, \sigma - k/2 - s) - \frac{1}{s} - \frac{c(\sigma) 8^{\sigma-k/2-1}}{s - \sigma + k/2 + 1} - \frac{C_k^{-1}}{\sigma - k/2 - s} - \frac{C_k^{-1} c(\sigma) 2^{\sigma-k/2-1}}{1-s}.
\end{aligned}$$

Hence, we get the following functional equation (cf. [3]).

$$\Phi_{\sigma,k}(\sigma - k/2 - s) = 2^{2s-\sigma+k/2} C_k \Psi_{\sigma,k}(s) = 2^{2s-2\sigma+k+1/2} (-1)^{(k^2-1)/8} \Psi_{\sigma,k}(s).$$

3.2

Now, we shall apply functional equations between $\Phi_{\sigma,k}(s)$ and $\Psi_{\sigma,k}(s)$ to those between $D^*(s, \delta)$ and $D(s, \delta)$. This can be done by calculating the Mellin transforms term by term and by specializing σ and k. The Mellin transform of $\tau_d(y, (\sigma - k)/2, \sigma/2)$ has been given in [3]. For each α, $\beta \in R$ and $s \in C$, we consider the following integral:

$$I(s, \alpha, \beta) = \int_0^\infty \frac{(1+u)^{\alpha-1} u^{\beta-1}}{(1+2u)^s} du.$$

This integral converges absolutely when $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(s) > \mathrm{Re}(\alpha) + \mathrm{Re}(\beta) - 1$. This is related with the usual hypergeometric functions. This can be shown as follows. For any non negative integer n and a variable a, we put $(a)_n = \Gamma(a+n)/\Gamma(a)$. The Gauss hypergeometric function $F(a, b, c; z)$ is by definition

$$F(a, b, c; z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

and has an integral expression

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt$$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|z| < 1$. Now, in our integral, substituting the variable u by $u = v/(2-2v)$, we get

$$\begin{aligned}
I(s, \alpha, \beta) &= \frac{1}{2} \int_0^1 v^{\beta-1} (1-v)^{s-\alpha-\beta} (1 - \frac{v}{2})^{\alpha-1} \, dv \\
&= \frac{\Gamma(\beta)\Gamma(s-\alpha-\beta+1)}{\Gamma(s-\alpha+1)} F(1-\alpha, \beta, s-\alpha+1; 1/2).
\end{aligned}$$

Now, we assume that $\mathrm{Re}(\sigma) > 0$, $\mathrm{Re}(\sigma) > k$, and $\mathrm{Re}(s) > \mathrm{Re}(\sigma) - k/2 - 1$. Then, the Mellin transforms of the Whittaker functions are given by

$$\int_0^\infty \tau_d(y, \frac{\sigma-k}{2}, \frac{\sigma}{2}) y^{s-1} dy = \frac{e(k/8)(2\pi)^{\sigma-k/2}}{\Gamma((\sigma-k)/2))\Gamma(\sigma/2)} \times \begin{cases} d^{\sigma-k/2-1}(2\pi d)^{-s}\Gamma(s) I(s, \frac{\sigma-k}{2}, \frac{\sigma}{2}), & \text{if } d > 0, \\ |d|^{\sigma-k/2-1}(2\pi d)^{-s}\Gamma(s) I(s, \frac{\sigma}{2}, \frac{\sigma-k}{2}), & \text{if } d < 0. \end{cases}$$

Hence, if $d = (-1)^{(k+1)/2} d_K f^2 \neq 0$, we get

$$\begin{aligned} \int_0^\infty c_d(y) y^{s-1} dy &= e(\frac{k}{8}) C((-1)^{(k+1)/2} d_K f^2, \sigma, k)(|d_K| f^2)^{\sigma-k/2-1-s} \\ &\quad \times \frac{(2\pi)^{\sigma-k/2-s}\Gamma(s)}{\Gamma((\sigma-k)/2))\Gamma(\sigma/2)} \times \begin{cases} I(s, (\sigma-k)/2, \sigma/2), & \text{if } d > 0 \\ I(s, \sigma/2, (\sigma-k)/2), & \text{if } d < 0 \end{cases} \end{aligned}$$

If $\mathrm{Re}(s) > -k/2 - 1$, the right hand side can be continued holomorphically to $\sigma = 0$. Indeed, under this condition on s and k, $I(s, \sigma/2, (\sigma-k)/2))$ is holomorphic at $\sigma = 0$, and also

$$\frac{1}{\Gamma(\sigma/2)} I(s, \frac{\sigma-k}{2}, \frac{\sigma}{2}) = \frac{1}{\Gamma(\sigma/2)(e^{\pi i \sigma} - 1)} \int_\infty^{(0+)} \frac{(u+1)^{(\sigma-k)/2-1} u^{\sigma/2-1}}{(2u+1)^s} du,$$

and this function is 1 at $\sigma = 0$.

For the sake of simplicity, we put

$$Z^*(k, \sigma, s, d_K) = |d_K|^{\sigma-k/2-1-s} \frac{L(\sigma - (k+1)/2, \chi_K)\zeta(2s)\zeta(2s - 2\sigma + k + 2)}{\zeta(2\sigma - k - 1) L(2s - \sigma + (k+3)/2, \chi_K)},$$

and

$$C_{\sigma,k} = (-1)^{(k^2-1)/8} \frac{2^{k/2-\sigma+3/2}\pi^{\sigma-k/2}}{\Gamma((\sigma-k)/2)\Gamma(\sigma/2)}.$$

We write the above Dirichlet series as $Z^*(k, \sigma, s, d_K) = \sum_{n=1}^\infty a(n) n^{-s}$ and put

$$Z(k, \sigma, s, d_K) = \sum_{n=1}^\infty \frac{a(4n)}{n^s}.$$

Then, we get

$$\begin{aligned} &\Phi_{\sigma,k}(s) = \\ &\quad C_{\sigma,k}(2\pi)^{-s}\Gamma(s) \\ &\quad \times (\sum_{(-1)^{(k+1)/2} d_K > 0} Z^*(k, \sigma, s, d_K) I(s, \frac{\sigma-k}{2}, \frac{\sigma}{2}) + \sum_{(-1)^{(k+1)/2} d_K < 0} Z^*(k, \sigma, s, d_K) I(s, \frac{\sigma}{2}, \frac{\sigma-k}{2})) \end{aligned}$$

and

$$\Psi_{\sigma,k}(s) =$$
$$C_{\sigma,k}(2\pi)^{-s}\Gamma(s)$$
$$\times(\sum_{(-1)^{(k+1)/2}d_K>0} Z(k,\sigma,s,d_K)I(s,\frac{\sigma-k}{2},\frac{\sigma}{2}) + \sum_{(-1)^{(k+1)/2}d_K<0} Z(k,\sigma,s,d_K)I(s,\frac{\sigma}{2},\frac{\sigma-k}{2}))$$

Case (I): We put $k=-(n+1)$ and $\sigma=0$. We assume that n is even and $n \geq 4$.
By definition, we have

$$\begin{aligned}\sum_{(-1)^{n/2}d_K>0} Z^*(-(n+1),0,s,d_K) &= \sum_{(-1)^{n/2}d_K>0} |d_K|^{(n-1)/2-s}\frac{L(n/2,\chi_K)\zeta(2s)\zeta(2s-n+1)}{\zeta(n)L(2s-n/2+1,\chi_K)} \\ &= 2^{-1}(2\pi)^{n/2}(-1)^{[n/4]}\Gamma(n/2)^{-1}\zeta(n)^{-1}D_n^*(s,1).\end{aligned}$$

Hence, we get

$$\Phi_{0,-n-1}(s) = (-1)^{n(n+2)/8+[n/4]}\pi^{n+1/2}\Gamma((n+1)/2)^{-1}\Gamma(n/2)^{-1}\zeta(n)^{-1}(2\pi)^{-s}\Gamma(s)D_n^*(s,1).$$

In the same way, we get

$$\Psi_{0,-n-1}(s) = (-1)^{n(n+2)/8+[n/4]}\pi^{n+1/2}\Gamma((n+1)/2)^{-1}\Gamma(n/2)^{-1}\zeta(n)^{-1}(2\pi)^{-s}\Gamma(s)D_n(s,1).$$

Hence, we get

$$(2\pi)^{-(n+1)/2+s}\Gamma(\frac{n+1}{2}-s)D_n^*(\frac{n+1}{2}-s,1) = 2^{2s-(n+1)/2}(2\pi)^{-s}\Gamma(s)D_n(s,1)C_k,$$

where $C_k = 2^{-n/2}(-1)^{n(n+2)/8}$. Since

$$\Gamma((n+1)/2-s)^{-1} = (-1)^{n/2}\pi^{-1}(\cos\pi s)\Gamma(s-(n-1)/2),$$

we get

Proposition 3.2 *For any even positive integer $n \geq 4$, we have*

$$D_n^*(\frac{n+1}{2}-s,1) = (-1)^{n(n-2)/8}2^{-n/2}\pi^{-2s+(n-1)/2}\Gamma(s)\Gamma(s-\frac{n-1}{2})(\cos\pi s)D_n(s,1).$$

When $n=2$ and $\delta=-1$, we need some modification of functional equation, since $L(1,\chi_K)=\infty$, if $K=Q\oplus Q$. But if $\delta=1$, the definition of D_2 itself is the same and we have

Proposition 3.3

$$D_2^*(3/2-s,1) = 2^{-1}\pi^{1/2-2s}\Gamma(s)\Gamma(s-1/2)(\cos(\pi s)D_2(s,1)-\zeta(2s-1)).$$

This proposition was first proved by Siegel [17] and Shintani [16] without using Eisenstein series. (Incidentally, in Shintani's notation, $D_2(s,1) = 2^{2s}\xi_{-}^{*}(s)$ and $D_2^{*}(s,1) = \xi_{-}(s)$.) Later, an alternative proof in the case $n = 2$ is obtained by Sturm [18]. His argument is to use the Eisenstein series above, noting that, in the above calculation, $I(s,0,3/2)$ is holomorphic for $s > 1/2$ and that $\zeta(\sigma+1)/\Gamma(\sigma/2)$ is holomorphic at $\sigma = 0$. As for the case $(n,\delta) = (2,-1)$, we need a modification of the definition, which will be explained in section 5.

Case(II): We put $k = -n+3$ and $\sigma = 2$ with even $n \geq 4$. By changing the variable u by $v = 2u(2u+1)^{-1}$, we get

$$I(s,1,\frac{n-1}{2}) = 2^{-(n-1)/2}\int_0^1 v^{(n-3)/2}(1-v)^{s-(n+1)/2}dv = 2^{-(n-1)/2}\Gamma(s)^{-1}\Gamma(\frac{n-1}{2})\Gamma(s-\frac{n-1}{2}).$$

On the other hand, the function $I(s,\frac{n-1}{2},1)$ can be continued meromorphically to the whole s plane and satisfies

$$I(s,\frac{n-1}{2},1)+I(\frac{n+1}{2}-s,\frac{n-1}{2},1) = 2^{-(n-1)/2}\Gamma((3-n)/2)^{-1}\Gamma(1-s)\Gamma(s-\frac{n-1}{2}).$$

This can be proved easily by using the Pochhammer integral formula for the beta function. Indeed, put $v = (1+2u)/(2+2u)$. Then,

$$I(s,\frac{n-1}{2},1) = 2^{-(n-1)/2}\int_{1/2}^1 v^{-s}(1-v)^{s-(n+1)/2}dv.$$

Now, for α, $\beta > 0$, put $f(x) = x^{\alpha-1}(1-x)^{\beta-1}$. Take a positive branch at $x = 1/2$ and take an analytic continuation along the path starting at $1/2$, encircling 1 and 0 in the positive direction, then encircling 1 and 0 in the negative direction. Then,

$$\int_{1/2}^{(1+,0+,1-,0-)} x^{\alpha}(1-x)^{\beta}dx = \frac{1}{(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})}B(\alpha,\beta),$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)^{-1}$ is the usual beta function. We also have

$$\int_{1/2}^1 v^{\alpha}(1-v)^{\beta}dv = \frac{1}{(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})}(\int_{1/2}^{(1+)} f(x)dx + \int_{1/2}^{(1-)} f(x)dx),$$

and

$$\int_0^{1/2} v^{\alpha}(1-v)^{\beta}dv = \frac{1}{(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})}(\int_{1/2}^{(0+)} f(x)dx + \int_{1/2}^{(0-)} f(x)dx),$$

and we get the above result. By virtue of the above integral expression, it is easy to see that $\Gamma(s-(n-1)/2)^{-1}I(s,\frac{n-1}{2},1)$ is an entire function and that $I(s,\frac{n-1}{2},1)$ has simple poles only at $s = (n+1)/2 - m$ ($m = 1, 2, \ldots$). The residue for each m is $(-1)^{m-1}2^{-(n-1)/2}\Gamma(m-(n-1)/2)\Gamma((3-n)/2)^{-1}\Gamma(m)^{-1}$.

Now, we get

$$\begin{aligned}\Phi_{2,-n+3}(s) &= (-1)^{n(n+2)/8+1+[n/4]}\pi^{n+1/2}\Gamma(n/2)^{-1}\Gamma((n-1)/2)^{-1}\zeta(n)^{-1}(2\pi)^{-s}\Gamma(s)\\ &\quad\times(D_n^*(s,1)I(s,\frac{n-1}{2},1)+D_n^*(s,-1)I(s,1,\frac{n-1}{2})),\end{aligned}$$

and

$$\begin{aligned}\Psi_{2,-n+3}(s) &= (-1)^{n(n+2)/8+1+[n/4]}\pi^{n+1/2}\Gamma(n/2)^{-1}\Gamma((n-1)/2)^{-1}\zeta(n)^{-1}(2\pi)^{-s}\Gamma(s)\\ &\quad\times(D_n(s,1)I(s,\frac{n-1}{2},1)+D_n(s,-1)I(s,1,\frac{n-1}{2})).\end{aligned}$$

Hence, we get

$$\begin{aligned}&(2\pi)^{s-(n+1)/2}\Gamma(\frac{n+1}{2})\\ &\times(D_n^*(\frac{n+1}{2}-s,1)I(\frac{n+1}{2}-s,\frac{n-1}{2},1)+D_n^*(\frac{n+1}{2}-s,-1)I(\frac{n+1}{2}-s,1,\frac{n-1}{2}))\\ =\ &2^{2s-n-1/2}(-1)^{n(n+2)/8+1}(2\pi)^{-s}\Gamma(s)\\ &\times(D_n(s,1)I(s,\frac{n-1}{2},1)+D_n(s,-1)I(s,1,\frac{n-1}{2})).\end{aligned}$$

Here, using the functional equation between $D_n^*(s,1)$ and $D_n(s,1)$, and the formulas for $I(s,1,\frac{n-1}{2})$ and $I(s,\frac{n-1}{2},1)$, we get

$$\begin{aligned}&(2\pi)^{s-(n+1)/2}2^{-(n-1)/2}\Gamma(\frac{n-1}{2})\Gamma(1-s)D_n^*(\frac{n+1}{2}-s,-1)\\ =\ &2^{2s-n-1/2}(-1)^{n(n+2)/8+1}(2\pi)^{-s}\Gamma(s)2^{-(n-1)/2}\\ &\times(\Gamma((3-n)/2)^{-1}D_n(s,1)\Gamma(1-s)\Gamma(s-\frac{n-1}{2})+\Gamma(\frac{n-1}{2})D_n(s,-1)\Gamma(s)^{-1}\Gamma(s-\frac{n-1}{2})).\end{aligned}$$

Since $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s$, and $\Gamma(\frac{n-1}{2})\Gamma(\frac{3-n}{2})=\pi/\sin(\frac{n-1}{2}\pi)=\pi(-1)^{n/2+1}$, we get

Proposition 3.4 *For even $n\geq 4$, we get*

$$\begin{aligned}D_n^*(\frac{n+1}{2}-s,-1) &= 2^{-n/2}\pi^{-2s+(n-1)/2}(-1)^{n(n+2)/8+1}\Gamma(s)\Gamma(s-\frac{n-1}{2})\\ &\quad\times((\sin\pi s)D_n(s,-1)+(-1)^{n/2+1}D_n(s,1)).\end{aligned}$$

As for the necessary modification in case $n=2$, see section 5.

For later use, we shall give poles and residues of $D_n^*(s,\delta)$ and $D_n(s,\delta)$ ($\delta=\pm1$.) If $\sigma=0$ and $k=-(n+1)$, then $c_0(y)=1$. So, $\Phi_{0,k}(s)$, or $\Psi_{0,k}(s)$ has simple poles only at $s=0$ and $(n+1)/2$.

Proposition 3.5 *The functions $D_n^*(s,1)$, or $D_n(s,1)$ is holomorphic except for a simple pole at $s=(n+1)/2$ with the residue $(-1)^{[n/4]}(2\pi)^{-n/2}\Gamma(n/2)\zeta(n)$, or $(-1)^{[n/4]}2^{n/2}\pi^{-n/2}\Gamma(n/2)\zeta(n)$, respectively. The functions $D_n^*(s,-1)$, or $D_n(s,-1)$ is holomorphic except for simple pole at $s=1$ and $s=(n+1)/2$. The residue at $s=1$ is $\pi^{-n-1/2}\Gamma(n/2)\Gamma((n-1)/2)\zeta(n-1)$, or $2\pi^{-n-1/2}\Gamma(n/2)\Gamma((n-1)/2)\zeta(n-1)$, and one at $s=(n+1)/2$ is $(-1)^{[n/4]}(2\pi)^{-n/2}\Gamma(n/2)\zeta(n)$ or $(-1)^{[n/4]}2^{n/2}\pi^{-n/2}\Gamma(n/2)\zeta(n)$, respectively.*

Proof. Since $\Phi_{0,-n-1}(s)$, or $\Psi_{0,-n-1}(s)$ has a simple pole at $s=0$, $\Phi_{0,-n-1}(s)/\Gamma(s)$, or $\Psi_{0,-n-1}(s)/\Gamma(s)$ is holomorphic at $s=0$. Hence, $D_n^*(s,1)$, or $D_n(s,1)$ is holomorphic for $s\neq(n+1)/2$. At $s=(n+1)/2$, the residue of $\Phi_{0,-n-1}(s)$, or $\Psi_{0,-n-1}(s)$ is $2^{-n-1/2}(-1)^{n(n+2)/8}$ or $2^{-1/2}(-1)^{n(n+2)/8}$. Hence we get our results for $D_n^*(s,\delta)$ and $D_n(s,\delta)$ for $\delta=1$. Next, we treat the case $\delta=-1$. By the results in section 3.1, for even $n\geq 4$, we see that $\Phi_{2,-n+3}(s)$ and $\Psi_{2,-n+3}(s)$ are holomorphic except for $s=0$, 1, $(n-1)/2$, and $(n+1)/2$ and $\Gamma(s)D_n^*(s,1)$ or $\Gamma(s)D_n(s,1)$ has simple poles only at $s=0$, $(n+1)/2$. Since $\Gamma(s-(n-1))/2)^{-1}I(s,(n-1)/2,1)$ is entire, it is easy to see that $D_n^*(s,-1)$ and $D_n(s,-1)$ are holomorphic at $s\neq 0$, 1, and $(n+1)/2$. Now, by Proposition 3.2 and the above result, we get

$$D_n^*(0,1)=D_n(0,1)=(-1)^{n(n+2)/8+1+[n/4]}\pi^{-n-1/2}\Gamma(n/2)\Gamma((n+1)/2))\zeta(n).$$

Since the residue of $\Phi_{2,-n+3}(s)$ and $\Psi_{2,-n+3}(s)$ at $s=0$ is -1 and $I(0,\frac{n-1}{2},1)=-2/(n-1)$, we can see that $D_n^*(s,-1)$ and $D_n(s,-1)$ are holomorphic at $s=0$. The residue of $\Phi_{2,-n+3}(s)$, or $\Psi_{2,-n+3}(s)$ at $s=(n+1)/2$ is $2^{-n-1/2}(-1)^{n(n+2)/8+1}$, or $2^{-1/2}(-1)^{(n(n+2)/8+1}$, and at $s=1$, it is $2^{-(n+1)/2}\zeta(n-1)\zeta(n)^{-1}$, or $2^{(1-n)/2}\zeta(n-1)\zeta(n)^{-1}$, respectively. Using $I((n+1)/2,(n-1)/2,1)=2(1-2^{(1-n)/2})/(n-1)$ and the estimates of residues of $D_n(s,1)$ and $D_n^*(s,1)$ at $s=(n+1)/2$ just obtained, we can easily get the residues for $\delta=-1$. q.e.d.

4 Proof of functional equations for $n\geq 3$

The functional equations for the case where $n\geq 3$, as well as the residues are easily obtained just from the well known facts on the Riemann zeta function and results on $D^*(s,\delta)$ and $D(s,\delta)$ in the last section. We carry out this calculation in this section. By using the following well known functional equation

$$\zeta(1-s)=\pi^{-s-1/2}(\cos\frac{\pi s}{2})\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})\zeta(s)=\pi^{-s}2^{1-s}(\cos\frac{\pi s}{2})\Gamma(s)\zeta(s),$$

we get

$$\begin{aligned}\zeta(2(\frac{n+1}{2}-s)-(2i-1)) &= \pi^{-2s+n-2i+1/2}\Gamma(s+i-\frac{n}{2})\Gamma(s+i-\frac{n+1}{2})\zeta(2s+2i-n-1)\\ &\quad\times\begin{cases}(-1)^{i-(n+1)/2}\cos\pi s & \text{if } n \text{ is odd,}\\ (-1)^{i-n/2}\sin\pi s & \text{if } n \text{ is even,}\end{cases}\end{aligned}$$

$$\begin{aligned}\zeta(2(\frac{n+1}{2}-s)-2i) &= \pi^{-2s+n-2i-1/2}\Gamma(s+i-\frac{n}{2})\Gamma(s+i-\frac{n-1}{2})\zeta(2s+2i-n) \\ &\quad \times \begin{cases} (-1)^{i-(n-1)/2}\sin\pi s & \text{if } n \text{ is odd,} \\ (-1)^{i-n/2}\cos\pi s & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

Hence, for even $n \geq 4$, we get

$$\begin{aligned}A_n(\frac{n+1}{2}-s) &= (-1)^{n(n-2)/8}\pi^{-(n-2)s+\frac{n(n-3)}{4}+\frac{1}{2}}(\cos\pi s)^{\frac{n}{2}-1}\frac{\gamma_n(s-\frac{n+1}{2})}{\Gamma(s)\Gamma(s-\frac{n-1}{2})}A_n(s) \\ B_n(\frac{n+1}{2}-s) &= (-1)^{n(n-2)/8}\pi^{-ns+\frac{n(n-1)}{4}}\gamma_n(s-\frac{n+1}{2})(\sin\pi s)^{\frac{n}{2}}B_n(s).\end{aligned}$$

Using Proposition 3.2 and 3.3, we get

$$\begin{aligned}D_n^*(\frac{n+1}{2}-s,1)A_n(\frac{n+1}{2}-s) &= 2^{-n/2}\pi^{-ns+\frac{n(n-1)}{4}}(\cos\pi s)^{n/2} \\ &\quad \times\gamma_n(s-\frac{n+1}{2})A_n(s)D_n(s,1), \\ D_n^*(\frac{n+1}{2}-s,-1)A_n(\frac{n+1}{2}-s) &= 2^{-n/2}\pi^{-ns+\frac{n(n-1)}{4}}\gamma_n(s-\frac{n+1}{2})A_n(s) \\ &\quad \times((-1)^{n^2/4+1}(\cos\pi s)^{n/2-1}(\sin\pi s)D_n(s,-1) \\ &\quad +(\cos\pi s)^{n/2-1}D_n(s,1)).\end{aligned}$$

For odd $n \geq 3$, we get

$$\begin{aligned}Q_n(\frac{n+1}{2}-s) &= \pi^{-ns+n(n-1)/4}2^{1-s}(-1)^{(n^2-1)/8}\gamma_n(s-\frac{n+1}{2})(\cos\frac{\pi s}{2})(\cos\pi s)^{(n-1)/2}R_n(s), \\ R_n(\frac{n+1}{2}-s) &= \pi^{-ns+n(n-1)/4}2^{(n+1)/2-s}\gamma_n(s-\frac{n+1}{2})(\sin\pi s)^{(n-1)/2}Q_n(s) \\ &\quad \times\begin{cases}\cos\frac{\pi s}{2} & \text{if } n\equiv 1 \bmod 4, \\ \sin\frac{\pi s}{2} & \text{if } n\equiv 3 \bmod 4.\end{cases}\end{aligned}$$

Now, the functional equations in section 1 are proved as follows. We put $\epsilon_0 = \epsilon\delta^{(n+1)/2}$.

When n is odd, then for $L = L_n^*$ or L_n, the zeta function $\zeta(s, L, \epsilon, \delta)$ depends only on ϵ_0. For the sake of simplicity, we put $\zeta(s, L, \epsilon, \delta) = \zeta^+(s, L)$, or $\zeta^-(s, L)$, if $\epsilon = 1$, or -1, respectively. We have

$$\begin{aligned}\zeta(\frac{n+1}{2}-s, L_n^*, \delta, \epsilon) &= b_n 2^{(n-1)((n+1)/2-s)}(Q_n(\frac{n+1}{2}-s)+\epsilon_0(-1)^{(n^2-1)/8}R_n(\frac{n+1}{2}-s)) \\ &= \pi^{-ns+(n(n-1)/4}2^{-ns+(n^2+1)/2}\gamma(s-\frac{n+1}{2}) \\ &\quad \times(\cos\frac{\pi s}{2}(\cos\pi s)^{(n-1)/2}R_n(s) \\ &\quad +(-1)^{(n^2-1)/8}2^{(n-1)/2}\epsilon_0(\sin\pi s)^{(n-1)/2}f(s)Q_n(s)),\end{aligned}$$

where we denote $f(s) = \cos\frac{\pi s}{2}$, or $\sin\frac{\pi s}{2}$, if $n \equiv 1 \bmod 4$, or $3 \bmod 4$, respectively. We get the following equalities easily from the definitions.

$$\begin{aligned} Q_n(s) &= b_n^{-1}2^{-(n+1)/2}(\zeta^+(s,L_n)+\zeta^-(s,L_n)), \\ R_n(s) &= b_n^{-1}2^{-1}(\zeta^+(s,L_n)-\zeta^-(s,L_n)). \end{aligned}$$

Hence, we get

$$\begin{aligned} \zeta^{\pm}(\frac{n+1}{2}-s,L_n^*) &= (2\pi)^{-ns}\pi^{n(n-1)/4}2^{(n^2-1)/2}\gamma_n(s-\frac{n+1}{2}) \\ &\times(\cos\frac{\pi s}{2}(\cos\pi s)^{(n-1)/2} \pm (-1)^{(n^2-1)/8}(\sin\pi s)^{(n-1)/2}f(s))\zeta^+(s,L_n) \\ &+(-\cos\frac{\pi s}{2}(\cos\pi s)^{(n-1)/2} \pm (-1)^{(n^2-1)/8}(\sin\pi s)^{(n-1)/2}f(s))\zeta^-(s,L_n). \end{aligned}$$

Thus, we prove Theorem 1.

Next, we treat the case of even n. We prove the functional equation for each δ and n.

Case(I): We take $\delta = 1$ and $n \equiv 0 \bmod 4$ with $n \geq 4$. Then, $\delta_n = 1$ and

$$\begin{aligned} b_n(-1)^{[n/4]}D_n(s,1)A_n(s) &= 2^{-1}(\zeta(s,L_n,1,1)+\zeta(s,L_n,1,-1)), \\ b_n(-1)^{n(n+2)/8}\frac{2|B'_{n/2}|}{n}B_n(s) &= 2^{-n/2-1}(\zeta(s,L_n,1,1)-\zeta(s,L_n,1,-1)). \end{aligned}$$

So, we get

$$\begin{aligned} &\zeta(\frac{n+1}{2}-s,L_n^*,1,\epsilon) = \\ &2^{-ns+\frac{n^2}{2}-1}\pi^{-ns+\frac{n(n-1)}{4}}\gamma_n(s-\frac{n+1}{2}) \times \\ &((\cos\pi s)^{n/2} \times (\zeta(s,L_n,1,1)+\zeta(s,L_n,1,-1)) \\ &+\epsilon(-1)^{n(n-2)/8}(\sin\pi s)^{n/2} \times (\zeta(s,L_n,1,1)-\zeta(s,L_n,1,-1))). \end{aligned}$$

Case(II): We take $\delta = -1$ and $n \equiv 0 \bmod 4$. Then $\delta_n = 0$ and the zeta functions do not depend on ϵ.

$$\begin{aligned} &\zeta(\frac{n+1}{2}-s,L_n^*,-1,\epsilon) = \\ &2^{-ns-n^2/2}\pi^{-ns+\frac{n(n-1)}{4}}\gamma_n(s-\frac{n+1}{2}) \\ &\times((-1)^{n^2/4+1}(\cos\pi s)^{n/2-1}(\sin\pi s)\zeta(s,L_n,-1,\epsilon) \\ &+(\cos\pi s)^{n/2-1}2^{-1}(\zeta(s,L_n,1,1)-\zeta(s,L_n,1,-1)). \end{aligned}$$

Case(III): We take $\delta = 1$, $n \equiv 2 \bmod 4$. Then, the zeta functions do not depend on ϵ. In this case, we have

$$\zeta(\frac{n+1}{2}-s,L_n^*,1,\epsilon) = 2^{n(\frac{n+1}{2}-s)}2^{-n/2}\pi^{-ns+\frac{n(n-1)}{4}} \times(\cos\pi s)^{n/2}\gamma_n(s-\frac{n+1}{2})\zeta(s,L_n,1,\epsilon).$$

Case(IV): We put $\delta=-1$ and $n\equiv 2 \bmod 4$. In this case, $\delta_n=1$, and

$$\begin{aligned}
&\zeta(\frac{n+1}{2}-s,L_n^*,-1,\epsilon)=\\
&2^{-ns-n^2/2}\pi^{-ns+\frac{n(n-1)}{4}}\gamma_n(s-\frac{n+1}{2})\\
&\times((-1)^{n^2/4+1}2^{-1}(\cos\pi s)^{n/2-1}(\sin\pi s)(\zeta(s,L_n,-1,1)+\zeta(s,L_n,-1,-1)\\
&+(-1)^{n(n-2)/8}2^{-1}\epsilon(\sin\pi s)^{n/2}(\zeta(s,L_n,-1,1)-\zeta(s,L_n,-1,-1))\\
&+(\cos\pi s)^{n/2-1}\zeta(s,L_n,1,\epsilon)).
\end{aligned}$$

Hence, we proved Theorem 3.

The results on residues are easily obtained as follows. As is well known, $\zeta(s)$ has a simple pole at $s=1$ with residue 1 and is holomorphic for the other s. Also we have $\zeta(-2m)=0$ for all natural integers m. Theorem 2 follows very easily by these facts and the explicit formula of the zeta functions. To prove Theorem 4, we use Proposition 3.5.

5 Functional equation for $n=2$

When $n=2$, we need some modification of the definition of our zeta functions, since the volume $\mu(x)$ in the definition of zeta functions for $n\geq 3$ is infinite for some $x\in V$ when $n=2$. Shintani has given a modified definition of the zeta function and has shown the functional equations for such zeta functions. Later, F. Sato [11] gave a more theoretical definition of the above Shintani's modification by using a kind of Eisenstein series of $O(2,1)$ (a kind of zeta function of two variables) and gave an alternative proof of the functional equation. Here we give the third alternative proof of Shintani's results above by using real analytic Eisenstein series of half integral weight. That is, we give a natural modification of the definition from our stand point, show the functional equation of our zeta functions, and give the location of poles and the principal parts of the Laurent expansion there. As for the zeta function of definite symmetric matrices (i.e. $\zeta_2(s,L,1,1)$), this approach has been already used by J. Sturm [18]. So the main point is the case for indefinite symmetric matrices (i.e. $\zeta(s,L,-1,-1)$.) Our idea is as follows. When we fix an odd integer k, the functional equation of the Mellin transforms of Eisenstein series in the previous section is valid as a meromorphic function of s and σ for whole $\mathbf{C}^2$ plane. Now, we fix $k=1$ from now on throughout this section. Then each side of the functional equation have a pole at $\sigma = 2$ and this is the reason that our original definition of our zeta function cannot be applied in case $n = 2$. That is, the most part of the coefficients of the above Dirichlet series with respect to s have definite values at $\sigma = 2$, which form the main part of our zeta function, but the modified terms come from the coefficients which have pole at $\sigma = 2$. That is, we can compare the constant terms of the Laurent expansion of both sides of the functional equation associated with Eisenstein series at $\sigma = 2$, and get the modification of zeta functions and the functional equation at the same time. By the way, it is easy to see that the residue of both sides at $\sigma = 2$ is

$$C_{2,1}\zeta(2)^{-1}2^{s-2}\pi^{1/2-s}\Gamma(s-1/2)\zeta(2s-1).$$

and this gives you nothing new. We define the main part of our zeta functions. Define four Dirichlet series

$$\begin{aligned}
\xi_+^M(s) &= \zeta(2s)\sum_{d=1}^{\infty} h(d)\log(\epsilon_d)d^{-s}, \\
\xi_+^{*,M}(s) &= \zeta(2s)\{\sum_{d=1}^{\infty} h(4d)\log(\epsilon_{4d})(4d)^{-s} + 2^{-2s}\sum_{d=1}^{\infty} h(4d+1)\log(\epsilon_{4d+1})(4d+1)^{-s}\}, \\
\xi_-(s) &= 2\zeta(2s)\sum_{d=1}^{\infty} h(-d)w_{-d}^{-1}|d|^{-s}, \\
\xi_-^*(s) &= 2\zeta(2s)\{\sum_{d=1}^{\infty} h(-4d)w_{-4d}^{-1}|4d|^{-s} + 2^{-2s}\sum_{d=1}^{\infty} h(-4d+1)w_{-4d+1}^{-1}(4d-1)^{-s}\},
\end{aligned}$$

where $h(d)$ is the class number in the narrow sense of the order $\mathcal{O}_d$ of conductor d in a quadratic field and ϵ_d is the fundamental unit of norm 1 of $\mathcal{O}_d$ and w_d is the number of units of $\mathcal{O}_d$.

Then, by the classical class number formula, we can show that

$$\begin{aligned}
\xi_+^M(s) &= \sum_{d_K>0, d_K\neq 1} \zeta(2)Z^*(1,2,s,d_K), \\
\pi\xi_-(s) &= \sum_{d_K<0} \zeta(2)Z^*(1,2,s,d_K).
\end{aligned}$$

Now, put

$$\begin{aligned}
h(\sigma,s) &= \frac{\zeta(2s)\zeta(2s-2\sigma+3)}{\zeta(2\sigma-2)\zeta(2s-\sigma+2)} I(s,\sigma/2,(\sigma-1)/2) \\
h^*(\sigma,s) &= \frac{\zeta(2s)\zeta(2s-2\sigma+3)}{\zeta(2\sigma-2)\zeta(2s-\sigma+2)} \frac{(1-2^{\sigma-2}+2^{2\sigma-3}-2^{2s+2\sigma-3})}{(1-2^{2s+\sigma-2})} \times I(s,\sigma/2,(\sigma-1)/2).
\end{aligned}$$

We define two functions $g(\sigma,s)$ and $g^*(\sigma,s)$ so that the following equalities hold.

$$\begin{aligned}
\Phi_{\sigma,1}(s) &= C_{\sigma,1}(2\pi)^{-s}\Gamma(s)\{g(\sigma,s)+\zeta(\sigma-1)h(\sigma,s)\} \\
\Psi_{\sigma,1}(s) &= C_{\sigma,1}(2\pi)^{-s}\Gamma(s)\{g^*(\sigma,s)+\zeta(\sigma-1)h^*(\sigma,s)\}.
\end{aligned}$$

Then, we get

$$\begin{aligned} g(2,s) &= \zeta(2)^{-1}(\xi_+^M(s)I(s,1,1/2)+\pi\xi_-(s)I(s,1/2,1)) \\ g^*(2,s) &= 2^{2s}\zeta(2)^{-1}(\xi_+^{*,M}(s)I(s,1,1/2)+\pi\xi_-^*(s)I(s,1/2,1)). \end{aligned}$$

To obtain the functional equation which coincides with Shintani's original version, it is convenient to multiply by

$$\epsilon(\sigma,s) = C_{\sigma,1}^{-1}(2\pi)^{\sigma-s-1/2}2^{(\sigma-1)/2}\zeta(2\sigma-2)\Gamma((\sigma-1)/2)^{-1}\Gamma(1-s)^{-1}$$

both sides of the functional equation between $\Phi_{\sigma,1}(s)$ and $\Psi_{\sigma,1}(s)$ in the previous section. To rewrite the functional equation in this form, we put

$$\begin{aligned} A(\sigma,s) &= 2^{(\sigma-1)/2}\zeta(2\sigma-2)\Gamma((\sigma-1)/2)^{-1} \\ &\quad \times\frac{\Gamma(\sigma-s-1/2)}{\Gamma(1-s)}g(\sigma,\sigma-s-1/2) \\ A^*(\sigma,s) &= 2^{(\sigma-1)/2}\zeta(2\sigma-2)\Gamma((\sigma-1)/2)^{-1} \\ &\quad \times(2\pi)^{\sigma-2s-1/2}2^{2s-2\sigma+3/2}\frac{\Gamma(s)}{\Gamma(1-s)}g^*(\sigma,s) \\ B(\sigma,s) &= \frac{\Gamma(\sigma-s-1/2)}{\Gamma((\sigma+1)/2-s)}\frac{\zeta(2\sigma-1-2s)\zeta(2-2s)}{\zeta(\sigma+1-2s)} \\ &\quad \times F_2(\sigma,s) \\ B^*(\sigma,s) &= 2^{-\sigma+1}\pi^{\sigma-2s-1/2}\frac{\Gamma(s)\Gamma(s-\sigma+3/2)}{\Gamma(1-s)\Gamma(s-\sigma/2+1)} \\ &\quad \times\frac{\zeta(2s)\zeta(2s-2\sigma+3)}{\zeta(2s-\sigma+2)}\times\frac{1-2^{\sigma-2}+2^{2\sigma-3}-2^{-2s+2\sigma-3}}{1-2^{-2s+\sigma-2}} \\ &\quad \times F_1(\sigma,s), \end{aligned}$$

where, for the sake of simplicity, we put

$$\begin{aligned} F_1(\sigma,s) &= F(1-\frac{\sigma}{2},\frac{\sigma-1}{2};s-\frac{\sigma}{2}+1;\frac{1}{2}), \\ F_2(\sigma,s) &= F(1-\frac{\sigma}{2},\frac{\sigma-1}{2},\frac{\sigma+1}{2}-s;\frac{1}{2}). \end{aligned}$$

Then, the functional equation reads as

$$A(\sigma,s)+B(\sigma,s)\zeta(\sigma-1) = A^*(\sigma,s)+\zeta(\sigma-1)B^*(\sigma,s).$$

Comparing the constant terms of the both sides at $\sigma=2$, we get

$$\begin{aligned} & A(2,s)+\frac{d}{d\sigma}B(\sigma,s)|_{\sigma=2} \\ =\; & A^*(2,s)+\frac{d}{d\sigma}B^*(\sigma,s)|_{\sigma=2}. \end{aligned}$$

As is easily shown, we get

$$\begin{aligned}
A(2,s) &= \xi_+^M(3/2-s) \\
&\quad +\sqrt{2\pi}\frac{\Gamma(3/2-s)}{\Gamma(1-s)}\xi_-(3/2-s)I(3/2-s,1/2,1), \\
A^*(2,s) &= 2^{2s-1}\pi^{1/2-2s}\Gamma(s)\Gamma(s-1/2) \\
&\quad \times\{\sin(\pi s)\xi_+^{*,M}(s)+\pi^{3/2}\sqrt{2}\frac{\xi_-^*(s)}{\Gamma(s-1/2)\Gamma(1-s)}I(s,1/2,1)\}.
\end{aligned}$$

It has been already proved in the previous section that

$$\begin{aligned}
\xi_-(3/2-s) &= 2^{2s-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\cos(\pi s)\xi_-^*(s) \\
&\quad -2^{-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1).
\end{aligned}$$

Using this functional equation and the well known fact

$$\cos(\pi s) = -\pi/\Gamma(s-1/2)\Gamma(3/2-s),$$

we get

$$\begin{aligned}
A(2,s) &= \xi_+^M(3/2-s) \\
&\quad -2^{2s-1/2}\pi^{2-2s}\Gamma(s)\Gamma(1-s)^{-1}\xi_-^*(s)I(3/2-s,1/2,1) \\
&\quad -2^{-1/2}\pi^{1-2s}\Gamma(s)\Gamma(s-1/2)\Gamma(3/2-s)\Gamma(1-s)^{-1} \\
&\quad \times\zeta(2s-1)I(3/2-s,1/2,1).
\end{aligned}$$

Hence, by using the formula

$$I(s,1/2,1)+I(3/2-s,1/2,1) = \frac{\Gamma(1-s)\Gamma(s-1/2)}{\sqrt{2\pi}},$$

and applying the functional equation, we get

$$\begin{aligned}
&\xi_+^M(3/2-s)+\frac{d}{d\sigma}B(\sigma,s)|_{\sigma=2} \\
= \ &2^{2s-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\{\sin(\pi s)\xi_+^{*,M}(s)+\pi\xi_-^*(s)\} \\
&+\frac{\pi^{1-2s}}{\sqrt{2}}\frac{\Gamma(s)\Gamma(s-1/2)\Gamma(3/2-s)}{\Gamma(1-s)}\zeta(2s-1)I(\frac{3}{2}-s,\frac{1}{2},1)+\frac{d}{d\sigma}B^*(\sigma,s)|_{\sigma=2}.
\end{aligned}$$

Now, the problem is to calculate the difference

$$\frac{d}{d\sigma}B^*(\sigma,s)|_{\sigma=2}-\frac{d}{d\sigma}B(\sigma,s)|_{\sigma=2}.$$

When we differentiate $B(\sigma,s)$ and $B^*(\sigma,s)$, roughly we regard these as products of functions of σ which are powers of two, the shifted Riemann zeta functions, gamma factors, and hypergeometric functions. We get

$$
\begin{aligned}
\frac{d}{d\sigma}B(\sigma,s)|_{\sigma=2} &= \zeta(2-2s)\{\frac{\zeta^{'}(3-2s)}{\zeta(3-2s)}-\frac{\zeta^{'}(2-2s)}{\zeta(2-2s)}\}+\zeta^{'}(2-2s) \\
&+\frac{1}{2}\zeta(2-2s)\frac{\Gamma^{'}(3/2-s)}{\Gamma(3/2-s)} \\
&+\zeta(2-2s)\frac{d}{d\sigma}F_2(\sigma,s)|_{\sigma=2}.
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{d\sigma}B^{*}(\sigma,s)|_{\sigma=2} &= \pi^{3/2-2s}\frac{\Gamma(s-1/2)}{\Gamma(1-s)}\{\zeta(2s-1)\{\frac{\zeta^{'}(2s)}{\zeta(2s)}-\frac{\zeta^{'}(2s-1)}{\zeta(2s-1)}\}-\zeta^{'}(2s-1)\} \\
&+(\log\pi)\pi^{3/2-2s}\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(1-s)} \\
&+(\log 2)\pi^{3/2-2s}\frac{\Gamma(s-1/2)}{\Gamma(1-s)}\frac{\zeta(2s-1)}{2(1-2^{-2s})} \\
&+2^{-1}\pi^{1/2-2s}\zeta(2s-1)\Gamma(s)\Gamma(s-1/2)(\sin\pi s)\{\frac{\Gamma^{'}(s)}{\Gamma(s)}-\frac{\Gamma^{'}(s-1/2)}{\Gamma(s-1/2)}\} \\
&-2^{-1}\pi^{1/2-2s}\zeta(2s-1)\Gamma(s)\Gamma(s-1/2)(\sin\pi s)\frac{\Gamma^{'}(s-1/2)}{\Gamma(s-1/2)} \\
&+\pi^{3/2-2s}\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(1-s)}\frac{d}{d\sigma}F_1(\sigma,s)|_{\sigma=2}.
\end{aligned}
$$

By the functional equation of the Riemann zeta function and the well known formula $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$, we get

$$
\begin{aligned}
\zeta(2-2s) &= \pi^{3/2-2s}\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(1-s)} \\
&= \pi^{1/2-2s}(\sin\pi s)\zeta(2s-1)\Gamma(s)\Gamma(s-1/2).
\end{aligned}
$$

Now we would like to calculate

$$
\zeta(2-2s)\frac{d}{d\sigma}(F_1(\sigma,s)-F_2(\sigma,s))|_{\sigma=2}.
$$

The following formula of the hypergeometric functions is well known.

$$
\begin{aligned}
F(a,b,c,z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z) \\
&+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z).
\end{aligned}
$$

So, we get

$$\begin{aligned}F_1(\sigma,s) &= \frac{\Gamma(s-\sigma/2+1)\Gamma(s-(\sigma-1)/2)}{\Gamma(s)\Gamma(s-\sigma+3/2)}F_2(\sigma,s)\\ &\quad +2^{-s+(\sigma-1)/2}\frac{\Gamma(s-\sigma/2+1)\Gamma(-s+(\sigma-1)/2)}{\Gamma(1-\sigma/2)\Gamma((\sigma-1)/2)}\\ &\quad \times F(s,s-\sigma+3/2,s-(\sigma-3)/2;\frac{1}{2}).\end{aligned}$$

Now, we differentiate at $\sigma=2$ the both sides of the above formula. Since $\Gamma(1-\sigma/2)^{-1}$ has a simple zero at $\sigma=2$ and $\frac{d}{d\sigma}\Gamma(1-\sigma/2)^{-1}|_{\sigma=2}=-1/2$, the differential coefficient at $\sigma=2$ of the second term of the right hand side is

$$-2^{-1/2-s}\frac{\Gamma(s)\Gamma(1/2-s)}{\Gamma(1/2)}F(s,s-1/2,s+1/2;1/2),$$

but we see

$$\begin{aligned}F(s,s-\frac{1}{2},s+\frac{1}{2}:\frac{1}{2}) &= \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s-\frac{1}{2})}\int_0^1 t^{s-\frac{3}{2}}(1-\frac{1}{2}t)^{-s}dt\\ &= 2^{s-\frac{1}{2}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s-\frac{1}{2})}\int_{1/2}^1(1-v)^{s-\frac{3}{2}}v^{-s}dt\\ &= 2^s\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s-\frac{1}{2})}I(s,\frac{1}{2},1).\end{aligned}$$

Hence we get

$$\begin{aligned}\frac{d}{d\sigma}(F_1-F_2)|_{\sigma=2} &= -\frac{1}{2}(\frac{\Gamma'(s)}{\Gamma(s)}-\frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)})\\ &\quad -(2\pi)^{-1/2}\frac{\Gamma(s)\Gamma(1/2-s)\Gamma(s+1/2)}{\Gamma(s-1/2)}I(s,1/2,1).\end{aligned}$$

Using the notation of Shintani, we define

$$\begin{aligned}\xi_+(s) &= \xi_+^M(s)+\zeta(2s-1)(\frac{\zeta'(2s)}{\zeta(2s)}-\frac{\zeta'(2s-1)}{\zeta(2s-1)}),\\ \xi_+^*(s) &= \xi_+^{*,M}(s)+2^{1-2s}\zeta(2s-1)(\frac{\zeta'(2s)}{\zeta(2s)}-\frac{\zeta'(2s-1)}{\zeta(2s-1)}),\\ &\quad +2^{-2s}(\log 2)(1-2^{-2s})^{-1}\zeta(2s-1).\end{aligned}$$

We prepare two more relations. By differentiating the usual functional equation of the Rieman zeta function, we get

$$\frac{\zeta'(2-2s)}{\zeta(2-2s)}=\log\pi-\frac{1}{2}\frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)}-\frac{1}{2}\frac{\Gamma'(1-s)}{\Gamma(1-s)}-\frac{\zeta'(2s-1)}{\zeta(2s-1)}.$$

On the other hand, differentiating the relations $\Gamma(s)\Gamma(1-s) = \pi/\sin\pi s$ and $\Gamma(s-1/2)\Gamma(3/2-s) = -\pi/\cos\pi s$, we get

$$\frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} = \frac{\pi}{(\cos\pi s)(\sin\pi s)} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(3/2-s)}{\Gamma(3/2-s)}.$$

By comparing all the equalities we obtained above, we can show that

$$\begin{aligned}\xi_+(3/2-s) &= 2^{2s-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)((\sin\pi s)\xi_+^*(s) + \pi\xi_-^*(s)) \\ &\quad +2^{-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1)(\sin\pi s)(\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)}).\end{aligned}$$

This is completely the same as Shintani's functional equation.

Next, for the sake of completeness, we shall calculate principal part of the Laurent expansion at poles of $\xi_+(s)$ and $\xi_+^*(s)$ by our method. For that purpose, we need more precise information on the location of poles and their residues of the functions which appear in $\Phi_{\sigma,1}(s)$ and $\Psi_{\sigma,1}(s)$.

We define

$$I_2(s) = \frac{\Gamma(3/2-s)}{2\sqrt{\pi}\Gamma(1-s)}\int_0^1 u^{-1/2}(1-u)^{-s}\log(1-u/2)\,du.$$

This function converges when $\mathrm{Re}(s) < 1$. Differentiating the integral expression of the beta function

$$B(t+\frac{1}{2}, 1-s) = \frac{\Gamma(t+1/2)\Gamma(1-s)}{\Gamma(t+3/2-s)} = \int_0^1 u^{t-1/2}(1-u)^{-s}\,du$$

at $t = 0$ under the integral sign, we get

$$\frac{\Gamma(3/2-s)}{\sqrt{\pi}\Gamma(1-s)}\int_0^1 u^{-1/2}(1-u)^{-s}\log(u)\,du = \frac{\Gamma'(1/2)\Gamma(3/2-s) - \Gamma(1/2)\Gamma'(3/2-s)}{\sqrt{\pi}\Gamma(3/2-s)}.$$

Using this relation, and differentiating under the integral sign, we get

$$\begin{aligned}\frac{d}{d\sigma}(F_1(\sigma,s))|_{\sigma=2} &= I_2(s) \text{ and} \\ \frac{d}{d\sigma}(F_2(\sigma,s))|_{\sigma=2} &= I_2(\frac{3}{2}-s).\end{aligned}$$

As before, for a non negative integer n and a variable number a, we put $(a)_n = \Gamma(a+n)/\Gamma(a)$. We define the Barnes generalized hypergeometric function ${}_3F_2$ as usual by

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{(\beta_1)_n(\beta_2)_n}\frac{z^n}{n!}.$$

Then, we also get

$$
\begin{aligned}
I_2(s) &= -\sum_{n=1}^{\infty}\frac{1}{2^{n+1}n}\frac{\Gamma(n+1/2)\Gamma(3/2-s)}{\sqrt{\pi}\Gamma(n+3/2-s)} \\
&= -\frac{\Gamma(3/2-s)}{2^3\Gamma(5/2-s)}{}_3F_2(1,1,3/2;2,5/2-s;1/2).
\end{aligned}
$$

Since $I_2(s)$ is holomorphic for $\mathrm{Re}(s) < 1$ and $I_2(3/2-s)$ for $\mathrm{Re}(s) > 1/2$, $I_2(s)$ is meromorphically continued to the whole s plane by virtue of the formula for $I_2(3/2-s)-I_2(s)$ we gave before. It is also easy to show that $I_2(s)$ has simple poles at $s = 1/2+m$ $(m = 1,2,\dots)$ with residues

$$
(-1)^{m+1}2^{-1/2-2m}(2m)!(m!)^{-2}mI(1/2+m,1/2,1),
$$

and holomorphic for the other s. We get

$$
\begin{aligned}
\frac{d}{d\sigma}B(\sigma,s)|_{\sigma=2} &= \zeta(2-2s)\Big(\frac{\zeta^{'}(3-2s)}{\zeta(3-2s)}-\frac{\zeta^{'}(2-2s)}{\zeta(2-2s)}\Big)+\zeta^{'}(2-2s) \\
&\quad +\frac{1}{2}\zeta(2-2s)\frac{\Gamma^{'}(3/2-s)}{\Gamma(3/2-s)}+\zeta(2-2s)I_2(s).
\end{aligned}
$$

It was shown in section 3 that the function $(\sigma-2)s(s-\sigma+1/2)(s-\sigma+3/2)\epsilon(\sigma,s)\Phi_{\sigma,1}(\sigma-1/2-s)$ is holomorphic on $(\sigma,s)\in U_{3/2}(2)\times\mathbf{C}$, where $U_r(a) = \{\sigma\in\mathbf{C};|\sigma-a|<r\}$. We write the Taylor expansion at $\sigma = 2$ of this function as $\sum_{n=0}^{\infty} f_i(s)(\sigma-2)^n$. The first two terms are calculated explicitly as follows.

$$
\begin{aligned}
f_0(s) &= s(s-1/2)(s-3/2)\zeta(2-2s). \\
f_1(s) &= s(s-1/2)(s-3/2)\Big(A(2,s)+\frac{d}{d\sigma}B(\sigma,s)|_{\sigma=2}+\gamma B(2,s)\Big) \\
&\quad -2s(s-1)B(2,s) \\
&= s(s-1/2)(s-3/2)\times\Big(\xi_+(3/2-s) \\
&\quad +\sqrt{2\pi}\frac{\Gamma(3/2-s)}{\Gamma(1-s)}\xi_-(3/2-s)I(3/2-s,1/2,1)+\zeta^{'}(2-2s) \\
&\quad +\frac{1}{2}\zeta(2-2s)\frac{\Gamma^{'}(3/2-s)}{\Gamma(3/2-s)}+\frac{1}{2}\zeta(2-2s)I_2(s)+\gamma\zeta(2-2s)\Big) \\
&\quad -2s(s-1)\zeta(2-2s).
\end{aligned}
$$

Now, notation being as in section 3, we put

$$
\begin{aligned}
\tilde{J}(\sigma,s) &= 2^{s+1/2-\sigma}\epsilon(\sigma,s) \\
&\quad \times\Big(\int_1^{\infty}(G_1(y/2)-c_0(y/2))y^{\sigma-3/2-s}\,dy+C_1\int_1^{\infty}(H_1(y/2)-c_0(y/8))y^{s-1}\,dy\Big).
\end{aligned}
$$

Then, $(\sigma-2)\Gamma(1-s)\tilde{J}(\sigma,s)$ is holomorphic on $(\sigma,s)\in U_{3/2}(2)\times\mathbf{C}$. By the formula in section 3, we get

$$\begin{aligned}A(\sigma,s)+\zeta(\sigma-1)B(\sigma,s) &= \tilde{J}(\sigma,s)-\pi^{1-s}\frac{\Gamma(\sigma-3/2)\zeta(2\sigma-3)}{\Gamma((\sigma-1)/2)\Gamma(1-s)}\times(\frac{2^{\sigma/2-1}}{s-\sigma+3/2}+\frac{2^{1-\sigma/2}}{1-s})\\ &\quad -\pi^{-s}\frac{\Gamma(\sigma/2)\zeta(2\sigma-2)}{\Gamma(1-s)}\times(\frac{2^{(\sigma-3)/2}}{s}+\frac{2^{(3\sigma-5)/2}}{\sigma-1/2-s}).\end{aligned}$$

We write the Laurent expansion of $\tilde{J}(\sigma,s)$ at $\sigma=2$ as $\tilde{J}(\sigma,s)=J_{-1}(s)/(\sigma-2)+J_0(s)+\cdots$. Calculating the first few terms of the Taylor expansion at $\sigma=2$ of the product of the right hand side of the above equality and $(\sigma-2)s(s-\sigma+1/2)(s-\sigma+3/2)$, we get the following expression of $f_i(s)$ $(i=0,1)$.

$$\begin{aligned}f_0(s) &= -2^{-2}\pi^{1-s}\frac{s(s-3/2)}{(1-s)\Gamma(1-s)}+s(s-1/2)(s-3/2)J_{-1}(s).\\ f_1(s) &= -2s(s-1)J_{-1}(s)+s(s-1/2)(s-3/2)J_0(s)\\ &\quad +\frac{\sqrt{2}\pi^{2-s}}{12}\frac{(s-1/2)(s+3/2)}{\Gamma(1-s)}\\ &\quad -\frac{\pi^{1-s}}{8(1-s)\Gamma(1-s)}(3\gamma s(s-3/2)+s(s-3/2)(1-4s)\log(2)+4s(1-s)),\end{aligned}$$

where γ is the Euler constant and $\gamma=-\Gamma'(1)$. First, we calculate the Laurent expansion of $\xi_+(s)$ at $s=1$. It is clear that the order of the pole of $\xi_+(s)$ at $s=1$ is at most two. We write $\xi_+(s)=a_{-2}(s-1)^{-2}+a_{-1}(s-1)^{-1}+O(1)$. Since $\zeta'(2-2s)=-(1-2s)^{-2}+O(1))$ and $-2s(s-1)\zeta(2-2s)=2^{-1}(1-2s)^{-1}+O(s-1/2)$, and since $f_1(s)$ is holomorphic, we get $-2^{-1}(a_{-2}-4^{-1})-4^{-1}=0$, hence $a_{-2}=-1/4$.

Comparing the values at $s=1/2$ of the both expression of $f_1(s)$, we can calculate a_{-1} as follows. From the first equality above, we get

$$\begin{aligned}J_{-1}(s) &= \zeta(2-2s)+\frac{2^{-2}\pi^{1-s}}{(1-s)(s-1/2)\Gamma(1-s)}\\ &= \frac{1}{2}(\gamma-\log(\pi))+1-\log(2)+O(s-1/2).\end{aligned}$$

We easily get

$$f_1(\frac{1}{2})=\frac{5}{8}\gamma-\frac{1}{4}\log(\pi)+\frac{1}{4}-\frac{5}{8}\log(2).$$

On the other hand, we have

$$s(s-1/2)(s-3/2)\xi_+(3/2-s)=+8^{-1}(s-1/2)^{-1}+(4^{-1}+2^{-1}a_{-1})+o(s-1/2),$$

$I(1,1/2,1)=\sqrt{2}\log(\sqrt{2}+1)$, $I_2(1/2)=-2^{-1}3\log(2)+\log(\sqrt{2}+1)$, and $\xi_{-}(3/2-s)=4^{-1}(s-2/1)^{-1}+O(1)$. Hence, we also get

$$f_1(1/2)=\frac{1}{4}+\frac{1}{2}a_{-1}+\frac{5}{8}\gamma-\frac{3}{8}\log(2).$$

Hence, we get $a_{-1}=-2^{-1}\log(2\pi)$. Now, $f_1(0)=-2^{-4}\sqrt{2}\pi^2$. Since $I(3/2,1/2,1)=2^{-1}\int_0^1(1-2^{-1}t)^{-1/2}dt=2-\sqrt{2}$ and the residue of $\xi_{-}(3/2-s)$ at $s=0$ is $-\pi/12$. Hence, if we denote the residue of $\xi_+(s)$ at $s=3/2$ by d, then $f_1(0)=-2^{-2}3d-2^{-5}\pi^2(2-\sqrt{2})$. Hence $d=\pi^2/12$. Now, we shall show that $\xi_+(s)$ is holomorphic for all $s\neq 1,3/2$. Since $\Gamma(1-s)^{-1}I(3/2-s,1/2,1)$ is entire and $I_2(s)$ has poles only at $s=1/2+m$ $(m=1,2,\ldots)$, we see $\xi_+(3/2-s)$ is holomorphic for $s\neq 0,1/2,1/2+m$. Since the residue of $\Gamma(1/2-s)$ at $s=m+1/2$ is $(-1)^m/\Gamma(m+1)$, we see that the residue at $s=m+1/2$ of $I_2(s)/2$ is

$$(-1)^m 2^{-2}\pi^{-1}\Gamma(m)^{-1}\Gamma(m+1/2)I(1/2+m,1/2,1).$$

The residue of $\Gamma'(3/2-s)/\Gamma(3/2-s)$ at $m+1/2$ is $-1/2$. By the functional equation for $\xi_{-}(s)$, we see that $\xi_{-}(1-m)=-2^{-1}(-1)^m\zeta(1-2m)$ for natural numbers $m\geq 2$ and $\xi_{-}(0)=2^{-1}\zeta(-1)+1/12$. We get $I(1-m,1/2,1)+I(m+1/2,1/2,1)=(2\pi)^{-1}\Gamma(1/2-m)\Gamma(m)$ and $I(3/2,1/2,1)=2-\sqrt{2}$, $I(0,1/2,1)=-2$. So, comparing the Taylor expansion at $s=m+1/2$ of the two expression of $f_1(s)$, it is easy to see that $\xi_+(3/2-s)$ is holomorphic also at $s=m+1/2$ for natural numbers m. Thus we obtained all the location of poles of $\xi_+(s)$ and its principal part of the Laurent expansion at each pole. As for $\xi_+^*(s)$, we have

$$\begin{aligned}
&2^{1-2s}\pi^{2s-3/2}\Gamma(1-s)\Gamma(s-1/2)^{-1}f_1(s)\\
=\;& s(s-1/2)(s-3/2)(\xi_+^*(s)+(2\pi)^{1/2}\frac{\Gamma(s)}{\Gamma(s-1/2)}\xi_-^*(s)I(s,1/2,1)\\
&-2^{1-2s}\zeta'(2s-1)+2^{1-2s}\log(\pi)\zeta(2s-1)+2^{1-2s}\gamma\zeta(2s-1)\\
&+2^{-2s}\zeta(2s-1)(\frac{\Gamma'(s)}{\Gamma(s)}-2\frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)})+2^{1-2s}\zeta(2s-1)\times I_2(3/2-s))\\
&-2s(s-1)2^{1-2s}\zeta(2s-1).
\end{aligned}$$

By the functional equation, we get $\xi_-^*(1-m)=(-1)^{m+1}2^{2m-2}\zeta(1-2m)$ for $m=2,3,\ldots$ and $\xi_-^*(0)=0$. Using this, we get the location of poles and the expansion of $\xi_+^*(s)$ at poles almost in the same way as before. We omit the details here. Thus, we get the alternative proof of the following results first obtained by Shintani [16] in somewhat more complicated method.

Proposition 5.1 (Shintani [16], F. Sato [11]) *Two zeta functions $\xi_+(s)$ and $\xi_+^*(s)$ are meromorphically continued to the whole s plane and satisfies the following functional equation.*

$$\begin{aligned}
\xi_+(3/2-s) \;=\;& 2^{2s-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)((\sin\pi s)\xi_+^*(s)+\pi\xi_-^*(s))\\
&+2^{-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1)(\sin\pi s)(\frac{\Gamma'(s)}{\Gamma(s)}-\frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)}).
\end{aligned}$$

Both zeta functions are holomorphic except for $s = 1$ and $s = 3/2$. The principal part of the Laurent expansions are given as follows. At $s = 1$, we get $\xi_+(s) = -4^{-1}(s-1)^{-2} - 2^{-1}\log(2\pi)(s-1)^{-1} + \cdots$ and $\xi_+^*(s) = -8^{-1}(s-1)^{-2} - 4^{-1}\log(2\pi)(s-1)^{-1} + \cdots$. At $s = 3/2$, we get $\xi_+(s) = 12^{-1}\pi^2(s-3/2) + \cdots$, and $\xi_+^*(s) = 24^{-1}\pi^2(s-3/2)^{-1} + \cdots$.

.

6 Special values of zeta functions and dimensions of Siegel cusp forms

The values $\zeta^+(1-m, L_n^*, 1, 1)$ for natural numbers m are related with dimension formulas of Siegel cusp forms and important. In this section, first we shall give the formulas for the values of $\zeta^+(s, L, 1, 1)$ at non positive integers for $L = L_n^*$, or L_n. These are rational, and described by Bernoulli numbers explicitly. Secondly we apply these results to the dimensions of Siegel cusp forms. The results in this section have been announced in [4] before, but we reproduce them with more detailed explanation and the proof.

For each non-negative integer m, we denote by B_m the Bernoulli number defined by

$$\frac{te^t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

Next, we shall give special values of the above zeta functions.

Theorem 5 *When n is odd (including the case $n = 1$), for each natural number m,*

$$\begin{aligned}
\zeta(1-m, L_n^*, 1, 1) &= \frac{(-1)^{(n+1)/2}|B_2B_4\cdots B_{n-1}|B_{m+\frac{n-1}{2}}B_{2m}B_{2m+2}\cdots B_{2m+n-3}}{2^{(2m+1)(n-1)/2}(\frac{n-1}{2})!m(m+1)\cdots(m+\frac{n-1}{2})}, \\
\zeta(1-m, L_n, 1, 1) &= 2^{(2m-1)(n-1)/2}\zeta(1-m, L_n^*, 1, 1).
\end{aligned}$$

Theorem 6 *When n is even, for each natural number m with $nm \neq 2$,*

$$\begin{aligned}
\zeta(1-m, L_n^*, 1, 1) &= \frac{(-1)^{[\frac{n}{4}]+\frac{nm}{2}}|B_2B_4\cdots B_{n-2}B_{n/2}|B_{2m}B_{2m+2}\cdots B_{2m+n-2}}{2^{nm+\frac{n-2}{2}}(\frac{n}{2})!m(m+1)\cdots(m+\frac{n-2}{2})}, \\
\zeta(1-m, L_n, 1, 1) &= 2^{nm-\frac{n}{2}}\zeta(1-m, L_n^*, 1, 1),
\end{aligned}$$

and $\zeta(0, L_2^*, 1, 1) = 1/48$, $\zeta(0, L_2, 1, 1) = 0$. In particular, if $n \equiv 2 \bmod 4$ and $n > 2$, then $\zeta(1-m, L_n^*, 1, 1) = \zeta(1-m, L_n, 1, 1) = 0$.

Proof of Theorem 5, 6. The case when n is odd is obvious, noting the well known relation $\zeta(1-m) = -B_m/m$ for the Riemann zeta function. Now, assume n is even. By Proposition 3.2 and 3.3, $D_n^*(s,1)$ and $D_n(s,1)$ are holomorphic at $1-m$ ($m=1,2,\ldots$). When $n=2$, we have $\zeta(s,L_2^*,1,1) = 2^{2s-1}D_2^*(s,1)$ and $\zeta(s,L_2,1,1) = 2^{-1}D_2(s,1)$. So, our results easily follow from Proposition 3.3. When $n \geq 4$, we have $A_n(1-m) = 0$ and

$$B_n(1-m) = (-1)^{n/2}\prod_{i=1}^{n/2} B_{2m+2i-2}/(2m+2i-2),$$

and we get the special values. q.e.d.

Remark. When $n=2$, these values have been essentially known by the functional equation. When $n=3$, as a special case of the above results, we get $\zeta(0,L_3^*) = \frac{1}{3456}$. This value has been obtained by an indirect method by Hashimoto and Tsushima. They compared two expressions of the dimension of Siegel cusp forms of degree 3, one by the Selberg trace formula and the other by Riemann Roch Theorem.

Next, we shall give an explicit value of the contribution of 'central' unipotent elements to the dimension of Siegel cusp forms. For each natural numbers n and N, we denote by $\Gamma_n(1) = Sp(n,Z)$ the full Siegel modular group of size $2n$ and $\Gamma_n(N)$ the principal congruence subgroup of level N of degree n:

$$\Gamma_n(N) = \{g \in Sp(n,Z); g \equiv 1_{2n} \bmod N\}.$$

For a natural number k, we denote by $S_k(\Gamma_n(N))$ the space of Siegel cusp forms of weight k belonging to $\Gamma_n(N)$. First we review the Godement dimension formula.

Theorem 7 (Godement) *For wny integer $k > 2n$, we get*

$$\dim S_k(\Gamma_n(N)) = \frac{a(k)}{\#Z(\Gamma_n)}\int_{\Gamma_n\backslash H_n}\sum_{\gamma\in\Gamma_n} H_\gamma(Z)dZ,$$

where $dZ = (\det(Y))^{-n-1}dXdY$,

$$H_\gamma(Z) = \det\left(\frac{Z-\gamma\bar{Z}}{2i}\right)^{-k}\det(C\bar{Z}+D)^{-k}(\det Y)^k,$$

and

$$\begin{aligned} a_n(k) &= \frac{1}{2^n(2\pi)^{n(n+1)/2}}\prod_{j=0}^{n-1}\frac{\Gamma(k-\frac{n-1}{2}+\frac{j}{2})}{\Gamma(k-n+\frac{j}{2})} \\ &= \frac{1}{2^{n^2+2n}\pi^{n(n+1)/2}}\prod_{1\leq i\leq j\leq n}(2k-2n+i+j-2). \end{aligned}$$

Now, for each natural number r with $0 \leq r \leq n$, let Π_r be the set of elements u of $\Gamma_n(N)$ such that u is $Sp(n, Z)$-conjugate to a matrix of the form $\begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix}$ where $x \in L_n$ and x is of rank r. When $r = 0$, this is nothing but 1. When $r \geq 1$, we call such unipotent elements 'central', because they are $Sp(n, Q)$-conjugate to some elements of the center of the unipotent radical of a maximal parabolic subgroup of $Sp(n, Q)$. Using Shintani's notation, we denote by $I_n(\Pi_r, N, k)$ the contribution of Π_r to $\dim S_k(\Gamma_n(N))$. That is, we put

$$I_n(\Pi_r, N, k) = \frac{a(k)}{\# Z(\Gamma_n)} \int_{\Gamma_n \backslash H_n} \sum_{\gamma \in \Pi_r} H_\gamma(Z) dZ.$$

Shintani [16] has given a formula to express each $I_n(\Pi_r, N, k)$ as a product of some elementary factors and a special value of $\zeta(s, L_n^*)$ as follows. For each natural number l, we put

$$\omega = \prod_{i=1}^{l} \zeta(2i),$$

and

$$U_l = \prod_{i=1}^{l} \frac{2\pi^i}{\Gamma(i)}.$$

Theorem 8 (Shintani [16]) *For each natural number n and each integer $k > 2n$, we get*

$$I_n(\Pi_0, N, k) \quad = \quad [\Gamma_n(1) : \Gamma_n(N)] \frac{\omega_n}{U_n (4\pi)^{n(n+1)/2}} \prod_{i=1}^{n} \prod_{j=0}^{i-1} (2k - n - i + 2j),$$

and for each r with $1 \leq r \leq n-1$, we get

$$I_n(\Pi_r, N, k) \quad = \quad [\Gamma_n(1) : \Gamma_n(N)] \frac{\omega_{n-r}}{U_n (4\pi)^{(n-r)(n-r+1)/2}} \prod_{i=1}^{n-r} \prod_{j=0}^{i-1} (2k - n - i + 2j) \zeta(r - n, L_r^*, 1, 1).$$

Remark. In Shintani's original theorem, he assumed that $k \geq 2n + 3$, but it is easy to show that this also holds for $k \geq 2n + 1$ by applying carefully the estimation he used. Incidentally, $\zeta(s, L_n^*, 1, 1) = \zeta_r^*(s)$ in his notation.

Using this theorem, we get an explicit formula for $I_n(\Pi_r, N, k)$ as a corollary to our Theorem 5 and 6. We state this below.

Before stating this result, we prepare more notations. For each non-negative integer r with $0 \leq r \leq n$, we set

$$C_{n-r} = C_{n-r}(k, N) = [Sp(n, Z) : \Gamma_n(N)] N^{-r(n - \frac{r-1}{2})} \prod_{t=1}^{n-r} \frac{t!}{(2t-1)!!} \begin{pmatrix} k - 1 - \frac{n}{2} + \frac{t}{2} \\ t \end{pmatrix},$$

where, for any natural number t, we denote by $\begin{pmatrix} \alpha \\ t \end{pmatrix}$ the binomial coefficient

$$\begin{pmatrix} \alpha \\ t \end{pmatrix} = \frac{\alpha(\alpha-1)\cdots(\alpha-t+1)}{t!},$$

and $(2t-1)!! = 1\cdot 3\cdots(2t-1) = 2^{-t}t!\begin{pmatrix} 2t \\ t \end{pmatrix}$. When $r = n$, the product part in the above definition of C_{n-r} is regarded as 1.

Theorem 9 *For each natural numbers n, N, k with $k > 2n$, the contribution $I_n(\Pi_r, k, N)$ is given as follows.*
(1) For odd r with $1 \leq r \leq n$,

$$I_n(\Pi_r, k, N) = \\ C_{n-r} \times \frac{(-1)^{[\frac{n+1}{2}]}|B_{n-\frac{r-1}{2}}B_2B_4\cdots B_{r-1}B_2B_4\cdots B_{2n-2r}B_{2n-2r+2}B_{2n-2r+4}\cdots B_{2n-r-1}|}{2^{n+\frac{r-1}{2}}(n-\frac{r-1}{2})!(\frac{r-1}{2})!}.$$

In particular, if $n - \frac{r-1}{2}$ is an odd number greater than 1, then $I_n(\Pi_r, k, N) = 0$.
(2) For even r with $0 \leq r \leq n$,

$$I_n(\Pi_r, k, N) = \\ C_{n-r} \times \frac{(-1)^{r(1+\delta_{nr})/2}|B_2B_4\cdots B_{r-2}B_{\frac{r}{2}}|\cdot|B_2B_4\cdots B_{2n-2r}B_{2n-2r+2}B_{2n-2r+4}\cdots B_{2n-r}|}{2^{2n-\frac{r}{2}}(\frac{r}{2})!(n-\frac{r}{2})!},$$

where δ_{nr} is the Kronecker symbol. In particular, for r such that $r \equiv 2 \bmod 4$ with $r \neq 2$, we get $I_n(\Pi_r, k, N) = 0$.

The proof is straightforward, if we note $\pi^{-i}\Gamma(i)\zeta(2i) = \frac{2^{i-1}}{(2i-1)!!}\pi^i|B_{2i}|$.

When $N \geq 3$, it is conjectured, independently by several mathematicians, that any contribution to $\dim S_k(\Gamma_n(N))$ of conjugacy classes other than central unipotent conjugacy classes and the identity element would vanish. So, we are naturally led to the following conjecture. The point of our conjecture is that everything is *explicit*.

Conjecture *When $N \geq 3$ and $k > 2n$,*

$$\dim S_k(\Gamma_n(N)) = \sum_{r=0}^{n} I_n(\Pi_r, k, N),$$

where each $I_n(\Pi_r, k, N)$ is as given explicitly in Theorem 5 above.

This conjecture is known to be true for $n \leq 3$ by Christian [1], Morita [7], Yamazaki [20] and Tsushima [19].

Numerical examples of the conjecture:
We assume $N \geq 3$ hereafter.
(1) When $n = 4$ and $k > 8$, the conjecture reads as

$$\begin{aligned}
&\dim S_k(\Gamma_4(N)) = [Sp(4, Z) : \Gamma_4(N)] \times \\
&\quad \{\frac{1}{2^{25} \cdot 3^8 \cdot 5^4 \cdot 7^2}(2k-2)(2k-3)(2k-4)^2(2k-5)^2(2k-6)^2(2k-7)(2k-8) \\
&\quad + \frac{1}{N^4} \times \frac{1}{2^{17} \cdot 3^7 \cdot 5^3 \cdot 7}(2k-3)(2k-4)(2k-5)^2(2k-6)(2k-7) \\
&\quad - \frac{1}{N^7} \times \frac{1}{2^{15} \cdot 3^5 \cdot 5 \cdot 7}(2k-4)(2k-5)(2k-6) + \frac{1}{N^{10}} \times \frac{1}{2^{12} \cdot 3^4 \cdot 5}\}.
\end{aligned}$$

(2) When $n = 5$ and $k > 10$, the conjecture reads as

$$\begin{aligned}
&\dim S_k(\Gamma_5(N)) = [Sp(5, Z) : \Gamma_5(N)] \times \\
&\quad \{\frac{(2k-2)(2k-3)(2k-4)^2(2k-5)^2(2k-6)^3(2k-7)^2(2k-8)^2(2k-9)(2k-10)}{2^{33} \cdot 3^{12} \cdot 5^5 \cdot 7^3 \cdot 11} \\
&\quad - \frac{1}{N^9} \times \frac{1}{2^{23} \cdot 3^7 \cdot 5^3 \cdot 7}(2k-4)(2k-5)(2k-6)^2(2k-7)(2k-8) \\
&\quad - \frac{1}{N^{12}} \times \frac{1}{2^{17} \cdot 3^7 \cdot 5^2 \cdot 7}(2k-5)(2k-6)(2k-7) \\
&\quad + \frac{1}{N^{14}} \times \frac{1}{2^{16} \cdot 3^6 \cdot 5 \cdot 7}(2k-6)\}.
\end{aligned}$$

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