All Abelian Quotient C.I.-Singularities Admit Projective Crepant Resolutions in All Dimensions

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Abstract

For Gorenstein quotient spaces $\mathbb{C}^d/G$, a direct generalization of the classical McKay correspondence in dimensions $d \geq 4$ would primarily demand the existence of projective, crepant desingularizations. Since this turned out to be not always possible, Reid asked about special classes of such quotient spaces which would satisfy the above property. We prove that the underlying spaces of all Gorenstein abelian quotient singularities, which are embeddable as complete intersections of hypersurfaces in an affine space, have torus-equivariant projective crepant resolutions in all dimensions. We use techniques from toric and discrete geometry.

1. Introduction

Up to isomorphism, the underlying spaces $\mathbb{C}^d/G$, $d \geq 2$, of Gorenstein quotient singularities can be always realized by finite subgroups G of $\mathrm{SL}(d,\mathbb{C})$ acting linearly on $\mathbb{C}^d$. For $d = 2$, these Gorenstein quotient spaces are embeddable in $\mathbb{C}^3$ as hypersurfaces of A-D-E type. In 1979, McKay [18] observed a remarkable connection between the representation theory of the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and the Dynkin diagrams of certain irreducible root systems. This was the starting-point for Gonzalez-Sprinberg and Verdier [8], and Knörrer [15] to construct a purely geometric, direct correspondence

$$\Psi : \mathrm{Irr}^0(G) \to \mathrm{Exc}(f)$$

"of McKay-type" between the set $\mathrm{Irr}^0(G)$ of non-trivial irreducible representations of G, and the set $\mathrm{Exc}(f)$ of exceptional prime divisors of the minimal desingularization $f : X \to \mathbb{C}^2/G$, (or, equivalently, between $\mathrm{Irr}^0(G)$ and the members of the natural basis of the cohomology ring $H^*(X,\mathbb{Q})$). The bijection Ψ induces, in fact, an isomorphism between the graph of $\mathrm{Irr}^0(G)$ and the dual (resolution-) graph w.r.t. f, i.e., the product of the images of two distinct elements of $\mathrm{Irr}^0(G)$ under Ψ is mapped onto the exceptional prime divisor corresponding to the "right" graph-vertex. (For various applications of the geometry of Kleinian singularities and more recent results, including also "quiver-theoretic" methods, we refer to Slodowy [28]).

There are several difficulties (not only of group-theoretic nature) to generalize (even partially) McKay-type correspondences in higher dimensions. Already for $d = 3$, many things change drastically. For instance, the minimal embedding dimension of $\mathbb{C}^3/G$ is in many cases very high, and its singular locus is only rarely a singleton. In addition, *crepant* desingularizations (i.e., the high-dimensional analogues of the above f, cf. [22, 24]) are unique only up to "isomorphism in codimension 1", and there are lots of examples of non-projective f's. Nevertheless, Markushevich [17], Ito [10, 11, 12] and Roan [26, 27] proved case-by-case the *existence* of crepant resolutions of $\mathbb{C}^3/G$, for all possible finite subgroups G of $\mathrm{SL}(3,\mathbb{C})$, by making use of Blichfeldt's classification results, and Ito & Reid [13] established a canonical one-to-one correspondence between the conjugacy classes of the Tate twist $G(-1)$ with age $=1$ and the crepant discrete valuations on $\mathbb{C}^d/G$ for any $d \geq 2$. It is worth mentioning that the existence of *terminal* Gorenstein quotient singularities for $d \geq 4$ (cf. [19, 20]) means automatically that *not all* Gorenstein quotient singularities can be resolved by crepant birational morphisms. On the other hand, Batyrev [2] and Kontsevich [16] recently announced proofs of the invariance of *all* cohomology dimensions of the overlying spaces of all "full" crepant desingularizations for arbitrary $d \geq 2$. (For the case of abelian acting groups, see [3, 5.4]). For these reasons, to proceed the investigation of these quotient spaces in dimensions $d \geq 4$ one has *either* to work with at most *partial* crepant morphisms (and lose the above quoted cohomology dimension invariance) *or* to determine *special classes* of quotients for which this classical type of McKay-correspondence makes sense. As there are only very few known examples to follow the latter approach to the problem (cf. Hirzebruch-Höfer [9, p. 257] and Roan [26, § 5]), Reid asked about conditions on G's which would guarantee the existence of the required "full" crepant (preferably projective) resolutions of the corresponding quotient spaces (see [25], [13, § 4.5]). We believe that one significant class of Gorenstein quotient spaces which enjoys this property is that of complete intersections ("c.i.'s").

Conjecture 1.1. *For all finite subgroups G of $\mathrm{SL}(d,\mathbb{C})$, for which $\mathbb{C}^d/G$ is minimally embeddable as complete intersection (in an affine space $\mathbb{C}^r, r \geq d+1$), the quotient space $\mathbb{C}^d/G$ admits crepant, projective desingularizations for* all $d \geq 2$.

In this paper, we prove the following:

Theorem 1.2. *Conjecture* 1.1 *is true for all* abelian *finite groups $G \subset \mathrm{SL}(d,\mathbb{C})$ (for which $\mathbb{C}^d/G$ is a* "c.i."*).*

For other, non-c.i. abelian quotient spaces having such desingularizations, we refer to [5]. Our proof makes use of toric geometry and Watanabe's classification result [32], and is motivated by the so-called "nice polyhedral subdivisions" of Knudsen & Mumford [14, Ch. III]. More precisely, the paper is organized as follows: In § 2 we introduce those notions of the theory of toric varieties which will enable a convenient combinatorial characterization of the inner structure of $\mathbb{C}^d/G$. Useful tools and arguments coming from convex geometry and the theory of triangulations of polyhedral complexes are presented in § 3. In section 4 we explain why the existence of projective, crepant resolutions can be deduced from the existence of b.c.b.-triangulations of the junior simplex. The method of how one passes from "special data" (parametrizing the defining groups of c.i.-quotient spaces) to "Watanabe forests" and to their lattice-geometric realization via "Watanabe simplices" is described explicitly in § 5. Section 6 provides the inductive, constructive procedure leading to the desired "nice" b.c.b.-triangulations of all Watanabe simplices. Finally, in § 7 we give concrete formulae for the computation of the dimensions of the cohomology groups of the spaces desingularizing $\mathbb{C}^d/G$'s (which are "c.i") by any crepant morphism.

Concerning the general terminology, we always work with normal complex varieties, i.e., with normal, integral, separated schemes defined over $\mathbb{C}$. A *partial desingularization* of such an X is a proper birational morphism $f : Y \to X$ on X, in which Y is assumed to be normal. $f : Y \to X$ is called a *full* (or *overall*) *desingularization of* X if, in addition, we have $\mathrm{Sing}(Y) = \varnothing$, i.e., if the overlying space Y is smooth. For brevity's sake, the word *desingularization* of X (or the phrase *resolution of singularities of* X) is sometimes used as synonymous with the phrase "full (or overall) desingularization of X". When we refer to partial desingularizations, we mention it explicitly. (By the word *singularity* we intimate either a singular point or the germ of a singular point, but the meaning will be in each case clear from the context.)

A partial desingularization $f : Y \to X$ of a $\mathbb{Q}$-Gorenstein complex variety X with global index j is called *non-discrepant* or simply *crepant* if $\omega_X^{[j]} \cong f_*\left(\omega_Y^{\otimes j}\right)$, or, in other words, if the (up to rational equivalence uniquely determined) difference $jK_Y - f^*(jK_X)$ contains exceptional prime divisors which have vanishing multiplicities. (ω_X, K_X and ω_Y, K_Y denote the dualizing sheaves and the canonical divisors of X and Y respectively). Furthermore, $f : Y \to X$ is *projective* if Y admits an f-ample Cartier divisor.

2. A brief toric glossary

We recall some basic facts from the theory of toric varieties and fix the notation which will be used in the sequel. For details the reader is referred to the standard textbooks [6] [7] [14] [21].

(a) For a set A of vectors of $\mathbb{R}^d$, the *linear hull*, the *positive hull*, the *affine hull* and *the convex hull* of A is

$$\begin{aligned}
\mathrm{lin}(A) &= \left\{ \sum_{i=1}^{k} \mu_i \mathbf{x}_i \;\middle|\; \mathbf{x}_i \in A,\ \mu_i \in \mathbb{R},\ k \in \mathbb{N} \right\}, \\
\mathrm{pos}(A) &= \left\{ \sum_{i=1}^{k} \mu_i \mathbf{x}_i \;\middle|\; \mathbf{x}_i \in A,\ \mu_i \in \mathbb{R},\ \mu_i \geq 0,\ k \in \mathbb{N} \right\}, \\
\mathrm{aff}(A) &= \left\{ \sum_{i=1}^{k} \mu_i \mathbf{x}_i \;\middle|\; \mathbf{x}_i \in A,\ \mu_i \in \mathbb{R},\ \sum_{i=1}^{k} \mu_i = 1,\ k \in \mathbb{N} \right\}, \quad \text{and} \\
\mathrm{conv}(A) &= \left\{ \sum_{i=1}^{k} \mu_i \mathbf{x}_i \;\middle|\; \mathbf{x}_i \in A,\ \mu_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^{k} \mu_i = 1,\ k \in \mathbb{N} \right\},
\end{aligned}$$

respectively. Moreover, we define the *integral affine hull* of A as

$$\mathrm{aff}_{\mathbb{Z}}(A) = \left\{ \sum_{i=1}^{k} \mu_i \mathbf{x}_i \mid \mathbf{x}_i \in A,\ \mu_i \in \mathbb{Z},\ \sum_{i=1}^{k} \mu_i = 1,\ k \in \mathbb{N} \right\}.$$

For a set $A \subset \mathbb{R}^d$ the dimension of its affine hull is denoted by $\dim(A)$.

(b) Let $N \cong \mathbb{Z}^d$ be a free $\mathbb{Z}$-module of rank $d \geq 1$. N can be regarded as a *lattice* in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$. (We shall represent the elements of N by column vectors). If $\{n_1, \ldots, n_d\}$ is a $\mathbb{Z}$-basis of N, then

$$\det(N) := |\det(n_1, \ldots, n_d)|$$

is the *lattice determinant*. An $n \in N$ is *primitive* if $\mathrm{conv}(\{\mathbf{0}, n\}) \cap N$ contains no other lattice points except $\mathbf{0}$ and n.

Let $N \cong \mathbb{Z}^d$ be a lattice as above, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual, $N_{\mathbb{R}}, M_{\mathbb{R}}$ their real scalar extensions, and $\langle .,. \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}$-bilinear pairing. A subset σ of $N_{\mathbb{R}}$ is a *strongly convex polyhedral cone* (*s.c.p. cone*, for short), if there exist $n_1, \ldots, n_k \in N_{\mathbb{R}}$, such that $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$, and $\sigma \cap (-\sigma) = \{\mathbf{0}\}$. Its *dimension* $\dim(\sigma)$ is defined to be the dimension of the $\mathbb{R}$-vector space $\mathrm{lin}(\sigma)$ and its *relative interior* $\mathrm{int}(\sigma)$ is the usual topological interior of it, considered as a subset of $\mathrm{lin}(\sigma)$. The *dual cone* of an s.c.p. cone σ is defined by

$$\sigma^{\vee} := \{\mathbf{x} \in M_{\mathbb{R}} \mid \langle \mathbf{x}, \mathbf{y} \rangle \geq 0, \quad \text{for all } \mathbf{y} \in \sigma\}$$

and satisfies: $\sigma^{\vee} + (-\sigma^{\vee}) = M_{\mathbb{R}}$ and $\dim(\sigma^{\vee}) = d$. A subset τ of a s.c.p. cone σ is called a *face* of σ (notation: $\tau \prec \sigma$), if $\tau = \{\mathbf{y} \in \sigma \mid \langle m_0, \mathbf{y} \rangle = 0\}$, for some $m_0 \in \sigma^{\vee}$. A s.c.p. cone σ is *simplicial* (resp. *rational*) if $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$, where the vectors $n_1, \ldots, n_k$ are $\mathbb{R}$-linearly independent (resp. if $n_1, \ldots, n_k \in N_{\mathbb{Q}}$). If $\sigma \subset N_{\mathbb{R}}$ is a rational s.c.p. cone, then σ is "pointed" and the subsemigroup $\sigma \cap N$ of N is a monoid having the origin $\mathbf{0}$ as its neutral element. Using its dual $M \cap \sigma^{\vee}$, one constructs a finitely generated, normal $\mathbb{C}$-subalgebra $\mathbb{C}[M \cap \sigma^{\vee}]$ of $\mathbb{C}[M]$ and a d-dimensional affine complex variety

$$U_{\sigma} := \text{Max-Spec}\left(\mathbb{C}\left[M \cap \sigma^{\vee}\right]\right) .$$

(c) For $N \cong \mathbb{Z}^d$ we define an algebraic torus $T_N \cong (\mathbb{C}^*)^d$ by $T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Every $m \in M$ assigns a character $\mathbf{e}(m) : T_N \to \mathbb{C}^*$ with $\mathbf{e}(m)(t) := t(m)$, for all $t \in T_N$. We have

$$\mathbf{e}\left(m + m'\right) = \mathbf{e}(m) \cdot \mathbf{e}\left(m'\right), \quad \text{for } m, m' \in M, \qquad \text{and } \mathbf{e}(\mathbf{0}) = 1.$$

Moreover, for each $n \in N$ we define an 1-parameter subgroup of T_N by

$$\gamma_n : \mathbb{C}^* \to T_N \quad \text{with} \quad \gamma_n(\lambda)(m) := \lambda^{\langle m, n \rangle}, \quad \text{for} \quad \lambda \in \mathbb{C}^*, \ m \in M,$$

($\gamma_{n+n'} = \gamma_n \circ \gamma_{n'}$, for $n, n' \in N$). We can therefore identify M with the character group of T_N and N with the group of 1-parameter subgroups of T_N. Furthermore, U_{σ} (as above) can be identified with the set of semigroup homomorphisms:

$$U_{\sigma} = \left\{u : M \cap \sigma^{\vee} \to \mathbb{C}^* \ \middle|\ u(\mathbf{0}) = 1, \ u\left(m + m'\right) = u(m) \cdot u\left(m'\right), \ \text{ for all } m, m' \in M \cap \sigma^{\vee}\right\},$$

where $\mathbf{e}(m)(u) := u(m)$, for all $m \in M \cap \sigma^{\vee}$ and for all $u \in U_{\sigma}$.

(d) A *fan* Δ w.r.t. $N \cong \mathbb{Z}^d$ is a finite collection of rational s.c.p. cones in $N_{\mathbb{R}}$ such that:

(i) any face τ of $\sigma \in \Delta$ belongs to Δ, and

(ii) for $\sigma_1, \sigma_2 \in \Delta$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

The union $|\Delta| := \bigcup \{\sigma \mid \sigma \in \Delta\}$ is called the *support* of Δ. Furthermore, we define

$$\Delta(i) := \{\sigma \in \Delta \mid \dim(\sigma) = i\}, \text{ for } 0 \leq i \leq d.$$

If $\varrho \in \Delta(1)$, then there exists a unique primitive vector $n(\varrho) \in N \cap \varrho$ with $\varrho = \mathbb{R}_{\geq 0}\ n(\varrho)$ and each cone $\sigma \in \Delta$ can be therefore written as

$$\sigma = \sum_{\substack{\varrho \in \Delta(1) \\ \varrho \prec \sigma}} \mathbb{R}_{\geq 0}\ n(\varrho).$$

The set $\mathrm{Sk}^1(\sigma) := \{n(\varrho) \mid \varrho \in \Delta(1), \varrho \prec \sigma\}$ is called the (*pure*) *first skeleton* of σ. The *toric variety* $X(N, \Delta)$ associated to a fan Δ w.r.t. the lattice N is by definition the identification space

$$X(N, \Delta) := \Big(\bigcup_{\sigma \in \Delta} U_{\sigma}\Big) \Big/ \sim$$

with $U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2}$ if and only if there is a $\tau \in \Delta$, such that $\tau \prec \sigma_1 \cap \sigma_2$ and $u_1 = u_2$ within U_τ. As a complex variety, $X(N,\Delta)$ turns out to be irreducible, normal, Cohen-Macaulay and to have at most rational singularities (cf. [7, p. 76] and [21, Thm. 1.4, p. 7 and Cor. 3.9, p. 125]). $X(N,\Delta)$ admits a canonical T_N-action which extends the group multiplication of $T_N = U_{\{\mathbf{0}\}}$:

$$T_N \times X(N,\Delta) \ni (t,u) \longmapsto t \cdot u \in X(N,\Delta) \tag{2.1}$$

where, for $u \in U_\sigma$, $(t \cdot u)(m) := t(m) \cdot u(m)$, for all $m \in M \cap \sigma^\vee$. The orbits w.r.t. the action (2.1) are parametrized by the set of all the cones belonging to Δ. For a $\tau \in \Delta$, we denote by $\mathrm{orb}(\tau)$ (resp. by $V(\tau)$) the orbit (resp. the closure of the orbit) which is associated to τ. The spaces $\mathrm{orb}(\tau)$ and $V(\tau)$ have the following properties (cf. [7, pp. 52-55], [21, § 1.3]):

(i) $V(\tau) = \coprod \{\mathrm{orb}(\sigma) \mid \sigma \in \Delta, \tau \prec \sigma\}$ and $\mathrm{orb}(\tau) = V(\tau) \smallsetminus \bigcup \{V(\sigma) \mid \tau \precneqq \sigma\}$.

(ii) $V(\tau) = X(N(\tau), \mathrm{Star}(\tau))$ is itself a toric variety w.r.t.

$$N(\tau) := N/N_\tau, \quad N_\tau := N \cap \mathrm{lin}(\tau), \quad \mathrm{Star}(\tau) := \{\overline{\sigma} \mid \sigma \in \Delta, \ \tau \prec \sigma\},$$

where $\overline{\sigma} := (\sigma + (N_\tau)_{\mathbb{R}}) \ / \ (N_\tau)_{\mathbb{R}}$ denotes the image of σ in $N(\tau)_{\mathbb{R}} = N_{\mathbb{R}} \ / \ (N_\tau)_{\mathbb{R}}$.

(iii) For $\tau \in \Delta$, $V(\tau)$ has a natural affine open covering $\{U_\sigma(\tau) \mid \tau \prec \sigma\}$ consisting of "intermediate" spaces

$$U_\tau(\tau) = \ \mathrm{orb}(\tau) \hookrightarrow U_\sigma(\tau) \hookrightarrow U_\sigma$$

being defined by: $U_\sigma(\tau) := \text{Max-Spec}(\mathbb{C}[\overline{\sigma}^\vee \cap \ M(\tau)])$, with $M(\tau)$ denoting the dual of $N(\tau)$. It should be pointed out, that every T_N-invariant subvariety of U_σ has the form $U_\sigma(\tau)$ and that $\dim(U_\sigma(\tau)) = \dim(\sigma) - \dim(\tau)$.

(e) Let $X(N,\Delta)$ be a d-dimensional (not necessarily compact) toric variety. A function $\psi : |\Delta| \to \mathbb{R}$ is an (*integral*) Δ-*support function* if $\psi(N \cap |\Delta|) \subset \mathbb{Z}$ and $\psi|_\sigma$ is linear for all $\sigma \in \Delta$. This means that for all $\sigma \in \Delta$ there has to be some $m_\sigma \in M$ such that

$$\psi(\mathbf{x}) = \langle m_\sigma, \mathbf{x} \rangle \quad \text{for all } \mathbf{x} \in \sigma.$$

We denote by $\mathrm{SF}(N,\Delta)$ the additive group of all Δ-support functions. A $\psi \in \mathrm{SF}(N,\Delta)$ is *strictly upper convex* if for every $\sigma \in \Delta$ the m_σ can be chosen such that $\psi(\mathbf{x}) \le \langle m_\sigma, \mathbf{x} \rangle$, with equality if and only if $\mathbf{x} \in \sigma$. Let

$$\mathrm{SUCSF}(N,\Delta) := \{\psi \in \mathrm{SF}(N,\Delta) \mid \psi \ \text{ strictly upper convex}\}.$$

The group T_N-$\mathrm{CDiv}(X(N,\Delta))$ of T_N-invariant Cartier divisors on $X(N,\Delta)$ has the $\mathbb{Z}$-basis $\{\mathbb{Z}\, V(\varrho) \mid \varrho \in \Delta(1)\}$ (cf. [21, p. 68]).

Theorem 2.1. *The relationship between the Δ-support functions and the T_N-invariant Cartier divisors on $X(N,\Delta)$ is given by the following bijections:*

$$\begin{array}{ccccccc} \psi & \in & \mathrm{SF}(N,\Delta) & & \supset & \mathrm{SUCSF}(N,\Delta) \\ \updownarrow & & \updownarrow \ 1{:}1 & & & \updownarrow \ 1{:}1 \\ D_\psi & \in & T_N\text{-}\mathrm{CDiv}(X(N,\Delta)) & = \bigoplus_{\varrho \in \Delta(1)} \mathbb{Z}\, V(\varrho) & \supset & \left\{ \begin{array}{c} \text{ample divisors} \\ \text{on } \ X(N,\Delta) \end{array} \right\} \end{array} \tag{2.2}$$

with

$$D_h := - \sum_{\varrho \in \Delta(1)} \psi(n(\varrho)) \ V(\varrho).$$

Proof. For the first bijection see [21, Prop. 2.1, p. 68-69]. The proof of the second bijection of (2.2), in the case in which $X(N,\Delta)$ is compact, is given in [21, Cor. 2.14, p. 83]. The general case is treated in [14, Ch. I, Thm. 9, p. 42, and Thm. 13, p. 48]. $\square$

Corollary 2.2. *Let $X(N,\Delta)$ be a toric variety (resp. a compact toric variety). Then $X(N,\Delta)$ is quasiprojective (resp. projective) if and only if* $\mathrm{SUCSF}(N,\Delta) \neq \varnothing$.

(f) A *map of fans* $\varpi : (N',\Delta') \to (N,\Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varpi : N' \to N$ whose extension $\varpi_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ satisfies the property

$$\text{for all } \sigma' \in \Delta' \text{ there is some } \sigma \in \Delta \text{ such that } \varpi_{\mathbb{R}}(\sigma') \subseteq \sigma.$$

Every map of fans $\varpi : (N',\Delta') \to (N,\Delta)$ induces an holomorphic map

$$\varpi_* : X\left(N',\Delta'\right) \to X(N,\Delta)$$

which is equivariant w.r.t. the actions of $T_{N'}$ and T_N on the toric varieties $X(N',\Delta')$, $X(N,\Delta)$.

Theorem 2.3. *If $\varpi : (N',\Delta') \to (N,\Delta)$ is a map of fans, ϖ_* is* proper *if and only if $\varpi^{-1}(|\Delta|) = |\Delta'|$. In particular, if $N = N'$ and Δ' is a refinement of Δ, i.e., if each cone of Δ is a union of cones of Δ', then $\mathrm{id}_* : X(N,\Delta') \to X(N,\Delta)$ is* proper *and* birational.

Proof. See [21, Thm. 1.15, p. 20, and Cor. 1.18, p. 23]. $\square$

(g) Let $N \cong \mathbb{Z}^d$ be a lattice of rank d and $\sigma \subset N_{\mathbb{R}}$ a simplicial, rational s.c.p. cone of dimension $k \leq d$. σ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_k$, for distinct 1-dimensional cones $\varrho_1, \ldots, \varrho_k$. We denote by

$$\mathbf{Par}(\sigma) := \left\{ \mathbf{x} \in (N_\sigma)_{\mathbb{R}} \;\middle|\; \mathbf{x} = \sum_{j=1}^{k} \varepsilon_j \, n(\varrho_j), \text{ with } 0 \leq \varepsilon_j < 1 \text{ for } 1 \leq j \leq k \right\}$$

the *fundamental (half-open) parallelotope* which is associated to σ. The *multiplicity* $\mathrm{mult}(\sigma, N)$ of σ with respect to N is defined as

$$\mathrm{mult}(\sigma, N) := \#\left(\mathbf{Par}(\sigma) \cap N_\sigma\right) = \mathrm{Vol}\left(\mathbf{Par}(\sigma); N_\sigma\right),$$

where

$$\mathrm{Vol}\left(\mathbf{Par}(\sigma); N_\sigma\right) := \frac{\mathrm{Vol}\left(\mathbf{Par}(\sigma)\right)}{\det(N_\sigma)}$$

is the relative volume of $\mathbf{Par}(\sigma)$ w.r.t. N_σ. The affine toric variety U_σ is smooth if and only if $\mathrm{mult}(\sigma, N) = 1$.

Theorem 2.4. *Every toric variety $X(N,\Delta)$ admits a T_N-equivariant desingularization*

$$f = \mathrm{id}_* : X\left(N,\Delta'\right) \to X(N,\Delta)$$

by a suitable refinement Δ' of Δ.

In fact, one can always make Δ simplicial without introducing additional rays. (This step relies on Carathéodory's theorem, cf. [21, p. 175]). In a second step, this new simplicial Δ will be subdivided further into subcones of strictly smaller multiplicities than those of the cones of the starting-point. (Since for each $\sigma \in \Delta$, $\mathrm{mult}(\sigma, N)$ is a volume, $\mathrm{mult}(\widehat{\sigma}, N) < \mathrm{mult}(\sigma, N)$ for every simplicial subcone $\widehat{\sigma}$ of σ). Δ' is constructed after finitely many subdivisions of this kind (cf. [14, pp. 31-35]).

(h) For the germ $(U_\sigma, \mathrm{orb}(\sigma))$ of an affine d-dimensional toric variety w.r.t. a *singular point* $\mathrm{orb}(\sigma)$, (with $\dim(\sigma) = d = \dim(N)$), we define the *splitting codimension* $\mathrm{splcod}(\mathrm{orb}(\sigma); U_\sigma)$ of $\mathrm{orb}(\sigma)$ in U_σ as

$$\mathrm{splcod}(\mathrm{orb}(\sigma); U_\sigma) := \max \left\{ \varkappa \in \{2, \ldots, d\} \;\middle|\; \begin{array}{c} U_\sigma \cong U_{\sigma'} \times \mathbb{C}^{d-\varkappa}, \ \dim(\sigma') = \varkappa, \\ \text{and} \quad \mathrm{Sing}(U_{\sigma'}) \neq \varnothing \end{array} \right\}.$$

(For $\mathrm{orb}(\sigma)$ regular we can formally define $\mathrm{splcod}(\mathrm{orb}(\sigma); U_\sigma) = 0$.) If $\mathrm{splcod}(\mathrm{orb}(\sigma); U_\sigma) = d$, then $\mathrm{orb}(\sigma)$ will be called an *msc-singularity*, i.e., a singularity having the maximum splitting codimension. An msc-singularity $(U_\sigma, \mathrm{orb}(\sigma))$ is *absolutely unbreakable* if U_σ cannot be isomorphic even to the product of (at least two) *singular* affine toric varieties.

(i) For any abelian finite $G \subset \mathrm{GL}(d, \mathbb{C})$, $d \geq 2$, of order $l \geq 2$, which is *small* (i.e. which has no pseudoreflections), $\mathbb{C}^d/G$ is singular, with singular locus $\mathrm{Sing}(\mathbb{C}^d/G)$ containing at least the image $[\mathbf{0}]$ of the origin under the canonical quotient-map $\mathbb{C}^d \to \mathbb{C}^d/G$. If we fix a decomposition

$$G \cong (\mathbb{Z}/q_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/q_\kappa\mathbb{Z})$$

into cyclic groups, take primitive q_ith roots of unity ζ_{q_i}, and choose eigencoordinates $z_1, \ldots, z_d$, then the action of G on $\mathbb{C}^d$ is determined by

$$(\mathbb{Z}/q_i\mathbb{Z}) \times \mathbb{C}^d \ni (g_i, (z_1, \ldots, z_d)) \longmapsto \mathrm{diag}\left(\zeta_{q_i}^{\alpha_{i,1}} z_1, \ldots, \zeta_{q_i}^{\alpha_{i,d}} z_d\right) \in \mathbb{C}^d, \quad 1 \leq i \leq \kappa,$$

for generators $g_i = \mathrm{diag}(\zeta_{q_i}^{\alpha_{i,1}}, \ldots, \zeta_{q_i}^{\alpha_{i,d}})$, where the weights $(\alpha_{i,1}, \ldots, \alpha_{i,d})$ are unique up to only the usual conjugacy relations. Using the above toric glossary, we may identify $\mathbb{C}^d/G$ with the affine toric variety $U_{\sigma_0} = \text{Max-Spec}(\mathbb{C}[\sigma_0^\vee \cap M_G])$ being associated to the positive orthant $\sigma_0 = \mathrm{pos}(\{e_1, \ldots, e_d\})$ with respect to the lattice of weights

$$N_G = \mathbb{Z}^d + \sum_{i=1}^{\kappa} \mathbb{Z} \left(\frac{1}{q_i} (\alpha_{i,1}, \ldots, \alpha_{i,d})^\intercal \right) \quad \text{with} \ \det(N_G) = \frac{1}{l},$$

where M_G denote the dual of N_G, which can be viewed as the lattice parametrizing all G-invariant Laurent monomials. Formally, we identify $[\mathbf{0}]$ with $\mathrm{orb}(\sigma_0)$ and U_{σ_0} with the toric variety $X(N_G, \Delta_G)$ w.r.t. the fan Δ_G consisting of σ_0 itself, together with all its faces. In these terms, and since σ_0 is a simplicial s.c.p. cone, the singular locus of $X(N_G, \Delta_G)$ can be written as the union

$$\mathrm{Sing}(X(N_G, \Delta_G)) = \mathrm{orb}(\sigma_0) \cup \left(\bigcup \{ U_{\sigma_0}(\tau) \mid \tau \precneqq \sigma_0, \ \mathrm{int}(\mathbf{Par}(\tau)) \cap N_G \neq \varnothing \} \right)$$

For $0 \leq i \leq d$, $\Delta_G(i)$ has $\binom{d}{i}$ simplicial cones. In particular, $\Delta_G(0) = \{\mathbf{0}\}$, and for $1 \leq i \leq d$,

$$\Delta_G(i) = \{ \sigma_0(\nu_1, \ldots, \nu_i) := \mathrm{pos}(\{e_{\nu_1}, \ldots, e_{\nu_i}\}) \mid 1 \leq \nu_1 < \nu_2 < \cdots < \nu_i \leq d \}.$$

For any cone $\sigma_0(\nu_1, \ldots, \nu_i) \in \Delta_G(i)$, $U_{\sigma_0}(\sigma_0(\nu_1, \ldots, \nu_i))$ is nothing but

$$(\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_{\nu_1} = \cdots = z_{\nu_i} = 0 \})/G,$$

i.e., a $(d-i)$-dimensional coordinate-subspace of $\mathbb{C}^d$ divided by the inherited G-action.

Proposition 2.5. *For an abelian G (as above) the following conditions are equivalent* :

(i) $X(N_G, \Delta_G) = U_{\sigma_0} = \mathbb{C}^d/G$ *is Gorenstein,*

(ii) $G \subset \mathrm{SL}(d, \mathbb{C})$,

(iii) $\sum_{j=1}^{d} \alpha_{i,j} \equiv 0 \bmod q_i$, *for* $1 \leq i \leq \kappa$,

(iv) $\langle (1, 1, \ldots, 1), n \rangle \geq 1$, *for all* $n \in \sigma_0 \cap (N_G \smallsetminus \{\mathbf{0}\})$,

(v) $(U_{\sigma_0}, \mathrm{orb}(\sigma_0))$ *is a canonical singularity of index* 1.

Proof. (i) $\Leftrightarrow$ (ii) follows from [31] and (i) $\Leftrightarrow$ (v) from [22, 23]. All the other implications can be checked easily. $\square$

From now on, let $X(N_G, \Delta_G)$ be Gorenstein. The cone $\sigma_0 = \mathbb{R}_{\geq 0}\, \mathfrak{s}_G$ is supported by the so-called *junior lattice simplex* $\mathfrak{s}_G = \mathrm{conv}(\{e_1, \ldots, e_d\})$ (w.r.t. N_G; cf. [13] [3]). Note that up to $\mathbf{0}$ there is no other lattice point of $\sigma_0 \cap N_G$ lying "under" the affine hyperplane of $\mathbb{R}^d$ containing $\mathfrak{s}_G$. Moreover, the lattice points representing the $l-1$ non-trivial group elements are exactly those belonging to the intersection of a dilation $\lambda\, \mathfrak{s}_G$ of $\mathfrak{s}_G$ with $\mathbf{Par}(\sigma_0)$, for some integer λ, $1 \leq \lambda \leq d-1$. For c.i.-U_{σ_0}'s, our purpose is to construct crepant, projective (full) desingularizations

$$f : X\left(N_G, \widehat{\Delta_G}\right) \to X(N_G, \Delta_G) = U_{\sigma_0}$$

which are T_{N_G}-equivariant (with $T_{N_G} = (\mathbb{C}^*)^d/G$ denoting the algebraic torus embedded in U_{σ_0}), and have overlying (quasiprojective) spaces $X\left(N_G, \widehat{\Delta_G}\right)$ associated to fans $\widehat{\Delta_G}$ which refine appropriately Δ_G.

3. On b.c.b.-triangulations of lattice polytopes

In this section we introduce "b.c.b.-triangulations" and study their behaviour with respect to *joins* and *dilations*. (We shall mostly use the standard terminology from the theory of polyhedral complexes and polytopes, cf. [30] [33]).

(a) By $\mathrm{vert}(\mathcal{S})$ we denote the set of vertices of a polyhedral complex $\mathcal{S}$. By a *triangulation* $\mathcal{T}$ of a polyhedral complex $\mathcal{S}$ we mean a geometric simplicial subdivision of $\mathcal{S}$ with $\mathrm{vert}(\mathcal{S}) \subset \mathrm{vert}(\mathcal{T})$. If $\mathcal{T}$ is d-dimensional, then by $\mathcal{T}(i)$ we denote the set of i-dimensional simplices of $\mathcal{T}$. A polytope P will be frequently identified with the polyhedral complex consisting of P itself together with all its faces.

(b) A triangulation $\mathcal{T}$ of a polyhedral d-complex $\mathcal{S}$ is *coherent* (or *regular*) if there exists a strictly upper convex $\mathcal{T}$-support function $\psi : |\mathcal{T}| \to \mathbb{R}$, i.e., a real function on the underlying space $|\mathcal{T}|$ for which

$$\psi(t\,\mathbf{x} + (1-t)\mathbf{y}) \;\geq\; t\,\psi(\mathbf{x}) + (1-t)\psi(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in |\mathcal{T}| \text{ and } \; t \in \mathbb{R} \cap [0,1],$$

and $\psi|_{\mathbf{s}} \neq \psi|_{\mathbf{s}'}$ for any two maximal simplices $\mathbf{s}, \mathbf{s}', \mathbf{s} \neq \mathbf{s}'$, of $\mathcal{T}$. The set of all strictly upper convex $\mathcal{T}$-support functions is denoted by $\mathrm{SUCSF}(\mathcal{T})$.

Lemma 3.1 (Patching Lemma). *Let* $P \subset \mathbb{R}^d$ *be a* d*-polytope,* $\mathcal{T} = \{\mathbf{s}_i \mid i \in I\}$ *(with* I *a finite set) a coherent triangulation of* P*, and* $\mathcal{T}_i = \{\mathbf{s}_{i,j} \mid j \in J_i\}$ *(*J_i *finite, for all* $i \in I$*)*

a coherent triangulation of $\mathbf{s}_i$, *for all* $i \in I$. *If* $\psi_i : |\mathcal{T}_i| \to \mathbb{R}$ *denote strictly upper convex* $\mathcal{T}_i$*-support functions, such that*

$$\psi_i \left|_{\mathbf{s}_i \cap \mathbf{s}_{i'}} = \psi_{i'} \right|_{\mathbf{s}_i \cap \mathbf{s}_{i'}}$$

for all $(i, i') \in I \times I$, *then*

$$\widetilde{\mathcal{T}} := \{\text{all the simplices } \mathbf{s}_{i,j} \mid \ j \in J_i, \ i \in I\}$$

forms a coherent triangulation *of the initial polytope* P.

The above ψ_i's can be canonically "patched together" to construct an element ψ of $\mathrm{SUCSF}(\widetilde{\mathcal{T}})$; see [14, Cor. 1.12, p. 115] or [4, Lemma 3.2.2].

(c) A triangulation $\mathcal{T}$ of a d-dimensional simplex $\mathbf{s}$ (or, more generally, of any pure d-dimensional simplicial complex $\mathcal{S}$) is *balanced* if its graph can be $(d+1)$-coloured, i.e., if there is a function

$$\varphi : \mathrm{vert}\,(\mathcal{T}) \to \{0, 1, 2, \ldots, d\}$$

such that any two adjacent vertices receive different values ("colours") under φ. If $\mathcal{T}$ is a balanced triangulation of a d-polytope, then all facets of $\mathcal{T}$ receive all the colours, and the colouring function φ is unique (up to permutation of the colours).

Example 3.2. The first two 2-dimensional triangulations (**A**) and (**B**) of Figure 1 are balanced, whereas (**C**) and (**D**) are not.

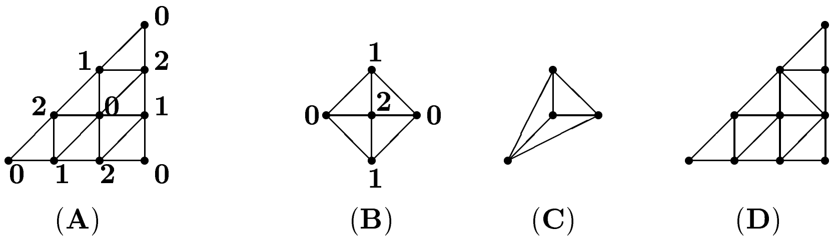

Figure 1

(d) Let N be a d-dimensional lattice. A *lattice polytope* (w.r.t. N) is a polytope in $N_{\mathbb{R}} \cong \mathbb{R}^d$ all of whose vertices belong to N. If $\{n_0, n_1, \ldots, n_k\}$ is a set of affinely independent lattice points $(k \leq d)$, $\mathbf{s}$ the k-simplex $\mathbf{s} = \mathrm{conv}(\{n_0, n_1, \ldots, n_k\})$, and $N_{\mathbf{s}} := \mathrm{lin}(\{n_1 - n_0, \ldots, n_k - n_0\}) \cap N$, then $\mathbf{s}$ is *basic* if it satisfies any one of the following equivalent conditions:

(i) $\mathrm{aff}_{\mathbb{Z}}(n_0, \ldots, n_k) = N \cap \mathrm{aff}(\mathbf{s})$,

(ii) $\{n_1 - n_0, n_2 - n_0, \ldots, n_k - n_0\}$ is a $\mathbb{Z}$-basis of $N_{\mathbf{s}}$,

(iii) $\{n_1 - n_0, n_2 - n_0, \ldots, n_k - n_0\}$ is a part of a $\mathbb{Z}$-basis of N,

(iv) $\mathbf{s}$ has relative volume $\mathrm{Vol}(\mathbf{s}; N_{\mathbf{s}}) = \dfrac{\mathrm{Vol}\,(\mathbf{s})}{\det\,(N_{\mathbf{s}})} = \dfrac{1}{k!}$ (w.r.t. $N_{\mathbf{s}}$).

A *lattice triangulation* $\mathcal{T}$ of a lattice polytope P (w.r.t. $N \cong \mathbb{Z}^d$) is a triangulation of P with vertices in N, and it is called *basic* if all its simplices are basic. If $\mathcal{T}$ is a basic triangulation of P w.r.t. N, then in particular we have

$$\mathrm{aff}_{\mathbb{Z}}(\mathrm{vert}((T)) \ = \ N \cap \mathrm{aff}(P).$$

Affine maps $\Phi : N_{\mathbb{R}} \to N_{\mathbb{R}}$ will be called *affine integral transformations* (w.r.t N) if they satisfy $N = \Phi(N)$, that is, if they have the form $\Phi\,(\mathbf{x}) = \mathcal{M}\mathbf{x} + \beta$, with $\mathcal{M} \in \mathrm{GL}_{\mathbb{Z}}\,(N) \cong \mathrm{GL}(d, \mathbb{Z})$ and $\beta \in N$ (i.e., if they are composed of N-unimodular transformations and N-integral translations).

Definition 3.3. From now on, *b.c.b.-triangulation* is used as an abbreviation for a basic, coherent, balanced triangulation of a lattice polytope.

Lemma 3.4 (Preservation of the "b.c.b.-property" under affine transformations). *Let $\mathcal{T}$ be a lattice triangulation of a lattice polytope $P \subset N_{\mathbb{R}}$ (w.r.t. $N \cong \mathbb{Z}^d$) and let $\Phi : N_{\mathbb{R}} \to N_{\mathbb{R}}$ be a (not necessarily integral) regular affine transformation. If $\mathcal{T}$ is a b.c.b.-triangulation w.r.t. N, then its image $\Phi(\mathcal{T}) := \{\Phi(\mathbf{s}) \mid \mathbf{s} \in \mathcal{T}\}$ under Φ is again a b.c.b.-triangulation of the transformed lattice d-polytope $\Phi(P)$ w.r.t. the lattice $\Phi(N)$.*

(e) If $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) is a triangulation of a d_1-polytope P_1 (resp. of a d_2-polytope P_2) in $\mathbb{R}^d$ such that $\dim(P_1 \cup P_2) = \dim(P_1) + \dim(P_2) + 1$, that is, such that $\mathrm{aff}(P_1)$ and $\mathrm{aff}(P_2)$ are skew affine subspaces of $N_{\mathbb{R}}$, then the *join* $\mathcal{T}_1 * \mathcal{T}_2$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ is defined by

$$\mathcal{T}_1 * \mathcal{T}_2 = \{\mathrm{conv}(\mathbf{s}_1 \cup \mathbf{s}_2) \mid \mathbf{s}_1 \in \mathcal{T}_1, \mathbf{s}_2 \in \mathcal{T}_2\}.$$

It is a triangulation of the usual join $P = \mathrm{conv}(P_1 \cup P_2)$ of P_1 with P_2, which is a polytope of dimension $d_1 + d_2 + 1$ (cf. [30, Ex. 7, p. 136] and [33, Ex. 9.9, p. 323]).

Theorem 3.5 (Preservation of the "b.c.b.-property" for joins). *Let $\mathcal{T}_1, \mathcal{T}_2$ be triangulations of polytopes P_1, $P_2 \subset \mathbb{R}^d$ of dimensions d_1 resp. d_2 in skew affine subspaces of $N_{\mathbb{R}}$, i.e., satisfying $\dim(P_1 \cup P_2) = d_1 + d_2 + 1$.*

If $\mathcal{T}_1$, $\mathcal{T}_2$ are b.c.b. triangulations of P_1, P_2 w.r.t. N, then $\mathcal{T}_1 * \mathcal{T}_2$ is a b.c.b.-triangulation of the $(d_1 + d_2 + 1)$-polytope $P = \mathrm{conv}(P_1 \cup P_2)$ w.r.t. N if and only if

$$\mathrm{aff}_{\mathbb{Z}}\big(\mathrm{vert}(\mathcal{T}_1) \cup \mathrm{vert}(\mathcal{T}_2)\big) = N \cap \mathrm{aff}(P_1 \cup P_2).$$

Proof. By (e) it is obvious that $\mathcal{T}_1 * \mathcal{T}_2$ is a lattice triangulation of P w.r.t. the lattice N.
(i) Let $\mathbf{s}$ be a $(d_1 + d_2 + 1)$-simplex of $\mathcal{T}_1 * \mathcal{T}_2$. By definition it can be written as

$$\mathbf{s} = \mathrm{conv}(\{n_0^{(1)}, \ldots, n_{d_1}^{(1)}, n_0^{(2)}, \ldots, n_{d_2}^{(2)}\}),$$

where $\mathrm{conv}(n_0^{(i)}, \ldots, n_{d_i}^{(i)}) \in \mathcal{T}_i(d_i)$ is a basic simplex w.r.t. $N \cap \mathrm{aff}(P_i)$, for $i = 1, 2$. In particular, we have $\mathrm{aff}_{\mathbb{Z}}(n_0^{(1)}, \ldots, n_{d_1}^{(1)}, n_0^{(2)}, \ldots, n_{d_2}^{(2)}) = \mathrm{aff}_{\mathbb{Z}}(\mathcal{T}_1 * \mathcal{T}_2)$. Thus, $\mathbf{s}$ is basic w.r.t. $N \cap \mathrm{aff}(P_1 \cup P_2)$ if and only if

$$\mathrm{aff}_{\mathbb{Z}}\big(\mathrm{vert}(\mathcal{T}_1) \cup \mathrm{vert}(\mathcal{T}_2)\big) = N \cap \mathrm{aff}(P_1 \cup P_2).$$

(ii) If $\psi_1 \in \mathrm{SUCSF}(\mathcal{T}_1)$, $\psi_2 \in \mathrm{SUCSF}(\mathcal{T}_2)$, then the function Ψ defined by

$$\Psi(t\,\mathbf{x} + (1-t)\;\mathbf{y}) := t\;\psi_1(\mathbf{x}) + (1-t)\;\psi_2(\mathbf{y}), \text{ for all } \mathbf{x} \in P_1, \mathbf{y} \in P_2, \text{ and } t \in \mathbb{R} \cap [0,1],$$

belongs to $\mathrm{SUCSF}(\mathcal{T}_1 * \mathcal{T}_2)$.
(iii) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are both balanced and have dimensions d_1 and d_2, with colouring functions

$$\varphi_1 : \mathrm{vert}(\mathcal{T}_1) \to \{0, 1, 2, \ldots, d_1\} \quad \text{and} \quad \varphi_2 : \mathrm{vert}(\mathcal{T}_2) \to \{0, 1, 2, \ldots, d_2\},$$

then $\varphi_1 * \varphi_2 : \mathrm{vert}(\mathcal{T}_1 * \mathcal{T}_2) = \mathrm{vert}(\mathcal{T}_1) \cup \mathrm{vert}(\mathcal{T}_2) \to \{0, 1, \ldots, d_1 + d_2 + 1\}$ defined by

$$(\varphi_1 * \varphi_2)(\mathbf{x}) := \begin{cases} \varphi_1(\mathbf{x}), & \text{if } \mathbf{x} \in \mathrm{vert}(\mathcal{T}_1) \\ \varphi_2(\mathbf{x}) + d_1 + 1, & \text{if } \mathbf{x} \in \mathrm{vert}(\mathcal{T}_2) \end{cases}$$

is a colouring function for $\mathcal{T}_1 * \mathcal{T}_2$. □

(f) Let now $\mathcal{T}$ be a triangulation of d-polytope $P \subset \mathbb{R}^d$. If $\lambda P = \{\lambda \mathbf{x} \mid \mathbf{x} \in P\}$ is the λ-times *dilated* P, i.e., the image of P under the dilation map $d_\lambda : \mathbb{R}^d \ni \mathbf{x} \mapsto \lambda \mathbf{x} \in \mathbb{R}^d$, where $\lambda \in \mathbb{Z}$, $\lambda \geq 1$, then $\lambda \mathcal{T}$ will denote the corresponding triangulation of λP under the same dilation map, i.e., $\lambda \mathcal{T} = \{\lambda \mathbf{s} \mid \mathbf{s} \in \mathcal{T}\}$.

Lemma 3.6 (Preservation of the "c.b.-property" for dilations).
Let $\mathcal{T}$ denote a triangulation of a d-polytope $P \subset \mathbb{R}^d$, and $\lambda \geq 1$ an integer.

(i) *If $\mathcal{T}$ is coherent, then $\lambda \mathcal{T}$ is coherent.*

(ii) *If $\mathcal{T}$ is balanced, then $\lambda \mathcal{T}$ is balanced.*

(iii) *If P is a lattice polytope w.r.t. a lattice $N \subset \mathbb{R}^d$, and $\mathcal{T}$ a lattice triangulation, then $\lambda \mathcal{T}$ is a lattice triangulation too w.r.t. the same lattice.*

Proof. For (i), if $\psi \in \mathrm{SUCSF}(\mathcal{T})$, then $\psi \circ (d_\lambda)^{-1} \in \mathrm{SUCSF}(\lambda \mathcal{T})$. For (ii), if φ is a colouring function for $\mathcal{T}$, then $\varphi \circ (d_\lambda)^{-1}$ serves as a colouring function for $\lambda \mathcal{T}$. (iii) is obvious because λ is an integer. $\square$

Remark 3.7. If $\mathcal{T}$ is a basic triangulation of a lattice d-polytope $P \subset \mathbb{R}^d$, then for $\lambda \geq 2$, the dilation $\lambda \mathcal{T}$ is non-basic (see Figure 2). Nevertheless, as we shall see below in Prop. 6.1, any dilation of a b.c.b. triangulation admits a b.c.b. triangulation.

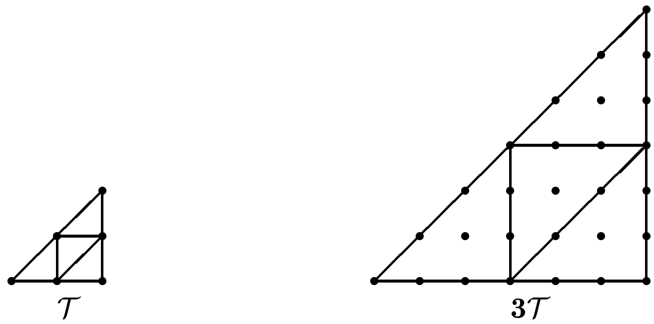

Figure 2

4. Torus-equivariant, crepant, projective resolutions

Reverting to (singular) Gorenstein abelian quotient spaces

$$X(N_G, \Delta_G) = U_{\sigma_0} = \mathbb{C}^d / G, \quad G \subset \mathrm{SL}(d, \mathbb{C}), \quad d \geq 2,$$

we explain how the desired desingularizations can be constructed by means of triangulations. Let $f = \mathrm{id}_* : X(N_G, \widehat{\Delta_G}) \to X(N_G, \Delta_G)$ be an arbitrary T_{N_G}-equivariant desingularization of $X(N_G, \Delta_G)$ (as in Thm. 2.4). There are one-to-one correspondences:

$$\begin{array}{ccc} \varrho & \in & \left(\widehat{\Delta_G}(1) \smallsetminus \{\mathbb{R}_{\geq 0}\, e_1, \ldots, \mathbb{R}_{\geq 0}\, e_d\} \right) \\ \updownarrow & & \updownarrow \\ n(\varrho) & \in & \left(\bigcup \left\{ \mathrm{Sk}^1(\sigma) \;\middle|\; \sigma \in \widehat{\Delta_G} \right\} \smallsetminus \{e_1, \ldots, e_d\} \right) \\ \updownarrow & & \updownarrow \\ D_{n(\varrho)} := V(\varrho) = V(\mathbb{R}_{\geq 0}\, n(\varrho)) & \in & \left\{ \begin{array}{c} \text{exceptional prime divisors} \\ \text{with respect to } f \end{array} \right\} \end{array} \tag{4.1}$$

Furthermore, $D_{e_i} := V(\mathbb{R}_{\geq 0}\, e_i)$, in $X(N_G, \widehat{\Delta_G})$, corresponds to the strict transform of

$$\left(\left\{\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_i = 0\right\}\right)/G$$

with respect to f, for $1 \leq i \leq d$.

Proposition 4.1. *A T_{N_G}-equivariant partial desingularization $f : X\left(N_G, \widehat{\Delta_G}\right) \to U_{\sigma_0}$ (defined by any fan $\widehat{\Delta_G}$ refining Δ_G w.r.t. N_G) is crepant if and only if*

$$\bigcup \left\{ \mathrm{Sk}^1(\sigma) \;\middle|\; \sigma \in \widehat{\Delta_G} \right\} \subset \left\{ \mathbf{x} = (x_1, \ldots, x_d)^\intercal \in (N_G)_{\mathbb{R}} \;\middle|\; \sum_{i=1}^{d} x_i = 1 \right\}. \tag{4.2}$$

Proof. Let $m_1, \ldots, m_d$ be $\mathbb{R}$-linearly independent vectors of M_G. The dualizing sheaf $\omega_{T_{N_G}}$ of T_{N_G} is generated by the rational differential form

$$\phi_{T_{N_G}} := \frac{\pm \det(M_G)}{\det\Big(\bigoplus_{1 \leq i \leq d} \mathbb{Z} m_i\Big)} \cdot \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_d}{u_d} \ ,$$

where $u_1 = \mathbf{e}(m_1), \ldots, u_d = \mathbf{e}(m_d)$ ($\phi_{T_{N_G}}$ is independent of the specific choice of $m_1, \ldots, m_d$). Moreover, for the dualizing sheaf $\omega_{X(N_G, \Delta_G)}$ of $X(N_G, \Delta_G)$ we have

$$\omega_{X(N_G,\Delta_G)} = \mathcal{O}_{X(N_G,\Delta_G)}\left(K_{X(N_G,\Delta_G)}\right) = \mathbb{C}\left[M_G \cap \ \mathrm{int}\left(\sigma_0^\vee\right)\right] \cdot \phi_{T_{N_G}}$$

(cf. [22, Lemma 3.3, p. 293]). On the other hand, conditions (i)-(v) of Prop. 2.5 are equivalent to the triviality of $\omega_{X(N_G,\Delta_G)}$, as one can easily verify by using Ishida's criteria [21, p. 126]. This means, in particular, that the semigroup ideal $M_G \cap \ \mathrm{int}(\sigma_0^\vee)$ of the semigroup ring $M_G \cap \sigma_0^\vee$ is principal. In fact, it is generated by the element $(1, 1, \ldots, 1, 1)$ and $\mathbb{C}[M_G \cap \ \mathrm{int}(\sigma_0^\vee)]$ by $\mathbf{e}((1, 1, \ldots, 1, 1))$, where $\mathrm{ord}_{V(\mathbb{R}_{\geq 0} e_i)}\left(((1, \ldots, 1)) \cdot \phi_{T_{N_G}}\right) = 1$ for $1 \leq i \leq d$.

f is crepant if and only if the difference

$$K_{X\left(N_G, \widehat{\Delta_G}\right)} - f^*\left(K_{X(N_G,\Delta_G)}\right)$$

between the canonical divisor of $X\left(N_G, \widehat{\Delta_G}\right)$ and the pull-back of the canonical (trivial) divisor of $X(N_G, \Delta_G)$ vanishes. In this case, $K_{X\left(N_G, \widehat{\Delta_G}\right)}$ is trivial too. Let

$$X(N_G, \widehat{\Delta_G}) = \Big(\coprod_{\sigma \in \widehat{\Delta_G}} \widehat{U}_\sigma\Big) \Big/ \sim,$$

$\sigma' \in \widehat{\Delta_G}(d)$ an arbitrary maximal cone of $\widehat{\Delta_G}$, ϱ a ray from $\left(\widehat{\Delta_G}(1) \smallsetminus \{\mathbb{R}_{\geq 0}\, e_1, \ldots, \mathbb{R}_{\geq 0}\, e_d\}\right)$ with $\varrho \prec \sigma'$, and $D_{n(\varrho)}$ the corresponding exceptional prime divisor (as in (4.1)). Since

$$H^0\left(\widehat{U}_{\sigma'}; \omega_{X\left(N_G, \widehat{\Delta_G}\right)}\right) = \mathbb{C}\left[M_G \cap \ \mathrm{int}\left((\sigma')^\vee\right)\right] \cdot \phi_{T_{N_G}} \ ,$$

by the same argument as above, if $\omega_{\widehat{U}_{\sigma'}}$ is trivial, then it is generated by $\mathbf{e}((1, 1, \ldots, 1, 1)) \cdot \phi_{T_{N_G}}$. The discrepancy of $D_{n(\varrho)}$ with respect to $f|_{\widehat{U}_{\sigma'}}$ is equal to $\mathrm{ord}_{D_{n(\varrho)}}\left(\mathbf{e}((1, \ldots, 1)) \cdot \phi_{T_{N_G}}\right) - 1$, where this vanishing order of $\left(\mathbf{e}((1, \ldots, 1)) \cdot \phi_{T_{N_G}}\right)$ along $D_{n(\varrho)}$ equals

$$\mathrm{ord}_{D_{n(\varrho)}}\left(\mathbf{e}((1, \ldots, 1)) \cdot \phi_{T_{N_G}}\right) \ = \ \langle (1, \ldots, 1), n(\varrho) \rangle \,,$$

by [7, lemma of p. 61] and the definition of $\phi_{T_{N_G}}$. Hence f is crepant if and only if the condition (4.2) is satisfied. $\square$

Corollary 4.2. *All T_{N_G}-equivariant partial crepant desingularizations of $U_{\sigma_0} = \mathbb{C}^d/G$ are of the form*

$$f_{\mathcal{T}} : X\left(N_G, \widehat{\Delta_G}(\mathcal{T})\right) \to U_{\sigma_0} \tag{4.3}$$

for some fan $\widehat{\Delta_G}(\mathcal{T}) = \{\sigma_{\mathbf{s}}, \mathbf{s} \in \mathcal{T}\}$, where $\sigma_{\mathbf{s}} := \{\gamma\mathbf{x} \in (N_G)_{\mathbb{R}} \mid \mathbf{x} \in \mathbf{s}, \gamma \in \mathbb{R}_{\geq 0}\}$ and $\mathcal{T}$ denotes a lattice triangulation of the junior simplex $\mathfrak{s}_G$.

Since a simplex $\mathbf{s} \in \mathcal{T}$ is basic if and only if $\operatorname{mult}(\sigma_{\mathbf{s}}) = 1$, i.e., if and only if the toric variety $U_{\sigma_{\mathbf{s}}}$ is smooth, we have:

Lemma 4.3. *$f_{\mathcal{T}} : X\left(N_G, \widehat{\Delta_G}(\mathcal{T})\right) \to U_{\sigma_0}$ is a crepant (full) desingularization if and only if $\mathcal{T}$ is basic.*

Example 4.4. Consider the 3-dim. Gorenstein cyclic quotient singularity of *type* $\frac{1}{7}(3,3,1)$ (in the sense of [24, § 4.2]). The junior simplex $\mathfrak{s}_G = \operatorname{conv}(\{e_1, e_2, e_3\})$ contains three additional inner lattice points, namely $n_1 = \frac{1}{7}(3,3,1)^{\intercal}$, $n_2 = \frac{1}{7}(2,2,3)^{\intercal}$, $n_3 = \frac{1}{7}(1,1,5)^{\intercal}$. The unique basic triangulation $\mathcal{T}$ of $\mathfrak{s}_G$ is drawn in Figure 3. The three exceptional prime divisors on $X\left(N_G, \widehat{\Delta_G}(\mathcal{T})\right)$ are isomorphic to

$$D_{n_1} \cong \mathbb{P}^2_{\mathbb{C}}, \quad D_{n_2} \cong \mathbb{F}_3, \quad \text{and} \quad D_{n_3} \cong \mathbb{F}_5,$$

respectively, where $\mathbb{F}_j := \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(j)\right), j \geq 0$, denote the Hirzebruch-surfaces.

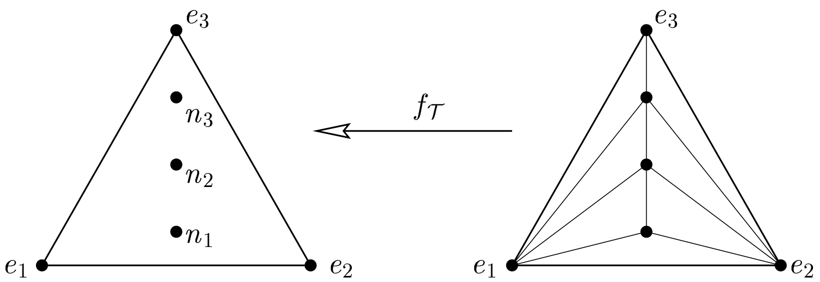

Figure 3

Coming back to (4.3), we see that there is a bijection

$$\operatorname{SUCSF}\left(N_G, \widehat{\Delta_G}(\mathcal{T})\right) \ni \psi \mapsto \psi|_{\mathcal{T}} \in \operatorname{SUCSF}(\mathcal{T})$$

(with inverse defined by taking the extension $\widetilde{\psi}$ of any $\psi \in \operatorname{SUCSF}(\mathcal{T})$, so that $\widetilde{\psi}(\gamma\mathbf{x}) = \psi(\mathbf{x})$, for all $\mathbf{x} \in \mathbf{s}$, $\mathbf{s} \in \mathcal{T}(d)$, and $\gamma \in \mathbb{R}_{\geq 0}$). Therefore, by 2.1, 2.2, we obtain:

Proposition 4.5. *$f_{\mathcal{T}}$ is a projective crepant morphism if and only if $\mathcal{T}$ is coherent.*

Hence, the existence of a projective, crepant (full) desingularization of $U_{\sigma_0} = \mathbb{C}^d/G$ is equivalent to the existence of a basic, coherent triangulation of the junior simplex $\mathfrak{s}_G$. Our use of balanced complexes is a requirement that is not needed from the toric point of view; however, it is our main technical tool for "gluing" triangulations together. In particular, it is not clear whether every dilation of a basic triangulation has a basic refinement; the same is proved for balanced basic triangulations, and for b.c.b. triangulations, below.

5. From special data to Watanabe simplices

In 1980 Watanabe [32, 1.7, 2.1 and 2.8] classified *all* abelian quotient c.i.-singularities in terms of "special data" that encode complete descriptions of the corresponding polynomial rings and acting groups. Although his method for proving this theorem is purely algebraic, he also indicated how could one represent these "special data" by certain graphs (which are actually forests, and will be henceforth called *Watanabe forests*). As we shall see in the present section, this graph-theoretic approach to the Classification Theorem 5.3 yields a very useful lattice-geometric interpretation. More precisely, we work out the following one-to-one correspondences:

$$\boxed{\begin{array}{ccc} \left\{\begin{array}{c}\text{special non-trivial}\\ \text{data } \mathbb{D}=(\mathfrak{D},\mathbf{w})\\ (\text{up to isomorphisms } \simeq)\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{c}\text{abelian quotient c.i.-singularities}\\ \left(\mathbb{C}^d/G,[\mathbf{0}]\right)\cong\left(\mathbb{C}^d/G_{\mathbb{D}},[\mathbf{0}]\right)\\ (\text{up to analytic isomorphisms } \cong)\end{array}\right\} \\ \updownarrow & & \updownarrow \\ \left\{\begin{array}{c}\text{non-trivial}\\ \text{Watanabe forests } \mathbf{W}_{\mathbb{D}}\\ (\text{up to graph-theoretic}\\ \text{isomorphisms } \cong_{\mathrm{wg}})\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{c}\text{Gorenstein, simplicial, singular, toric}\\ \text{varieties } X\left(N_G,\Delta_G\right)=U_{\sigma_0}\text{ for which}\\ \mathfrak{s}_G \text{ is a Watanabe simplex w.r.t. } N_G\\ (\text{up to affine integral transformations})\end{array}\right\} \end{array}}$$

Let us first formulate Watanabe's result.

Definition 5.1. Let $d \geq 2$ be an integer. A *special datum* $\mathbb{D}=(\mathfrak{D},\mathbf{w})$ (w.r.t. d) is a pair consisting of a set of non-empty subsets of $\{1,2,\ldots,d\}$ (i.e. $\mathfrak{D} \subseteq \mathbf{2}^{\{1,2,\ldots,d\}} \smallsetminus \{\varnothing\}$), together with a "weight-function" $\mathbf{w} : \mathfrak{D} \to \mathbb{N}$, such that:

(i) For each $i \in \{1,2,\ldots,d\}$ we have $\{i\} \in \mathfrak{D}$.

(ii) For every pair of index-subsets $J,\ J' \in \mathfrak{D}$, either $J \subseteq J'$, or $J' \subseteq J$, or $J \cap J' = \varnothing$.

(iii) If J is a maximal element with respect to the inclusion relation "$\subseteq$", then $\mathbf{w}(J) = 1$.

(iv) If $J,\ J' \in \mathfrak{D}$ and $J \subsetneqq J'$, then $\mathbf{w}(J) > \mathbf{w}(J')$ and $\mathbf{w}(J') \,|\, \mathbf{w}(J)$.

(v) For $J_1, J_2, J \in \mathfrak{D}$, with $J_1 \sqsubset J$, $J_2 \sqsubset J$, we have $\mathbf{w}(J_1) = \mathbf{w}(J_2)$.

For this the binary cover relation "$\sqsubset$" on the sets of $\mathfrak{D}$ is defined by:

$$J \sqsubset J' \underset{\text{def}}{\Longleftrightarrow} \left(J \subsetneqq J' \text{ and } \nexists J'',\ J'' \in \mathfrak{D} :\ J \subsetneqq J'' \subsetneqq J'\right).$$

$\mathbb{D}=(\mathfrak{D},\mathbf{w})$ is *non-trivial* if there is at least one subset $J \in \mathfrak{D}$ with cardinality $\#(J) \geq 2$.

Definition 5.2. Let $d \geq 2$ be an integer, and let $\mathbb{D}=(\mathfrak{D},\mathbf{w})$ be a special datum w.r.t. d. We define the polynomial ring $R_{\mathbb{D}}$ and the group $G_{\mathbb{D}}$ by

$$R_{\mathbb{D}} := \mathbb{C}\left[\mathfrak{x}_J \;\middle|\; J \in \mathfrak{D}\right] \quad \text{with} \quad \mathfrak{x}_J := \left(\prod_{i\in J} \mathfrak{x}_i\right)^{\mathbf{w}(J)}, \quad \text{and}$$

$$G_{\mathbb{D}} := \left\langle \left\{ \mathrm{diag}\Big(1,\ldots,1,\underbrace{\zeta_{\mathbf{w}}}_{i\text{-th pos.}},1,\ldots,1,\underbrace{\zeta_{\mathbf{w}}^{-1}}_{j\text{-th pos.}},1,\ldots,1\Big) \;\middle|\; \begin{array}{c} i \in J_1,\ j \in J_2 \\ \text{for all } J_1, J_2, J \in \mathfrak{D} \\ \text{with } J_1 \sqsubset J,\ J_2 \sqsubset J \\ \text{and } \mathbf{w} = \mathbf{w}(J_1) = \mathbf{w}(J_2) \end{array} \right\} \right\rangle$$

Here $\zeta_{\mathbf{w}}$ denotes a primitive $\mathbf{w}$th root of unity, and the diagonal matrices generating $G_{\mathbb{D}}$ are $(d \times d)$-matrices in $\mathrm{SL}(d,\mathbb{C})$.

Theorem 5.3 (Watanabe's Classification Theorem). *Let $d \geq 2$ be an integer and G be a finite, abelian subgroup of* $\mathrm{SL}(d,\mathbb{C})$ *of order* ≥ 2. *The following conditions are equivalent*:

(i) *The quotient space $\mathbb{C}^d/G$ is minimally embeddable as a complete intersection of hypersurfaces in an affine (complex) space.*

(ii) *There is a special datum $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ (w.r.t. d), such that*

$$\mathbb{C}^d/G \cong \text{Max-Spec}\,(R_{\mathbb{D}})$$

and G is conjugate to $G_{\mathbb{D}}$ (within $\mathrm{SL}(d,\mathbb{C})$*).*

In other words,

$$\left(\mathbb{C}^d/G, [\mathbf{0}]\right) \cong \left(\mathbb{C}^d/G_{\mathbb{D}}, [\mathbf{0}]\right),$$

i.e., up to an analytic isomorphism, all (germs of) abelian quotient c.i.*-singularities are parameterized by the set of non-trivial special data $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ (w.r.t. d).*

Remark 5.4. Let $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ be a special datum (w.r.t. d) and let $J_{\blacklozenge}^{[1]}, \ldots, J_{\blacklozenge}^{[\kappa]}$ denote the maximal elements of $\mathfrak{D}$ (w.r.t. "$\subseteq$"). By the properties (i) and (ii) in 5.1, we have:

(i) $\bigcup_{i=1}^{\kappa} J_{\blacklozenge}^{[i]} = \{1, 2, \ldots, d\}$ (set-theoretically).

(ii) For any $J \in \mathfrak{D}$ that is not a singleton ($\#(J) \geq 2$), there is a set-theoretic partition: $J = \dot{\bigcup} \ \{J' \in \mathfrak{D} \mid J' \sqsubset J\}$.

Remark 5.5. The elements of J's determine the positions of the roots of unity within the diagonals of the matrices generating $G_{\mathbb{D}}$. As there are many choices for the elements of J's leading to the same matrices (up to permutations of their entries), and since Theorem 5.3 gives the classification *up to isomorphism*, we shall define "isomorphisms" between special data in order to work only with convenient representatives of the corresponding equivalence-classes. For a special datum $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ (w.r.t. d) let

$$\mathfrak{D}\,(p) := \{J \in \mathfrak{D} \mid \#\,(J) = p\}, \quad 1 \leq p \leq d,$$

denote the subset of $\mathfrak{D}$ consisting of all index-sets of fixed cardinality p.

Two special data $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ and $\mathbb{D}' = (\mathfrak{D}', \mathbf{w}')$ (w.r.t. d) are *isomorphic* (and we denote this isomorphism by $\mathbb{D} \simeq \mathbb{D}'$) if there exists a bijection $\theta : \{1, \ldots, d\} \to \{1, \ldots, d\}$ for which

(i) $J \in \mathfrak{D} \Longleftrightarrow \Theta(J) \in \mathfrak{D}'$ (i.e., θ induces a bijection $\Theta : \mathfrak{D}(p) \leftrightarrow \mathfrak{D}'(p)$ for $1 \leq p \leq d$), and

(ii) $\mathbf{w}'\,(\Theta\,(J)) = \mathbf{w}\,(J)$, for all $J \in \mathfrak{D}$.

It is easy to verify the following equivalence-implications:

$$\mathbb{D} \simeq \mathbb{D}' \Longleftrightarrow \left(\begin{array}{c} G_{\mathbb{D}} \text{ and } G_{\mathbb{D}'} \text{ belong to the same} \\ \text{conjugacy class (within } \mathrm{SL}\,(d,\mathbb{C})\text{)} \end{array}\right) \Longleftrightarrow \left(\mathbb{C}^d/G_{\mathbb{D}}, [\mathbf{0}]\right) \cong \left(\mathbb{C}^d/G_{\mathbb{D}'}, [\mathbf{0}]\right)$$

Convention A. From now on, as representatives $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ of the equivalence-classes from

$$(\{\text{all special data (w.r.t. } d)\} \,/ \simeq),$$

we shall consider (without loss of generality) only $\mathfrak{D}$'s all of whose index subsets J have the form $J = \{\nu, \nu+1, \ldots, \xi-1, \xi\}$, $1 \leq \nu \leq \xi \leq d$, i.e., contiguous segments of $\{1, 2, \ldots, d\}$.

Convention B. We refer to the minimum and maximum elements of $J = \{\nu, \nu+1, \ldots, \xi-1, \xi\}$ with the notation $\nu = \nu_J$ and $\xi = \xi_J$.

Definition 5.6. Let $d \geq 2$ be an integer, $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ a special datum w.r.t. d, and

$$J_0 = \{\nu_0, \nu_0 + 1, \ldots, \xi_0 - 1, \xi_0\} \in \mathfrak{D}$$

a *fixed* index-set. We define the *subdatum* $\mathbb{D}_{J_0} = (\mathfrak{D}_{J_0}, \mathbf{w}_{J_0})$ of $\mathbb{D}$ *relative to* J_0 by

$$\mathfrak{D}_{J_0} := \{J \in \mathfrak{D} \mid J \subseteq J_0\}, \quad \text{and} \quad \mathbf{w}_{J_0}(J) := \frac{\mathbf{w}(J)}{\mathbf{w}(J_0)}, \text{ for all } J \in \mathfrak{D}_{J_0}.$$

The subdatum $\mathbb{D}_{J_0} = (\mathfrak{D}_{J_0}, \mathbf{w}_{J_0})$ can be viewed as an "autonomous" special datum $\mathbb{D}' = (\mathfrak{D}', \mathbf{w}')$ w.r.t. $\xi_0 - \nu_0 + 1$ via the bijection

$$\vartheta_0 : \mathfrak{D}_{J_0} \to \mathfrak{D}' = \{1, 2, \ldots, \xi_0 - \nu_0 + 1\}$$

defined by

$$\vartheta_0(\nu_0) = 1, \ \vartheta_0(\nu_0 + 1) = 2, \ \ldots \ , \vartheta_0(\xi_0) = \xi_0 - \nu_0 + 1,$$

(where $\mathbf{w}'(J) := \mathbf{w}_{J_0}\left(\vartheta_0^{-1}(J)\right)$, for all $J \in \mathfrak{D}'$).

Convention C. From now on we shall use a subdatum of a special datum $\mathbb{D}$ relative to some J_0 in both "roles" without referring explicitly to the identification map ϑ_0. (In the first case we shall emphasize the induced embedding $R_{\mathbb{D}_{J_0}} \hookrightarrow R_{\mathbb{D}}$ or $G_{\mathbb{D}_{J_0}} \subset G_{\mathbb{D}}$; in the latter, its own right to enjoy properties (i)-(v) of 5.1).

Next we recall some definitions concerning *trees*. We shall mostly use standard terminology. As usual, a *graph* is determined by the set of its *vertices* and the set of its *edges*. A vertex v is a *neighbour* of another vertex v', if v and v' are adjacent in the considered graph. A graph is *connected* if there is a path between any two vertices of it. A *cycle* in a graph is a simple path from a vertex to itself. By a *weighted graph* is meant a graph with weights assigned to its vertices. Two weighted graphs are *isomorphic* to each other (denoted by $\cong_{\text{wg}}$) if there exists a bijection between the sets of their vertices preserving both adjacency and weights.

A graph having no cycle is *acyclic*. A *tree* is a connected acyclic graph. (A *trivial tree* is a tree consisting of only one vertex). An arbitrary acyclic graph is called a *forest*. (So all connected components of a forest are trees). A *leaf* is a vertex of degree at most 1, i.e., a vertex being contained in at most one edge. A *rooted tree* distinguishes one vertex as its *root*. If v is a non-root vertex of a rooted tree, then its *parent* is the neighbour of v on the path connecting v with its root. The *children* of v are its other neighbours. In this case, the leaves are exactly the vertices without children.

A *rooted plane tree* is a rooted tree which is embedded in the plane, i.e., a rooted tree which is endowed with a left-to-right ordering specified for the children of each vertex. (By a *plane forest* we mean a forest having only rooted plane trees as connected components).

Definition 5.7. Let d be an integer ≥ 2 and $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ be a special datum w.r.t. d. The *Watanabe forest associated to* $\mathbb{D}$ is defined to be a weighted plane forest $\mathbf{W}_{\mathbb{D}}$ whose vertices are in one-to-one correspondence with the elements of $\mathfrak{D}$ and whose edges are to be drawn as follows: If J, J' are two elements of $\mathfrak{D}$, then the corresponding vertices of $\mathbf{W}_{\mathbb{D}}$, say v_J, $v_{J'}$, should be joinable by an edge of $\mathbf{W}_{\mathbb{D}}$ if and only if either $J \sqsubset J'$ or $J' \sqsubset J$. If $\mathbf{W}_{\mathbb{D}}$ happens to be a tree, then we shall speak of a *Watanabe tree* (being associated to $\mathbb{D}$). For a $J \in \mathfrak{D}$, we associate the weight $\mathbf{w}(v_J) := \mathbf{w}(J)$ to v_J. Moreover, for J, $J' \in \mathfrak{D}$, with $J \sqsubset J'$, using Watanabe's original *upside-down convention*, we shall join v_J and $v_{J'}$ in such a way, that $v_{J'}$ lies *over* v_J. It is therefore useful to regard the edge connecting v_J with $v_{J'}$ as "directed" and denote it by $\overrightarrow{v_J, v_{J'}}$. (For typographical reasons, in our figures we shall denote the vertices of the forest $\mathbf{W}_{\mathbb{D}}$ by $\boxed{v_J, \mathbf{w}(J)}$ enclosing also their weights.)

Remark 5.8. (i) By 5.1 (i) we have $\overset{\bullet}{\bigcup}\ \{J\ \in \mathfrak{D}\ \mid v_J\ \text{ leaf}\} = \{1,2,\ldots,d\}$ (set-theoretically).
(ii) By definition,

$$\mathbb{D} \simeq \mathbb{D}' \Longleftrightarrow \mathbf{W}_{\mathbb{D}} \cong_{\mathrm{wg}} \mathbf{W}_{\mathbb{D}'}$$

(iii) Let $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ be a special datum and assume that its Watanabe forest $\mathbf{W}_{\mathbb{D}}$ has connected components

$$\mathbf{W}_{\mathbb{D}^{[1]}}, \mathbf{W}_{\mathbb{D}^{[2]}}, \ldots, \mathbf{W}_{\mathbb{D}^{[\kappa-1]}}, \mathbf{W}_{\mathbb{D}^{[\kappa]}}$$

whose vertices are in one-to-one correspondence with the elements of

$$\mathfrak{D}^{[1]}, \mathfrak{D}^{[2]}, \ldots, \mathfrak{D}^{[\kappa-1]}, \mathfrak{D}^{[\kappa]}$$

for a partition $\mathfrak{D} = \bigcup^{\bullet}_{1\leq i\leq\kappa} \mathfrak{D}^{[i]}$ of $\mathfrak{D}$. Then each component $\mathbf{W}_{\mathbb{D}^{[i]}}$ has root $v_{J^{[i]}_{\blacklozenge}}$, with $J^{[i]}_{\blacklozenge}$ as in 5.4, and $\mathbb{D}^{[i]} := \left(\mathfrak{D}^{[i]}, \mathbf{w}\,|_{\mathfrak{D}^{[i]}}\right)$ is to be identified with the subdatum $\mathbb{D}_{J^{[i]}_{\blacklozenge}}$ (in the sense of 5.6), for $1 \leq i \leq \kappa$. Furthermore, the group $G_{\mathbb{D}}$ splits into the direct product

$$G_{\mathbb{D}} = G_{\mathbb{D}_1} \times G_{\mathbb{D}_2} \times \cdots \times G_{\mathbb{D}_{\kappa-1}} \times G_{\mathbb{D}_\kappa}.$$

Remark 5.9. Let $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ be a special datum (w.r.t. an integer $d \geq 2$). Then we have:
(i) $\left(\mathbb{C}^d/G_{\mathbb{D}}, [\mathbf{0}]\right) \cong (U_{\sigma_0}, \mathrm{orb}(\sigma_0))$ is a singularity if and only if $\mathbf{W}_{\mathbb{D}}$ contains at least one non-trivial tree. In this case,

$$\mathrm{splcod}\left(\mathrm{orb}(\sigma_0); U_{\sigma_0}\right) = d - \#\ \{\text{all trivial trees of }\ \mathbf{W}_{\mathbb{D}}\}\ .$$

(ii) $\left(\mathbb{C}^d/G_{\mathbb{D}}, [\mathbf{0}]\right)$ is a d-dimensional msc-singularity if and only if all connected components $\mathbf{W}_{\mathbb{D}^{[i]}}$ of $\mathbf{W}_{\mathbb{D}}$ are non-trivial trees. In this case, for $d \geq 3$, all $G_{\mathbb{D}}$'s are abelian, *non-cyclic* groups.
(iii) $\left(\mathbb{C}^d/G_{\mathbb{D}}, [\mathbf{0}]\right)$ is a d-dimensional absolutely unbreakable msc-singularity if and only if $\mathbf{W}_{\mathbb{D}}$ itself is a non-trivial Watanabe tree. If, in addition, this singularity is a hypersurface-singularity, then Theorem 5.3 can be simplified as follows:

Proposition 5.10. *Let G be a finite abelian subgroup of $\mathrm{SL}(d,\mathbb{C})$, $d \geq 2$, of order $l \geq 2$. Then $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ is an absolutely unbreakable hypersurface-msc-singularity if and only if G is conjugate (within $\mathrm{SL}(d,\mathbb{C})$) to a group of the form*

$$G(d;k) := \left\langle \left\{ \mathrm{diag}\left(1,1,\ldots,1, \underbrace{\zeta_k}_{i\text{-th position}}, \underbrace{\zeta_k^{-1}}_{(i+1)\text{-position}}, 1,\ldots,1,1\right) \;\middle|\; 1 \leq i \leq d-1 \right\} \right\rangle,$$

for a $k \geq 2$. In this case, $l = k^{d-1}$, and one has the following analytic germ isomorphisms

$$(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0)) \cong \left(\mathbb{C}^d/G(d;k), [\mathbf{0}_{\mathbb{C}^d}]\right)$$

and

$$\left(\mathbb{C}^d/G(d;k), [\mathbf{0}_{\mathbb{C}^d}]\right) \cong \left(\left\{ (z_0,..,z_d) \in \mathbb{C}^{d+1} \;\middle|\; z_0^k = \prod_{i=1}^{d} z_i \right\}, \mathbf{0}_{\mathbb{C}^{d+1}}\right). \tag{5.1}$$

In particular, $G(d;k) = G_{\mathbb{D}} \cong (\mathbb{Z}/k\mathbb{Z})^{d-1}$, *where* $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ *denotes the special datum with*

$$\mathfrak{D} = \{\{1\},\{2\},\ldots,\{d\},\{1,2,\ldots,d\}\}, \quad \mathbf{w}(\{j\}) = k,\ 1 \le j \le d, \quad \text{and} \quad \mathbf{w}(\{1,2,\ldots,d\}) = 1.$$

Its associated Watanabe tree is given in Figure 4.

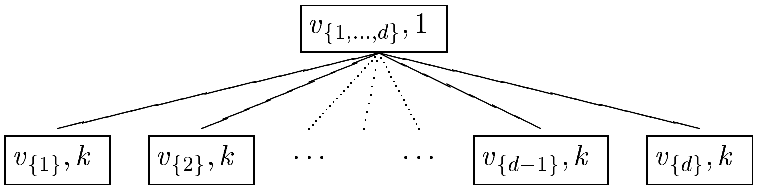

Figure 4

Definition 5.11. We shall call the above singularities (5.1) *$(d;k)$-hypersurface-singularities*, or simply *$(d;k)$-hypersurfaces.*

Remark 5.12. (i) Using the projection map

$$\left\{ (z_0, z_1, \ldots, z_d) \in \mathbb{C}^{d+1} \;\middle|\; z_0^k = \prod_{i=1}^{d} z_i \right\} \ni (z_0, z_1, \ldots, z_d) \longmapsto (z_1, \ldots, z_d) \in \mathbb{C}^d$$

one may regard $(\mathbb{C}^d/G(d;k), [\mathbf{0}_{\mathbb{C}^d}])$ as the total space of a k-sheeted covering of $(\mathbb{C}^d, \mathbf{0}_{\mathbb{C}^d})$ having the union of all coordinate hyperplanes of $\mathbb{C}^d$ as its branching locus.

(ii) Removing the assumption for $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ to be absolutely unbreakable, we obtain a direct product of such hypersurface-singularities.

To present a fairly short proof of our main Theorem 1.2 (by avoiding to work *simultaneously* with forests *and* triangulations), we introduce a new term, under the name "Watanabe simplex."

Definition 5.13. Let $d \ge 0$ be an integer and N a d-dimensional lattice in $N_{\mathbb{R}} \cong \mathbb{R}^d$. The *Watanabe simplices* w.r.t. N are the lattice simplices $\mathbf{s}$ satisfying

$$\mathrm{aff}_{\mathbb{Z}}(\mathbf{s}) \quad = \quad \mathrm{aff}(\mathbf{s}) \cap N$$

that arise in the following manner:

(i) Every 0-dimensional lattice simplex $\mathbf{s} = \{n\}$, $n \in N$, is a Watanabe simplex.

(ii) A lattice simplex $\mathbf{s} \subset N_{\mathbb{R}}$ of dimension $d' \ge 1$ is a Watanabe simplex if and only if

• *either* $\mathbf{s} = \mathbf{s}_1 * \mathbf{s}_2$, where $\mathbf{s}_1$, $\mathbf{s}_2$ are Watanabe simplices of dimensions d_1, $d_2 \ge 0$ *with* $d' = d_1 + d_2 + 1$, , with respect to sublattices $N_1 \subset \mathrm{aff}(\mathbf{s}_1)$, $N_2 \subset \mathrm{aff}(\mathbf{s}_2)$ of N, such that $N = \mathrm{aff}_{\mathbb{Z}}(N_1 \cup N_2)$,

• *or* $\mathbf{s}$ is a lattice translate of some dilation $\lambda\mathbf{s}'$, where $\lambda \ge 2$ is an integer, and $\mathbf{s}'$ is a d'-dimensional Watanabe simplex w.r.t. N.

(These conditions are mutually exclusive; with this definition every affine integral transformation that preserves N also preserves the Watanabe simplices of N.)

Example 5.14. The 2-dimensional lattice simplices (**A**) and (**B**) (w.r.t. $\mathbb{Z}^2$) of Figure 5 are Watanabe simplices, whereas (**C**) and (**D**) are not Watanabe simplices.

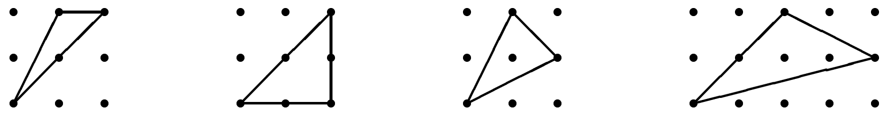

Figure 5

We are now going to prove the following:

Theorem 5.15 (Reduction-Theorem).
Let $d \geq 2$ be an integer, $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ a non-trivial special datum (w.r.t. d), $\mathbf{W}_{\mathbb{D}}$ the associated Watanabe forest, and $(\mathbb{C}^d/G_{\mathbb{D}}, [\mathbf{0}])$ the corresponding abelian quotient c.i.-singularity. If we identify the underlying space $\mathbb{C}^d/G_{\mathbb{D}}$ with the toric variety $X(N_{G_{\mathbb{D}}}, \Delta_{G_{\mathbb{D}}}) = U_{\sigma_0}$ (as in § 2, (i)), then the junior simplex $\mathfrak{s}_{G_{\mathbb{D}}}$ is a (non-basic) Watanabe simplex; and conversely, every (non-basic) Watanabe simplex w.r.t. an $N \cong \mathbb{Z}^d$ is (up to an affine integral transformation) the junior simplex corresponding to some abelian quotient c.i.-singularity.

The proof will be done in four steps. We begin with the case in which $\mathbf{W}_{\mathbb{D}}$ is a Watanabe tree.

• *First step.* Let $\mathbf{W}_{\mathbb{D}}$ denote a non-trivial Watanabe tree. At first we explain how the labeled weights naturally lead to "free parameters". By 5.4 (i), $J_\blacklozenge = \{1, 2, \ldots, d\}$ is the maximal element of $\mathfrak{D}$ (w.r.t. inclusion "$\subseteq$"). For any $J \in \mathfrak{D}$, $J \subsetneqq J_\blacklozenge$, there exists a unique chain of oversets

$$J_0 = J \sqsubset J_1 \sqsubset J_2 \sqsubset \cdots \sqsubset J_{\rho-1} \sqsubset J_\rho = J_\blacklozenge \quad (\text{with } \rho \geq 1)$$

which, in fact, corresponds to the directed path

$$\left[\overrightarrow{v_{J_0}, v_{J_1}},\ \overrightarrow{v_{J_1}, v_{J_2}},\ \ldots\ ,\ \overrightarrow{v_{J_{\rho-2}}, v_{J_{\rho-1}}},\ \overrightarrow{v_{J_{\rho-1}}, v_{J_\rho}}\right]$$

connecting v_J with the root $v_{J_\blacklozenge}$ of $\mathbf{W}_{\mathbb{D}}$. By 5.1 (iii)-(iv), the weights of the vertices of $\mathbf{W}_{\mathbb{D}}$ can be written in the form

$$\mathbf{w}(v_J) = \mathbf{w}(J) = \begin{cases} 1 & \text{if } J = J_\blacklozenge \\ \prod\limits_{i=0}^{\rho-1} k_{J_i, J_{i+1}} & \text{otherwise} \end{cases}$$

for integers $k_{J_0,J_1}, k_{J_1,J_2}, \ldots, k_{J_{\rho-1},J_\rho}$ (which are ≥ 2), assigned to the edges

$$\overrightarrow{v_{J_0}, v_{J_1}},\ \overrightarrow{v_{J_1}, v_{J_2}},\ \ldots\ ,\ \overrightarrow{v_{J_{\rho-2}}, v_{J_{\rho-1}}},\ \overrightarrow{v_{J_{\rho-1}}, v_{J_\rho}}\ .$$

Definition 5.16. By the above procedure we assign an integer $k_{J,J'} \geq 2$ to *every* edge $\overrightarrow{v_J, v_{J'}}$ of $\mathbf{W}_{\mathbb{D}}$ ($J \sqsubset J'$). These integers will be called the *free parameters* of $\mathbb{D}$ (or of $\mathbf{W}_{\mathbb{D}}$). For fixed J', the value $k_{J,J'}$ is the *same* for *all* J's for which $J \sqsubset J'$ (see 5.1 (v)). Thus for $k_{J,J'}$ we can use the alternative notation

$$k_{J,J'} =: k_{\nu_{J'}, \xi_{J'}}$$

which indicates the "free parameter source" (with $\nu_{J'}$ and $\xi_{J'}$ as defined in 5.5). For example, the free parameter k of $(d;k)$-hypersurfaces equals $k_{1,d}$.

Example 5.17. Up to an analytic isomorphism, all four-dimensional absolutely unbreakable, msc-, quotient c.i.-singularities are *either* $(4;k)$-hypersurfaces *or* singularities corresponding to exactly one of the four special data $\mathbb{D}$, whose associated Watanabe trees $\mathbf{W}_{\mathbb{D}}$ and free parameters $k_{\ldots}$ are depicted in Figures 6a)-6d).

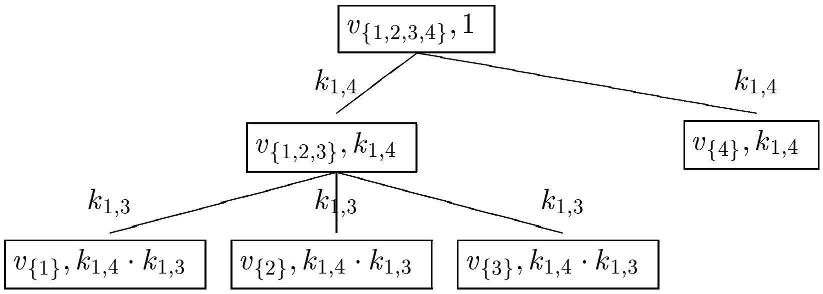

Figure 6a)

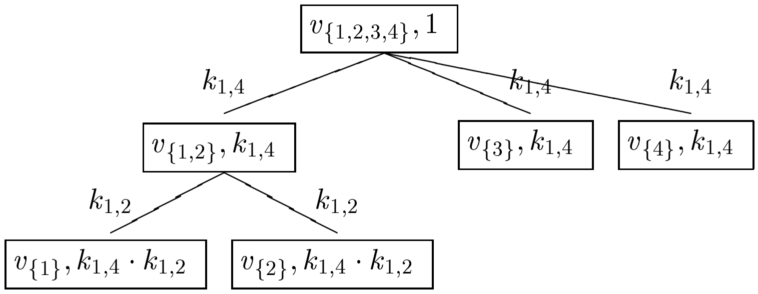

Figure 6b)

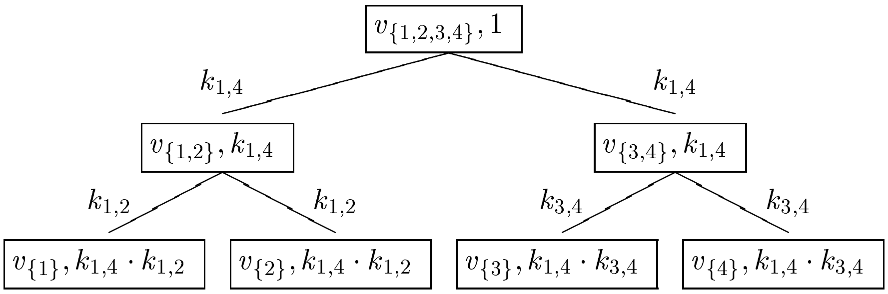

Figure 6c)

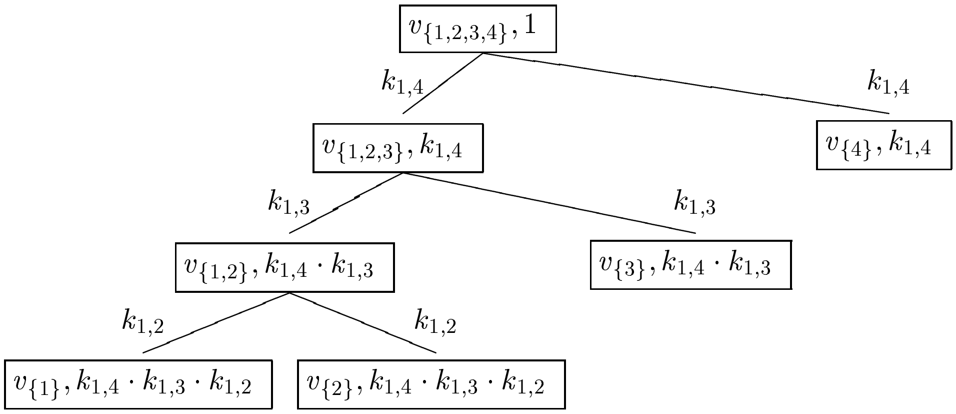

Figure 6d)

• *Second step.* Let $\mathbf{W}_{\mathbb{D}}$ be, as before, a non-trivial Watanabe tree associated to a special datum $\mathbb{D} = (\mathfrak{D}, \mathbf{w})$ w.r.t. a $d \geq 2$. Keeping conventions A- C (of 5.5, 5.6) in mind, we fix an enumeration, say $J_1, \ldots, J_p$, of all members of $\{J \in \mathfrak{D} \mid J \sqsubset J_{\blacklozenge}\}$, so that

$$J_i = \{\nu_i, \nu_i + 1, \ldots, \xi_i - 1, \xi_i\} \quad \text{for } 1 \leq i \leq p,$$

where

$$\nu_1 = 1 \leq \xi_1, \quad \nu_i = \xi_{i-1} + 1, \quad \text{for } 2 \leq i \leq p \ \text{ and } \ \nu_p \leq \xi_p = d,$$

and denote by $\mathcal{W}_1, \ldots, \mathcal{W}_p$ the subtrees of $\mathbf{W}_{\mathbb{D}}$ which begin with $v_{J_1}, \ldots, v_{J_p}$ as in Figure 7.

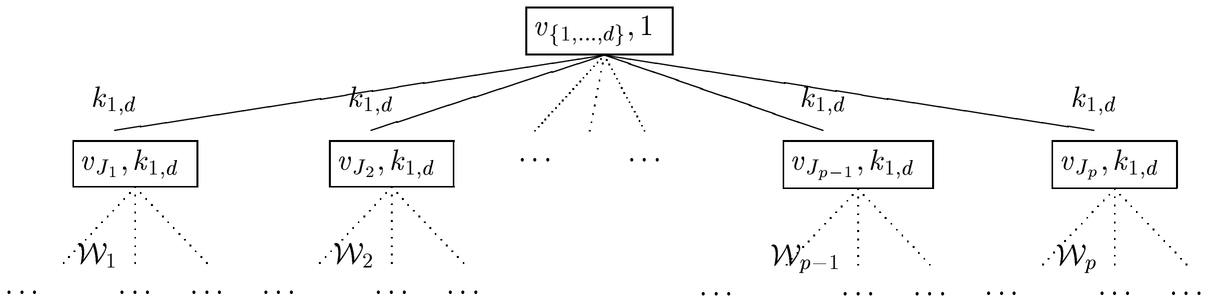

Figure 7

To the subtrees $\mathcal{W}_1, \ldots, \mathcal{W}_p$ we assign the "autonomous" Watanabe trees $\mathbf{W}_{\mathbb{D}_1}, \ldots, \mathbf{W}_{\mathbb{D}_p}$, induced by the subdata $\mathbb{D}_i = (\mathfrak{D}_i, \mathbf{w}_i)$ of $\mathbb{D}$, where

$$\mathfrak{D}_i := \mathfrak{D}_{J_i}, \quad \mathbf{w}_i := \mathbf{w}_{J_i}, \quad \text{for} \quad 1 \leq i \leq p,$$

(in the notation introduced in 5.6). $\mathbf{W}_{\mathbb{D}_i}$ is derived from $\mathbf{W}_{\mathbb{D}}$ after the killing of the parent $v_{J_{\blacklozenge}}$ of v_{J_i} (w.r.t. $\mathbf{W}_{\mathbb{D}}$), the christening of v_{J_i} as its root, and the adoption of the appropriate weights. In other words, the children of the root of $\mathbf{W}_{\mathbb{D}}$ become autonomous roots (and parents).

• In the arguments of the next four lemmas we shall use induction on the cardinality number d of $J_{\blacklozenge}$, i.e., on the dimension of the singularity. (That all assertions are correct for $d = 2$ can be checked easily, and will be therefore omitted). Our "induction hypothesis" is that all assertions (formulated below) are true for all Watanabe trees having roots corresponding to maximal index-sets of cardinality $< d$, and, in particular, true for $\mathbf{W}_{\mathbb{D}_1}, \ldots, \mathbf{W}_{\mathbb{D}_p}$. (We do not exclude the possibility that some of the $\mathbf{W}_{\mathbb{D}_i}$'s are trivial trees.)

Lemma 5.18. (i) *If we define*

$$\mathfrak{Y}_{\mathbb{D}} := \bigcup_{J \in \mathfrak{D}} \left\{ \mathrm{diag}\Big(1, \ldots, 1, \underbrace{\zeta_{\mathbf{w}(J')}}_{\nu_J\text{-th pos.}}, 1, \ldots, 1, \underbrace{\zeta_{\mathbf{w}(J')}^{-1}}_{\nu_{J'}\text{-th pos.}}, 1, \ldots, 1\Big) \;\middle|\; \begin{array}{l} \text{for all } J' \in \mathfrak{D} \\ \text{with } \ J' \sqsubset J \\ \text{and } \nu_J < \nu_{J'} \end{array} \right\},$$

then $G_{\mathbb{D}} = \langle \mathfrak{Y}_{\mathbb{D}} \rangle$, *and* $\mathfrak{Y}_{\mathbb{D}}$ *is a minimal generating system for* G.
(ii) *Identifying* $\mathbb{C}^d / G_{\mathbb{D}}$ *with* $X\left(N_{G_{\mathbb{D}}}, \Delta_{G_{\mathbb{D}}}\right) = U_{\sigma_0}$ *as in* §2(i), *we obtain*

$$N_{G_{\mathbb{D}}} = \sum_{j=1}^{d} \ \mathbb{Z}\ e_j + \mathbb{Z}\ \mathfrak{N}_{\mathbb{D}},$$

where $\{e_1,\ldots,e_d\}$ *are the unit vectors of* $(N_{G_{\mathbb{D}}})_{\mathbb{R}} \cong \mathbb{R}^d$, *and*

$$\mathfrak{N}_{\mathbb{D}} := \bigcup_{J\in\mathfrak{D}} \left\{ \frac{1}{\mathbf{w}(J')} \left(e_{\nu_J} - e_{\nu_{J'}}\right) \;\middle|\; \begin{array}{l} \text{for all } J' \in \mathfrak{D} \\ \text{with } \ J' \sqsubset J \\ \text{and } \nu_J < \nu_{J'} \end{array} \right\} .$$

(iii) *Furthermore,* $\#(\mathfrak{Y}_{\mathbb{D}}) = \#(\mathfrak{N}_{\mathbb{D}}) = d-1$.

Proof. (i) Obviously, each element of the generating system given in the definition 5.2 of $G_{\mathbb{D}}$ can be written as product of elements of $\mathfrak{Y}_{\mathbb{D}}$, and any group generated by a proper subset of $\mathfrak{Y}_{\mathbb{D}}$ is a proper subgroup of $G_{\mathbb{D}}$.
(ii) This follows from the usual representation of the group elements of $\mathfrak{Y}_{\mathbb{D}}$ by lattice points in $N_{G_{\mathbb{D}}}$.
(iii) By induction hypothesis, $\#(\mathfrak{Y}_{\mathbb{D}_i}) = \#(\mathfrak{N}_{\mathbb{D}_i})$ equals $\nu_i - \xi_i$, for all i, $1 \le i \le p$. Hence,

$$\begin{aligned} \#(\mathfrak{Y}_{\mathbb{D}}) &= \#(\mathfrak{N}_{\mathbb{D}}) = \sum_{i=1}^{p} (\nu_i - \xi_i) + (p-1) = \sum_{i=1}^{p} \nu_i - \sum_{i=2}^{p} \xi_i + (p-2) \\ &= \sum_{i=1}^{p} \nu_i - \sum_{i=2}^{p} (\nu_{i-1}+1) + (p-2) = d-1, \end{aligned}$$

and the proof is completed. □

In the following we denote by $\left\{n_2^{(\mathbb{D})}, n_3^{(\mathbb{D})}, \ldots, n_d^{(\mathbb{D})}\right\}$ an enumeration of the $d-1$ elements of $\mathfrak{N}_{\mathbb{D}}$ such that

$$n_{\nu_i}^{(\mathbb{D})} = \frac{1}{k_{1,d}} \left(e_1 - e_{\nu_i}\right) \quad \text{for } 2 \le i \le p. \tag{5.2}$$

Furthermore, we assume that we have chosen an enumeration of the elements $\left\{n_{\nu_i+1}^{(\mathbb{D}_i)}, \ldots, n_{\xi_i}^{(\mathbb{D}_i)}\right\}$ of $\mathfrak{N}_{\mathbb{D}_i}$ such that

$$n_{\nu_i+1}^{(\mathbb{D})} = \frac{1}{k_{1,d}}\, n_{\nu_i+1}^{(\mathbb{D}_i)},\;\; n_{\nu_i+2}^{(\mathbb{D})} = \frac{1}{k_{1,d}}\, n_{\nu_i+2}^{(\mathbb{D}_i)},\;\; \ldots,\;\; n_{\xi_i}^{(\mathbb{D})} = \frac{1}{k_{1,d}}\, n_{\xi_i}^{(\mathbb{D}_i)}, \quad \text{for } 1 \le i \le p.$$

Lemma 5.19.

(i) $\left\{e_1, n_2^{(\mathbb{D})}, n_3^{(\mathbb{D})}, \ldots, n_d^{(\mathbb{D})}\right\}$ *is a* $\mathbb{Z}$*-basis of the lattice* $N_{G_{\mathbb{D}}}$.

(ii) $\left\{n_2^{(\mathbb{D})}, n_3^{(\mathbb{D})}, \ldots, n_d^{(\mathbb{D})}\right\}$ *forms a* $\mathbb{Z}$*-basis of* $N_{G_{\mathbb{D}}} \cap \operatorname{lin}(\{e_1-e_2, e_1-e_3, \ldots, e_1-e_d\})$.

Proof. (i) By definition of $N_{G_{\mathbb{D}}}$ it suffices to show that $\mathbb{Z}^d \subset \mathbb{Z}\{e_1, n_2^{(\mathbb{D})}, n_3^{(\mathbb{D})}, \ldots, n_d^{(\mathbb{D})}\}$. By induction hypothesis, we may assume

$$\mathbb{Z}\, e_{\nu_i} + \mathbb{Z}\, e_{\nu_i+1} + \cdots + \mathbb{Z}\, e_{\xi_i} \subset \mathbb{Z}\left\{e_{\nu_i}, n_{\nu_i+1}^{(\mathbb{D}_i)}, \ldots, n_{\xi_i}^{(\mathbb{D}_i)}\right\}, \quad \text{for } 1 \le i \le p.$$

Since $k_{1,d} \in \mathbb{Z}$ we also have

$$\mathbb{Z}\, e_{\nu_i} + \mathbb{Z}\, e_{\nu_i+1} + \cdots + \mathbb{Z}\, e_{\xi_i} \subset \mathbb{Z}\left\{e_1, n_{\nu_i}^{(\mathbb{D})}, n_{\nu_{i+1}}^{(\mathbb{D})}, \ldots, n_{\xi_i}^{(\mathbb{D})}\right\}$$

and thus

$$\bigcup_{i=2}^{p}\left\{e_1, n^{(\mathbb{D})}_{\nu_i}, n^{(\mathbb{D})}_{\nu_i+1}, \ldots, n^{(\mathbb{D})}_{\xi_i}\right\} \quad = \quad \left\{e_1, n^{(\mathbb{D})}_{2}, n^{(\mathbb{D})}_{3}, \ldots, n^{(\mathbb{D})}_{d}\right\}$$

is indeed a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

(ii) By (i), it suffices to prove that each vector $n^{(\mathbb{D})}_i$, $2 \leq i \leq d$, belongs to the subspace $\mathrm{lin}(\{e_1 - e_2, \ldots, e_1 - e_d\})$. By (5.2) this is true for all $n^{(\mathbb{D})}_{\nu_i}$, $2 \leq i \leq p$. By induction hypothesis, we may assume that

$$n^{(\mathbb{D}_i)}_{\nu_i+j} \in \mathrm{lin}\left(\left\{e_{\nu_i} - e_{\nu_i+1}, \ldots, e_{\nu_i} - e_{\xi_i}\right\}\right), \quad 1 \leq i \leq p,\ 1 \leq j \leq \xi_i - \nu_i,$$

and hence that

$$n^{(\mathbb{D})}_{\nu_i+j} \in \mathrm{lin}\left(\left\{e_{\nu_i} - e_{\nu_i+1}, \ldots, e_{\nu_i} - e_{\xi_i}\right\}\right)$$

too. Since $e_{\nu_i} - e_{\nu_i+j} = (e_1 - e_{\nu_i+j}) - (e_1 - e_{\nu_i})$, we are done. $\square$

• *Third step.* Since the lattice $N_{G_\mathbb{D}}$ is "skew" and d-dimensional, it is not so convenient to work directly with it. For this reason, we shall perform two affine transformations (cf. Lemma 3.4), such that $N_{G_\mathbb{D}} \cap \mathrm{aff}(\mathfrak{s}_{G_\mathbb{D}}) = N_{G_\mathbb{D}} \cap \mathrm{aff}(e_1, \ldots, e_d)$ becomes the standard lattice $\sum_{i=2}^{d} \mathbb{Z}e_i \cong \mathbb{Z}^{d-1} \subset \mathbb{R}^d$.

Define first $\Phi^{\mathbb{D}}_1 : (N_{G_\mathbb{D}})_\mathbb{R} \to (N_{G_\mathbb{D}})_\mathbb{R}$ by

$$\Phi^{\mathbb{D}}_1(\mathbf{x}) := e_1 - \mathbf{x}\ .$$

Obviously,

$$\Phi^{\mathbb{D}}_1\left(\mathfrak{s}_{G_\mathbb{D}}\right) = \mathrm{conv}\left(\{\mathbf{0}, e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_d\}\right),$$

and $\Phi^{\mathbb{D}}_1(N_{G_\mathbb{D}}) = N_{G_\mathbb{D}}$. After that define the affine (linear) transformation (cf. Lemma 5.19) $\Phi^{\mathbb{D}}_2 : (N_{G_\mathbb{D}})_\mathbb{R} \to (N_{G_\mathbb{D}})_\mathbb{R}$ by setting $\Phi^{\mathbb{D}}_2(e_1) = e_1$ and

$$\Phi^{\mathbb{D}}_2\left(\frac{1}{\mathbf{w}(J')}\ \left(e_{\nu_J} - e_{\nu_{J'}}\right)\right) := e_{\nu_{J'}},\quad \text{for all elements}\quad \frac{1}{\mathbf{w}(J')}\ \left(e_{\nu_J} - e_{\nu_{J'}}\right)\quad \text{of}\ \ \mathfrak{N}_\mathbb{D}\ .$$

Now let $\mathbf{\Phi}^{\mathbb{D}} : (N_{G_\mathbb{D}})_\mathbb{R} \to (N_{G_\mathbb{D}})_\mathbb{R}$ be the decomposition $\mathbf{\Phi}^{\mathbb{D}} := \Phi^{\mathbb{D}}_2 \circ \Phi^{\mathbb{D}}_1$ and let

$$\widetilde{\mathfrak{s}_{G_\mathbb{D}}} := \mathbf{\Phi}^{\mathbb{D}}(\mathfrak{s}_{G_\mathbb{D}}) \quad \text{and} \quad \widetilde{N^{\mathfrak{s}}_{G_\mathbb{D}}} := \mathbf{\Phi}^{\mathbb{D}}\left(N_{G_\mathbb{D}} \cap \mathrm{aff}(\mathfrak{s}_{G_\mathbb{D}})\right).$$

Observe that $\widetilde{N^{\mathfrak{s}}_{G_\mathbb{D}}} = \sum_{i=2}^{d} \mathbb{Z}e_i \cong \mathbb{Z}^{d-1} \subset \mathbb{R}^d$. To provide a convenient description of the vertices of $\Phi^{\mathbb{D}}(\mathfrak{s}_{G_\mathbb{D}})$ incorporating our inductive argumentation, we denote the vertex set of $\widetilde{\mathfrak{s}_{G_{\mathbb{D}_i}}} = \mathbf{\Phi}^{\mathbb{D}_i}(\mathfrak{s}_{G_{\mathbb{D}_i}}) \subset \sum_{j=\nu_i+1}^{\xi_i} \mathbb{Z}e_j$ corresponding to the tree $\mathbf{W}_{\mathbb{D}_i}$ by

$$\mathrm{vert}\left(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_i}}}\right) = \left\{\mathbf{0}, \mathfrak{y}^{(\mathbb{D}_i)}_{\nu_i+1}, \mathfrak{y}^{(\mathbb{D}_i)}_{\nu_i+2}, \ldots, \mathfrak{y}^{(\mathbb{D}_i)}_{\xi_i}\right\},$$

in such a way that

$$\mathfrak{y}^{(\mathbb{D}_i)}_{\nu_i+j} := \mathbf{\Phi}^{\mathbb{D}_i}\left(e_{\nu_i+j}\right) = \Phi^{\mathbb{D}_i}_2\left(e_{\nu_i} - e_{\nu_i+j}\right), \quad 1 \leq i \leq p, 1 \leq j \leq \nu_i - \xi_i.$$

Analogously, we label the vertices in $\mathrm{vert}(\widetilde{\mathfrak{s}_{G_\mathbb{D}}}) = \{\mathbf{0}, \mathfrak{y}^{(\mathbb{D})}_2, \mathfrak{y}^{(\mathbb{D})}_3, \ldots, \mathfrak{y}^{(\mathbb{D})}_d\}$ with

$$\mathfrak{y}^{(\mathbb{D})}_{j+1} := \mathbf{\Phi}^{\mathbb{D}}\left(e_{j+1}\right) = \Phi^{\mathbb{D}}_2\left(e_1 - e_{j+1}\right) \quad \text{for } 1 \leq j \leq d-1.$$

Lemma 5.20. *Using the notation introduced above,*

$$\mathfrak{y}^{(\mathbb{D})}_{\nu_i} = k_{1,d}\cdot e_{\nu_i}, \quad 2\le i\le p, \tag{5.3}$$

$$\mathfrak{y}^{(\mathbb{D})}_{\nu_1+j} = k_{1,d}\cdot \mathfrak{y}^{(\mathbb{D}_1)}_{\nu_1+j}, \quad 1\le j\le \xi_1-\nu_1, \tag{5.4}$$

$$\mathfrak{y}^{(\mathbb{D})}_{\nu_i+j} = k_{1,d}\cdot\left(\mathfrak{y}^{(\mathbb{D}_i)}_{\nu_i+j}+e_{\nu_i}\right), \quad 2\le i\le p,\ 1\le j\le \xi_j-\nu_j. \tag{5.5}$$

Proof. (5.3) follows from (5.2) because

$$\mathfrak{y}^{(\mathbb{D})}_{\nu_i} = \Phi^{\mathbb{D}}_2\,(e_1-e_{\nu_i}) = k_{1,d}\cdot\Phi^{\mathbb{D}}_2\left(n^{(\mathbb{D})}_{\nu_i}\right) = k_{1,d}\cdot e_{\nu_i}\ .$$

On the other hand,

$$\Phi^{\mathbb{D}}_2\,(e_{\nu_i}-e_{\nu_i+j}) = k_{1,d}\cdot\Phi^{\mathbb{D}_i}_2\,(e_{\nu_i}-e_{\nu_i+j}) = k_{1,d}\cdot\mathfrak{y}^{(\mathbb{D}_i)}_{\nu_i+j}$$

and (5.4) is clear by setting $i=1$ ($\nu_1=1$). Finally, to get (5.5) we write

$$\begin{aligned}\Phi^{\mathbb{D}}_2\,(e_1-e_{\nu_i+j}) &= \Phi^{\mathbb{D}}_2\,((e_1-e_{\nu_i})+(e_{\nu_i}-e_{\nu_i+j}))\\ &= \Phi^{\mathbb{D}}_2\,(e_1-e_{\nu_i})+\Phi^{\mathbb{D}}_2\,(e_{\nu_i}-e_{\nu_i+j}) = k_{1,d}\cdot e_{\nu_i}+k_{1,d}\cdot\mathfrak{y}^{(\mathbb{D}_i)}_{\nu_i+j}\end{aligned}$$

and we are done. □

Lemma 5.21. *The image $\widetilde{\mathfrak{s}_{\mathbb{D}}}$ of the junior simplex $\mathfrak{s}_{G_{\mathbb{D}}}$ under $\Phi^{\mathbb{D}}$ satisfies the following:*

(i)
$$\sum_{j=2}^{d}\mathbb{Z}e_j = \widetilde{N^{\mathfrak{s}}_{G_{\mathbb{D}}}} = \mathrm{aff}_{\mathbb{Z}}\left(\widetilde{N^{\mathfrak{s}}_{G_{\mathbb{D}_1}}}\cup\left(\widetilde{N^{\mathfrak{s}}_{G_{\mathbb{D}_2}}}+e_{\nu_2}\right)\cup\ldots\cup\left(\widetilde{N^{\mathfrak{s}}_{G_{\mathbb{D}_p}}}+e_{\nu_p}\right)\right).$$

(ii)
$$d-1 = \dim\left(\widetilde{\mathfrak{s}_{G_{\mathbb{D}}}}\right) = \dim\left(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_1}}}\right)+\sum_{i=2}^{p}\dim\left(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_i}}}+e_{\nu_i}\right)+p-1,$$

(iii) *The simplex $\widetilde{\mathfrak{s}_{G_{\mathbb{D}}}}$ can be expressed as a dilation of $p-1$ simplex-joins*

$$\widetilde{\mathfrak{s}_{G_{\mathbb{D}}}} = k_{1,d}\ \left(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_1}}}*\left(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_2}}}+e_{\nu_2}\right)*\cdots*\left(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_p}}}+e_{\nu_p}\right)\right)\subset\sum_{j=2}^{d}\mathbb{Z}e_j\cong\mathbb{Z}^{d-1}.$$

Proof. By construction we have $\widetilde{N^{\mathfrak{s}}_{G_{\mathbb{D}_i}}} = \sum_{j=\nu_i+1}^{\xi_i}\mathbb{Z}e_j$ which shows (i). Obviously, $\dim(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_i}}}) = \xi_i-\nu_i$ and since the simplices $\widetilde{\mathfrak{s}_{G_{\mathbb{D}_1}}}$, $(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_2}}}+e_{\nu_2}),\ldots,(\widetilde{\mathfrak{s}_{G_{\mathbb{D}_p}}}+e_{\nu_p})$ are contained in complementary subspaces, (ii) and (iii) are immediate consequences of Lemma 5.20. □

Example 5.22. If we consider the 7-dimensional absolutely unbreakable, msc-, quotient c.i.-singularity corresponding to the special datum $\mathbb{D}$ whose Watanabe tree $\mathbf{W}_{\mathbb{D}}$ shown in Figure 8, then the image $\widetilde{\mathfrak{s}_{\mathbb{D}}}$ of the junior simplex $\mathfrak{s}_{\mathbb{D}}$ under $\boldsymbol{\Phi}^{\mathbb{D}}$ is the Watanabe simplex defined as convex hull of the seven vectors:

$(0,0,0,0,0,0,0)^{\intercal}$,
$(0,k_{1,7}k_{1,2},0,0,0,0,0)^{\intercal}$,
$(0,0,k_{1,7},0,0,0,0)^{\intercal}$,
$(0,0,0,k_{1,7},0,0,0)^{\intercal}$,
$(0,0,0,0,k_{1,7},k_{1,7}k_{4,7}k_{4,5},0)^{\intercal}$,
$(0,0,0,k_{1,7},0,k_{1,7}k_{4,7},0)^{\intercal}$,
$(0,0,0,k_{1,7},0,k_{1,7}k_{4,7},k_{1,7}k_{4,7}k_{4,5})^{\intercal}$.

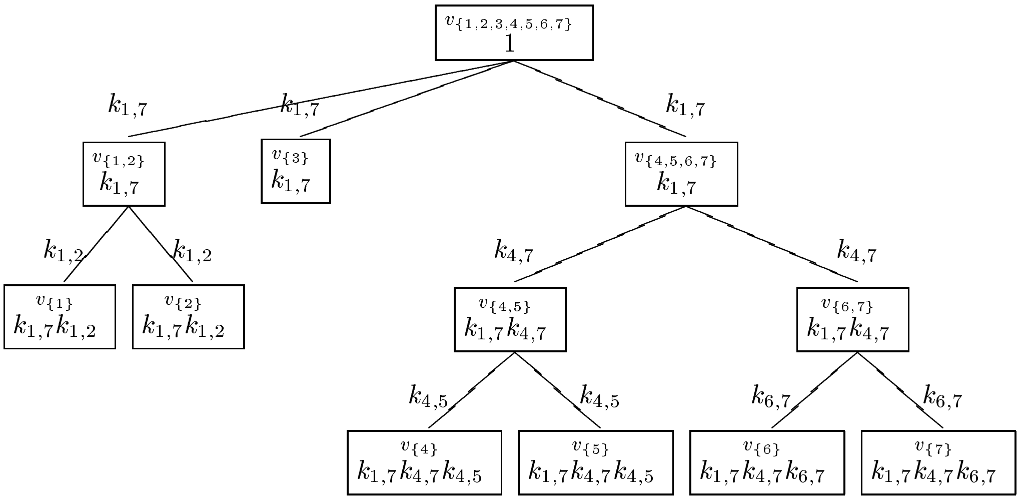

Figure 8

• *Fourth step.* By Lemma 5.21 and by construction, it is now clear that, up to an affine integral transformation, $\mathfrak{s}_{G_{\mathbb{D}}}$ is a Watanabe simplex whenever $\mathbf{W}_{\mathbb{D}}$ is a Watanabe tree. If $\mathbf{W}_{\mathbb{D}}$ is a Watanabe forest, then we recall the "superscripts in square brackets" from 5.8 (iii): Taking its connected components to be $\mathbf{W}_{\mathbb{D}[i]}$, $1 \leq i \leq \kappa$, the group $G_{\mathbb{D}}$ splits into the direct product of $G_{\mathbb{D}[i]}$'s, and by arguments similar to those of the previous steps we can show that $\mathfrak{s}_{G_{\mathbb{D}}}$ is again a transformed join of $\mathfrak{s}_{G_{\mathbb{D}[1]}}$ with the junior simplices $\mathfrak{s}_{G_{\mathbb{D}[i]}}$, $2 \leq i \leq \kappa$, corresponding to the other components, after translating by unit vectors. (We do not need an extra dilation in this case). Hence, the one direction of the Reduction Theorem 5.15 is completely proved. The proof of the converse statement is based on the "backtracking method", and on the fact that any toric affine variety being associated to a rational s.c.p. cone supported by a (non-basic) simplex, is the underlying space of an abelian quotient singularity; it is therefore omitted.

Example 5.23. Up to isomorphism, the only four-dimensional abelian, msc-, quotient c.i.-singularity which "breaks" is that corresponding to the datum $\mathbb{D}$ and Watanabe forest $\mathbf{W}_{\mathbb{D}}$ of Figure 9. (In fact, it splits into two Hirzebruch-Jung singularities of types $A_{k_{1,2}-1}$ and $A_{k_{3,4}-1}$, respectively).

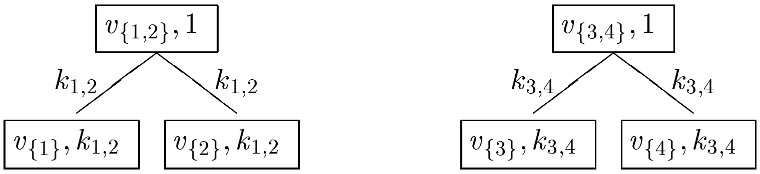

Figure 9

The junior tetrahedron $\mathfrak{s}_{G_{\mathbb{D}}}$ (which is the join of two 1-dimensional Watanabe simplices), is drawn in Figure 10, for the values of free parameters $k_{1,2} = 3, \;\; k_{3,4} = 5$.

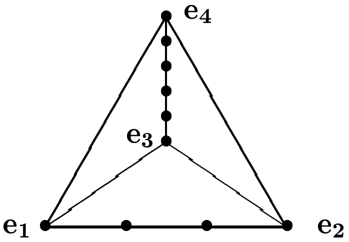

Figure 10

6. Final step: Basic, coherent triangulations of the junior simplex

In this section we give the proof of our Main Theorem 1.2. To obtain the desired basic, coherent triangulations of the junior simplices $\mathfrak{s}_G$ of all abelian quotient c.i.-singularities, it suffices (by 3.4, 5.15) to construct b.c.b.-triangulations for all Watanabe simplices.

Proposition 6.1. *All dilations $\lambda\,\mathbf{s}$ for integral $\lambda \geq 2$, of a basic lattice d-simplex $\mathbf{s}$ in $\mathbb{R}^d$, have b.c.b.-triangulations.*

Proof. Up to an affine integral transformation, we may assume that $\mathbf{s}$ equals $\mathbf{s}_d$ (w.r.t. $\mathbb{Z}^d$), where

$$\mathbf{s}_d := \text{conv}\left(\{\mathbf{0}, e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_d\}\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid 0 \leq x_d \leq x_{d-1} \leq \ldots \leq x_1 \leq 1\right\}.$$

The affine hyperplane arrangement $\mathbb{H}_d$ (of type $\widetilde{\mathcal{A}}_d$) consisting of the union of hyperplanes

$$\mathcal{H}_i\left(k\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid x_i = k\right\}, \quad \text{for } 1 \leq i \leq d, \quad k \in \mathbb{Z}$$

and

$$\mathcal{H}_{i,j}\left(k\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid x_i - x_j = k\right\} \quad \text{for } 1 \leq i < j \leq d, \quad k \in \mathbb{Z}$$

is infinite, but only the hyperplanes

$$\mathcal{H}_i\left(k\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid x_i = k\right\}, \quad \text{for } 1 \leq i \leq d, \quad 1 \leq k \leq \lambda - 1$$

and

$$\mathcal{H}_{i,j}\left(k\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid x_i - x_j = k\right\} \quad \text{for } 1 \leq i < j \leq d, \quad 1 \leq k \leq \lambda - 1$$

intersect the interior of $\lambda\,\mathbf{s}_d$. The hyperplanes $\mathcal{H}_i\left(k\right), 1 \leq i \leq d, k \in \mathbb{Z}$, subdivide $\mathbb{R}^d$ into unit cubes. Given such a cube

$$C\left(\boldsymbol{\mu}\right) := [0,1]^d + \boldsymbol{\mu} \qquad (\text{where } \boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d)$$

and fixing i, j with $1 \leq i < j \leq d$, we see that there is only one of the hyperplanes $\mathcal{H}_{i,j}\left(k\right)$ intersecting the interior of $C\left(\boldsymbol{\mu}\right)$, namely $\mathcal{H}_{i,j}\left(\mu_i - \mu_j\right)$. The hyperplanes

$$\left\{\mathcal{H}_{i,j}\left(\mu_i - \mu_j\right) \mid 1 \leq i < j \leq d\right\}$$

provide the "usual" basic triangulation of $C\left(\boldsymbol{\mu}\right)$ into $d!$ basic subsimplices of the form

$$s\left(\boldsymbol{\mu},\theta\right) = \boldsymbol{\mu} + \text{conv}\left(\left\{\mathbf{0}, e_{\theta(1)}, e_{\theta(1)+\theta(2)}, \ldots, e_{\theta(1)+\theta(2)+\cdots+\theta(d)}\right\}\right), \text{ for all permutations } \theta \in \mathfrak{S}_d.$$

Thus, $\mathbb{H}_d$ defines a basic triangulation $\mathfrak{T}_{(d)}$ of the entire space $\mathbb{R}^d$ (cf. [14, Ch. III]). This triangulation is also coherent because

$$\mathbb{R}^d \ni \mathbf{x} \longmapsto \overline{\psi}\left(\mathbf{x}\right) = -\sum_{0 \leq i < j \leq d} \left\{ \sum_{0 \leq k \leq x_j - x_i} H(x_j - x_i - k) + \sum_{x_j - x_i \leq k \leq 0} H(k - x_j + x_i) \right\} \in \mathbb{R}$$

with $x_0 := 0$, using the Heaviside function $H : \mathbb{R} \to \mathbb{R}$

$$H(x) := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is a strictly upper convex function for this triangulation. Finally, it is balanced via the colouring function φ_d:

$$\mathbb{Z}^d \ni \boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \overset{\varphi_d}{\mapsto} (\mu_1 + \cdots + \mu_d) \mod (d+1) \in \{0, 1, \ldots, d\}$$

which $(d+1)$-colours all the facets $s(\boldsymbol{\mu}, \theta)$ of $\mathfrak{T}_{(d)}$.

Since the bounding hyperplanes of $\lambda\, \mathbf{s}_d$ are contained in $\mathbb{H}_d$, the restriction $\mathfrak{T}_{[d;\lambda]} := \mathfrak{T}_{(d)}\,|_{\lambda\, \mathbf{s}_d}$ is a b.c.b.-triangulation of $\lambda\, \mathbf{s}_d$. □

Example 6.2. Figure 11 provides the "nice" basic triangulation $\mathfrak{T}_{[2;4]}$ of $4\, \mathbf{s}_2$ inherited from the affine hyperplane arrangement $\mathbb{H}_2$.

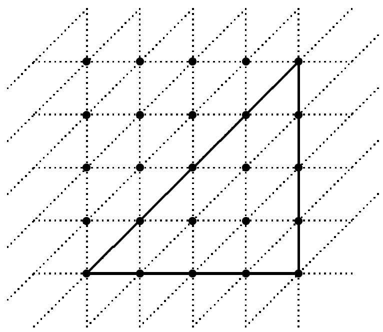

Figure 11

Corollary 6.3. *Theorem* 1.2 *is true for all* $(d; k)$*-hypersurface-singularities.*

Proof. In this case the junior simplex $\mathfrak{s}_G$ equals (up to an affine integral transformation) $k\, \mathbf{s}$, where

$$\mathbf{s} = \text{conv}\left(\left\{\frac{1}{k}\, e_1, \ldots, \frac{1}{k}\, e_d\right\}\right)$$

w.r.t. $N_{G(d;k)}$. But $\mathbf{s}$ is obviously basic. □

Remark 6.4. Corollary 6.3 generalizes Roan's result in [26, § 5]. (Roan proved the existence of crepant desingularizations of $(d; 2)$-hypersurfaces by successively blowing up the singular parts of the branching locus of the corresponding double covering of $\mathbb{C}^d$; see Remark 5.12(i)).

By Theorem 5.15, and what we have already explained in § 4, our Main Theorem 1.2 is now a consequence of the following:

Theorem 6.5. *Every Watanabe simplex has a b.c.b.-triangulation.*

Proof. Trivially, every Watanabe 0-simplex has a b.c.b.-triangulation. If $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) denotes a b.c.b.-triangulation of a Watanabe d_1-simplex $\mathbf{s}_1$ (resp. of a Watanabe d_2-simplex $\mathbf{s}_2$), then $\mathcal{T}_1 * \mathcal{T}_2$ is a b.c.b.-triangulation of $\mathbf{s}_1 * \mathbf{s}_2$ by Theorem 3.5. Hence, it suffices to show that for a given Watanabe simplex $\mathbf{s}$, being equipped with a b.c.b.-triangulation, say $\mathcal{T}$, having a colouring function $\varphi : \mathbf{s} \cap \mathbb{Z}^d \to \{0, 1, \ldots, d\}$ and a strictly upper convex function $\psi : |\mathbf{s}| \to \mathbb{R}$, the λ-times dilation of $\mathbf{s}$ possesses itself a b.c.b.-triangulation. Every facet F of $\mathcal{T}$ gives rise to a *unique* affine integral transformation $\Phi_F : \mathbf{s}_d \to F$ ($\mathbf{s}_d$ as before in the proof of 6.1) which respects the colourings relatively to $\mathbf{s}_d$, i.e., which sends every vertex of $\mathbf{s}_d$ to the vertex of F that has same colour. Φ_F maps $\lambda\,\mathbf{s}_d$ onto $\lambda\,F$. Thus, the image

$$\widetilde{\mathcal{T}_{\lambda F}} := \Phi_F\left(\mathfrak{T}_{[d;\lambda]}\right) = \lambda\,\mathfrak{T}_{(d)}\,|_{\lambda F}$$

constitutes a triangulation of $\lambda\,F$ (which is a b.c.b.-triangulation by 3.6, 3.4, and 6.1). The triangulations $\left\{\widetilde{\mathcal{T}_{\lambda F}} \mid F \text{ facets of } \mathcal{T}\right\}$ fit together to give a basic triangulation $\widehat{\mathcal{T}}$ of $\lambda\,\mathbf{s}$. The triangulation $\widehat{\mathcal{T}}$ is also coherent by Patching Lemma 3.1. Finally, $\widehat{\mathcal{T}}$ is balanced because $\widehat{\varphi}: |\widehat{\mathcal{T}}| \to \mathbb{R}$ (with $\widehat{\varphi}\,|_F = \widehat{\varphi_F}$) defined by

$$\lambda\,F \ni \mathbf{x} \mapsto \widehat{\varphi_F}\,(\mathbf{x}) := \varphi_d\left(\left(\Phi_F\right)^{-1}(\mathbf{x})\right) \in \mathbb{R}$$

on *all* facets F of $\mathcal{T}$, is a colouring map. □

Example 6.6. Let $\mathcal{T}$ denote the unique b.c.b.-triangulation of the 2-dimensional Watanabe simplex

$$\mathbf{s} = \operatorname{conv}\left(\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \quad (\text{w.r.t. } \mathbb{Z}^2)$$

with the facets $F_1 = \operatorname{conv}(\{\binom{-1}{1}, \binom{-1}{0}, \binom{0}{0}\})$ and $F_2 = \operatorname{conv}(\{\binom{-1}{-1}, \binom{-1}{0}, \binom{0}{0}\})$. Figure 12 illustrates how one "glues together"

$$\widetilde{\mathcal{T}_{3F_i}} = \Phi_{F_i}\left(\mathfrak{T}_{[2;3]}\right) = 3\,\mathfrak{T}_{(2)}\,|_{3\,F_i}\;, \quad i = 1, 2,$$

to obtain the b.c.b.-triangulation $\widehat{\mathcal{T}}$. The affine integral transformations Φ_{F_i} are given by

$$\Phi_{F_1}(\mathbf{x}) = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1\\-1\end{pmatrix}, \quad \Phi_{F_2}(\mathbf{x}) = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1\\1\end{pmatrix}, \quad \text{for } \mathbf{x} \in \mathbb{R}^2.$$

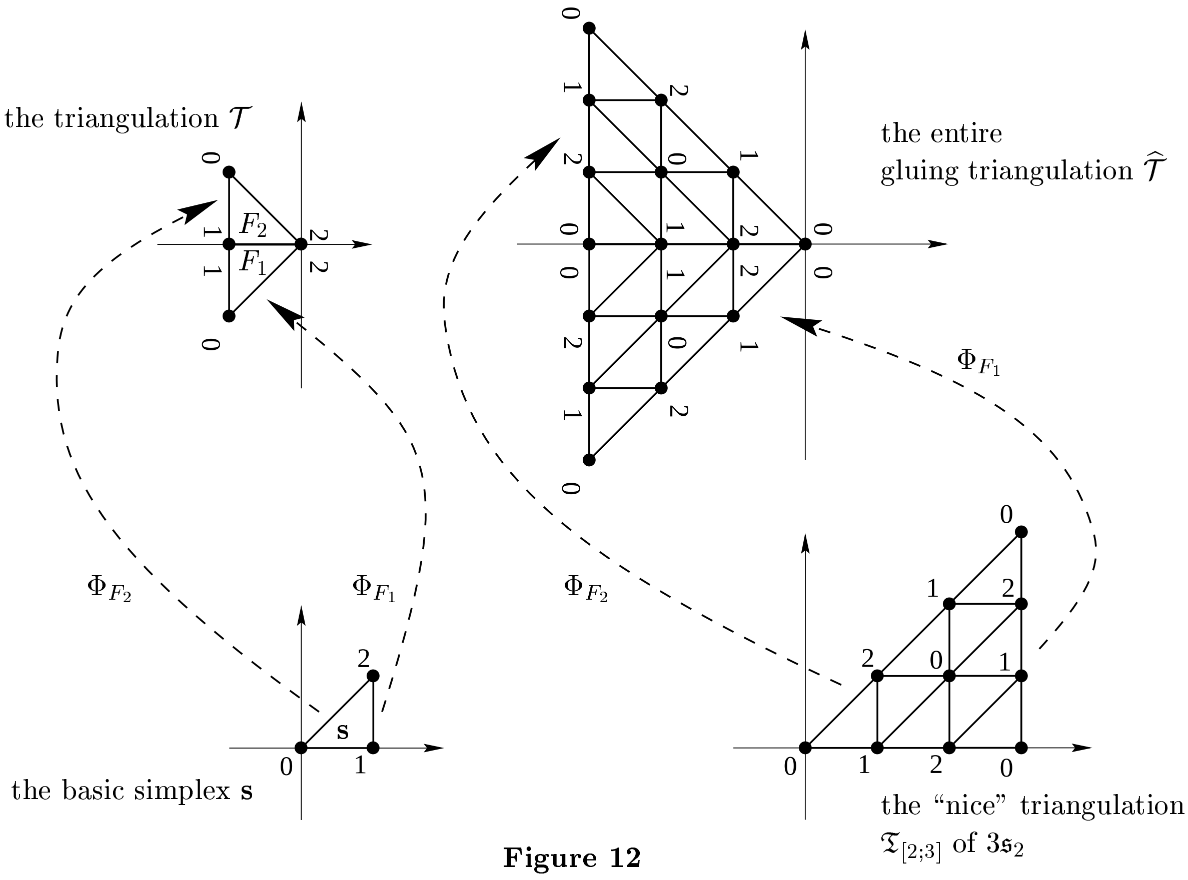

Figure 12

7. On the computation of cohomology group dimensions

Let $\mathbb{C}^d/G = U_{\sigma_0} = X(N_G, \Delta_G)$ be again the underlying space of an abelian quotient c.i.-singularity and

$$f_{\mathcal{T}} : X\left(N_G, \widehat{\Delta_G}(\mathcal{T})\right) \to X(N_G, \Delta_G) \tag{7.1}$$

any partial crepant desingularization induced by a lattice triangulation $\mathcal{T}$ of the junior simplex $\mathfrak{s}_G$. It is easy to verify that the central fiber $\mathbf{F}_{\mathcal{T}} = (f_{\mathcal{T}})^{-1}([\mathbf{0}])$ of $f_{\mathcal{T}}$ is a strong deformation retract of the overlying space $X\left(N_G, \widehat{\Delta_G}(\mathcal{T})\right)$. Theorem 1.2 guarantees the existence of *at least one* b.c.b.-triangulation $\mathcal{T}$ of $\mathfrak{s}_G$, so that (7.1) is a projective, crepant, full desingularization. But even if we let $\mathcal{T}$ go through the entire class of *all* possible basic triangulations of $\mathfrak{s}_G$, and use [3, Thm. 5.4], [13, Cor. 1.5], we obtain the one-to-one McKay-type correspondence

$$\{\text{elements of } G\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{a basis of } H^*(\mathbf{F}_{\mathcal{T}}, \mathbb{Q}) \text{ consisting} \\ \text{of classes of algebraic cycles}\end{array}\right\}$$

and, in particular,

$$\{\text{elements of } G \text{ of age } 1\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{exceptional prime} \\ \text{divisors w.r.t. } f_{\mathcal{T}}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{a basis of } H^2(\mathbf{F}_{\mathcal{T}}, \mathbb{Q}) \\ \text{consisting of classes of} \\ \text{algebraic cycles}\end{array}\right\}$$

where the elements of age 1 are those having lattice-point-representatives lying on $\mathfrak{s}_G$. In fact, only the even cohomology groups of $\mathbf{F}_{\mathcal{T}}$ are non-trivial. To compute their dimensions we need some concepts from enumerative combinatorics.

For a lattice d-polytope $P \subset N_{\mathbb{R}}$ w.r.t. an $N \cong \mathbb{Z}^d$, and κ a positive integer, let

$$\mathbf{Ehr}_N(P, \kappa) = \mathbf{a}_0(P) + \mathbf{a}_1(P)\ \kappa + \cdots + \mathbf{a}_{d-1}(P)\ \kappa^{d-1} + \mathbf{a}_d(P)\ \kappa^d \ \in\ \mathbb{Q}[\kappa]$$

denote the *Ehrhart polynomial* of P (w.r.t. N), where $\mathbf{Ehr}_N(P, \kappa) := \#\ (\kappa\, P \cap N)$, and

$$\mathbf{Ehr}_N(P; \mathfrak{t}) := 1 + \sum_{\kappa=1}^{\infty} \mathbf{Ehr}_N(P, \kappa)\ \mathfrak{t}^{\kappa} \in \mathbb{Q}[\![\mathfrak{t}]\!]$$

the corresponding *Ehrhart series.* Writing $\mathbf{Ehr}_N(P; \mathfrak{t})$ as

$$\mathbf{Ehr}_N(P; \mathfrak{t}) = \frac{\boldsymbol{\delta}_0(P) + \ \boldsymbol{\delta}_1(P)\ \mathfrak{t} + \cdots + \ \boldsymbol{\delta}_{d-1}(P)\ \mathfrak{t}^{d-1} + \ \boldsymbol{\delta}_d(P)\ \mathfrak{t}^d}{(1-\mathfrak{t})^{d+1}}$$

we get the so-called $\boldsymbol{\delta}$-*vector* $\boldsymbol{\delta}(P) = (\boldsymbol{\delta}_0(P), \boldsymbol{\delta}_1(P), \ldots, \boldsymbol{\delta}_{d-1}(P), \boldsymbol{\delta}_d(P))$ of P.

Definition 7.1. For any integer $d \geq 0$ we introduce the *transfer* $\mathbf{a}$-$\boldsymbol{\delta}$-*matrix* $\mathcal{M}_d \in \mathrm{GL}(d+1, \mathbb{Q})$ (depending only on d) to be defined as

$$\mathcal{M}_d := (\mathfrak{R}_{i,j})_{0 \le i,j \le d} \quad \text{with} \quad \mathfrak{R}_{i,j} := \frac{1}{d!}\left\{\sum_{p=i}^{d} \begin{bmatrix} d \\ p \end{bmatrix} \binom{p}{i}\ (d-j)^{p-i}\right\}$$

where $\begin{bmatrix} d \\ p \end{bmatrix}$ denotes the Stirling number (of the first kind) of d over p.

The following lemma can be proved easily.

Lemma 7.2. *For a lattice d-polytope $P \subset N_{\mathbb{R}}$ w.r.t. an $N \cong \mathbb{Z}^d$, we have*

$$(\mathbf{a}_0(P), \mathbf{a}_1(P), \ldots, \mathbf{a}_{d-1}(P), \mathbf{a}_d(P)) = (\boldsymbol{\delta}_0(P), \boldsymbol{\delta}_1(P), \ldots, \boldsymbol{\delta}_{d-1}(P), \boldsymbol{\delta}_d(P)) \cdot (\mathcal{M}_d)^{\intercal}.$$

Theorem 7.3. *For any integer $d \geq 2$, and for any crepant (full) desingularization (7.1) of an abelian quotient c.i.-space $\mathbb{C}^d/G$, the non-trivial cohomology dimensions of $\mathbf{F}_{\mathcal{T}}$ (or of $X\left(N_G, \widehat{\Delta_G}(\mathcal{T})\right)$, for any basic $\mathcal{T}$), can be determined inductively (via the coefficients of Ehrhart polynomials) as follows :*

(i) *If $\mathfrak{s}_G$ is the join of two Watanabe simplices $\mathbf{s}_1$, $\mathbf{s}_2$, of dimensions d_1, d_2, $(d_1 + d_2 = d - 2)$, then*

$$\dim_{\mathbb{Q}} H^{2i}(\mathbf{F}_{\mathcal{T}}, \mathbb{Q}) =$$

$$= \sum_{\substack{0 \leq p,q \leq d-1 \\ \text{with } p+q=i}} \left(\left(\left[(\mathcal{M}_{d-1})^{-1} \right]_{p+1} \cdot \begin{pmatrix} \mathbf{a}_0(\mathbf{s}_1) \\ \mathbf{a}_1(\mathbf{s}_1) \\ \vdots \\ \mathbf{a}_{d-1}(\mathbf{s}_1) \end{pmatrix} \right) \times \left(\left[(\mathcal{M}_{d-1})^{-1} \right]_{q+1} \cdot \begin{pmatrix} \mathbf{a}_0(\mathbf{s}_2) \\ \mathbf{a}_1(\mathbf{s}_2) \\ \vdots \\ \mathbf{a}_{d-1}(\mathbf{s}_2) \end{pmatrix} \right) \right) \tag{7.2}$$

(ii) *If $\mathfrak{s}_G$ (up to an affine integral transformation) is the dilation $\lambda\,\mathbf{s}$, $\lambda \geq 2$, of a Watanabe simplex $\mathbf{s}$, then*

$$\dim_{\mathbb{Q}} H^{2i}(\mathbf{F}_{\mathcal{T}}, \mathbb{Q}) = \left[(\mathcal{M}_{d-1})^{-1} \right]_{i+1} \cdot \begin{pmatrix} \mathbf{a}_0(\mathbf{s}) \\ \lambda\,\mathbf{a}_1(\mathbf{s}) \\ \vdots \\ \lambda^{d-1}\,\mathbf{a}_{d-1}(\mathbf{s}) \end{pmatrix} \tag{7.3}$$

for all i, $0 \leq i \leq d-1$, where for all ρ, $1 \leq \rho \leq d$, $\left[(\mathcal{M}_{d-1})^{-1} \right]_{\rho}$ denotes the ρ-th row-vector of $(\mathcal{M}_{d-1})^{-1}$.

Proof. By [3, Thm. 4.4], $\dim_{\mathbb{Q}} H^{2i}(\mathbf{F}_{\mathcal{T}}, \mathbb{Q})$ equals the i-th component of the $\boldsymbol{\delta}$-vector of $\mathfrak{s}_G$. That the computation can be done inductively follows from the converse statement in Theorem 5.15 and Theorem 1.2. In fact, for any G being conjugate to a $G_{\mathbb{D}}$, the cohomology dimensions can be read off from the free parameters of $\mathbf{W}_{\mathbb{D}}$.

(i) Since $\mathcal{T}$ is basic, the $\boldsymbol{\delta}$-vector of $\mathfrak{s}_G$ is equal to its h-vector (see Stanley [29, 2.5]). Hence, (7.2) follows from the formula which provides the h-vector of the join of two simplices.

(ii) (7.3) is obvious because $\mathbf{a}_j(\lambda\,\mathbf{s}) = \lambda^j\,\mathbf{a}_j(\mathbf{s})$ for all j, $0 \leq j \leq d-1$. □

Corollary 7.4. *Let G be conjugate to $G(d;k)$ (within $\mathrm{SL}(d,\mathbb{C})$). Then the non-trivial cohomology dimensions of any crepant resolution (7.1) of the $(d;k)$-hypersurfaces equals*

$$\dim_{\mathbb{Q}} H^{2i}\left(\mathbf{F}_{\mathcal{T}},\mathbb{Q}\right)=\left[\left(\mathcal{M}_{d-1}\right)^{-1}\right]_{i+1}\cdot\begin{pmatrix}1\\ \vdots\\ \underbrace{\dfrac{k^j}{(d-1)!}\ \left\{\sum_{p=j}^{d-1}\begin{bmatrix}d-1\\ p\end{bmatrix}\binom{p}{j}\ (d-1)^{p-j}\right\}}_{\text{entry of the column at }(j+1)\text{-position }(0\leq j\leq d-1)}\\ \vdots\\ \dfrac{k^{d-1}}{(d-1)!}\end{pmatrix}\tag{7.4}$$

for $0\leq i\leq d-1$. *In particular, the Euler-Poincaré characteristic* $\chi\left(\mathbf{F}_{\mathcal{T}}\right)$ *of* $\mathbf{F}_{\mathcal{T}}$ *equals*

$$\chi\left(\mathbf{F}_{\mathcal{T}}\right)=\chi\left(X\left(N_{G(d;k)},\widehat{\Delta_{G(d;k)}}\left(\mathcal{T}\right)\right)\right)=\left|G\left(d;k\right)\right|=k^{d-1}\tag{7.5}$$

Proof. In this case (up to an affine integral transformation) $\mathfrak{s}_G$ equals $k\,\mathbf{s}$, where

$$\mathbf{s}=\mathrm{conv}\left(\left\{\frac{1}{k}\ e_1,\ldots,\frac{1}{k}\ e_d\right\}\right)$$

w.r.t. $N_{G(d;k)}$. Since

$$\mathbf{Ehr}_{N_{G(d;k)}}\left(\mathbf{s},\kappa\right)=\binom{\kappa+d-1}{\kappa}=\binom{\kappa+d-1}{d-1}=\frac{1}{(d-1)!}\sum_{p=0}^{d-1}\begin{bmatrix}d-1\\ p\end{bmatrix}\ (\kappa+d-1)^p\ ,$$

or, alternatively, since

$$\boldsymbol{\delta}_0\left(\mathbf{s}\right)=1\quad\text{ and }\quad\boldsymbol{\delta}_1\left(\mathbf{s}\right)=\cdots=\boldsymbol{\delta}_{d-1}\left(\mathbf{s}\right)=0,$$

we get

$$\mathbf{a}_j\left(\mathbf{s}\right)=\left\{\begin{array}{c}(j+1)\text{-entry of the}\\ \text{first column of}\\ \text{the matrix }\mathcal{M}_{d-1}\end{array}\right\}=\frac{1}{(d-1)!}\ \left\{\sum_{p=j}^{d-1}\begin{bmatrix}d-1\\ p\end{bmatrix}\binom{p}{j}\ (d-1)^{p-j}\right\}$$

for all j, $0\leq j\leq d-1$, and (7.4) follows from (7.3). Finally, formula (7.5) follows from the equality $\chi\left(\mathbf{F}_{\mathcal{T}}\right)=(d-1)!\,\mathbf{a}_{d-1}\left(\mathbf{s}\right)$. $\square$

8. Comments and open problems.

(i) All partial, crepant, projective desingularizations (7.1) of $\left(\mathbb{C}^d/G,[\mathbf{0}]\right)=\left(U_{\sigma_0},\mathrm{orb}\left(\sigma_0\right)\right)$ can be studied by means of the *secondary polytope* of the junior simplex $\mathfrak{s}_G$, whose vertices parametrize all coherent triangulations $\mathcal{T}$ of $\mathfrak{s}_G$. Passing from one vertex of this polytope to another, we perform a finite series of flops. It should be mentioned that for $d\geq 4$, it is possible to start from a vertex corresponding to a basic $\mathcal{T}$, and arrive at another maximal, but non-basic triangulation $\mathcal{T}'$. In the language of "toric MMP" (see Reid [23]), our b.c.b.-triangulations lead to *smooth* minimal models. (Note that all morphisms $f_{\mathcal{T}}$ can be written, by [23, 0.2-0.3], as compositions of finite sequences of more elementary toric contraction-morphisms).

(ii) For any $d \geq 3$, the general hyperplane section of the above $(\mathbb{C}^d/G, [\mathbf{0}])$ through $[\mathbf{0}]$ is either a rational or an elliptic Gorenstein singularity (see [24, § 3.10]). Already for $d = 3$, both possibilities occur. For example, the general hyperplane section of the $(3; k)$-hypersurfaces is a compound Du Val singularity (of type cD_4) for $k = 2$, and an elliptic Gorenstein surface singularity whenever $k \geq 3$. It might be interesting to investigate the class of Gorenstein elliptic singularities obtained by this procedure by exploiting the inductive character of Watanabe's classification, essentially via the forests $\mathbf{W}_{\mathbb{D}}$. (What is the relationship between these general-hyperplane-section singularities and the free parameters of the starting-point singularities for $d \geq 4$?)

(iii) Could Theorem 1.2 be generalized for the underlying spaces of Gorenstein, toric, non-quotient, c.i.-singularities? More precisely, what would be the geometric analogue of joins and dilations describing the structure of lattice polytopes which support the Gorenstein cones in this case?

(iv) Theorem 1.2 has various applications to global geometrical constructions. For instance, every "well-stratified" Calabi-Yau variety which is locally a complete intersection, and has at most abelian quotient singularities, possesses global, crepant, full resolutions in all dimensions. (Nevertheless, to check the projectivity of these globally desingularizing morphisms one needs to apply the Nakai-Moishezon criterion, and this is only possible if one has some extra information available about the intrinsic geometry of the varieties being under consideration).

(v) A special class of such Calabi-Yau varieties, which is of particular interest, is that of compactified hypersurfaces $\overline{Z_f} \subset Y_P$ being embedded in a toric variety Y_P, associated to a *reflexive, simple* lattice polytope P. Assuming that $\overline{Z_f}$ is P-*regular* (in Batyrev's sense [1]), and that the polar lattice polytope $P^\star$ of P has only Watanabe simplices as faces of codimension ≥ 2, there exist always global, crepant, full desingularizations $\widehat{Z_f} \to \overline{Z_f}$ of $\overline{Z_f}$. One method to construct at least one of them is to triangulate the faces of $P^\star$ as in the present paper and then to join the single interior point of $P^\star$ with them. For example, the mirror-partner $\overline{Z_g}$ of a marginally deformed Fermat-hypersurface $\overline{Z_f}$ (or of any smooth hypersurface) of degree d in $\mathbb{P}^{d-1}_{\mathbb{C}}$ has global, crepant, full desingularizations in *all* dimensions. (In this particular case, one can, in addition, easily construct a globally projective desingularizing morphism). Hence, the so-called "string-theoretic" Hodge numbers of this $\overline{Z_g}$ (cf. [1] [3]) are nothing but the *usual* Hodge-numbers of an always existing *smooth, projective* $\widehat{Z_g}$; in particular, the corresponding monomial-divisor mirror-map provides the usual dualism between the polynomial first-order deformations of $\overline{Z_f}$ within $H^{d-3,1}(\overline{Z_f})$ on the one hand, and the toric part of $H^{1,1}\left(\widehat{Z_g}\right)$, on the other. (So there are concrete exceptional divisors, and there is no need here to work in the category of singular spaces). This motivates the formulation of a purely combinatorial problem: to *classify* all reflexive, simplicial polytopes, at least in dimension 5, having only Watanabe simplices as faces of codimension at least 2.

Acknowledgements. The first author benefited greatly from the stimulating mathematical atmosphere of the Max-Planck-Institute in Bonn, where he spent the last part of his graduate studies under the direction of Prof. F. Hirzebruch, towards the end of which many of the ideas contained in his contribution here were conceived in their first form. Subsequent visits were also in more than one way beneficial. He also would like to thank Prof. M. Reid for his interest in our project concerning the existence of crepant resolutions, to DFG for the support by an one-year-fellowship, and to Mathematics Institute of Bonn University for hospitality.
The second and third author wish to acknowledge the long-term support of their research by a DFG Gerhard-Hess Forschungsförderpreis.

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