Nonvanishing of motivic L-functions

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Introduction

Let E be a modular elliptic curve over $\mathbb{Q}$ and let $L(E,s)$ denote the associated L-function. The Birch and Swinnerton-Dyer conjecture then predicts that $L(E,s)$ has a zero at $s=1$ of order precisely equal to the rank of the Mordell-Weil group $E(\mathbb{Q})$. According to Waldspurger's theorem [Wa] we know that there exists a real quadratic character χ such that the twisted L-function $L(E,\chi,s)$ does not vanish at $s=1$. Recently V.A. Kolyvagin [Ko] has proved that $E(\mathbb{Q})$ is finite if $L(E,1)\neq 0$ and there exists a suitable real quadratic character χ_t such that $L(E,\chi_t,s)$ has a simple zero at $s=1$. The latter condition was proved to be true for infinitely many t [BFH][MuMu]. H. Iwaniec [Iw] and A. Perelli and J. Pomykała [PP] have proved a quantitative results on this condition; J. Pomykała [P] has generalized it to n-th derivative of $L(E,\chi,s)$.

Variants and generalizations are possible. For instance, S. Friedberg and H. Hoffstein [FH] established a non-vanishing theorem for quadratic twists of the L-series of an arbitrary cuspidal automorphic form on GL(2) over any number field. In some cases such a result can be applied to the construction of l-adic representations associated to modular forms over imaginary quadratic fields [Ta]. Further examples as well as some perspective concerning the higher rank case is discussed in an excellent survey article [BFH2].

We should also mention that there are examples of irreducible cuspidal automorphic representations of GL(2) over a number field such that the corresponding twisted L-function vanishes at the center of the critical strip for all quadratic characters [Ro].

Here is our contribution to the above picture.

Let M be a pure motive over $\mathbb{Q}$, and let $L(M,s)$ denote the corresponding L-function. In this paper we prove, under certain assumptions, (a quantitative version of) a non-vanishing theorem for n-th derivatives of quadratic twists of $L(M,s)$ at the center of the critical strip (Theorem 3). Also, assuming the Birch and Swinnerton-Dyer conjecture for abelian varieties over $\mathbb{Q}$, we obtain a bound for the order of the (twisted) Shafarevich-Tate group (Theorem 4).

All the results of this paper except section 5 could be stated (and proved) without using any motivic language. Almost all we need to know about motives concerns the conjectural description of their L-functions (see section 1).

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0. Statement of the results

Let M be a pure motive over $\mathbb{Q}$ of weight $w = w(M)$ and of the global conductor $c(M)$. For any Dirichlet character χ one can define the twisted motive M_χ; let $L(M_\chi, s)$ denote the corresponding L-function. Also, let $L^{(n)}(M_\chi, s)$ denote the n-th derivative of motivic L-function. Assume that $L^{(n)}(M_\chi, s)$ have good analytic properties (see section 1.1). Write

$$L(M, s) = \sum_{m=1}^{\infty} a_m m^{-s}, \quad \mathrm{Re}(s) \gg 0.$$

Let

$$D(M, s) = \sum_{m=kl^2} b_m m^{-s}, \quad \mathrm{Re}(s) \gg 0,$$

where k has prime factors in $4c(M)$, $(l, 4c(M)) = 1$ and

$$b_m = a_m \prod_{p|m,\ (p,4c(M))=1} \left(1 + \frac{1}{p}\right)^{-1}.$$

Such a series arises naturally when one try to follow Iwaniec's work [Iw]. In a case of the motive attached to a newform f of weight k the series $D(M(f), s)$ has meromorphic continuation to all the complex plane and does not vanish at $s = \frac{k}{2}$ (see [Iw] for $k = 2$; in a case of arbitrary weight one can use the same trick). In general we do not expect for $D(M, s)$ to have any "nice" analytical properties. All we need (and assume) is that $D(M, s)$ has analytic continuation to $\mathrm{Re}(s) > \frac{w}{2} - \epsilon$ ($\epsilon > 0$). Also, for our purposes, it is important to have an information about the behaviour at $s = \frac{w+1}{2}$. The next non-trivial example (communicated to us by D. Zagier) is $M = \mathrm{Sym}^2 M(\Delta)$, where Δ is the unique normalized cusp form of weight 12 and level 1: in this case $D(M, s)$ can be analytically prolonged to (at least) $\mathrm{Re}(s) > 11 - \epsilon$ and does not vanish at $s = \frac{23}{2}$. Note however that, in general, $D(M, s)$ may have a pole at $s = \frac{w+1}{2}$ (see Appendix).

Let $\mathcal{C}$ be the set of primitive quadratic characters χ of conductors $c(\chi)$ which are relatively prime to $c(M)$. Let

$$\mathcal{C}(T) = \{\chi \in \mathcal{C} :\ c(\chi) \leq T\}.$$

In a case $\chi = \chi_t$, $\chi_t(\cdot) = \left(\frac{-t}{\cdot}\right)$ (the Kronecker symbol) we sometimes write $M_t = M_{\chi_t}$, etc.

Let $\epsilon(M)$ be the rooth number in the functional equation (1.1) for $L(M,s)$. Our next restriction on M is that $M = M^\vee(-w)$. Then of course $\epsilon(M) = \pm 1$, and more precisely

$$\epsilon(M_\chi) = (-1)^{\mathrm{ord}_{s=\frac{w+1}{2}} L(M_\chi, s)}$$

for all quadratic characters χ of conductors relatively prime to $c(M)$. It is certainly the case for modular forms of trivial characters, abelian varieties, etc.

Let $\mathcal{T}$ be defined as follows

$$\mathcal{T} := \{t \text{ square free} : 0 < t \equiv -\nu^2 \mathrm{mod}(4c(M)) \text{ for some } \nu \text{ prime to } 4c(M)\}.$$

The proof of the following crucial result depends on Iwaniec's fundamental work [Iw].

Theorem 1. *Assume that $n \neq 0$ or $\epsilon(M) = -1$. Then we have the following asymptotic equality ($T \to \infty$)*

$$\sum_{\chi_t \in \mathcal{C}(T),\, t \in \mathcal{T}} L^{(n)}\left(M_t, \frac{w+1}{2}\right) F\left(\frac{t}{T}\right) \sim C(M,n) \int F(u)du\, T(\log T)^n,$$

where $C(M,n) \neq 0$ and F is a smooth function compactly supported in $\mathbb{R}^+$ with positive mean value.

We carry over the method of the paper [HB] to prove the following estimate, which also have important implications in our work on distribution of zeros [DP].

Theorem 2. *Let $\epsilon > 0$ and assume that M is of rank ≤ 2. Then, for $T > T_0(M,\epsilon)$ and $|s - \frac{w+1}{2}| \leq 2(\log T)^{-1}$, it holds*

$$\sum_{\chi \in \mathcal{C}(T)} |L(M_\chi, s)|^2 \ll T^{1+\epsilon}. \tag{0.1}$$

We expect the above estimate to hold in general. The case of rank ≥ 2 presents the difficulties analogous to those for higher moments of classical Dirichlet L-series.

Let

$$N_n(T,M) = \sharp\{\chi \in \mathcal{C}(T) :\ L^{(n)}(M_\chi, \frac{w+1}{2}) \neq 0\}.$$

Assuming (0.1) for all (pure) motives we derive from Theorem 1 the following estimate.

Theorem 3. *We have*

$$N_n(T,M) \gg T^{1-\epsilon}.$$

It is perhaps time to recall the empirical distribution of the ranks of elliptic curves given by D. Zagier and G. Kramarz [ZK] (for the family $x^3 + y^3 = m$: about 23.3% have rank 2 or greater) and by A. Brumer and O. McGuinness [BM] (for elliptic curves of prime conductor: about 20% have even rank ≥ 2). It may suggest that the expectation that half of all elliptic curves have rank 0 and half rank 1, with higher ranks occuring asymptotically for only 0% of all curves (with respect to any natural ordering) is wrong. (We refer to section 6 for further comments.)

Our application concerns the bound for the order of the twisted Shafarevich-Tate groups. We generalize the main result of [MaMu] (where the case of an elliptic curve was treated) as follows.

Theorem 4. *Let A be an abelian variety over $\mathbb{Q}$. Assume that the Birch and Swinnerton-Dyer conjecture holds true for the quadratic twists of A. Assuming (5.1), there exist infinitely many t such that A_t has analytic rank 0 and*

$$\sharp Ш(A_t) \gg c(A_t)^{\frac{1}{4}-\epsilon}.$$

1. Preliminaries on motivic L-functions

1.1. Motivic L-functions. We will define a pure motive M over $\mathbb{Q}$ of weight $w = w(M)$ and rank $d = d(M)$, by specifying Betti, de Rham and l-adic realisations (for each prime l)

$$H_B(M), \quad H_{DR}(M), \quad H_l(M)$$

which are, respectively, vector spaces over $\mathbb{Q}$, $\mathbb{Q}$ and $\mathbb{Q}_l$ of dimension d, and which are endowed with the additional structures and comparison isomorphisms [BK][CP-R][D]. In particular $H_B(M)$ admits an involution ρ_B, $H_l(M)$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, and there is a Hodge decomposition into $\mathbb{C}$-vector spaces

$$H_B(M) \otimes \mathbb{C} = \oplus_{i+j=w} H^{i,j}(M),$$

where, letting ρ_B act on the vector space on the left via the first factor in the tensor product, we have

$$\rho_B(H^{i,j}(M)) = H^{j,i}.$$

The L-function of M is defined for $\mathrm{Re}(s) \gg 0$ as the following Euler product

$$L(M,s) = \prod_p L_p(M, p^{-s}),$$

extended over all primes p and where

$$\begin{aligned} L_p(M,X)^{-1} &= \det\left(1 - X \cdot r_l(Fr^{-1})|_{H_l(M)^{I_p}}\right) \\ &= (1-\alpha_p^{(1)}X)...(1-\alpha_p^{(d)}X); \end{aligned}$$

here r_l is the representation giving $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module structure on $H_l(M)$, and $Fr_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the Frobenius element at p, defined modulo conjugation and modulo the inertia subgroup $I_p \subset G_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the decomposition group

G_p. We make the standard hypothesis that the coefficients of $L_p(M,X)^{-1}$ do not depend on $l \neq p$.

We next state the conjectural functional equation for $L(M,s)$. First let's recall the definition of the Euler factor $L_\infty(M,s)$ at infinity. We suppose that if $w(M)$ is even and $H^{\frac{w}{2},\frac{w}{2}} \neq 0$, the involution ρ_B acts on it by a scalar ± 1. Put

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s), \quad h(i,j) = \dim_{\mathbb{C}} H^{i,j}(M),$$

and, if $w(M)$ is even and $H^{\frac{w}{2},\frac{w}{2}}(M) \neq 0$, we put

$$r^\alpha(M) = \dim_{\mathbb{C}}\{x \in H^{\frac{w}{2},\frac{w}{2}}(M) : \ \rho_B(x) = \alpha(-1)^{\frac{w}{2}}x\}, \quad \alpha = \pm 1.$$

We define

$$L_\infty(M,s) = L_{\infty,\mathbb{R}}(M,s)L_{\infty,\mathbb{C}}(M,s),$$

where

$$L_{\infty,\mathbb{R}}(M,s) = \Gamma_{\mathbb{R}}(s-\frac{w}{2})^{r^+(M)}\Gamma_{\mathbb{R}}(s+1-\frac{w}{2})^{r^-(M)},$$

and

$$L_{\infty,\mathbb{C}}(M,s) = \prod_{i<j}\Gamma_{\mathbb{C}}(s-i)^{h(i,j)}.$$

We put

$$\Lambda(M,s) = L_\infty(M,s)L(M,s).$$

We also assume the existence of a finite set of bad primes S such that, for each prime l and each $r \notin S \cup \{l\}$, the inertia group I_r acts trivially on $H_l(M)$. For each $r \in S$, and each $l \neq r$, we additionally assume that the conductor of $H_l(M)$ as a representation of the local Galois group D_r does not depend on l. The global conductor $c(M)$ of M is then defined as the product of local conductors at primes $r \in S$.

For each $p \notin S$, we assume that the reciprocal complex roots of $L_p(M,X)^{-1}$ have absolute value equal to $p^{\frac{w}{2}}$; according to the monodromy conjecture, a more refined version of the purity should hold also at bad primes. It implies, that $L(M,s)$ converges in the complex half-plane $\mathrm{Re}(s) > \frac{w}{2}+1$.

Conjecture [D][CP-R]. *The function $\Lambda(M,s)$ has a meromorphic continuation over the whole complex plane, and satisfies the functional equation*

$$\Lambda(M,s) = \epsilon(M)c(M)^{\frac{w+1}{2}-s}\Lambda(M^\vee(-w), w+1-s), \tag{1.1}$$

where $\epsilon(M)$ is a complex number which is independent of s. Moreover, $L(M,s)$ is holomorphic unless both $w(M)$ is even and $\mathbb{Q}(-\frac{w}{2})$ is a direct summand of M, in which case $L(M,s)$ may possibly have a pole at the point $s = \frac{w}{2}+1$.

Remark. Let us recall that in a case $M = M^\vee(-w)$ the root number $\epsilon(M)$ can be given by the analytic formula

$$\epsilon(M) = (-1)^{\mathrm{ord}_{s=0}L(M,s)}. \tag{1.2}$$

Let χ be a real quadratic character of conductor $c(\chi)$. To such χ one can associate a motive $[\chi]$ over $\mathbb{Q}$ of weight 0 and rank 1 so that $L([\chi],s) = L(s,\chi)$. Also given any (pure) motive M, we can define $M_\chi = M \otimes [\chi]$; it is again a motive over $\mathbb{Q}$ of the same weight and rank so that we have

$$L(M_\chi, s) = \prod_p L_p(M, \chi(p)p^{-s}).$$

Lemma 1 [CP-R]. *Assume that* $(c(\chi), c(M)) = 1$. *Then*

$$c(M_\chi) = c(M)c(\chi)^{d(M)}.$$

1.2. The conjecture of Birch and Swinnerton-Dyer. Let A be an abelian variety over $\mathbb{Q}$, and let S be a set of rational primes where A has bad reduction. The L-function $L(A,s) = L_S(A,s)$ is defined, for $\mathrm{Re}(s) > \frac{3}{2}$, by

$$L(A,s) = \prod_{p \notin S} L_p(H^1(A), p^{-s}).$$

Let Ш(A) be the Shafarevich-Tate group of A, and let $A(\mathbb{Q})$, $A'(\mathbb{Q}) = \mathrm{Pic}^o(A)$ (resp. $A(\mathbb{Q}_p)$) are the groups of global (resp. local) points. Let R denote the determinant of the height pairing

$$A(\mathbb{Q}) \times A'(\mathbb{Q}) \to \mathbb{R}.$$

Also put $\pi_\infty(A) = \mathrm{vol}(A(\mathbb{R}))$, $\pi_{bad}(A) = \mathrm{vol}(\prod_{p\in S} A(\mathbb{Q}_p))$.

Conjecture (Birch and Swinnerton-Dyer, Tate [T]). *$L(A,s)$ has a zero of order $r = \mathrm{rank}A(\mathbb{Q})$ at $s = 1$ and*

$$\lim_{s\to 1} L(A,s)(s-1)^{-r} = \frac{\sharp Ш(A)\cdot R}{\sharp(A(\mathbb{Q})_{tors})\sharp\mathrm{Pic}^o(A)_{tors})}\,\pi_\infty(A)\pi_{bad}(A).$$

S. Bloch [Bl] proved that, in a case $r = 0$, the Birch and Swinnerton-Dyer conjecture is equivalent to the Tamagawa number conjecture. Later, S. Bloch and K. Kato [BK] defined the notion of the Shafarevich-Tate group, the groups of local and global points etc., and formulated the Tamagawa number conjecture in a general motivic situation.

2. Proof of Theorem 1

Put $i_0 = \min\{i : \ i \le j, \ h(i,j) \ne 0\}$. For $n \in \mathbb{Z}$ we introduce the generalized incomplete gamma function

$$V_n(M,X) = \frac{1}{2\pi i}\int_{(3/4)} L_\infty(M, s + i_0 - \frac{w-1}{2})\, s^{-n} X^{-s} ds,$$

where we integrate over the line $\mathrm{Re}(s) = \frac{3}{4}$.

Put $d_0 = \frac{1}{2}h(\frac{w}{2}, \frac{w}{2})$, $d_1 = \sum_{i<j} h(i,j)$. Consider the approximate functions

$$A_n(M_\chi, X) = \sum_{m=1}^{\infty} a_m \chi(m) m^{-\frac{w+1}{2}} V_n\left(M, \frac{\pi^{d_0}(2\pi)^{d_1} m}{X}\right).$$

By the functional equation (1.1) we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(-3/4)} L(M_\chi, s+\frac{w+1}{2})\frac{L_\infty(M, s+i_0-\frac{w-1}{2})}{s^n}\left(\frac{X}{\pi^{d_0}(2\pi)^{d_1}}\right)^s ds\\
&= \frac{\epsilon(M_\chi)}{2\pi i}\int_{(-3/4)} L(M_\chi, \frac{w+1}{2}-s)\frac{L_\infty(M, \frac{w+1}{2}-s)}{L_\infty(M, s+\frac{w+1}{2})}\\
&\qquad \times \frac{L_\infty(M, s+i_0-\frac{w-1}{2})}{s^n}\left(\frac{\pi^{d_0}(2\pi)^{d_1}X}{c(\chi)^d c(M)}\right)^s ds\\
&= (-1)^{n+1}\frac{\epsilon(M_\chi)}{2\pi i}\int_{(3/4)} L(M_\chi, s+\frac{w+1}{2})\frac{L_\infty(M, s+i_0-\frac{w-1}{2})}{s^n}\\
&\qquad \left(\frac{\pi^{d_0}(2\pi)^{d_1}}{c(\chi)^d c(M)/X}\right)^{-s} ds\\
&= (-1)^{n+1}\epsilon(M_\chi)A_n(M_\chi, c(\chi)^d c(M)/X).
\end{aligned}$$

Put

$$G_n(M_\chi, s, X) = L(M_\chi, s+\frac{w+1}{2})\frac{L_\infty(M, s+i_0-\frac{w-1}{2})}{s^n}\left(\frac{X}{\pi^{d_0}(2\pi)^{d_1}}\right)^s.$$

We obtain by the Cauchy theorem

(2.0)

$$\begin{aligned}
\mathrm{res}_{s=0} G_n(M_\chi, s, \sqrt{c(\chi)^d c(M)}) &= A_n(M_\chi, \sqrt{c(\chi)^d c(M)})\\
&\quad + (-1)^n \epsilon(M_\chi) A_n(M_\chi, \sqrt{c(\chi)^d c(M)}).
\end{aligned}$$

Consider the Laurent expansions

$$L_\infty(M, s+i_0-\frac{w-1}{2}) = \sum_{m=-r}^{\infty} \alpha_m(M) s^m, \quad \alpha_{-r}(M) \neq 0,$$

where $r = r(M) = h(\frac{w}{2}, \frac{w}{2}) + \sum_{i<j} h(i,j)$, and

$$\left(\frac{X}{\pi^{d_0}(2\pi)^{d_1}}\right)^s = \sum_{k=0}^{\infty} b_k(M, X) s^k.$$

It is plain to see that

$$\text{(2.1)} \qquad \operatorname{res}_{s=0} G_n(M_\chi, s, X) = \sum_{k=1}^{r} d_{-k}(M,X) \frac{L^{(n+k-1)}(M_\chi, \frac{w+1}{2})}{(n+k-1)!} + \sum_{l=0}^{n-1} c_l(M,X) \frac{L^{(n-1-l)}(M_\chi, \frac{w+1}{2})}{(n-1-l)!},$$

where

$$d_{-r+m}(M,X) = \sum_{t=0}^{m} \alpha_{-r+t}(M) b_{m-t}(M,X),$$

and

$$c_l(M,X) = \sum_{t=0}^{l} \alpha_{l-t}(M) b_t(M,X) + \sum_{m=0}^{r-1} \alpha_{-r+m}(M) b_{r-m+l}(M,X).$$

In particular, since

$$b_t(M,X) = \frac{1}{t!} \left(\log \frac{X}{\pi^{d_0} (2\pi)^{d_1}} \right)^t,$$

we obtain
(2.2)

$$\operatorname{res}_{s=0} G_n(M_\chi, s, X) \sim \alpha_{-r}(M) \sum_{l=0}^{n+r-1} b_l(M,X) \frac{L^{(n+r-1-l)}(M_\chi, \frac{w+1}{2})}{(n+r-1-l)!}, \quad X \to \infty.$$

Now the problem reduces to the question about the asymptotics of the sum

$$2 \sum_{t \in \mathcal{T}}{}' A_n \left(M_t, \sqrt{c(\chi_t)^d c(M)} \right) F\left(\frac{t}{T}\right).$$

Following Iwaniec's method [Iw] we obtain that the main contribution is equal to

$$S_0 = \frac{\gamma(M) T}{2c(M)} \sum_{m = kl^2} \frac{a_m \phi(l)}{l} m^{-\frac{w+1}{2}} \times \sum_{a \le T^{3/14}, (a, 4lc(M))=1} \mu(a) a^{-2} \int F(u) V_n \left(M, \frac{\pi^{d_0} (2\pi)^{d_1} m}{uT\sqrt{c(M)}} \right) du,$$

where

$$\gamma(M) := \sharp\{ d (\operatorname{mod} 4c(M)) : \ d \equiv -\nu^2 (\operatorname{mod} 4c(M)), \ (\nu, 4c(M)) = 1 \},$$

k has prime factors in $4c(M)$ and $(l, 4c(M)) = 1$.

The sum over $a > T^{3/14}$ contributes to $T^{11/14+\epsilon}$.

Now

$$\frac{a_m}{m^{\frac{w+1}{2}}}\frac{\phi(l)}{l}\sum_{(a,4c(M)l)=1}\frac{\mu(a)}{a^2}=$$

$$=\frac{a_m}{m^{\frac{w+1}{2}}}\prod_p\left(1-\frac{1}{p^2}\right)\prod_{p|4c(M)l}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|l}\left(1-\frac{1}{p}\right).$$

Since $(l,4c(M))=1$ we conclude the above is equal to

$$\frac{6}{\pi^2}\frac{a_m}{m^{\frac{w+1}{2}}}\prod_{p|4c(M)}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|l,\ (p,4c(M))=1}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|l,\ (p,4c(M))=1}\left(1-\frac{1}{p}\right)=$$

$$=\frac{6}{\pi^2}\frac{a_m}{m^{\frac{w+1}{2}}}\prod_{p|4c(M)}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|l,\ (p,4c(M))=1}\left(1+\frac{1}{p}\right)^{-1}.$$

Hence

(2.3)

$$S_0=\frac{3\gamma(M)T}{\pi^2c(M)}\prod_{p|4c(M)}\left(1-\frac{1}{p^2}\right)^{-1}\sum_{m=kl^2}\frac{b_m}{m^{\frac{w+1}{2}}}\int F(u)V_n\left(M,\frac{\pi^{d_0}(2\pi)^{d_1}m}{uT\sqrt{c(M)}}\right)du$$

$$=C_MT\int F(u)B_n(M,uT\sqrt{c(M)})du+O(T^{11/14+\epsilon});$$

here

$$B_n(M,X)=\sum_{m=kl^2}b_m m^{-\frac{w+1}{2}}V_n\left(M,\frac{\pi^{d_0}(2\pi)^{d_1}m}{X}\right),$$

with

$$b_m=a_m\prod_{p|m,\ (p,4c(M))=1}\left(1+\frac{1}{p}\right)^{-1},$$

where k has prime factors in $4c(M)$ and $(l,4c(M))=1$.

Assume that $D(M,s)$ has analytic continuation to Re $>\frac{w}{2}-\epsilon$. First, applying the Mellin transformation we see that

(2.4)

$$B_n(M,X)=\frac{1}{2\pi i}\int_{(3/4)}D(M,s+\frac{w+1}{2})\frac{L_\infty(M,s+i_0-\frac{w-1}{2})}{s^n}\left(\frac{X}{\pi^{d_0}(2\pi)^{d_1}}\right)^s.$$

Consider

$$H_n(M,s,X):=\mathrm{res}_{s=0}D(M,s+\frac{w+1}{2})\frac{L_\infty(M,s+i_0-\frac{w+1}{2})}{s^n}\left(\frac{X}{\pi^{d_0}(2\pi)^{d_1}}\right)^s.$$

Assume for the moment that $D(M,s)$ has no pole and does not vanish at $s=\frac{w+1}{2}$. Then by the analogue of (2.2) with $L(M_\chi,s)$ replaced by $D(M,s)$ we conclude

$$B_n(M,X)\sim\alpha_{-r}(M)b_{n-1+r}(M,X)\ D(M,\frac{w+1}{2}),\quad X\to\infty, \tag{2.5}$$

and combinying this with (2.3) we obtain the asymptotic equality ($T \to \infty$)

$$\text{(2.6)} \qquad \frac{C_M}{(n-1+r)!} D(M, \frac{w+1}{2}) \int F(u)du\, T(\log T)^{n-1+r}$$
$$\sim \sum_{t \le T} \sum_{l=0}^{n-1+r} b_l(M, t\sqrt{c(M)}) \frac{L^{(n-1+r-l)}(M_t, \frac{w+1}{2})}{(n-1+r-l)!} F\left(\frac{t}{T}\right),$$

with $C_M = \frac{3\gamma(M)}{\pi^2 c(M)} \prod_{p|4c(M)} (1-p^{-2})^{-1}$. To complete the proof of Theorem 1 in this case one takes $C(M,0) = 0$ if $\epsilon(M) = 1$, $C(M,0) = C_M$ if $\epsilon(M) = -1$ and $C(M,n) = (-1)^{\epsilon(M)}(-2)^{n-1} C_M$ if $n \neq 0$, and uses the induction in n.

In a general case we still can write down an analogue of (2.5) and, consequently of (2.6) (with a different constant term on the left).

3. Proof of Theorem 2

We slightly modify Heath-Brown's arguments ([HB], section 10). The main tool is the following large sieve type estimate ([HB], Corollary 3)

$$\text{(3.1)} \qquad \sum_{\chi \in \mathcal{C}(D)} |\sum_{n \le N} a_n \chi(n)|^2 \ll (DN)^\epsilon N(N+D) \max_n |a_n|^2.$$

Let A denote the number of $\Gamma_{\mathbb{R}}$'s plus twice number of $\Gamma_{\mathbb{C}}$'s in $L_\infty(M,s)$. Theorem 2 follows from the following result. (Here we do not assume that $M = M^\vee(-w)$.)

Theorem 2′. *Let $\epsilon > 0$ and assume that $d = d(M) \le 2$. Then we have for $\frac{w+1}{2} \le \sigma \le \frac{w+2}{2}$ and $D > D_0(M, \epsilon)$*

$$\sum_{\chi \in \mathcal{C}(D)} |L(M_\chi, \sigma + it)|^2 \ll \left(D + (D(|t|+1)^A)^{w+2-2\sigma}\right)(D|t|+1)^\epsilon.$$

Proof. Let $s = \sigma + it$,

$$S^+(D,s) = \sum_{\chi \in \mathcal{C}(D)} |L(M_\chi, s)|^2,$$

$$S^-(D,s) = \sum_{\chi \in \mathcal{C}(D)} |L(M_\chi^\vee(-w), s)|^2$$

and $\nu^\pm(\sigma)$ be the infimum of those exponents $\nu^\pm$ respectively for which

$$S^\pm(D, \sigma+it) \ll \left(D + (DT^A)^{w+1-2\sigma}\right)(DT^A)^{\nu^\pm}$$

uniformly in D and t, where $T = |t| + 1$. We will show that $\nu^{\pm} \leq 0$. Similarly as in [HB] p.268 we have for any $X > 0$ and $\frac{w+1}{2} \leq \alpha \leq \sigma \leq \frac{w+2}{2}$

(3.2)

$$S^{+}(D,s) \ll \sum_{\chi \in \mathcal{C}(D)} |\sum_{n} a_n \chi(n) n^{-s} e^{-\frac{n}{X}}|^2 + X^{2(\alpha-\sigma)} \int_{-\infty}^{\infty} S^{+}(D, \alpha+iu) e^{-|u-t|} du.$$

Using the functional equation (1.1) we obtain

$$S^{+}(D, \alpha+iu) = \sum_{\chi \in \mathcal{C}(D)} |U(D, \alpha+iu)|^2 |L(M_{\chi}^{\vee}(-w), w+1-\alpha-iu)|^2$$

where

(3.3)

$$U = U(D, \alpha+iu) = \epsilon(M_{\chi}) c(M_{\chi})^{\frac{w+1}{2}-(\alpha+iu)} L_{\infty}(M_{\chi}^{\vee}(-w), w+1-s) L_{\infty}(M_{\chi}, s)^{-1}.$$

The last quotient contains the Γ-factors of type

$$\Gamma\left(\frac{1}{2}(w+1-s+a)\right)\left(\Gamma(\frac{1}{2}(s+a)\right)^{-1}$$

or the similar ones with double arguments. In the second case we use the duplication formula

$$\Gamma(s) = \Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) 2^{s-1} \pi^{-\frac{1}{2}}$$

to have $\frac{s}{2}$ in all arguments of Γ-factors. If A denotes the total number of them, then using the equality

$$\Gamma(\sigma+it) = \sqrt{2\pi} |t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \left(1 + O(|t|^{-1})\right)$$

valid for $\sigma_0 \leq \sigma \leq \sigma_1$ and $t > t_0(\sigma_1, \sigma_2)$, we deduce that the ratio of the L_{∞}-terms in (3.3) is $\ll_{\sigma} |t|^{A(\frac{w+1}{2}-\sigma)}$. Hence by Lemma 1 ($\epsilon(M_{\chi})$ does not depend on D) we have

$$\begin{aligned} |U|^2 &\ll c(\chi)^{d(w+1-2\alpha)} |t|^{A(w+1-2\alpha)} \\ &\ll \left(D^d |t|^A\right)^{w+1-2\alpha}. \end{aligned}$$

The summation over $n > X(\log DT)^2$ in (3.2) is negligible due to the decay of $\exp\left(-\frac{n}{X}\right)$. We split the remaining part of the sum into the intervals of type $(N, 2N]$ and apply (3.1) to obtain

$$\sum_{\chi \in \mathcal{C}(D)} |\sum_{N \leq n \leq 2N} a_n \chi(n) n^{-s} e^{-\frac{n}{X}}|^2 \ll (D + X^{w+2-2\sigma})(DX)^{\epsilon}$$

for $\frac{w+1}{2} \leq \sigma \leq \frac{w+2}{2}$. Let $\Delta = \sigma - \frac{w+1}{2}$ and $\eta > 0$ be sufficiently small and $X = (DT^A)^{1+\eta}$. By the definition of $\nu^{\pm}$ and the above estimate we obtain

$$\begin{aligned} S^{+}(D,s) &\ll \\ &\ll \left((DT^A)^{w+2-2\sigma} + D\right)(DT^A)^{\eta+3\epsilon} \\ &+ (DT^A)^{2(\alpha-\sigma)(1+\eta)} (D^d T^A)^{w+1-2\alpha} (D + (DT^A)^{2\alpha-w}) (DT^A)^{\nu^{-}(w+1-\alpha)+\epsilon}. \end{aligned}$$

If $\frac{w+1}{2}+\eta\le\sigma\le\frac{w+2}{2}$ then we choose $\alpha=w+1-\sigma$ getting

$$
\begin{aligned}
S^+(D,s) \ll & \left((DT^A)^{w+2-2\sigma}+D\right)(DT^A)^{\eta+3\epsilon} \\
& + (DT^A)^{-4\Delta(1+\eta)}(D^dT^A)^{2\Delta}D(DT^A)^{\nu^-(w+1-\alpha)+\epsilon}
\end{aligned}
\tag{3.4}
$$

and by the inequality $d\le 2$ we obtain, by comparying the exponents that $\nu^+(\sigma)\le\max(\eta+3\epsilon,-4\eta^2+\epsilon+\nu^-(\sigma))$. Writting the same for the dual motive we conclude by the symmetry that $\nu^-(\sigma)\le\max(\eta+3\epsilon,-4\eta^2+\epsilon+\nu^+(\sigma))$. Choosing $2\eta>\sqrt{\epsilon}$ we conclude that $\nu^+(\sigma)\le 0$ in this case.

If $\frac{w+1}{2}\le\sigma\le\frac{w+1}{2}+\eta$ then choosing $\alpha=\frac{w+1}{2}-\eta$ we have by the above that $\nu^+(w+1-\alpha)\le 0$ and by (3.4) we obtain

$$
\begin{aligned}
S^+(D,s) &\ll \left((DT^A)^{w+2-2\sigma}+D\right)(DT^A)^{\eta+3\epsilon}+(D^dT^A)^{w+1-2\alpha}(D+(DT^A)^{2\alpha-w}) \\
&\ll \left((DT^A)^{w+2-2\sigma}+D\right)(DT^A)^{\eta+3\epsilon}+D(DT^A)^{4\eta+\epsilon},
\end{aligned}
$$

hence in this case also $\nu^+(\sigma)\le 0$. Analogously we prove that $\nu^-(\sigma)\le 0$. This completes the proof of Theorem 2.

4. Proof of Theorem 3

By (0.1) we see that

$$\sum_{\chi\in\mathcal{C}(D)}|L(M_\chi,s)|^2\ll D^{1+\epsilon}$$

uniformly for $|s-s_0|\le 2(\log D)^{-1}$, where $s_0=\frac{w+1}{2}$.

Also we have, by the Cauchy integral formula, that

$$L^{(n)}(M_\chi,s)=\frac{n!}{2\pi i}\int_{|s-\xi|=r}\frac{L(M_\chi,\xi)}{(\xi-s)^{n+1}}d\xi$$

with $r=(\log D)^{-1}$. Hence

$$
\begin{aligned}
&\sum_{\chi\in\mathcal{C}(D)}|L^{(n)}(M_\chi,s_0)|^2 \\
&\quad\ll(\log D)^{2n+2}\sum_{\chi\in\mathcal{C}(D)}\left(\int_{|\xi-s_0|=r}|L^{(n)}(M_\chi,\xi)|^2d\xi\right)\left(\int_{|\xi-s_0|=r}d\xi\right) \\
&\quad\ll(\log D)^{2n+1}\sum_{\chi\in\mathcal{C}(D)}\int_{|\xi-s_0|=r}|L^{(n)}(M_\chi,\xi)|^2d\xi \\
&\quad\ll(\log D)^{2n}\sup_{|\xi-s_0|\le(\log D)^{-1}}\sum_{\chi\in\mathcal{C}(D)}|L(M_\chi,\xi)|^2 \\
&\quad\ll D^{1+\epsilon}.
\end{aligned}
$$

Now from Theorem 1 and the above we obtain, by application of the Cauchy-Schwarz inequality, that

$$D(\log D)^n \ll \sum_{\chi\in\mathcal{C}(D)} |L^{(n)}(M_\chi, \frac{w+1}{2})| \leq (\sum_{\chi\in\mathcal{C}(D)} |L^{(n)}(M_\chi, \frac{w+1}{2})|)^{\frac{1}{2}} N_n(D,M)^{\frac{1}{2}}.$$

Hence

$$N_n(D,M) \gg D^{1-2\epsilon}$$

and Theorem 3 follows.

5. Applications to a bound for the order of the (twisted) Shafarevich-Tate groups

Let A be an abelian variety over $\mathbb{Q}$ of rank d and conductor $c(A)$; let p be a prime dividing $c(A)$. Let $\mathcal{A}_p$ be a Néron model of A over $\mathbb{Q}_p$ (see [BLR] for definition). We are interested in a bound for the order of the group $\Phi_p(A)$ of connected components of $\mathcal{A}_p$.

In a case of an elliptic curve E with integral j-invariant we have $\sharp\Phi_p(E) \leq 4$ (see [T1]).

In a general case W.G. McCallum [Mc] proved the following result. Assume that a Néron model $\mathcal{A}_p$ of A has no multiplicative component in its reduction; then such a bound exists and depends only on the dimension d of A: $\sharp\Phi_p(A) \leq C_d$.

Let A_χ denote the twist of A by quadratic character χ; it is again an abelian variety over $\mathbb{Q}$. Assume that for any p dividing $c(A)$ a Néron model $\mathcal{A}_p$ has no multiplicative component in its reduction; if $(c(\chi), c(A)) = 1$ the same holds for any A_χ. Hence, in this case

$$\prod_{p|c(A_t)} \sharp\Phi_p(A_t) \ll C_d^{\psi(t)},$$

where $A_t = A_{\chi_t}$, and $\psi(t)$ denotes the number of prime divisors of t. As a consequence, we obtain

$$\prod_{p|c(A_{\chi_t})} \sharp\Phi_p(A_{\chi_t}) = O(t^\epsilon). \tag{5.1}$$

Proof of Theorem 4. We start with the following estimate for the real period of A:

$$\pi_\infty(A_t) \ll t^{-\frac{d}{2}}. \tag{5.2}$$

Theorem 3 implies that there exist infinitely many t such that $L(A_t, 1) \gg 1$. Assuming the Birch and Swinnerton-Dyer conjecture for all such A_t we obtain

$$\sharp Ш(A_t) \gg \frac{1}{\pi_\infty(A_t)\pi_{bad}(A_t)}.$$

Now, the assertion follows from (5.1), (5.2) and Lemma 1.

Remark. If M is a pure motive of weight -1, then the same arguments (at least conjecturally) may be used to state the analogue of Theorem 4 in this case.

6. Concluding remarks

6.1. Let E be an elliptic curve over $\mathbb{Q}$ of conductor $N \to \infty$. It is conjectured [GS] that

$$\sharp Ш(E) = O(N^{\frac{1}{2}+\epsilon}) \ . \tag{6.1}$$

Let's also remark the following consequence of [GS]: if $\sharp Ш_l(E) \neq 1$ for some prime $l \gg N^{\frac{1}{2}+\epsilon}$, then either (6.1) is false or the Birch-Swinnerton-Dyer conjecture is false or the Shimura-Taniyama-Weil conjecture is false. Note however that the last conjecture has been proved by A. Wiles for all semistable elliptic curves.

It is an important question whether there exist elliptic curves with arbitrary high rank and what is the frequency of these high ranks. The work of F. Gouvea and B. Mazur [GM] shows that

$$\begin{aligned}&\sharp\{D \in \mathbb{Z} : \ D \equiv 1 (\mathrm{mod}\ 8),\ 1 \le D \le X,\ D \text{ square free},\ \mathrm{ord}_{s=1} L(E_D, s) \geq 2\}\\ &\quad \gg X^{\frac{1}{2}-\epsilon}.\end{aligned}$$

On the other hand we have the following conjecture due to R. Greenberg [G].

Conjecture. *Let Σ be a finite set of primes. Let f vary over all normalized newforms of weight 2 for $\Gamma_0(q)$, where q is divisible only by primes in Σ. Then* $\mathrm{ord}_{s=1} L(f,s)$ *is 0 or 1, except for at most finitely many such f's.*

6.2. The following possible application was suggested by ([BFH2], p.169). Suppose that f and g are two newforms of the same weight k on some $\Gamma_0(q)$. One asks whether there are infinitely many quadratic characters χ such that

$$L(M(f)_\chi, \frac{k}{2})\ L(M(g)_\chi, \frac{k}{2}) \neq 0.$$

In a case f and g correspond to elliptic curves E_f and E_g respectively, it would then imply, combined with the Kolyvagin's work, that there are infinitely fundamental discriminants t such that the ranks of the Mordell-Weil groups of the twisted elliptic curves $E_f^{(t)}$ and $E_g^{(t)}$ are both zero. By this method it reduces to a problem concerning meromorphic continuality of $D(M(f) \oplus M(g), s)$ to $\mathrm{Re}(s) > \frac{k-1}{2} - \epsilon$, and certainly depends on a variant of Rankin-Selberg method for GL(4).

Appendix

It is quite easy to continue $D(\mathbb{Q}(0)^{\oplus r}, s)$ meromorphically to $\mathrm{Re}(s) > \frac{1}{2} - \epsilon$. Here the procedure is explicit and gives information about the behaviour of our series at $s = \frac{1}{2}$: it always has a pole at this point. Let us perform the calculations for $r = 1, 2$.

(i) Consider $D(\mathbb{Q}(0), s)$. We have

$$D(\mathbb{Q}(0), s) = (1 + 2^{-s}) \sum_{l=1}^{\infty} \prod_{p|l,\ p\neq 2} \left(1 + \frac{1}{p}\right)^{-1} l^{-2s}, \quad \mathrm{Re}(s) > \frac{1}{2}.$$

Let f be a multiplicative function, defined on powers of primes by

$$f(1) = 1, \qquad f(2^k) = 1,\ k \geq 1, \qquad f(p^k) = \left(1 + \frac{1}{p}\right)^{-1},\ p \neq 2,\ k \geq 1.$$

One checks that

$$f = \phi_1 \star \epsilon,$$

where ϕ is the Euler function, $\phi_1(n) := n^{-1}\phi(n)$ and ϵ is a multiplicative function defined by

$$\epsilon(1) = 1, \quad \epsilon(2^k) = 2^{-k},\ k \geq 1, \quad \epsilon(p^k) = \frac{1}{p^k(p+1)},\ p \neq 2,\ k \geq 1.$$

Therefore

$$D(\mathbb{Q}(0), s) = (1 + 2^{-s}) \left(\sum_{n=1}^{\infty} \phi(n) n^{-2s-1}\right) \left(\sum_{n=1}^{\infty} \epsilon(n) n^{-2s}\right).$$

Now

$$\sum_{n=1}^{\infty} \epsilon(n) n^{-2s} = (1 - 2^{-1-2s})^{-1} \prod_{p\neq 2} \left(1 + \frac{1}{p+1} \frac{p^{-1-2s}}{1 - p^{-1-2s}}\right)$$

is convergent and non-zero at $s = \frac{1}{2}$, and

$$\sum_{n=1}^{\infty} \phi(n) n^{-2s-1} = \frac{\zeta(2s)}{\zeta(2s+1)}.$$

Hence $D(\mathbb{Q}(0), s)$ has a simple pole at $s = \frac{1}{2}$.

(ii) Let $d(n)$ denote the number of divisors of n. We have

$$\begin{aligned} D(\mathbb{Q}(0) \oplus \mathbb{Q}(0), s) &= \sum_{r=0}^{\infty} \sum_{l \text{ odd}} d(2^r l^2) \prod_{p|l} \left(1 + \frac{1}{p}\right)^{-1} 2^{-rs} l^{-2s} \\ &= \sum_{r=0}^{\infty} d(2^r) 2^{-rs} \sum_{n \text{ odd}} d(n^2) \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1} n^{-2s}, \quad \mathrm{Re}(s) > \frac{1}{2}. \end{aligned}$$

The first sum is convergent and non-zero for $s = \frac{1}{2}$. Let ϵ be a multiplicative function defined by $\epsilon(1) = 1$, $\epsilon(2) = -1$, $\epsilon(2^r) = 0$, $(r \geq 2)$ and

$$\epsilon(p) = \frac{2p-1}{p+1},\ p \neq 2, \quad \epsilon(p^r) = \frac{2p}{p+1},\ p \neq 2,\ r \geq 2.$$

Let f be defined by $f(1) = 1$ and

$$f(2^r) = 0,\ r \geq 1, \quad f(p^r) = \frac{p}{p+1} d(p^{2r}),\ p \neq 2,\ r \geq 1.$$

Then $f = \epsilon \star 1$ and we obtain

$$\sum_{n \text{ odd}} d(n^2) \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1} n^{-2s} = (1-2^{-2s})^2 \zeta(2s)^2 \prod_{p \neq 2} \left(1 - (\frac{2}{p+1} + p^{-2s}) p^{-2s}\right).$$

Hence $D(\mathbb{Q}(0) \oplus \mathbb{Q}(0), s)$ has a double pole at $s = \frac{1}{2}$.

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