Semi-infinite Weil complex and superconformal algebras II.

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Contents.

0. Introduction.

This work is a continuation of [P2]. Recall that in [P2] we considered the semi-infinite Weil complex $W^{\frac{\infty}{2}+*}(\mathfrak{g})$ of a graded Lie algebra $\mathfrak{g}$. We proved that in the case when $\mathfrak{g}$ is the loop algebra of a complex finite-dimensional Lie algebra, $W^{\frac{\infty}{2}+*}(\mathfrak{g})$ is a module over the $N=2$ superconformal algebra. The $N=2$ superconformal algebra also acts on the semi-infinite cohomology of $\mathfrak{g}$ with coefficients in its "adjoint semi-infinite symmetric powers" $H^{\frac{\infty}{2}+*}(\mathfrak{g},S^{\frac{\infty}{2}+*}(\mathfrak{g}))$.

In this work we show that there is a superconformal algebra from the classification given in [K-VDL], which contains the $N=2$ superconformal algebra as a subalgebra, and acts on the relative semi-infinite Weil complex and relative semi-infinite cohomology.

In section 1 we review a definition of the relative semi-infinite Weil complex and the relative semi-infinite cohomology $H^{\frac{\infty}{2}+*}(\mathfrak{g},\mathfrak{g}_0,S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ of a graded Lie algebra $\mathfrak{g}$.

In section 2 we consider the relative semi-infinite Weil complex of the loop algebra of $\mathfrak{sl}(2)$. We describe the relative semi-infinite cohomology in terms of the Lie algebra (co)homology and give the proofs of the results announced in [P1]. We observe some $\mathfrak{sl}(2)$-module structures and an action of the $Witt$ algebra on this complex.

In section 3 we recall a definition of the superconformal algebra $S(2,0)$. The algebra

$S(2,0)$ has a simple ideal $S'(2,0)=[S(2,0),S(2,0)]$. We prove that the exterior derivations of $S'(2,0)$ form an $\mathfrak{sl}(2)$.

In section 4, using the structures described in section 2, we prove that the relative semi-infinite Weil complex of the loop algebra of $\mathfrak{sl}(2)$ (and the relative semi-infinite cohomology) is a module over a central extension $\hat{S}(2,0)$ of $S(2,0)$. We construct $S(2,0)$ using the quadratic expansions on the generators of Heisenberg and Clifford algebras related to $\mathfrak{g}$.

In section 5 we consider a complex finite-dimensional Lie algebra $\mathfrak{g}$ with a non-degenerate invariant symmetric form. We obtain a representation of $\hat{S}(2,0)$ with central charge $3dim\mathfrak{g}$ on the relative semi-infinite Weil complex of the loop algebra $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, and on the relative semi-infinite cohomology $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}_0,S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$. Moreover, there is an action of a central extension of the Lie superalgebra $DerS'(2,0)$ of derivations of $S'(2,0)$ on these spaces. In particular, the exterior derivations of $S'(2,0)$ define the $\mathfrak{sl}(2)$-module structure on the relative semi-infinite cohomology, which was observed in [FGZ] as an analogue of the classical structure on the de Rham cohomology in Kähler geometry.

Through this paper we use the notations and definitions given in section 1 of [P2].

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1. Relative semi-infinite Weil complex.

Let $\mathfrak{g}=\oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ be a graded Lie algebra over $\mathbb{C}$. In [P2] we reviewed a construction of the semi-infinite Weil complex

$$\{S^{\frac{\infty}{2}+*}(\mathfrak{g})\otimes\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g}),d+\mathrm{h}\}. \tag{1.1}$$

Recall that the semi-infinite symmetric power space $S^{\frac{\infty}{2}+*}(\mathfrak{g})$ is a module over the Heisenberg algebra $H(\mathfrak{g})$, and the semi-infinite exterior power space $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ is a module over the Clifford algebra $C(\mathfrak{g})$. Denote the actions of $H(\mathfrak{g})$ and $C(\mathfrak{g})$ on these modules by $\beta(x),\gamma(x')$ and $\tau(x),\varepsilon(x')$, respectively, for $x\in\mathfrak{g}$, $x'\in\mathfrak{g}'$.

Recall the definition of the relative semi-infinite Weil complex (cf. [FGZ]). Let $\rho(x)$ and $\pi(x)$ be the projective representations of $\mathfrak{g}$ in $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ and $S^{\frac{\infty}{2}+*}(\mathfrak{g})$, respectively. Let

$$\theta(x)=\rho(x)+\pi(x) \tag{1.2}$$

for $x\in\mathfrak{g}$. Since the cocycles corresponding to $\rho(x)$ and $\pi(x)$ cancel, the representation θ

of $\mathfrak{g}$ in $S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ is well-defined. Let

$$C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g})) = \{c \in S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^{\frac{\infty}{2}+*}(\mathfrak{g}) | \tau(x)c = 0, \theta(x)c = 0 \text{ for all } x \in \mathfrak{g}_0\}, \tag{1.3}$$

The differential d preserves this subcomplex since

$$\{d, \tau(x)\} = d\tau(x) + \tau(x)d = \theta(x), \tag{1.4}$$

and

$$[d, \theta(x)] = 0 \tag{1.5}$$

for any $x \in \mathfrak{g}$. The complex (1.3) is called *the relative semi-infinite Weil complex*. Its cohomology (with respect to the differential d) is called *the relative semi-infinite cohomology* $H^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$. Let $Deg\ \varepsilon(x') = Deg\ \gamma(x') = 1$, $Deg\ \tau(x) = Deg\ \beta(x) = -1$ for $x \in \mathfrak{g}, x' \in \mathfrak{g}'$. Then

$$\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g}) = \oplus_{j\in\mathbb{Z}} \Lambda^{\frac{\infty}{2}+j}(\mathfrak{g}), S^{\frac{\infty}{2}+*}(\mathfrak{g}) = \oplus_{l\in\mathbb{Z}} S^{\frac{\infty}{2}+l}(\mathfrak{g}). \tag{1.6}$$

Accordingly, for any $j, l \in \mathbb{Z}$ we obtain a group $H^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g}))$.

Let $deg\ \tau(x) = deg\ \beta(x) = n, deg\ \varepsilon(x') = deg\ \gamma(x') = -n$ for $x \in \mathfrak{g}_n, x' \in \mathfrak{g}'_n$. Since d preserves the graded structure *deg* of the Weil complex, the cohomology groups also have a graded structure. Notice that they are finite-dimensional in each level:

$$H^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g})) = \oplus_{n\in\mathbb{Z}} H^{\frac{\infty}{2}+j,n}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g})). \tag{1.7}$$

Recall that given $N \in \mathbb{Z}$, a polarization of $\mathfrak{g}$ is defined: $\mathfrak{n}_+ = \oplus_{n>N}\mathfrak{g}_n$, $\mathfrak{n} = \oplus_{n\leq N}\mathfrak{g}_n$. We fix $N = 0$. Thus $\mathfrak{n}_+ = \oplus_{n>0}\mathfrak{g}_n$. Let $\mathfrak{n}_- = \oplus_{n<0}\mathfrak{g}_n$.

For a homogeneous element in $\Lambda^a(\mathfrak{n}'_+) \wedge \Lambda^b_\infty(\mathfrak{n}'_-)$, a is the number of added elements, and b is the number of missing elements with respect to the vacuum vector

$$\mathbb{1}_\Lambda = e'_{i_1} \wedge e'_{i_1-1} \wedge \dots . \tag{1.8}$$

Let

$$\Lambda^{\frac{\infty}{2}+*}_{rel}(\mathfrak{g}) = \oplus_{a\geq 0, b\geq 0} \Lambda^a(\mathfrak{n}'_+) \wedge \Lambda^b_\infty(\mathfrak{n}'_-). \tag{1.9}$$

Let

$$C^{a,b}(\mathfrak{g}) = [S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^a(\mathfrak{n}'_+) \wedge \Lambda^b_\infty(\mathfrak{n}'_-)]^{\mathfrak{g}_0}. \tag{1.10}$$

Thus

$$C^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g})) = \oplus_{a-b=j} C^{a,b}(\mathfrak{g}). \tag{1.11}$$

The first natural example is the Witt algebra:

$$\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n, \tag{1.12}$$

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{1.13}$$

It was studied by B. Feigin and E. Frenkel in details in [FF]. In particular,

$$H^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g})) = S^*(\mathbb{C}L_0). \tag{1.14}$$

Thus the cohomology exists only in $deg = 0$. The next example is the loop algebra of $\mathfrak{sl}(2)$.

2. Relative semi-infinite Weil complex of $\mathfrak{sl}(2)\otimes\mathbb{C}[t,t^{-1}]$.

2.1. Relative semi-infinite cohomology. Let $\mathfrak{g} = \mathfrak{sl}(2)\otimes\mathbb{C}[t,t^{-1}]$, $\mathfrak{g}_n = \mathfrak{sl}(2)\otimes t^n$, let $\{e,h,f\}$ be the standard basis of $\mathfrak{sl}(2)$, $\{e_n,h_n,f_n\}$ be a basis of $\mathfrak{g}_n$, $\{e'_n,h'_n,f'_n\}$ be the dual basis.

Let $V^a\subset\Lambda^a(\mathfrak{n}'_+)$ be an irreducible $\mathfrak{g}_0$-module with highest vector $f'_1\wedge f'_2\wedge\ldots\wedge f'_a$. Let V^b be an irreducible $\mathfrak{g}_0$-module with highest vector $e_{-1}\wedge e_{-2}\wedge\ldots\wedge e_{-b}$. We identify V^b with a subspace of $\Lambda^b_\infty(\mathfrak{n}'_-)$. Let

$$\varphi : S^{\frac{\infty}{2}+*}(\mathfrak{g})\otimes V^a\otimes V^b \to S^{\frac{\infty}{2}+*}(\mathfrak{g})\otimes V^b\otimes V^a \tag{2.1.1}$$

be the natural isomorfism. The following statement was announced in [P1].

Theorem 2.1.1. The representatives of the elements of $H^{\frac{\infty}{2}+j}(\mathfrak{g},\mathfrak{g}_0,S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ in $C^{\frac{\infty}{2}+j}(\mathfrak{g},\mathfrak{g}_0,S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ are the arbitrary linear combinations of the elements

$$x\in[S^{\frac{\infty}{2}+*}(\mathfrak{g})\otimes V^a\otimes V^b]^{\mathfrak{g}_0}, \text{ where } a-b=j, \tag{2.1.2}$$

such that $x\in H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g}))\otimes V^b$, and $\varphi x\in H_b(\mathfrak{n}_-, S^{\frac{\infty}{2}+*})\otimes V^a$.

To prove this theorem we will use some Hochschild-Serre spectral sequences. We will use the notation $C^{\frac{\infty}{2}+*}(\mathfrak{g}) = C^{\frac{\infty}{2}+*}(\mathfrak{g},\mathfrak{g}_0,S^{\frac{\infty}{2}+*}(\mathfrak{g}))$.

Spectral sequence for $H^{\frac{\infty}{2}+*}(\mathfrak{g},\mathfrak{g}_0,S^{\frac{\infty}{2}+*}(\mathfrak{g}))$.

We define a filtration

$$\ldots\supset F^pC^{\frac{\infty}{2}+*}(\mathfrak{g})\supset F^{p+1}C^{\frac{\infty}{2}+*}(\mathfrak{g})\supset\ldots. \tag{2.1.3}$$

corresponding to the number of missing elements:

$$F^pC^{\frac{\infty}{2}+*}(\mathfrak{g}) = \{c\in\oplus_{a\ge 0, b\le -p}C^{a,b}(\mathfrak{g})\}. \tag{2.1.4}$$

We follow the usual construction of the Hochschild-Serre spectral sequence (see [Fu], [GM]) and set

$$Z_r^{p,q} = \{c \in F^p C^{\frac{\infty}{2}+p+q}(\mathfrak{g}) |\ dc \in F^{p+r} C^{\frac{\infty}{2}+p+q+1}(\mathfrak{g})\}. \tag{2.1.5}$$

Then set

$$E_r^{p,q} = Z_r^{p,q} / (Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}). \tag{2.1.6}$$

The differential d induces the differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{2.1.7}$$

and $E_{r+1}^{p,q} = H^{p,q}(E_r)$. Let

$$E_r = \oplus_{p,q} E_r^{p,q}. \tag{2.1.8}$$

Denote by d_r the corresponding differential. Thus we have

$$E_0 = \oplus_{a,b \geq 0} [S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^a(\mathfrak{n}'_+) \wedge \Lambda_\infty^b(\mathfrak{n}'_-)]^{\mathfrak{g}_0}, \tag{2.1.9}$$

and

$$E_1 = \oplus_{a,b \geq 0} [H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes H_\infty^b(\mathfrak{n}_-, \mathbb{C})]^{\mathfrak{g}_0}. \tag{2.1.10}$$

Notice that $H_\infty^b(\mathfrak{n}_-, \mathbb{C})$ coincides with the usual homology $H_b(\mathfrak{n}_-)$ of the Lie algebra $\mathfrak{n}_-$ with coefficients in the trivial representation.

Lemma 2.1.1. $H_b(\mathfrak{n}_-)$ is an irreducible $\mathfrak{sl}(2)$-module with highest weight $2b$, and highest vector

$$e_{-1} \wedge e_{-2} \wedge \ldots \wedge e_{-b}. \tag{2.1.11}$$

Proof. We will prove an equivalent statement for $H^i(\mathfrak{n}_-)$ using the generalized Bott-Kostant theorem ([GL]). Consider the affine Kac-Moody Lie algebra $A_1^{(1)}$:

$$A_1^{(1)} = \oplus_n \tilde{\mathfrak{g}}_n, \text{ where} \tag{2.1.12}$$

$$\tilde{\mathfrak{g}}_n = \mathfrak{sl}(2) \otimes t^n \text{ for } n \neq 0, \tag{2.1.13}$$

$$\tilde{\mathfrak{g}}_0 = \mathfrak{sl}(2) \oplus \mathbb{C}c \oplus \mathbb{C}\mathfrak{d}. \tag{2.1.14}$$

Recall that

$$\mathfrak{g}_0 = \mathfrak{sl}(2) =< e, h, f >, \tag{2.1.15}$$

where $\{e, h, f\}$ is the standard basis. Let the corresponding roots be $\pm\alpha$. The abelian subalgebra

$$\tilde{h}_0 = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}\mathfrak{d} \tag{2.1.16}$$

is identified with its dual by means of the nonsingular symmetric bilinear form on $A_1^{(1)}$. Let

$$h_0 = c - \alpha, h_1 = \alpha. \tag{2.1.17}$$

Let W be the Weyl group of $A_1^{(1)}$. Recall that W is generated by the linear automorphisms r_k of $\tilde{h}_0$, where $k = 0, 1$, defined by the condition

$$r_k \varphi = \varphi - < \varphi, h_k > h_k \text{ for all } \varphi \in \tilde{h}_0. \tag{2.1.18}$$

Let

$$\Delta_+ = \{\alpha + nc \text{ for } n \geq 0, \quad -\alpha + nc \text{ for } n > 0, \quad nc \text{ for } n > 0\} \tag{2.1.19}$$

$$\Delta_- = \{-\alpha + nc \text{ for } n \leq 0, \quad \alpha + nc \text{ for } n < 0, \quad nc \text{ for } n < 0\} \tag{2.1.20}$$

$$\Delta(\mathfrak{n}_-) = \{-\alpha + nc \text{ for } n < 0, \quad \alpha + nc \text{ for } n < 0, \quad nc \text{ for } n < 0\}. \tag{2.1.21}$$

According to the generalized Bott-Kostant theorem, the irreducible $\mathfrak{g}_0$-submodules of $H^i(\mathfrak{n}_-)$ correspond exactly to the elements w of W which satisfy the following conditions:

$$w(\Delta_+) \cap \Delta_- \subset \Delta(\mathfrak{n}_-), \tag{2.1.22}$$

$$\#(w(\Delta_+) \cap \Delta_-) = i, \tag{2.1.23}$$

where $\#$ denotes the number of the elements in the set.
Define $\rho \in \tilde{h}_0$ by

$$< \rho, h_k >= 1 \text{ for } k = 0, 1. \tag{2.1.24}$$

Then the corresponding highest weights are given by

$$w(\rho) - \rho. \tag{2.1.25}$$

One can check that in our case

$$\rho = 1/2\alpha + 2\mathfrak{d}, \tag{2.1.26}$$

and the only $w \in W$ which satisfies (2.1.22) and (2.1.23) is (of length i):

$$w = (r_0 r_1)^{i/2} \qquad \text{if } i \text{ is even,} \tag{2.1.27}$$

$$(r_0 r_1)^{[i/2]} r_0 \qquad \text{if } i \text{ is odd.} \tag{2.1.28}$$

Then

$$w(\rho) - \rho = -i(i+1)/2c + i\alpha. \tag{2.1.29}$$

Thus $H^i(\mathfrak{n}_-)$ is an irreducible $\mathfrak{g}_0$-module with highest weight $2i$.

$\blacksquare$

Corollary 2.1.1. Let $x \in H_b(\mathfrak{n}_-)$. We can choose a representative x such that

$$degx = -b(b+1)/2. \tag{2.1.30}$$

Corollary 2.1.2. $H^a(\mathfrak{n}_+)$ is an irreducible $\mathfrak{sl}(2)$-module with highest weight $2a$, and highest vector

$$f_1' \wedge f_2' \wedge \ldots \wedge f_a'. \tag{2.1.31}$$

Spectral sequence for $H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$.
Let

$$C^a(\mathfrak{g}) = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^a(\mathfrak{n}_+'). \tag{2.1.32}$$

Let $v \in S^{\frac{\infty}{2}+*}(\mathfrak{g}), \quad w_+ \in \Lambda^a(n_+')$. Define

$$fdeg(v \otimes w_+) = degv - degw_+. \tag{2.1.33}$$

Consider a spectral sequence associated with the filtration

$$\ldots \supset F^pC^a(\mathfrak{g}) \supset F^{p+1}C^a(\mathfrak{g}) \supset \ldots, \tag{2.1.34}$$

defined as follows:

$$F^pC^a(\mathfrak{g}) = \{c \in C^a(\mathfrak{g}) | fdegc \geq p\}. \tag{2.1.35}$$

Then

$$E_0 = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^a(\mathfrak{n}_+'), \tag{2.1.36}$$

and

$$E_1 = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes H^a(\mathfrak{n}_+, \mathbb{C}). \tag{2.1.37}$$

According to corollary 2.1.2, $H^a(\mathfrak{n}_+, \mathbb{C})$ and V^a are isomorphic as $\mathfrak{g}_0$-modules. Thus the elements of $S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^a$ are the representatives of the elements of E_1 in the space of cochains. Notice that

$$deg(V^a) = -a(a+1)/2. \tag{2.1.38}$$

Recall that the differential preserves the grading deg. Thus

$$H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) = \oplus_{n\in\mathbb{Z}} H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g}))_{deg=n}. \tag{2.1.39}$$

Consider the filtration defined by (2.1.35) in the space of a-cochains of $deg = n$. Then the representatives of the term E_1 of the corresponding spectral sequence constitute

$$[S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^a]_{deg=n}. \tag{2.1.40}$$

Notice that

$$fdeg([S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^a]_{deg=n}) = n + a(a+1). \tag{2.1.41}$$

Thus

$$E_1^{p,a-p} = 0, \text{ if } p \neq n + a(a+1). \tag{2.1.42}$$

Recall that the differential d_r acts as follows :

$$E_r^{p,a-p} \xrightarrow{d_r} E_r^{p+r,a-p-r+1}. \tag{2.1.43}$$

Thus

$$d_r = 0, \text{ if } r \neq 2a+2. \tag{2.1.44}$$

Thus E_∞ is determined by the complex :

$$\ldots \xrightarrow{d_{2a-2}} S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^{a-1} \xrightarrow{d_{2a}} S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^a \xrightarrow{d_{2a+2}} S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^{a+1} \xrightarrow{d_{2a+4}} \ldots. \tag{2.1.45}$$

Spectral sequence for $H_b(\mathfrak{n}_-, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$.
Let

$$C^b(\mathfrak{g}) = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^b(\mathfrak{n}_-). \tag{2.1.46}$$

Let $v \in S^{\frac{\infty}{2}+*}(\mathfrak{g}), \quad w_- \in \Lambda^b(n_-)$. Define

$$fdeg'(w_- \otimes v) = degw_- - degv. \tag{2.1.47}$$

Define the filtration

$$F^pC^b(\mathfrak{g}) = \{c \in C^b(\mathfrak{g}) | fdeg'c \geq p\}. \tag{2.1.48}$$

Then the first term of the corresponding spectral sequence is (cf. (2.1.37))

$$E_1 = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes H_b(\mathfrak{n}_-, \mathbb{C}). \tag{2.1.49}$$

According to lemma 2.1.1, $H_b(\mathfrak{n}_-, \mathbb{C})$ and V^b are isomorphic as $\mathfrak{g}_0$-modules. Thus the elements of $S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^b$ are the representatives of the elements of E_1 in the space of cochains. By analogy with (2.1.44) we obtain that

$$d_r = 0, \text{ if } r \neq 2b. \tag{2.1.50}$$

Thus E_∞ is determined by the complex

$$\dots \longrightarrow S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^{b+1} \xrightarrow{d_{2b+2}} S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^{b} \xrightarrow{d_{2b}} S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^{b-1} \longrightarrow \dots . \tag{2.1.51}$$

Proof of theorem 2.1.1.

We use the spectral sequence corresponding to the filtration (2.1.4). Recall that

$$E_1^{-b,a} = [H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes H^b_\infty(\mathfrak{n}_-, \mathbb{C})]^{\mathfrak{g}_0}. \tag{2.1.52}$$

First we will show that the representatives of the elements of $E_2^{-b,a}$ are the elements

$$x \in [S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^a \otimes V^b]^{\mathfrak{g}_0}, \tag{2.1.53}$$

such that $x \in H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes V^b$, and $\varphi x \in H_b(\mathfrak{n}_-, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes V^a$.

Recall that the term $E_2^{-b,a}$ is determined by the complex

$$\dots \xrightarrow{d_1} [H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes H^{b+1}_\infty(\mathfrak{n}_-, \mathbb{C})]^{\mathfrak{g}_0} \xrightarrow{d_1} [H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes H^{b}_\infty(\mathfrak{n}_-, \mathbb{C})]^{\mathfrak{g}_0} \xrightarrow{d_1}$$

$$\xrightarrow{d_1} [H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes H^{b-1}_\infty(\mathfrak{n}_-, \mathbb{C})]^{\mathfrak{g}_0} \xrightarrow{d_1} \dots . \tag{2.1.54}$$

According to (2.1.45), the elements of $H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ have representatives in $S^{\frac{\infty}{2}+*}(\mathfrak{g})\otimes V^a$. Hence the elements of $E_1^{-b,a}$ have representatives in $[S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^a \otimes V^b]^{\mathfrak{g}_0}$. Let

$$x = \Sigma_k \Sigma_i v^{k,i} \otimes w_+^k \otimes w_-^{k,i}, \text{ where } v^{k,i} \in S^{\frac{\infty}{2}+*}(\mathfrak{g}), w_+^k \in V^a, w_-^{k,i} \in V^b, \tag{2.1.55}$$

be a representative of an element of $E_1^{-b,a}$. Thus $x \in H^a(\mathfrak{n}_+, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes V^b$. Notice that

$$d_1 x = \varphi^{-1}(\Sigma_k d_{2b}(\Sigma_i v^{k,i} \otimes w_-^{k,i}) \otimes w_+^k). \tag{2.1.56}$$

Thus x is a representative of an element of $E_2^{-b,a}$ if and only if

$$\varphi x \in H_b(\mathfrak{n}_-, S^{\frac{\infty}{2}+*}(\mathfrak{g})) \otimes V^a. \tag{2.1.57}$$

Next we will show that

$$E_2^{-b,a} = E_\infty^{-b,a}. \tag{2.1.58}$$

Let x be a representative of an element of $E_r^{-b,a}$, where $r \geq 2$, in $Z_r^{-b,a}$. Then

$$dx \in F^{-b+r} C^{\frac{\infty}{2}+a-b+1}(\mathfrak{g}). \tag{2.1.59}$$

Hence

$$dx \in \oplus_{a_1 \geq 0, b_1 \leq b-2} C^{a_1,b_1}(\mathfrak{g}). \tag{2.1.60}$$

On the other hand,

$$d = d_+ + d_-, \tag{2.1.61}$$

where

$$d_+ : C^{a,b}(\mathfrak{g}) \to C^{a+1,b}(\mathfrak{g}), \tag{2.1.62}$$

$$d_- : C^{a,b}(\mathfrak{g}) \to C^{a,b-1}(\mathfrak{g}). \tag{2.1.63}$$

Thus if $x \in [S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes V^a \otimes V^b]^{\mathfrak{g}_0}$, then

$$dx \in C^{a+1,b}(\mathfrak{g}) \oplus C^{a,b-1}(\mathfrak{g}). \tag{2.1.64}$$

Then according to (2.1.60) and (2.1.64), we have

$$dx = 0. \tag{2.1.65}$$

■

Remark 2.1.1. Notice that to compute the relative semi-infinite cohomology one can use as well the filtration defined by the number of added elements :

$$\tilde{F}^p C^{\frac{\infty}{2}+*}(\mathfrak{g}) = \{c \in \oplus_{a \geq p, b \geq 0} C^{a,b}(\mathfrak{g})\}. \tag{2.1.66}$$

The first term of the corresponding spectral sequence is

$$\tilde{E}_1 = \oplus_{a,b\geq 0}[H^a(\mathfrak{n}_+, \mathbb{C}) \otimes H^b_\infty(\mathfrak{n}_-, S^{\frac{\infty}{2}+*}(\mathfrak{g}))]^{\mathfrak{g}_0}. \tag{2.1.67}$$

Both spectral sequences corresponding to the filtrations (2.1.4) and (2.1.66) collapse in the second term.

2.2. An action of the Witt algebra.

We will show that the relative semi-infinite cohomology admits an action of the Witt algebra.

Let u run through the standard basis $\{e, h, f\}$ of $\mathfrak{sl}(2)$, let $\{u_n\}$ be a basis of $\mathfrak{g}_n$, $\{u'_n\}$ be the dual basis. Let

$$L_n = -\Sigma_u \Sigma_m m : \tau(u_{m+n})\varepsilon(u'_m) : -\Sigma_u \Sigma_m m : \beta(u_m)\gamma(u'_{m-n}) :, \tag{2.2.1}$$

We obtain the Witt algebra $L = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n$, $[L_n, L_m] = (n-m)L_{n+m}$. Recall that $\mathbb{1}_S$ is the vacuum vector in $S^{\frac{\infty}{2}+*}(\mathfrak{g})$, and $\mathbb{1}_\Lambda$ is the vacuum vector in $\Lambda_{rel}^{\frac{\infty}{2}+*}(\mathfrak{g})$. Thus

$$\mathbb{1} = \mathbb{1}_S \otimes \mathbb{1}_\Lambda \tag{2.2.2}$$

is the vacuum vector in $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$. Notice that $L_n\mathbb{1} = 0$ if and only if $n \geq -1$.

Lemma 2.2.1. For any $j, l \in \mathbb{Z}$, $C^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g}))$ is a module over L.

Proof. Notice that

$$[L_n, \tau(u_m)] = -m\tau(u_{m+n}), \tag{2.2.3}$$

$$[L_n, \varepsilon(u'_m)] = (m-n)\varepsilon(u'_{m-n}); \tag{2.2.4}$$

$$[L_n, \beta(u_m)] = -(m+n)\beta(u_{m+n}), \tag{2.2.5}$$

$$[L_n, \gamma(u'_m)] = m\gamma(u'_{m-n}). \tag{2.2.6}$$

It follows from (2.2.3)-(2.2.6) that $S^{\frac{\infty}{2}+l}(\mathfrak{g}) \otimes \Lambda^{\frac{\infty}{2}+j}(\mathfrak{g})$ is a module over L. Notice that according to (2.2.3)

$$[L_n, \tau(\mathfrak{g}_0)] = 0, \tag{2.2.7}$$

Let us show that

$$[L_n, \theta(u_m)] = -m\theta(u_{m+n}). \tag{2.2.8}$$

Recall that

$$\theta(u_m) = \rho(u_m) + \pi(u_m). \tag{2.2.9}$$

We will show that

$$[L_n, \rho(u_m)] = -m\rho(u_{m+n}), \tag{2.2.10}$$

$$[L_n, \pi(u_m)] = -m\pi(u_{m+n}). \tag{2.2.11}$$

In fact, according to (2.2.3) and (2.2.4)

$$\begin{aligned}
&[L_n, \rho(u_m)] = [L_n, \Sigma_i : \tau([u_m, v_i])\varepsilon(v'_i) :] = [L_n, \Sigma_i : \tau([u,v]_{m+i})\varepsilon(v'_i) :] = \\
&\Sigma_i(: -(m+i)\tau([u,v]_{m+i+n})\varepsilon(v'_i) : + : \tau([u,v]_{m+i})(i-n)\varepsilon(v'_{i-n}) :) = \\
&\Sigma_i(: -(m+i)\tau([u,v]_{m+i+n})\varepsilon(v'_i) : + : \tau([u,v]_{m+i+n})i\varepsilon(v'_i) :) = \\
&-m\Sigma_i : \tau([u,v]_{m+i+n})\varepsilon(v'_i) := -m\rho(u_{m+n}).
\end{aligned} \tag{2.2.12}$$

According to (2.2.5) and (2.2.6)

$$[L_n, \pi(u_m)] = [L_n, \Sigma_i : \beta([u_m, v_i])\gamma(v'_i) :] = [L_n, \Sigma_i : \beta([u,v]_{m+i})\gamma(v'_i) :] =$$
$$\Sigma_i(: -(m+i+n)\beta([u,v]_{m+i+n})\gamma(v'_i) : + : \beta([u,v]_{m+i}) i\gamma(v'_{i-n}) :) =$$
$$\Sigma_i(: -(m+i+n)\beta([u,v]_{m+i+n})\gamma(v'_i) : + : \beta([u,v]_{m+i+n})(i+n)\gamma(v'_i) :) =$$
$$- m\Sigma_i : \beta([u,v]_{m+i+n})\gamma(v'_i) := -m\pi(u_{m+n}). \tag{2.2.13}$$

Thus (2.2.8) follows. From (2.2.8) we obtain

$$[L_n, \theta(\mathfrak{g}_0)] = 0. \tag{2.2.14}$$

According to (2.2.7) and (2.2.14) $C^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g}))$ is a module over L.

■

Corollary 2.2.1. For any $j, l \in \mathbb{Z}$, $H^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g}))$ is a module over L.
Proof. Notice that

$$[L_n, d] = 0, \tag{2.2.15}$$

see corollary 2.1 in [P2].

2.3 An $\mathfrak{sl}(2)$-module structure.

We will show that the relative semi-infinite Weil complex is a module over an $\mathfrak{sl}(2)$. Define $SL(2) =< E, H, F >$ as follows:

$$E = -\frac{1}{2} i\Sigma_m[\gamma(e'_m)\gamma(f'_{1-m}) + \gamma(f'_m)\gamma(e'_{1-m}) + 2\gamma(h'_m)\gamma(h'_{1-m})], \tag{2.3.1}$$
$$H = -\Sigma_m[: \gamma(e'_m)\beta(e_m) : + : \gamma(f'_m)\beta(f_m) : + : \gamma(h'_m)\beta(h_m) :], \tag{2.3.2}$$
$$F = -\frac{1}{2} i\Sigma_m[\beta(e_m)\beta(f_{1-m}) + \beta(f_m)\beta(e_{1-m}) + (1/2)\beta(h_m)\beta(h_{1-m})]. \tag{2.3.3}$$

Thus

$$\begin{gathered}[H, E] = 2E, [H, F] = -2F, [E, F] = H, \\ E(F) : S^{\frac{\infty}{2}+l}(\mathfrak{g}) \to S^{\frac{\infty}{2}+l\pm 2}(\mathfrak{g}), \\ H \text{ is the multiplication by } l.\end{gathered} \tag{2.3.4}$$

Clearly, the action of $SL(2)$ on $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ is trivial.
Lemma 2.3.1. For any $j \in \mathbb{Z}$, $C^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ is a module over $SL(2)$.

Proof. Notice that

$$\begin{aligned}
[E, \beta(e_m)] &= -i\gamma(f'_{1-m}), \\
[E, \beta(f_m)] &= -i\gamma(e'_{1-m}), \\
[E, \beta(h_m)] &= -2i\gamma(h'_{1-m}), \\
[E, \gamma(u'_m)] &= 0;
\end{aligned} \tag{2.3.5}$$

$$\begin{aligned}
[H, \gamma(u'_m)] &= \gamma(u'_m), \\
[H, \beta(u_m)] &= -\beta(u_m);
\end{aligned} \tag{2.3.6}$$

$$\begin{aligned}
[F, \gamma(e'_m)] &= i\beta(f_{1-m}), \\
[F, \gamma(f'_m)] &= i\beta(e_{1-m}), \\
[F, \gamma(h'_m)] &= \frac{1}{2}i\beta(h_{1-m}), \\
[F, \beta(u_m)] &= 0.
\end{aligned} \tag{2.3.7}$$

Obviously,

$$[SL(2), \tau(\mathfrak{g})] = 0, \tag{2.3.8}$$

$$[SL(2), \rho(\mathfrak{g})] = 0. \tag{2.3.9}$$

One can check that

$$[SL(2), \pi(\mathfrak{g})] = 0. \tag{2.3.10}$$

Hence

$$[SL(2), \theta(\mathfrak{g})] = 0. \tag{2.3.11}$$

It follows from (2.3.8) and (2.3.11) that $C^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ is a module over $SL(2)$.

■

Corollary 2.3.1. For any $j \in \mathbb{Z}$, $H^{\frac{\infty}{2}+j}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ is a module over $SL(2)$.
Proof. It follows from (2.3.10) that

$$[SL(2), d] = 0. \tag{2.3.12}$$

2.4. Another $\mathfrak{sl}(2)$-module structure.

There exists another $\mathfrak{sl}(2)$-module structure on the relative semi-infinite Weil complex and relative semi-infinite cohomology. Define $\mathcal{SL}(2) =< \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} >$ as follows :

$$\begin{aligned}
&\boldsymbol{\mathcal{E}} = i\Sigma_{m>0} m[\varepsilon(e'_{-m})\varepsilon(f'_m) + \varepsilon(f'_{-m})\varepsilon(e'_m) + 2\varepsilon(h'_{-m})\varepsilon(h'_m)], && (2.4.1)\\
&\boldsymbol{\mathcal{H}} = -\Sigma_{u,m\neq 0} : \tau(u_m)\varepsilon(u'_m) :, && (2.4.2)\\
&\boldsymbol{\mathcal{F}} = -i\Sigma_{m>0}\frac{1}{m}[\tau(f_m)\tau(e_{-m}) + \tau(e_m)\tau(f_{-m}) + \frac{1}{2}\tau(h_m)\tau(h_{-m})]. && (2.4.3)
\end{aligned}$$

Thus

$$\begin{aligned}
&[\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{E}}] = 2\boldsymbol{\mathcal{E}}, [\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}}] = -2\boldsymbol{\mathcal{F}}, [\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}] = \boldsymbol{\mathcal{H}},\\
&\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{F}}) : \Lambda_{rel}^{\frac{\infty}{2}+j}(\mathfrak{g}) \to \Lambda_{rel}^{\frac{\infty}{2}+j\pm 2}(\mathfrak{g}), \qquad (2.4.4)\\
&\boldsymbol{\mathcal{H}} \text{ is the multiplication by } j.
\end{aligned}$$

Clearly, the action of $\mathcal{SL}(2)$ on $S^{\frac{\infty}{2}+*}(\mathfrak{g})$ is trivial.

Lemma 2.4.1. For any $l \in \mathbb{Z}$, $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g}))$ is a module over $\mathcal{SL}(2)$.

Proof. Notice that

$$\begin{aligned}
&[\boldsymbol{\mathcal{E}}, \tau(e_m)] = im\varepsilon(f'_{-m}),\\
&[\boldsymbol{\mathcal{E}}, \tau(f_m)] = im\varepsilon(e'_{-m}), \qquad (2.4.5)\\
&[\boldsymbol{\mathcal{E}}, \tau(h_m)] = 2im\varepsilon(h'_{-m}),\\
&[\boldsymbol{\mathcal{E}}, \varepsilon(u'_m)] = 0;
\end{aligned}$$

$$\begin{aligned}
&[\boldsymbol{\mathcal{H}}, \varepsilon(u'_m)] = \varepsilon(u'_m), \qquad (2.4.6)\\
&[\boldsymbol{\mathcal{H}}, \tau(u_m)] = -\tau(u_m);
\end{aligned}$$

$$\begin{aligned}
&[\boldsymbol{\mathcal{F}}, \varepsilon(e'_m)] = \frac{i}{m}\tau(f_{-m}),\\
&[\boldsymbol{\mathcal{F}}, \varepsilon(f'_m)] = \frac{i}{m}\tau(e_{-m}), \qquad (2.4.7)\\
&[\boldsymbol{\mathcal{F}}, \varepsilon(h'_m)] = \frac{i}{2m}\tau(h_{-m}),\\
&[\boldsymbol{\mathcal{F}}, \tau(u_m)] = 0.
\end{aligned}$$

It follows from (2.4.5) that

$$[\mathcal{SL}(2), \tau(\mathfrak{g}_0)] = 0. \qquad (2.4.8)$$

Clearly,

$$[\mathcal{SL}(2), \pi(\mathfrak{g})] = 0. \tag{2.4.9}$$

One can check that

$$[\mathcal{SL}(2), \rho(\mathfrak{g}_0)] = 0. \tag{2.4.10}$$

Hence

$$[\mathcal{SL}(2), \theta(\mathfrak{g}_0)] = 0. \tag{2.4.11}$$

It follows from (2.4.8) and (2.4.11) that $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g}))$ is a module over $\mathcal{SL}(2)$.

■

Corollary 2.4.1. For any $l \in \mathbb{Z}$, $H^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+l}(\mathfrak{g}))$ is a module over $\mathcal{SL}(2)$.
Proof. Notice that $\mathcal{SL}(2)$ does not commute with d. In fact,

$$[\boldsymbol{\mathcal{F}}, d] \neq 0. \tag{2.4.12}$$

Nevertheless, an $\mathcal{SL}(2)$-module structure on the relative semi-infinite cohomology *does exist*, as it was pointed out in [FGZ]. We will explain this in lemma 5.1.

3. The superconformal algebra $S(2,0)$.

Recall the definition of superconformal algebras ([K-VDL]). Let

$$W(N) = der\mathbb{C}[t^{-1}, t, \theta_1, \dots, \theta_N], \tag{3.1}$$

where $\theta_1, \dots, \theta_N$ are odd variables. Let

$$S(N, \alpha) = \{D \in W(N) | Div(t^\alpha D) = 0\} \text{ for } \alpha \in \mathbb{C}, \tag{3.2}$$

where

$$Div(f\partial/\partial t + \Sigma_i f_i \partial/\partial\theta_i) = \partial f/\partial t + \Sigma_i (-1)^{p(f_i)} \partial f_i/\partial\theta_i. \tag{3.3}$$

Assume that $N > 1$. Notice that $S(N, \alpha)$ is simple, if $\alpha \notin \mathbb{Z}$; if $\alpha \in \mathbb{Z}$, then $S(N, \alpha)$ contains the simple ideal $S'(N, \alpha) = [S(N, \alpha), S(N, \alpha)]$:

$$0 \to S'(N, \alpha) \to S(N, \alpha) \to \mathbb{C}t^{-\alpha}\theta_1 \dots \theta_N \partial/\partial t \to 0. \tag{3.4}$$

Notice that $S(N, \alpha)$ has a nontrivial central extension if and only if $N = 2$ (see [K-VDL]), which is important for our consideration. In what follows $N = 2$ and $\alpha = 0$. Let

$$\{\mathcal{L}_n, E_n, H_n, F_n, \mathfrak{h}_n, \mathfrak{p}_n, \mathfrak{x}_n, \mathfrak{y}_n\}_{n \in \mathbb{Z}} \tag{3.5}$$

be the basis of $S'(2,0)$ defined as follows:

$$\mathfrak{L}_n = -t^n(t\partial/\partial t + (1/2)(n+1)(\Sigma_i \theta_i \partial/\partial\theta_i)), \tag{3.6}$$
$$E_n = t^n \theta_2 \partial/\partial\theta_1, \tag{3.7}$$
$$H_n = t^n(\theta_2\partial/\partial\theta_2 - \theta_1\partial/\partial\theta_1), \tag{3.8}$$
$$F_n = t^n\theta_1\partial/\partial\theta_2, \tag{3.9}$$
$$\mathfrak{h}_n = t^n\theta_2\partial/\partial t - nt^{n-1}\theta_1\theta_2\partial/\partial\theta_1, \tag{3.10}$$
$$\mathfrak{p}_n = -t^{n+1}\partial/\partial\theta_2, \tag{3.11}$$
$$\mathfrak{x}_n = t^{n+1}\partial/\partial\theta_1, \tag{3.12}$$
$$\mathfrak{y}_n = t^n\theta_1\partial/\partial t + nt^{n-1}\theta_1\theta_2\partial/\partial\theta_2. \tag{3.13}$$

The nonvanishing commutation relations are

$$[\mathfrak{L}_n, \mathfrak{L}_m] = (n-m)\mathfrak{L}_{n+m}, \tag{3.14}$$
$$[E_n, F_k] = H_{n+k}, [H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k}, \tag{3.15}$$
$$[\mathfrak{L}_n, E_k] = -kE_{n+k}, [\mathfrak{L}_n, H_k] = -kH_{n+k}, [\mathfrak{L}_n, F_k] = -kF_{n+k}, \tag{3.16}$$
$$[\mathfrak{L}_n, \mathfrak{h}_k] = (1/2)(n-2k+1)\mathfrak{h}_{n+k}, [\mathfrak{L}_n, \mathfrak{p}_k] = (1/2)(n-2k-1)\mathfrak{p}_{n+k}, \tag{3.17}$$
$$[\mathfrak{L}_n, \mathfrak{x}_k] = (1/2)(n-2k-1)\mathfrak{x}_{n+k}, [\mathfrak{L}_n, \mathfrak{y}_k] = (1/2)(n-2k+1)\mathfrak{y}_{n+k}, \tag{3.18}$$
$$[E_n, \mathfrak{y}_k] = \mathfrak{h}_{n+k}, [F_n, \mathfrak{h}_k] = \mathfrak{y}_{n+k}, [H_n, \mathfrak{h}_k] = \mathfrak{h}_{n+k}, [H_n, \mathfrak{y}_k] = -\mathfrak{y}_{n+k}, \tag{3.19}$$
$$[E_n, \mathfrak{p}_k] = \mathfrak{x}_{n+k}, [F_n, \mathfrak{x}_k] = \mathfrak{p}_{n+k}, [H_n, \mathfrak{x}_k] = \mathfrak{x}_{n+k}, [H_n, \mathfrak{p}_k] = -\mathfrak{p}_{n+k}, \tag{3.20}$$
$$[h_n, \mathfrak{x}_k] = (k+1-n)E_{n+k}, [p_n, \mathfrak{y}_k] = (k-n-1)F_{n+k}. \tag{3.21}$$
$$[\mathfrak{h}_n, \mathfrak{p}_k] = \mathfrak{L}_{n+k} - (1/2)(k-n+1)H_{n+k}, [\mathfrak{x}_n, \mathfrak{y}_k] = -\mathfrak{L}_{n+k} + (1/2)(k-n-1)H_{n+k}, \tag{3.22}$$

Let

$$\boldsymbol{\mathcal{F}} = -\theta_1\theta_2\partial/\partial t. \tag{3.23}$$

Then

$$\mathbb{C}\boldsymbol{\mathcal{F}} \ltimes S'(2,0) = S(2,0), \tag{3.24}$$

and the additional nonvanishing commutation relations are

$$[\boldsymbol{\mathcal{F}}, \mathfrak{x}_k] = \mathfrak{h}_{k+1}, [\boldsymbol{\mathcal{F}}, \mathfrak{p}_k] = \mathfrak{y}_{k+1}. \tag{3.25}$$

There exists only one (up to equivalence) cocycle ψ on $S(2,0)$ ([K-VDL]):

$$\psi(\mathfrak{L}_m, \mathfrak{L}_n) = (1/12)(m^3 - m)\delta_{m,-n}, \tag{3.26}$$

$$\psi(\mathsf{h}_m, \mathsf{p}_n) = (1/6)m(m-1)\delta_{m,-n}, \tag{3.27}$$

$$\psi(\mathsf{x}_m, \mathsf{y}_n) = -(1/6)m(m+1)\delta_{m,-n}, \tag{3.28}$$

$$\psi(H_m, H_n) = (1/3)m\delta_{m,-n}, \tag{3.29}$$

$$\psi(E_m, F_n) = (1/6)m\delta_{m,-n}, \tag{3.30}$$

and it is zero otherwise. Denote the corresponding central extension of $S(2,0)$ by $\hat{S}(2,0)$. Thus

$$\hat{S}(2,0) = S(2,0) \oplus < \mathsf{C} >, \tag{3.31}$$

$$[s_1, s_2]_{|\hat{S}(2,0)} = [s_1, s_2]_{|S(2,0)} + \psi(s_1, s_2)\mathsf{C}, \text{ for } s_1, s_2 \in S(2,0), \tag{3.32}$$

The cocycle ψ also defines the central extension $\hat{S}'(2,0)$ of $S'(2,0)$

Remark 3.1. The superalgebra $S'(2,0)$ can be described as follows:

$$S'(2,0) = S'(2,0)_{\bar{0}} \oplus S'(2,0)_{\bar{1}}; \tag{3.33}$$

where

$$S'(2,0)_{\bar{0}} = Witt \ltimes \hat{SL}(2), \tag{3.34}$$

$$Witt =< \mathfrak{L}_n >_{n\in\mathbb{Z}}, \hat{SL}(2) =< E_n, H_n, F_n >_{n\in\mathbb{Z}}, \tag{3.35}$$

$$S'(2,0)_{\bar{1}} =< \mathsf{h}_n, \mathsf{y}_n >_{n\in\mathbb{Z}} \oplus < \mathsf{x}_n, \mathsf{p}_n >_{n\in\mathbb{Z}} . \tag{3.36}$$

Thus $S'(2,0)_{\bar{1}}$ is a direct sum of two standard (odd) $\hat{SL}(2)$-modules.

Remark 3.2. The $N = 2$ superconformal algebra is isomorphic to a subalgebra of $\hat{S}(2,0)$.

Recall that *the* $N = 2$ *superconformal algebra* is spanned by the generators $\mathfrak{L}_n$ of Witt algebra, two fermionic fields $G_r^{\pm}$, a U(1) current H_n, and a central element C, where $n \in \mathbb{Z}, r \in \mathbb{Z} + 1/2$ ([FST]). The commutation relations are

$$[\mathfrak{L}_n, \mathfrak{L}_m] = (n-m)\mathfrak{L}_{n+m} + \frac{\mathsf{C}}{12}(n^3 - n)\delta_{n,-m}, \tag{3.37}$$

$$[\mathfrak{L}_n, H_m] = -mH_{n+m}, \tag{3.38}$$

$$[\mathfrak{L}_n, G_r^{\pm}] = (\frac{n}{2} - r)G_{n+r}^{\pm}, \tag{3.39}$$

$$[G_r^+, G_s^-] = 2\mathfrak{L}_{r+s} + (r-s)H_{r+s} + \frac{\mathsf{C}}{3}(r^2 - \frac{1}{4})\delta_{r,-s}, \tag{3.40}$$

$$[H_n, H_m] = \frac{\mathsf{C}}{3}n\delta_{n,-m}, \tag{3.41}$$

$$[H_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}, \tag{3.42}$$

and they are zero otherwise. Thus the $N=2$ superconformal algebra is isomorphic to the subalgebra of $\hat{S}(2,0)$ spanned by $\mathfrak{L}_n, H_n, \mathsf{h}_n, \mathsf{p}_n$, and C. The isomorphism is given as follows:

$$\mathfrak{L}_n \to \mathfrak{L}_n, \tag{3.43}$$
$$H_n \to H_n, \tag{3.44}$$
$$G^+_{n-\frac{1}{2}} \to \sqrt{2}\mathsf{h}_n, \tag{3.45}$$
$$G^-_{n+\frac{1}{2}} \to \sqrt{2}\mathsf{p}_n, \tag{3.46}$$
$$\mathsf{C} \to \mathsf{C}. \tag{3.47}$$

Let $DerS'(2,0)$ be the Lie superalgebra of all derivations of $S'(2,0)$, and $Der_{ext}S'(2,0)$ be the exterior derivations of $S'(2,0)$. In other words,

$$Der_{ext}S'(2,0) = DerS'(2,0)/\{ads|s \in S'(2,0)\}. \tag{3.48}$$

Theorem 3.1. $Der_{ext}S'(2,0)$ has the structure of the $\mathfrak{SL}(2) =< \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} >$, where

$$[\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{E}}] = 2\boldsymbol{\mathcal{E}}, [\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}}] = -2\boldsymbol{\mathcal{F}}, [\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}] = \boldsymbol{\mathcal{H}}. \tag{3.49}$$

The action of $\mathfrak{SL}(2)$ on $S'(2,0)_{\bar{0}}$ is trivial and on $S'(2,0)_{\bar{1}}$ is given as follows:

$$[\boldsymbol{\mathcal{E}}, \mathsf{h}_k] = \mathsf{x}_{k-1}, [\boldsymbol{\mathcal{F}}, \mathsf{x}_k] = \mathsf{h}_{k+1}, [\boldsymbol{\mathcal{H}}, \mathsf{x}_k] = \mathsf{x}_k, [\boldsymbol{\mathcal{H}}, \mathsf{h}_k] = -\mathsf{h}_k, \tag{3.50}$$
$$[\boldsymbol{\mathcal{E}}, \mathsf{y}_k] = \mathsf{p}_{k-1}, [\boldsymbol{\mathcal{F}}, \mathsf{p}_k] = \mathsf{y}_{k+1}, [\boldsymbol{\mathcal{H}}, \mathsf{p}_k] = \mathsf{p}_k, [\boldsymbol{\mathcal{H}}, \mathsf{y}_k] = -\mathsf{y}_k. \tag{3.51}$$

Proof. Recall that (see [Fu])

$$Der_{ext}S'(2,0) \cong H^1(S'(2,0), S'(2,0)), \tag{3.52}$$

Notice that the superalgebra $S'(2,0)$ has the following $\mathbb{Z}$-grading deg:

$$deg\mathfrak{L}_n = -n, degE_n = -n-1, degF_n = -n+1, degH_n = -n, \tag{3.53}$$
$$deg\mathsf{h}_n = -n, deg\mathsf{p}_n = -n, deg\mathsf{x}_n = -n-1, deg\mathsf{y}_n = -n+1. \tag{3.54}$$

Then, accordingly, there is the gradind deg in $Der_{ext}S'(2,0)$. Let

$$L_n = -\mathfrak{L}_{-n} + (1/2)(1+n)H_{-n}. \tag{3.55}$$

Notice that for a homogeneous element $s \in S'(2,0)$

$$[L_0, s] = -(degs)s. \tag{3.56}$$

Let us consider the natural action of $S'(2,0)$ on $Der_{ext}S'(2,0)$. Then for a homogeneous element $D \in Der_{ext}S'(2,0)$

$$L_0(D) = -(degD)D. \tag{3.57}$$

On the other hand, since the action of a Lie superalgebra on its cohomology is trivial, then for any $D \in H^1(S'(2,0), S'(2,0))$ one must have

$$L_0(D) = 0. \tag{3.58}$$

Hence the nonzero elements of $Der_{ext}S'(2,0)$ must have $deg = 0$, and one can check that these elements are the arbitrary linear combinations of $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}}$ defined in (3.50) and (3.51).

■

4. An action of $\hat{S}(2,0)$ on the relative semi-infinite cohomology of $\mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}]$.

Let $\mathfrak{g} = \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}]$.

Theorem 4.1. There is a representation of $\hat{S}(2,0)$ in $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ with central charge 9.

Proof. We will construct a representation of $\hat{S}'(2,0)$ in $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ using the quadratic expansions on the generators of the Heisenberg algebra $H(\mathfrak{g})$ and Clifford algebra $C(\mathfrak{g})$. Then we will extend it to a representation of $\hat{S}(2,0)$ using the $\mathcal{SL}(2)$ introduced in section 2.4.

Let us write down the elements of the basis (3.5) of $S'(2,0)$ in terms of the quadratic expansions as follows:

$$L_n = -\Sigma_{u,m} m : \tau(u_{m+n})\varepsilon(u'_m) : -\Sigma_{u,m} m : \beta(u_m)\gamma(u'_{m-n}) :, \tag{4.1}$$

$$E_n = -(1/2)i\Sigma_m[\gamma(e'_m)\gamma(f'_{1-m+n}) + \gamma(f'_m)\gamma(e'_{1-m+n}) + 2\gamma(h'_m)\gamma(h'_{1-m+n})], \tag{4.2}$$

$$H_n = -\Sigma_{u,m} : \gamma(u'_{m+n})\beta(u_m) :, \tag{4.3}$$

$$F_n = -(1/2)i\Sigma_m[\beta(e_m)\beta(f_{1-m-n}) + \beta(f_m)\beta(e_{1-m-n}) + (1/2)\beta(h_m)\beta(h_{1-m-n})], \tag{4.4}$$

$$\mathfrak{h}_n = \Sigma_{u,m}\gamma(u'_{m+n})\tau(u_m), \tag{4.5}$$

$$\mathfrak{p}_n = \Sigma_{u,m} m\beta(u_{m-n})\varepsilon(u'_m), \tag{4.6}$$

$$\mathfrak{y}_n = i\Sigma_m[\beta(e_m)\tau(f_{1-m-n}) + \beta(f_m)\tau(e_{1-m-n}) + (1/2)\beta(h_m)\tau(h_{1-m-n})], \tag{4.7}$$

$$\mathfrak{x}_n = -i\Sigma_m[(1-m+n)(\varepsilon(e'_{1-m+n})\gamma(f'_m) + \varepsilon(f'_{1-m+n})\gamma(e'_m) + 2\varepsilon(h'_{1-m+n})\gamma(h'_m))], \tag{4.8}$$

Set

$$\mathcal{L}_n = -L_{-n} + (1/2)(1-n)H_n. \tag{4.9}$$

One can check that the commutation relations (3.14)-(3.22) are satisfied.

Notice that $L =< L_n >_{n\in\mathbb{Z}}$ is the Witt algebra described in section 2.2, and $\hat{SL}(2) =< E_n, H_n, F_n >_{n\in\mathbb{Z}}$ is the loop algebra of $SL(2) =< E, H, F >$ defined in section 2.3:

$$E_0 = E, H_0 = H, F_0 = F. \tag{4.10}$$

Let us show that the elements (4.1)-(4.8) act on $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$.

Notice that these elements commute with $\tau(\mathfrak{g}_0)$. Recall that

$$[L_n, \theta(\mathfrak{g}_0)] = [SL(2), \theta(\mathfrak{g}_0)] = 0, \tag{4.11}$$

see (2.2.14) and (2.3.11). Then it follows from the commutation relations

$$[L_n, E_k] = (1+n+k)E_{k-n}, [L_n, F_k] = (k-n-1)F_{k-n}, [E_n, F_k] = H_{n+k} \tag{4.12}$$

that

$$[\hat{SL}(2), \theta(\mathfrak{g}_0)] = 0. \tag{4.13}$$

Let us show that

$$[\mathsf{h}_n, \theta(\mathfrak{g}_0)] = 0. \tag{4.14}$$

Recall that

$$\theta(u_i) = \rho(u_i) + \pi(u_i) = \Sigma_{v,j}(: \tau([u_i, v_j])\varepsilon(v'_j) : + : \beta([u_i, v_j])\gamma(v'_j) :), \tag{4.15}$$

where u, v run through the standard basis $\{e, h, f\}$ of $\mathfrak{sl}(2)$, and $i, j \in \mathbb{Z}$. Notice that

$$[\mathsf{h}_n, \varepsilon(u'_m)] = \gamma(u'_{m+n}), \tag{4.16}$$

$$[\mathsf{h}_n, \beta(u_m)] = \tau(u_{m-n}). \tag{4.17}$$

Thus for any $i \in \mathbb{Z}$

$$[\mathsf{h}_n, \theta(u_i)] = -\Sigma_{v,j} : \tau([u,v]_{i+j})\gamma(v'_{j+n}) : +\Sigma_{v,j} : \tau([u,v]_{i+j-n})\gamma(v'_j) := 0. \tag{4.18}$$

In particular, we have (4.14). Let us show that

$$[\mathsf{p}_n, \theta(\mathfrak{g}_0)] = 0. \tag{4.19}$$

Notice that

$$[\mathfrak{p}_n, \tau(u_m)] = m\beta(u_{m-n}), \tag{4.20}$$

$$[\mathfrak{p}_n, \gamma(u_m)] = -(m+n)\varepsilon(u'_{m+n}). \tag{4.21}$$

Thus for any $i \in \mathbb{Z}$

$$[\mathfrak{p}_n, \theta(u_i)] = \Sigma_{v,j} : (i+j)\beta([u,v]_{i+j-n})\varepsilon(v'_j) : +\Sigma_{v,j} : \beta([u,v]_{i+j})(-(j+n))\varepsilon(v'_{j+n}) := \\ \Sigma_{v,j} : i\beta([u,v]_{i+j-n})\varepsilon(v'_j) : . \tag{4.22}$$

In particular, we have (4.19).

Similarly, one can show that

$$[\mathfrak{y}_n, \theta(\mathfrak{g}_0)] = 0, [\mathfrak{x}_n, \theta(\mathfrak{g}_0)] = 0. \tag{4.23}$$

To show that we obtain a representation of $\hat{S}'(2,0)$ in $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$, one should check the commutation relations on the vacuum vector $\imath$ (see theorem 2.1. of [P2]). Then we get the central charge 9.

To extend this representation to a representation of $\hat{S}(2,0)$, recall that

$$0 \to \hat{S}'(2,0) \to \hat{S}(2,0) \to \mathbb{C}\boldsymbol{\mathcal{F}} \to 0, \tag{4.24}$$

where $\boldsymbol{\mathcal{F}}$ is a derivation of $S'(2,0)$ defined by

$$[\boldsymbol{\mathcal{F}}, \mathfrak{x}_k] = \mathfrak{h}_{k+1}, [\boldsymbol{\mathcal{F}}, \mathfrak{p}_k] = \mathfrak{y}_{k+1}. \tag{4.25}$$

Let the action of $\boldsymbol{\mathcal{F}}$ on $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ be given by the quadratic expansion (2.4.3). Notice that the commutators of the element (2.4.3) with $\mathfrak{x}_k$ and $\mathfrak{p}_k$, differ from $\mathfrak{h}_{k+1}$ and $\mathfrak{y}_{k+1}$, respectively, by some linear combinations of the terms, which involve $\tau(u_0)$. Since $\tau(\mathfrak{g}_0)c = 0$ for any $c \in C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$, then a representation of $\hat{S}(2,0)$ in $C^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ is well-defined.

∎

Corollary 4.1. There is a representation of $\hat{S}(2,0)$ in $H^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ with central charge 9.

Proof. We will prove a more general statement in theorem 5.3.

Conjecture. (cf. [P1]). The space $H^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ is generated by the vacuum vector $\imath$ under the action of $\hat{S}(2,0)$.

5. The case of the loop algebra of a Lie algebra with a non-degenerate invariant symmetric form.

Let us consider a more general situation. Let V be a complex finite-dimensional vector space, $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$. Thus $\tilde{V} = \oplus_{n \in \mathbb{Z}} \tilde{V}_n$, where $\tilde{V}_n = V \otimes t^n$. Using the polarization

$$\tilde{V} = \tilde{V}_+ + \tilde{V}_-, \tag{5.1}$$

where

$$\tilde{V}_+ = \oplus_{n>0} \tilde{V}_n, \tilde{V}_- = \oplus_{n \leq 0} \tilde{V}_n \tag{5.2}$$

one can define the semi-infinite symmetric and exterior power spaces $S^{\frac{\infty}{2}+*}(\tilde{V})$ and $\Lambda^{\frac{\infty}{2}+*}(\tilde{V})$. Let

$$W^{\frac{\infty}{2}+*}(\tilde{V}) = S^{\frac{\infty}{2}+*}(\tilde{V}) \otimes \Lambda^{\frac{\infty}{2}+*}(\tilde{V}). \tag{5.3}$$

The following statement generalizes the theorem 3.1 of [P2].

Theorem 5.1. The space $W^{\frac{\infty}{2}+*}(\tilde{V})$ is a module over the superconformal algebra $\hat{S}'(2,0)$ with central charge $3dimV$.

Proof. Let element u run through a fixed basis of V. Let $\{u_n\}$ be a basis of $\tilde{V}_n$, $\{u'_n\}$ be the dual basis. Define

$$L_n = -\Sigma_{u,m}[m : \tau(u_{m+n})\varepsilon(u'_m) : +m : \beta(u_{m+n})\gamma(u'_m) :] \tag{5.4}$$

$$E_n = -(1/2)i\Sigma_{u,m}\gamma(u'_m)\gamma(u'_{1-m+n}), \tag{5.5}$$

$$H_n = -\Sigma_{u,m} : \gamma(u'_{m+n})\beta(u_m) : \tag{5.6}$$

$$F_n = -(1/2)i\Sigma_{u,m}\beta(u_m)\beta(u_{1-m-n}), \tag{5.7}$$

$$\mathfrak{h}_n = \Sigma_{u,m}\gamma(u'_{m+n})\tau(u_m), \tag{5.8}$$

$$\mathfrak{p}_n = \Sigma_{u,m} m\beta(u_{m-n})\varepsilon(u'_m), \tag{5.9}$$

$$\mathfrak{y}_n = i\Sigma_{u,m}\beta(u_m)\tau(u_{1-m-n}), \tag{5.10}$$

$$\mathfrak{x}_n = -i\Sigma_{u,m}(1-m+n)\varepsilon(u'_{1-m+n})\gamma(u'_m). \tag{5.11}$$

Set

$$\mathfrak{L}_n = -L_{-n} + \frac{n+1}{2}H_n. \tag{5.12}$$

■

Suppose that $\mathfrak{g}$ is a complex finite-dimensional Lie algebra, ϕ is a representation of $\mathfrak{g}$ in V, and $<,>$ is a non-degenerate $\mathfrak{g}$-invariant symmetric form on V.

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \oplus_{n\in\mathbb{Z}} \tilde{\mathfrak{g}}_n$. We extend ϕ to a representation of $\tilde{\mathfrak{g}}$ in $\tilde{V}$:

$$\phi(g \otimes t^n)(v \otimes t^k) = (\phi(g)v) \otimes t^{n+k}, \text{ for } g \in \mathfrak{g}, v \in V. \tag{5.13}$$

Let u, v run through a fixed basis in V. One can define the projective representations ρ and π of $\tilde{\mathfrak{g}}$ in $\Lambda^{\frac{\infty}{2}+*}(\tilde{V})$ and $S^{\frac{\infty}{2}+*}(\tilde{V})$, respectively:

$$\rho(x) = \Sigma_{u,i\in\mathbb{Z}} : \tau((\phi(x)u_i))\varepsilon(u'_i) :, \tag{5.14}$$

$$\pi(x) = \Sigma_{u,i\in\mathbb{Z}} : \beta(\phi(x)u_i))\gamma(u'_i) :, \tag{5.15}$$

where $x \in \tilde{\mathfrak{g}}$. The normal ordering operation is defined by

$$: \tau(u_i)\varepsilon(v'_j) := \left\{ \begin{array}{c} \tau(u_i)\varepsilon(v'_j) \text{ if } j \leq 0 \\ -\varepsilon(v'_j)\tau(u_i) \text{ if } j > 0 \end{array} \right\} \tag{5.16}$$

$$: \beta(u_i)\gamma(v'_j) := \left\{ \begin{array}{c} \beta(u_i)\gamma(v'_j) \text{ if } j \leq 0 \\ \gamma(v'_j)\beta(u_i) \text{ if } j > 0 \end{array} \right\}. \tag{5.17}$$

Since the cocycles corresponding to ρ and π cancel, then $\theta(x) = \rho(x) + \pi(x)$ is a representation of $\tilde{\mathfrak{g}}$ in $W^{\frac{\infty}{2}+*}(\tilde{V})$. Let

$$W^{\frac{\infty}{2}+*}_{rel}(\tilde{V}) = \{c \in W^{\frac{\infty}{2}+*}(\tilde{V}) | \tau(x)c = 0 \text{ for all } x \in \tilde{V}_0, \theta(x)c = 0 \text{ for all } x \in \tilde{\mathfrak{g}}_0\}. \tag{5.18}$$

By theorem 3.1,

$$DerS'(2,0) =< \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} > \ltimes S'(2,0). \tag{5.19}$$

Let $\hat{D}erS'(2,0)$ be a central extension of $DerS'(2,0)$. Thus

$$\hat{D}erS'(2,0) =< \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} > \ltimes \hat{S}'(2,0). \tag{5.20}$$

Theorem 5.2. The space $W^{\frac{\infty}{2}+*}_{rel}(\tilde{V})$ is a module over $\hat{D}erS'(2,0)$ with central charge $3dimV$.

Proof. Let $\{v_i\}$ be a basis of V so that $< v_i, v_j >= \delta_{i,j}$. Let u run through this basis. Let $\{u_n\}$ be a basis of $\tilde{V}_n$, $\{u'_n\}$ be the dual basis. Let us define $S'(2,0)$ using the quadratic expansions (5.4)-(5.12). By Theorem 5.1, there is a representation of $\hat{S}'(2,0)$ in $W^{\frac{\infty}{2}+*}(\tilde{V})$. Since the form $<,>$ is $\mathfrak{g}$-invariant, then there is an action of $\hat{S}'(2,0)$ on $W^{\frac{\infty}{2}+*}_{rel}(\tilde{V})$.

Let us define $\mathcal{SL}(2) =< \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} >$ as follows

$$\boldsymbol{\mathcal{E}} = i\Sigma_{u,m>0} m\varepsilon(u'_{-m})\varepsilon(u'_m), \tag{5.21}$$

$$\boldsymbol{\mathcal{H}} = -\Sigma_{u,m\neq 0} : \tau(u_m)\varepsilon(u'_m) :, \tag{5.22}$$

$$\boldsymbol{\mathcal{F}} = -i\Sigma_{u,m>0}(1/m)\tau(u_m)\tau(u_{-m}), \tag{5.23}$$

Notice that $\mathcal{SL}(2)$ acts on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{V})$. One can check that

$$[\boldsymbol{\mathcal{H}}, \mathfrak{h}_n] = -\Sigma_{u,m\neq 0}\gamma(u'_{m+n})\tau(u_m) = -\mathfrak{h}_n + \Sigma_u \gamma(u'_n)\tau(u_0), \tag{5.24}$$

$$[\boldsymbol{\mathcal{H}}, \mathfrak{p}_n] = \mathfrak{p}_n, \tag{5.25}$$

$$[\boldsymbol{\mathcal{H}}, \mathfrak{x}_n] = \mathfrak{x}_n, \tag{5.26}$$

$$[\boldsymbol{\mathcal{H}}, \mathfrak{y}_n] = -i\Sigma_{u,m\neq 1-n}\beta(u_m)\tau(u_{1-m-n}) = -\mathfrak{y}_n + i\Sigma_u\beta(u_{1-n})\tau(u_0), \tag{5.27}$$

$$[\boldsymbol{\mathcal{E}}, \mathfrak{h}_n] = \mathfrak{x}_{n-1}, \tag{5.28}$$

$$[\boldsymbol{\mathcal{E}}, \mathfrak{y}_n] = \mathfrak{p}_{n-1}, \tag{5.29}$$

$$[\boldsymbol{\mathcal{F}}, \mathfrak{x}_n] = \Sigma_{u,m\neq 0}\gamma(u'_{m+n+1})\tau(u_m) = \mathfrak{h}_{n+1} - \Sigma_u\gamma(u'_{n+1})\tau(u_0), \tag{5.30}$$

$$[\boldsymbol{\mathcal{F}}, \mathfrak{p}_n] = i\Sigma_{u,m\neq -n}\beta(u_m)\tau(u_{-m-n}) = \mathfrak{y}_{n+1} - i\Sigma_u\beta(u_{-n})\tau(u_0). \tag{5.31}$$

Notice that the relations (5.24)-(5.31) coinside with the relations (3.50), (3.51) up to some terms which involve the elements $\tau(u_0)$. Since the action of any element $\tau(u_0)$ on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{V})$ is trivial, then a representation of $\hat{D}erS'(2,0)$ in $W_{rel}^{\frac{\infty}{2}+*}(\tilde{V})$ is well-defined.

■

Lemma 5.1. Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric form $<,>$. Let $\{v_i\}$ be a basis of $\mathfrak{g}$ so that $< v_i, v_j >= \delta_{i,j}$. Let u run through this basis. Let $\{u_n\}$ be a basis of $\tilde{\mathfrak{g}}_n$, $\{u'_n\}$ be the dual basis. Let $\mathcal{SL}(2)$ be defined by (5.21)-(5.23). Then $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ is a module over $\mathcal{SL}(2)$.

Proof. Let $\mathfrak{l} = \oplus_{n\in\mathbb{Z}}\mathfrak{l}_n$ be a complex graded Lie algebra, so that there exists a 2-cocycle γ on $\mathfrak{l}$ such that $\gamma|_{\mathfrak{l}_n\times\mathfrak{l}_{-n}}$ is non-degenerate if $n \in \mathbb{Z}\backslash 0$ and zero otherwise.

It was shown in [FGZ] that in this case there is an $\mathcal{SL}(2)$-module structure on the relative semi-infinite cohomology of $\mathfrak{l}$. Namely, identifying $\mathfrak{l}'_n$ with $\mathfrak{l}_{-n}$, one obtains $\mathcal{SL}(2) =< \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} >$ where

$$\boldsymbol{\mathcal{E}} = (i/2)\Sigma_n\varepsilon(e_n)\varepsilon(e'_n), \tag{5.32}$$

$$\boldsymbol{\mathcal{F}} = (i/2)\Sigma_n\tau(e_n)\tau(e'_n), \tag{5.33}$$

$$\boldsymbol{\mathcal{H}} = -\Sigma_n : \tau(e_n)\varepsilon(e'_n) : . \tag{5.34}$$

Notice that in our case there exists a 2-cocycle γ on $\tilde{\mathfrak{g}}$ with the desired property:

$$\gamma(g_1\otimes t^n, g_2\otimes t^m) = n < g_1, g_2 > \delta_{n,-m}, \text{ for } g_1, g_2 \in \mathfrak{g}. \tag{5.35}$$

Then the formulas (5.32)-(5.34) give exactly the $\mathcal{SL}(2)$ defined by (5.21)-(5.23). Thus there is an action of $\mathcal{SL}(2)$ on $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$.

■

Theorem 5.3. Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric form. Then $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ is a module over $\hat{D}erS'(2,0)$ with central charge $3dim\mathfrak{g}$.

Proof. By definition, $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) = C^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$. Hence by theorem 5.2, there is a representation of $\hat{D}erS'(2,0)$ in $C^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$.

By theorem 3.1,

$$DerS'(2,0) =< \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} > \ltimes S'(2,0).$$

According to Lemma 5.1, the elements $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}}$ act on $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$. Recall that

$$S'(2,0)_{\bar{1}} =< \mathfrak{h}_n, \mathfrak{p}_n, \mathfrak{x}_n, \mathfrak{y}_n >_{n\in\mathbb{Z}} .$$

By corollary 2.1 of [P2],

$$[\mathfrak{h}_n, d] = [\mathfrak{p}_n, d] = 0. \tag{5.36}$$

Thus $\mathfrak{h}_n$ and $\mathfrak{p}_n$ act on $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$. Since

$$[\boldsymbol{\mathcal{E}}, \mathfrak{h}_n] = \mathfrak{x}_{n-1}, [\boldsymbol{\mathcal{F}}, \mathfrak{p}_n] = \mathfrak{y}_{n+1}, \tag{5.37}$$

then there is an action of $S'(2,0)_{\bar{1}}$ on $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$. Finally, one obtains an action of $S'(2,0)_{\bar{0}}$ on $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$, since

$$S'(2,0)_{\bar{0}} = [S'(2,0)_{\bar{1}}, S'(2,0)_{\bar{1}}]. \tag{5.38}$$

■

Corollary 5.1. Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric form. Then $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ is a module over $\hat{S}(2,0)$ with central charge $3dim\mathfrak{g}$.

In the conclusion we would like to propose a problem. In this work we described a representation of the superconformal algebra $\hat{S}(2,\alpha)$ in a relative semi-infinite Weil complex and relative semi-infinite cohomology in the case when $\alpha = 0$.

Problem. Is there an action of (central extension) of $S(2,\alpha)$ on (some) semi-infinite Weil complex for an arbitrary $\alpha \in \mathbb{C}$?

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