GENERIC CYCLES

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It follows immediately from a theorem of Mumford [Mum], that if the sheaf of holomorphic 2-forms on a smooth projective surface S is very ample then for any general point $p \in S$, there exists no point $q \neq p$, such that q is rationally equivalent to p.

This has been generalized by A.A.Roitman to the case of arbitrary projective varieties in [R1], where he got some upper bounds for dimensions of subvarieties of symmetric powers consisting of effective zero-cycles rationally equivalent to a sufficiently general fixed one.

The aim of this note is to present a refinement (see Proposition 2.2 below) of some results of [R1] using an approach of V.Srinivas [S], p.242.

In the first section we construct two versions of a class map on the Chow groups of varieties over a field K with a fixed model over a subfield $k \subset K$ killing all cycles on divisors defined over k, i.e., class maps on the quotient group of "extremely transcendental" cycles. Assuming some of motivic conjectures, we show that the latter class maps are injective modulo torsion.

In the second section one of these classes is applied to the study of zero-cycles. In particular, we get a version of a theorem of Roitman [R1] on dimensions of subvarieties of symmetric powers of a variety consisting of rationally equivalent effective 0-cycles.

In the last section we give a few examples where the constructed classes are injective and deduce some corollaries of this injectivity.

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0.1. **Notations and conventions.** If x is a point of a scheme X then $\overline{\{x\}}$ denotes the closure of the point x, which is a closed reduced irreducible subscheme of the scheme X.

For a scheme X and a non-negative integer q, the set of points of X whose closure has codimension (resp., dimension) q is denoted by X^q (resp., X_q).

For a vector space V over a field k the notation $V^\vee$ is used for the dual vector space $\mathrm{Hom}_k(k, V)$.

The Chow group $CH^q(X)$ of a scheme X is defined as the cokernel of the divisor map

$$\bigoplus_{x \in X^{q-1}} \kappa(x)^\times \longrightarrow \bigoplus_{x \in X^q} \mathbb{Z},$$

where $\kappa(x)^\times$ is the multiplicative group of non-zero functions on the subscheme $\overline{\{x\}}$ of X.

$F^\bullet$ and $N^\bullet$ will denote the Hodge and the coniveau filtrations on the de Rham cohomology, respectively.

1. Cycle classes $c_\bullet^F$ and $c_\bullet^N$

Let X be a smooth d-dimensional projective variety over an algebraically closed field k of characteristic 0. Then, using duality $H^q(X, \Omega_X^q) = H^{d-q}(X, \Omega_X^{d-q})^\vee$, one can define the class map $cl : CH^q(X) \longrightarrow H^q(X, \Omega^q_X)$ as $[Z] \longmapsto (\omega \mapsto \omega|Z)$, where $H^{d-q}(Z, \Omega^{d-q}_Z)$ is canonically identified with k.

Fix an auxiliary smooth projective variety Y over k with the field of rational functions K. Decomposition

$$\Omega^q_{X\times_k Y/k} = \sum_{r=0}^{q} \Omega^{q-r}_{X\times_k Y/Y} \otimes_{\mathcal{O}_{X\times_k Y}} \Omega^r_{X\times_k Y/X}$$

induces via class map a homomorphism (at this point we may forget that K was finitely generated over k)

$$\tilde{c}^F_q : CH^q(X \times_k K) \longrightarrow \bigoplus_{r=0}^{q} H^q(X, \Omega^r_X) \otimes_k \Omega^{q-r}_{K/k}. \tag{1}$$

Suppose W is a subvariety of X of codimension p. Then the composition of the natural map $CH^{q-p}(W \times_k K) \longrightarrow CH^q(X \times_k K)$ with the class map (1) factors through the subspace $\bigoplus_{r=p}^{q} H^q(X, \Omega^r_X) \otimes_k \Omega^{q-r}_{K/k}$, because for any cycle $Z \in CH^{q-p}(W \times_k Y)$ the linear map $[Z] : H^{d+\dim Y-q}(X \times_k Y, \Omega^{d+\dim Y-q}_{X\times_k Y/k}) \longrightarrow k$ factors through the restriction map $H^{d+\dim Y-q}(X \times_k Y, \Omega^{d+\dim Y-q}_{X\times_k Y/k}) \longrightarrow H^{d+\dim Y-q}(W \times_k Y, \Omega^{d+\dim Y-q}_{W\times_k Y/k})$, and therefore, the kernel of this restriction contains both subspaces

$$H^{d-q}(X, \Omega^{d-r}_{X/k}) \otimes H^{\dim Y}(Y, \Omega^{d+\dim Y-q}_{Y/k}) \subseteq H^{d+\dim Y-q}(X \times_k Y, \Omega^{r+\dim Y-q}_{X\times_k Y/k})$$

for any $r < p$. This shows that for any $0 \le q \le \dim X$, the sequence

$$\bigoplus_{x\in X^1} CH^{q-1}(\overline{\{x\}} \times_k K) \longrightarrow CH^q(X \times_k K) \xrightarrow{\widehat{c}^F_q} H^q(X, \mathcal{O}_X) \otimes_k H^q_{DR}(K/k), \tag{2}$$

where $\widehat{c}^F_q$ is induced by $\tilde{c}^F_q$, is a complex (this has been observed also by V.Srinivas [S]).

The following obvious lemma will allow us to define a class map $c^F_q : CH^q(k(X) \otimes_k K) \longrightarrow H^q(X, \mathcal{O}_X) \otimes_k H^q_{DR}(K/k)$.

Lemma 1.1. 1. *Let X be a reduced absolutely irreducible scheme over a field k. Then for any extension K of k the sequence*

$$\bigoplus_{x\in X^1} CH^{q-1}(\overline{\{x\}} \times_k K) \longrightarrow CH^q(X \times_k K) \longrightarrow CH^q(\mathbf{Spec}(k(X) \otimes_k K)) \longrightarrow 0$$

is exact. This means that $CH^*(\mathbf{Spec}(k(X) \otimes_k K))$ is the Chow ring of "extremely transcendental cycles" on X.

2. *If $q > \mathrm{degtr}(K/k)$ then $CH^q(\mathbf{Spec}(k(X) \otimes_k K)) = 0$, and thus the sequence (2) is obviously exact.*

Proof.

1. This is a limit case of the exact localization sequence for the Chow groups.
2. If the scheme X is finite-dimensional then Krull dimension of $\mathbf{Spec}(k(X) \otimes_k K)$ is the minimum of $\mathrm{degtr}(K/k)$ and $\dim X$. Otherwise, it is $\mathrm{degtr}(K/k)$. □

Version:

Using duality $H^{2q}(X) = H^{2d-2q}(X)^\vee$, one can define the class map $cl_H : CH^q(X) \longrightarrow H^{2q}(X)$ as $[Z] \longmapsto (\omega \mapsto \omega|Z)$, where $H^{2d}(Z)$ is canonically identified with $H^0(Z)$.

Composition of the class map cl_H with the Künneth decomposition induces a homomorphism

$$CH^q(X \times_k Y) \longrightarrow \bigoplus_{r=-q}^{q} H^{q+r}(X) \otimes H^{q-r}(Y). \tag{3}$$

Suppose W is a subvariety of X of codimension p. Then the composition of the natural map $CH^{q-p}(W \times_k Y) \longrightarrow CH^q(X \times_k Y)$ with the class map (3) commutes with composition of the corresponding class with X replaced by W with the Gysin map and factors through the subspace $\bigoplus_{r=2p-q}^{q} N^p H^{q+r}(X) \otimes H^{q-r}(Y)$, i.e., the following diagram commutes

$$\begin{tikzcd}
CH^{q-p}(W \times_k Y) \arrow[rr] \arrow[d] & & CH^q(X \times_k Y) \arrow[d] \\
H^{2(q-p)}(W \times_k Y) \arrow[r, "\mathrm{Gysin}\otimes id"] & \displaystyle\bigoplus_{r=2p-q}^{q} N^p H^{q+r}(X) \otimes H^{q-r}(Y) \arrow[r, "\subseteq"] & H^{2q}(X \times_k Y)
\end{tikzcd}$$

Therefore, interchanging the roles of X and Y, we can define the following class map (at this point, in the case $H^* = H^*_{DR}$, we may forget that K was finitely generated over k)

$$\tilde{c}_q^N : CH^q(X \times_k K) \longrightarrow \bigoplus_{r=-q}^{q} H^{q+r}(X) \otimes H^{q-r}(K/k).$$

This shows that for any $0 \leq q \leq \dim X$, the sequence

$$\bigoplus_{x \in X^1} CH^{q-1}(\overline{\{x\}} \times_k K) \longrightarrow CH^q(X \times_k K) \xrightarrow{\widehat{c}_q^N} H^q(k(X)/k) \otimes H^q(K/k), \tag{4}$$

where $\widehat{c}_q^N$ is induced by $\tilde{c}_q^N$, is a complex.

1.1. **Some consequences of motivic conjectures.** Under some assumptions, it is sometimes clear, that the last map in the sequence (2) is non-trivial, whenever $\Gamma(X, \Omega^q_{X/k}) \neq 0$ and the field K is big enough. In particular, one has the following

Lemma 1.2. *Let X be a smooth projective variety over a field k. Suppose that either*

- *for q as above there is a dominant rational map $f : X \longrightarrow S$ to a smooth projective q-dimensional variety S over k with $\Gamma(S, \Omega^q_S) \neq 0$, or*
- *the inverse Lefschetz isomorphism $H^{2d-q}(X)(d) \xrightarrow{\sim} H^q(X)(q)$ is induced by a correspondence on $X \times_k X$.*

Let K be an extension of the field of rational functions $k(X)$, and $\Gamma(X, \Omega^q_X) \neq 0$. Then the composition of the cycle-class map (1) with the projection to $H^q(X, \mathcal{O}_X) \otimes_k \Omega^q_{K/k}$-summand is non-trivial.

Proof.

- Since the problem is birational, we may suppose that $f : X \longrightarrow S$ is a morphism. Then we get a cycle $Z = X \times_S X$ on $X \times_k X$ of codimension q. Note that $[Z] = (f \times f)^*[\Delta_S] \in CH^q(X \times_k X)$ and therefore, the image of $[Z]$ under the map (2) coincides with the pull-back of the image of $[\Delta_S]$ under the map (2) for S instead of X. Note further, that the homomorphism $(f \times f)^* : H^q(S, \mathcal{O}_S) \otimes_k H^q_{DR}(k(S)/k) \longrightarrow H^q(X, \mathcal{O}_X) \otimes_k H^q_{DR}(k(X)/k)$ is injective and the image of $[\Delta_S]$ under the map (2) is "the identity element in the algebra of endomorphisms of $\Gamma(X, \Omega^d_X)$" (see Lemma 2.1 formula (7) for details), so the last map in the complex (2) is not trivial.
- Let $Z \subset X \times_k X$ be a cycle on $X \times_k X$ of codimension q such that in the commutative diagram

$$\begin{tikzcd} H^{2d-q}(X)(d) \arrow[r, "L^{-1}"] \arrow[d, "\mathrm{pr}_1^*"', "\subseteq"] & H^q(X)(q) \\ H^{2d-q}(X \times_k X)(d) \arrow[r, "\bigcap Z"] & H^{2d+q}(X \times_k X)(d+q) \arrow[u, "\mathrm{pr}_{2*}"'] \end{tikzcd}$$

the upper horizontal map L^{-1} is the inverse Lefschetz operator, that is, an isomorphism. Then its restriction induces an isomorphism

$$F^0 H^{2d-q}(X)(d) \cong H^{d-q}(X, \Omega^d_X) \xrightarrow{\sim} F^0 H^q(X)(q) \cong \Gamma(X, \Omega^q_X).$$

Note, however, that the class of this isomorphism in $H^q(X, \mathcal{O}_X) \otimes_k \Gamma(X, \Omega^q_X)$ coincides with the class of the cycle Z. $\square$

REMARK. If the space $H^{q,q}(X \times X) \bigcap H^{2q}(X \times X; \mathbb{Q}(q))$ is generated by algebraic cycles then the inverse Lefschetz isomorphism $H^{2d-q}(X)(d) \xrightarrow{\sim} H^q(X)(q)$ is induced by a correspondence on $X \times_k X$.

(*Proof.* Since the multiplication by $(d-q)$-th power of a hyperplane section gives an isomorphism $H^q(X)(q) \xrightarrow{\sim} H^{2d-q}(X)(d)$, we get an isomorphism

$$id \otimes L^{d-q} : H^q(X) \otimes H^q(X)(q) \xrightarrow{\sim} H^q(X) \otimes H^{2d-q}(X)(d) \cong \mathrm{End} H^q(X).$$

The identity of the algebra $\mathrm{End}H^q(X)$ is a rational element of type (d, d). By pulling this class back via the isomorphism $id \otimes L^{d-q}$, we get an element in $H^{q,q}(X \times X) \bigcap H^{2q}(X \times X; \mathbb{Q}(q))$ inducing the inverse Lefschetz isomorphism. $\square$)

It follows from the motivic conjectures, that the complex (2) is exact.

Namely, in [Mur] Murre states the following conjectures.

CONJECTURE A. For any smooth projective d-dimensional variety X there is a decomposition $\Delta_X = \sum_{i=0}^{2d} \pi_i^X$ such that elements $\pi_i^X \in CH^d(X \times_k X)_{\mathbb{Q}}$ are orthogonal projectors: $\pi_i^X \circ \pi_j^X = \delta_{i,j}\pi_i^X \in CH^d(X \times_k X)$ with classes $[\pi_i^X] \in H^{2d-i}(X) \otimes H^i(X) \subseteq H^{2d}(X \times X)$.

CONJECTURE B(q). Elements $\pi_0^X, \ldots, \pi_{q-1}^X$ and $\pi_{2q+1}^X, \ldots, \pi_{2d}^X$ act trivially on $CH^q(X)_{\mathbb{Q}}$.

CONJECTURE D(q). $CH^q_{hom}(X)_{\mathbb{Q}} = \mathrm{Ker}[\pi_{2q}^X : CH^q(X)_{\mathbb{Q}} \longrightarrow CH^q(X)_{\mathbb{Q}}]$.

In addition, we will need the following.

HODGE CONJECTURE H(q). The subspace of $H^{2q}(X; \mathbb{Q})$ generated by classes of algebraic cycles coincides with the space $H^{2q}(X; \mathbb{Q}) \cap F^q H^{2q}(X; \mathbb{C})$.

CONJECTURE H$(1, q)$(GENERALIZED HODGE CONJECTURE for coniveau 1). $N^1 H^q(X; \mathbb{Q})$ is the maximal Hodge substructure of $H^q(X; \mathbb{Q})$ in $F^1 H^q(X; \mathbb{C})$.

Proposition 1.1. 1. *Conjectures A, B(q) and D(q) imply triviality of the canonical map*

$$CH^q_{hom}(X \times_k Y) \longrightarrow CH^q(k(X) \otimes_k k(Y))_{\mathbb{Q}} \tag{5}$$

for any pair of k-varieties X and Y, and the existence of an embedding of $CH^q(k(X) \otimes_k k(Y))_{\mathbb{Q}}$ into a canonical quotient of the group $H^q(X;\mathbb{Q}) \otimes H^q(Y;\mathbb{Q}(q))$.

2. *Conjectures A, B(q), D(q) and H(q − 1) imply injectivity of the map*

$$c^N_q : CH^q(k(X) \otimes_k k(Y))_{\mathbb{Q}} \longrightarrow (H^q(X;\mathbb{Q})/N^1H^q(X;\mathbb{Q})) \otimes (H^q(Y;\mathbb{Q}(q))/N^1H^q(Y;\mathbb{Q})).$$

3. *Suppose c^N_q is injective and the generalized Hodge conjecture $H(1,q)$ holds. Then the map*

$$c^F_q : CH^q(k(X) \otimes_k k(Y))_{\mathbb{Q}} \longrightarrow H^q(X, \mathcal{O}_X) \otimes \Gamma(Y, \Omega^q_Y)$$

is injective.

In particular, assuming also $H(q)$, any embedding $\sigma : k \hookrightarrow \mathbb{C}$ gives rise to the following equivalence of the category $\Xi_d(k)$ of finitely generated fields of transcendence degree d over k (with morphisms given by $\mathrm{Hom}_{\Xi_d(k)}(K, L) = CH_0(K \otimes_k L)_{\mathbb{Q}}$) with a full subcategory of the category $\mathcal{H}^d_{\mathbb{Q}}$ of pure polarizable $\mathbb{Q}$-Hodge structures of weight d and length of the Hodge filtration $2d$ with no Hodge substructures of shorter Hodge filtration: for any smooth projective variety X over k

$$\Phi_\sigma : \Xi_d(k) \longrightarrow \mathcal{H}^d_{\mathbb{Q}}, \qquad k(X) \longmapsto H^d(X \times_{k,\sigma} \mathbb{C};\mathbb{Q})/N^1H^d(X \times_{k,\sigma} \mathbb{C};\mathbb{Q}).$$

Proof.

1. By the Conjecture D(q), $CH^q_{hom}(X \times_k Y) = \mathrm{Ker}(\sum_{s=0}^{2q} \pi^X_s \times \pi^Y_{2q-s})$. Clearly, the sequence

$$CH^*(X)_{\mathbb{Q}} \xrightarrow{\Delta-\pi_s} CH^*(X)_{\mathbb{Q}} \xrightarrow{\pi_s} CH^*(X)_{\mathbb{Q}}$$

is exact, so $CH^q_{hom}(X \times_k Y)_{\mathbb{Q}} = \mathrm{Im}(\Delta_{X\times Y} - \pi^{X\times Y}_{2q})$ is a subgroup of the group $\sum_{i=q}^{2q-1} \mathrm{Im}\pi^{X\times Y}_i = \sum_{i=q}^{2q-1}\sum_{k=0}^{i} \mathrm{Im}(\pi^X_k \times \pi^Y_{i-k})$.

It is not hard to see (modulo A and B(q)) that elements of types $\pi^X_0 \times \gamma, \dots, \pi^X_{q-1} \times \gamma$ and $\pi^X_{2q+1} \times \gamma, \dots, \pi^X_{2d} \times \gamma$ induce trivial maps $CH^q(X \times Y)_{\mathbb{Q}} \longrightarrow CH^q(X \times_k k(Y))_{\mathbb{Q}}$ for any cycle $\gamma \in CH^{d_Y}(Y \times Y)$. Indeed, let $\alpha \in CH^q(X \times Y)$ and β is the image of $(\Delta_X \times \gamma)(\alpha)$ in $CH^q(X \times_k k(Y))$. Then $\overline{(\pi^X_s \times \gamma)(\alpha)} = \pi^X_s(\beta) = 0$ for $0 \le s \le q-1$ and $2q+1 \le s \le 2d$.

Therefore, both k and $i-k$ should be in the set $\{q, \dots, 2q\}$ in order to induce non-trivial maps $CH^q_{hom}(X \times Y)_{\mathbb{Q}} \longrightarrow CH^q(k(X) \otimes_k k(Y))_{\mathbb{Q}}$. But then $i \ge 2q$, so $CH^q_{hom}(X \times Y)_{\mathbb{Q}} \longrightarrow CH^q(k(X) \otimes_k k(Y))_{\mathbb{Q}}$ is always trivial.

In the commutative diagram

$$\begin{array}{ccccc}
CH^q(X \times_k Y) & \xrightarrow{\pi^X_s \times \pi^Y_t} & CH^q(X \times_k Y) & \longrightarrow & CH^q(k(X) \otimes_k k(Y))_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \\
H^s(X) \otimes H^t(Y)(q) & \longrightarrow & \oplus_{r=-q}^{q} H^{q+r}(X) \otimes H^{q-r}(Y)(q) & &
\end{array}$$

$q \le s,t \le 2q$ if the composition of the maps in the first row is non-trivial and $q \le s+t \le 2q$ if $\pi_s^X \times \pi_t^Y$ is non-trivial, so $s=t=q$. We see then, that the action of $\pi_s^X \times \pi_t^Y$ is non-trivial on $CH^q(k(X)\otimes_k k(Y))_{\mathbb{Q}}$ only for $s=t=q$, and $\pi_s^X \times \pi_t^Y$ projects $H^*(X\times Y)$ onto $H^s(X)\otimes H^t(Y)$.

2. The group $CH^q(X\times Y)/\sim_{hom}$ of cycles modulo homological equivalence is embedded into $H^{2q}(X\times Y;\mathbb{Q}(q))$. The subgroup $N^1H^q(X;\mathbb{Q})\otimes H^q(Y;\mathbb{Q}(q))$ in the range of the latter embedding is generated by images under Gysin maps of $H^{q-2}(D;\mathbb{Q})\otimes H^q(Y;\mathbb{Q}(q-1))$ for all (desingularizations of) divisors D on X. If the image $\overline{\alpha}$ of an element $\alpha\in CH^q(X\times Y)$ in $H^{2q}(X\times Y;\mathbb{Q}(q))$ belongs in fact to this subgroup then, due to semi-simplicity of the category of pure polarizable Hodge structures and assuming the Hodge conjecture, $\overline{\alpha}$ is also the class of a cycle supported on a divisor of type $D\times_k Y$, so α should vanish in $CH^q(k(X)\otimes_k k(Y))_{\mathbb{Q}}$.
3. Let γ be an algebraic cohomology class in the subspace $F^1H^q(X;\mathbb{C})\otimes_{\mathbb{C}} H^q(Y;\mathbb{C})$ of $F^1H^{2q}(X\times Y;\mathbb{C})$. We may suppose that in the decomposition $\gamma=\sum_i \epsilon_i\otimes_{\mathbb{Q}}\delta_i$, where $\epsilon_i\in H^q(X;\mathbb{Q})$ and $\delta_i\in H^q(Y;\mathbb{Q})$, δ_i are $\mathbb{Q}$-linearly independent, and therefore, they are linearly independent over $\mathbb{C}$. After the complexification, the component of γ in $H^{0,q}(X)\otimes H^q(Y)$ should be trivial, so $\sum_i \epsilon_i^{0,q}\otimes_{\mathbb{Q}}\delta_i=0$ and, due to $\mathbb{C}$-linear independence of δ_i, all elements ϵ_i belong to $F^1H^*(X;\mathbb{C})$.
   In the same way, looking at the $H^q(X)\otimes H^{q,0}(Y)$-component, we conclude that elements δ_i are in the subspace $F^1H^q(Y;\mathbb{C})$, i.e., $\gamma\in F^1H^q(X;\mathbb{C})\otimes_{\mathbb{C}} F^1H^q(Y;\mathbb{C})$. γ defines a morphism of Hodge structures $H^{2\dim Y-q}(Y)(\dim Y)\xrightarrow{\alpha} H^q(X)(q)$. Since the image of α is a Hodge substructure in $F^1H^q(X;\mathbb{Q})$, the generalized Hodge conjecture for coniveau 1 asserts that the image of α is a subspace in $N^1H^q(X;\mathbb{Q})$, and finally we get that γ is in the subspace $N^1H^q(X;\mathbb{Q})\otimes_{\mathbb{Q}} F^1H^q(Y;\mathbb{Q})$. $\square$

2. 0-cycles

Lemma 2.1. *Let X be a smooth projective variety over an algebraically closed field k and K is an extension of $k(X)$. Then the set of 0-cycles on $X\times_k K$ carries a natural decreasing filtration of length d:*

$$F^s(X\times_k K)_0=\coprod_{s\le t\le q}\coprod_{x\in X^t}\mathrm{Hom}^{[0]}_{\mathrm{fields}/k}(k(x),K), \tag{6}$$

where $\mathrm{Hom}^{[0]}_{\mathrm{fields}/k}(k(x),K)=\{\sigma: k(x)\hookrightarrow\overline{K}\}/\mathrm{Gal}(\overline{K}/K)$*;* $F^0(X\times_k K)_0=(X\times_k K)_0$*,* $F^d(X\times_k K)_0=X^d$*,* $F^{d+1}(X\times_k K)_0=\emptyset$*.*

This filtration is multiplicative:

$$F^sCH^p(X\times_k K)\times F^tCH^q(Y\times_k K)\longrightarrow F^{s+t}CH^{p+q}(X\times_k Y\times_k K).$$

The group G of automorphisms of the field $\overline{K}$ over the field k acts transitively on each subset $\mathrm{Hom}^{[0]}_{\mathrm{fields}/k}(k(x),\overline{K})$ of the set $(X\times_k\overline{K})_0$.

Proof. Any reduced irreducible effective zero-cycle on $X\times_k K$ is determined by the kernel of a homomorphism of k-algebras $\mathcal{O}(U)\otimes_k K\longrightarrow\overline{K}$ identical on $k\otimes_k K$ for an appropriate open subset U in X.

Such homomorphism is given by some $\mathcal{O}(U)\xrightarrow{id\otimes 1}\mathcal{O}(U)\otimes_k K\longrightarrow\overline{K}$. Let $k(x)$ be the fraction field of the quotient of $\mathcal{O}(U)$ by the kernel of this homomorphism. Then we are reduced to homomorphisms $k(x) \otimes_k K \longrightarrow \overline{K}$ identical on $k \otimes_k K$ and such that $\overline{K}$ is an algebraic extension of the field of fractions of their images.

Let σ and σ' be some embeddings $k(x) \hookrightarrow \overline{K}$ with the same kernel of the corresponding homomorphisms $k(x) \otimes_k K \longrightarrow \overline{K}$.

First note that $\mathrm{Gal}(\overline{K}/K)$ obviously respects the kernels.

Secondly, we need to show that there exists an automorphism $\tau \in \mathrm{Gal}(\overline{K}/K)$ such that $\sigma' = \tau \circ \sigma$. For any element g of $k(x)$ let $\sum_i h_i y^i$ be a minimal polynomial of $\sigma'(g)$ with coefficients in the field K. Then $\sum_i g^i \otimes h_i \in \mathrm{Ker}(\sigma' \cdot id) = \mathrm{Ker}(\sigma \cdot id)$, and therefore $\sum_i h_i y^i$ is a minimal polynomial also of $\sigma(g)$ with coefficients in K, and thus, there is an element $\tau \in \mathrm{Gal}(\overline{K}/K)$ such that $\sigma' = \tau \circ \sigma$, so we get the identity (6). □

Lemma 2.2. *When $q = \dim X$, the class map (1) is compatible with the natural composition law*

$$\mathbb{Z}[\mathrm{Hom}^{[0]}_{\mathrm{fields}/k}(L, K)] \times \mathbb{Z}[\mathrm{Hom}^{[0]}_{\mathrm{fields}/k}(K, F)] \longrightarrow \mathbb{Z}[\mathrm{Hom}^{[0]}_{\mathrm{fields}/k}(L, F)],$$

and it is induced by

$$(x, \sigma) \longmapsto \left[\omega \longmapsto \sum_{\sigma' \in \mathrm{Gal}(\overline{K}/K)\text{-}orbit\ of\ \sigma} \sigma'\left(\omega|_{\overline{\{x\}}}\right)\right] \in H^d(X, \Omega^r_X) \otimes_k \Omega^{d-r}_{K/k}, \qquad (7)$$

where $x \in X^t$, $\sigma : k(x) \longrightarrow \overline{K}$, $\omega \in \Gamma(X, \Omega^{d-r}_{X/k})$, and the space $H^d(X, \Omega^r_X)$ is identified with the space $\Gamma(X, \Omega^{d-r}_{X/k})^\vee$.

Proof. Let (x, σ) correspond to a closed point of $X \times_k K$ and K is the function field of a smooth projective variety Y. Then the closure of (x, σ) is the graph of a rational map $Y \longrightarrow \overline{\{x\}}$. The rest is obvious. □

Now let X be a smooth projective variety over an algebraically closed field k of characteristic 0, and $k(X)$ be the field of rational functions on X.

The following proposition will not be used until some speculations in the next section.

Proposition 2.1. *(compare with* [R1] *Theorem 3,* [BS] *Prop.1,* [J] *Theorem 3.5...) If for an extension K of $k(X)$ all rational equivalence classes of 0-cycles come from r-subvarieties defined over k, i.e.,*

$$\bigoplus_{x \in X_r} CH_0(\overline{\{x\}} \times_k K) \longrightarrow\!\!\!\!\!\rightarrow CH_0(X \times_k K),$$

then $H^p(X) = N^1 H^p(X)$ (and therefore, $\Gamma(X, \Omega^p_X) = 0$) for all $p \geq r + 1$ and there is a finite set of cycles $S \subset X_r$ such that

$$\bigoplus_{x \in S} CH_0(\overline{\{x\}} \times_k K) \longrightarrow\!\!\!\!\!\rightarrow CH_0(X \times_k K).$$

The latter assertion holds if the groups $CH_0(\ \cdot\ )$ are replaced with the groups $CH_0(\ \cdot\ )/n$ for some positive integer n.

Proof. The diagonal $\Delta \in CH^{\dim X}(X \times_k X)$ acts on the cohomology as the identity. The kernel of the composition

$$CH^{\dim X}(X \times_k X) \longrightarrow\!\!\!\!\!\rightarrow CH^{\dim X}(X \times_k k(X)) \longrightarrow CH^{\dim X}(X \times_k K)$$

(at least, modulo torsion) consists of cycles supported on $X \times D$ for all divisors D. Denote by $\overline{\Delta}$ the image of Δ under this composition.

Under the assumptions as above, $\bigoplus_{x\in X_r} CH_0(\overline{\{x\}} \times_k K) \twoheadrightarrow CH_0(X \times_k K)$. So $\overline{\Delta}$ is a linear combination of cycles in $CH_0(X \times_k K)$ sitting on an r-cycle C defined over k. Then the restriction of $\overline{\Delta}$ to $CH_0((X - C) \times_k K)$ is a torsion element and the restriction of $\overline{\Delta}$ to $CH_0((X - C) \times_k k(X))$ is also a torsion element.

Thus (a multiple of) Δ is rationally equivalent to a sum $Z_1 + Z_2$, where Z_1 is supported on $C \times X$ and Z_2 is supported on $X \times D$ for an r-dimensional subscheme C and a divisor D. Clearly, the map $H^*(X) \longrightarrow H^*(X)$ induced by the correspondence Z_1 factors through the restriction $H^*(X) \longrightarrow H^*(Supp(C))$, and therefore, its image is in the subspace $N^1H^*(X)$ for $p > r$. The image of Z_2 is also in the same subspace $N^1H^*(X)$, so the same applies to the image of Δ.

Consider the (identity) map from CH_0(second X) to CH_0(first X) induced by $Z_1 + Z_2$. If the support of a cycle does not intersect D then it is sent to a cycle on C. Since any 0-cycle is rationally equivalent to another one with the support disjoint with the support of D this implies that $CH_0(C)$ generates $CH_0(X)$. $\square$

Corollary 2.1. *(*[R1]*) If $\Gamma(X, \Omega^r_X) \neq 0$ for some $r > 0$ and K contains an algebraic closure of $k(X)$, then the group $CH_0((X \times_k K)\backslash(\coprod_{x\in X_{r-1}} \overline{\{x\}} \times_k K))$ is of infinite rank (and non-trivial for any extension K of $k(X)$).*

Proof. This group is non-trivial, just because any embedding $\sigma : k(X) \longrightarrow K$ induces an embedding $\Gamma(X, \Omega^r_X) \longrightarrow H^r_{DR}(K/k)$.

If the group was only finite-dimensional, then all possible embeddings of fields $\sigma : k(X) \longrightarrow \overline{K}$ could span only a finite-dimensional k-vector subspace in the infinite-dimensional vector space $H^d(X, \Omega^{d-r}_X) \otimes_k H^r_{DR}(K/k)$. Therefore, after identifying $H^d(X, \Omega^{d-r}_X)$ with $\Gamma(X, \Omega^r_X)^\vee$, we see, that images of $\Gamma(X, \Omega^r_X)$ under all possible σ's span only a finite-dimensional k-subspace in $H^r_{DR}(K/k)$. But this is obviously false, when K contains an algebraic closure of $k(X)$. $\square$

2.1. **Dimensions of subvarieties consisting of rationally equivalent 0-cycles.** Before we proceed to the main result of this section we need the following lemma in linear algebra.

Lemma 2.3. *Let $V_1, \dots, V_N$ be a collection of subspaces of a K-space W. Fix an integer $r > 1$ and for each $1 \leq i \leq N$ fix a non-trivial element $\omega_i \in \wedge^r V_i$. Let P be a subspace of the sum $V_1 + \dots + V_N$ of these spaces such that $\sum_i \omega_i \in \wedge^r P$. Suppose that $t = \dim \mathrm{Ker}(V_1 \oplus \dots \oplus V_N \xrightarrow{\Sigma} W) \leq r - 2$. Then $\dim P \geq r \cdot N - t$.*

Remarks. 1. In the case $\dim V_1 = \dots = \dim V_N = r$ and $\sum V_i = W$ this bound is obviously precise.

2.If $N = 2$ and $t = r - 1$ then value r $(< r \cdot 2 - (r - 1) = r + 1)$ of $\dim P$ does occur.

Proof. First suppose that $t = 0$.

For each $1 \leq i \leq N$ we may choose a surjection $\pi_i : V_i \longrightarrow V'_i$ onto an r-dimensional K-space V'_i such that the r-vectors $\pi_i(\omega_i) \in \wedge^r V'_i$ are all non-trivial.

Let $P' = (\oplus_{i=1}^N \pi_i)(P) \subseteq \oplus_{i=1}^N V'_i$. Then the r-vector $\omega = \sum_{i=1}^N \pi_i(\omega_i) \in \wedge^r P'$ is non-trivial. In some dual bases $\{x_1^{(i)}, \dots, x_r^{(i)}\}$ and $\{\xi_1^{(i)}, \dots, \xi_r^{(i)}\}$ of the spaces V'_i and $V_i'^\vee$, respectively, we have $\pi_i(\omega_i) = x_1^{(i)} \wedge \dots \wedge x_r^{(i)}$ and the map

$$\oplus_{i=1}^N V_i'^\vee \xrightarrow{\langle\ \cdot\ ,\omega\rangle} \oplus_{i=1}^N \wedge^{r-1} V'_i \subset \wedge^{r-1}(\oplus_{i=1}^N V'_i),\ \xi_j^{(i)} \longmapsto x_1^{(i)} \wedge \dots \wedge \widehat{x_j^{(i)}} \wedge \dots \wedge x_r^{(i)},$$

is obviously injective (in fact, an isomorphism). But if $\dim P' < rN$ then there exists a non-zero element $v \in \oplus_{i=1}^N V_i'^\vee$ such that $\langle v, P'\rangle = 0$, and thus if $\omega \in \wedge^r P'$ then $\langle v, \omega\rangle = 0$, contradicting injectivity.

Sublemma 2.1. *Let $V_1, \dots, V_N$ be a collection of subspaces of a K-space W. Suppose that $r-1 > t = \dim \mathrm{Ker}(V_1 \oplus \dots \oplus V_N \longrightarrow W)$. Then the evident map $\oplus_{i=1}^N \wedge^{r-1} V_i \longrightarrow \wedge^{r-1} W$ is injective.*

Remark. If $t = r-1$ then the latter map can be not injective for any N and any spaces V_i of dimension $\geq r-1$.

Proof. Without loss of generality, we may suppose that $W = V_1 + \dots + V_N$. Let $\dim V_i = d_i$ and $V_i = V_i' + V_i''$, where $V_1 = V_1''$, $V_i' \subset \sum_{j=1}^{i-1} V_j$ and $V_i'' \cap (\sum_{j=1}^{i-1} V_j) = 0$. Then $\dim W = \sum_{i=1}^N d_i - t = \dim V_1'' + \dim V_2'' + \dim V_3'' + \dots + \dim V_N''$. If $\dim V_i'' \leq d_i - r + 1$ for some i then $\dim W \leq \sum_{i=1}^N d_i - (r-1) < \sum_{i=1}^N d_i - t$, so contradiction shows that $\dim V_i'' \geq d_i - r + 2$ for all i and therefore, $\dim V_i' \leq r-2$. This implies that in the decomposition

$$\wedge^{r-1} V_i = \wedge^{r-1} V_i' + \wedge^{r-2} V_i' \otimes V_i'' + \wedge^{r-3} V_i' \otimes \wedge^2 V_i'' + \wedge^{r-4} V_i' \otimes \wedge^3 V_i'' + \dots$$

$\wedge^{r-1} V_i' = 0$, and thus $\wedge^{r-1}(V_1 + \dots + V_{i-1}) \cap \wedge^{r-1} V_i = 0$. In other words,

$$\wedge^{r-1}(V_1 + \dots + V_{i-1}) \oplus \wedge^{r-1} V_i \hookrightarrow \wedge^{r-1}(V_1 + \dots + V_i).$$

Iterating the process we get a sequence of injections

$$\wedge^{r-1} V_1 \oplus \wedge^{r-1} V_2 \oplus \dots \oplus \wedge^{r-1} V_N \hookrightarrow \wedge^{r-1}(V_1 + V_2) \oplus \wedge^{r-1} V_3 \oplus \dots \oplus \wedge^{r-1} V_N \hookrightarrow \dots$$

$$\dots \hookrightarrow \wedge^{r-1}(V_1 + \dots + V_{N-1}) \oplus \wedge^{r-1} V_N \hookrightarrow \wedge^{r-1}(V_1 + \dots + V_N) = \wedge^{r-1} W.$$

$\square$

We use this Sublemma to reduce the case of general t to the case $t = 0$ as follows. Consider the commutative diagram (again, we suppose that $W = V_1 + \dots + V_N$)

$$\begin{tikzcd}
\oplus_{i=1}^N V_i'^\vee \arrow[r, "\subseteq"] \arrow[d, "\oplus\langle\,\cdot\,,\pi_i(\omega_i)\rangle"', "\cong"] & \oplus_{i=1}^N V_i^\vee \arrow[d, "\oplus\langle\,\cdot\,,\omega_i\rangle"] & W^\vee \arrow[l, "\supseteq"', "\text{codim}=t"] \arrow[d, "\langle\,\cdot\,,\sum\omega_i\rangle"] \\
\oplus_{i=1}^N \wedge^{r-1} V_i' \arrow[drr] & \oplus_{i=1}^N \wedge^{r-1} V_i \arrow[l, "onto"'] \arrow[r, "\alpha"] & \wedge^{r-1} W \arrow[d, "\beta"] \\
 & & \wedge^{r-1}(W/\sum_{i=1}^N \mathrm{Ker}(V_i \longrightarrow V_i'))
\end{tikzcd}$$

Let L be the subspace of $W^\vee$ of functionals vanishing on P. Then $L \subseteq \mathrm{Ker}\langle\,\cdot\,, \sum \omega_i\rangle$, and thus,

$$\dim P = \dim W - \dim L \geq \dim W - \dim \mathrm{Ker}\langle\,\cdot\,, \sum \omega_i\rangle = \dim \mathrm{Im}\langle\,\cdot\,, \sum \omega_i\rangle.$$

Note that $\mathrm{Ker}(\oplus_{i=1}^N V_i' \longrightarrow W/\sum_{i=1}^N \mathrm{Ker}(V_i \longrightarrow V_i'))$ is the image of $\mathrm{Ker}(\oplus_{i=1}^N V_i \longrightarrow W)$, so its dimension is $< r-1$, and thus we apply the Sublemma to see that the slope arrow is injective and the composition of $\langle\,\cdot\,,\oplus\omega_i\rangle$ with α and β is injective on the subspace $\oplus_{i=1}^N V_i'^{\vee}$, and therefore,

$$\dim \mathrm{Im}\langle\,\cdot\,,\sum\omega_i\rangle \geq \dim(W^{\vee}\cap(\oplus_{i=1}^N V_i'^{\vee})) \geq rN-t.$$

□

Proposition 2.2. *(compare with* [R1] *Theorem 1)*

1. *If the canonical sheaf Ω_X^d defines a rational map φ of X, then for any "generic" (i.e., in the complement to the union of all divisors defined over k) point $x\in X(K)$ and for any point y in its rational equivalence class φ is defined at y and $\varphi(x)=\varphi(y)$.*
2. *Suppose that $\Gamma(X,\Omega_X^r)\neq 0$ for some $r>1$. Fix such N subvarieties $Y_1,\dots,Y_N$ of X defined over k that each restriction map $\Gamma(X,\Omega_X^r)\longrightarrow\Gamma(Y_i,\Omega_{Y_i}^r)$ is not trivial. Let $y_1+\cdots+y_N$ be a sum of "generic" points on Y_i's corresponding to a point in the complement to all subvarieties in $Y_1\times_k\cdots\times_k Y_N$ of codimension $t+1<r$ defined over k. Then the set of effective 0-cycles on X in the rational equivalence class of $y_1+\cdots+y_N$ is either finite or countable union of subvarieties of codimension $\geq r\cdot N-t$ in the N-th symmetric power $\Sigma^N X$ of the variety X.*

Proof.

1. Due to Lemma 2.1, $\overline{K}$-valued points of X outside divisors defined over k are described as embeddings of fields $\sigma: k(X)\longrightarrow\overline{K}$.
   If the canonical sheaf defines a rational map of X, then the field of rational functions on X is an extension of the field L generated by fractions of global sections of the canonical sheaf.
   Therefore, restriction of any embedding of fields $\sigma: k(X)\longrightarrow\overline{K}$ to the field L is uniquely defined by the induced embedding $\Gamma(X,\Omega_{X/k}^{\dim X})\longrightarrow\Omega_{\overline{K}/k}^{\dim X}$, and its prolongations back to the field $k(X)$ are points of the fibre of the canonical map φ over the point $\sigma|_L$.
   By the second part of the same Lemma, this implies that for generic points (=embeddings of fields) in distinct fibres of φ their images under the last map in the sequence (2) are distinct and non-zero. On the other hand, the last map in the sequence (2) kills all non-generic (i.e., sitting on a divisor over k) points.
2. Fix N embeddings $\tau_i: k(Y_i)\longrightarrow K$ with images generating a field of transcendence degree $\sum_i d_i-t$ over k, where $d_i=\dim Y_i$. We may always suppose that the field k is countable. Now fix a holomorphic r-form ω on X with non-trivial restrictions to each of the subvarieties Y_i. If a cycle $\sum_i[\sigma_i: k(x_i)\longrightarrow K]$ is rationally equivalent to the fixed one then $\sum_i\tau_i(\omega|_{Y_i})=\sum_i\sigma_i(\omega|_{\overline{\{x_i\}}})$.
   We apply Lemma 2.3 as follows. Set $V_i=K\otimes_{\tau_i(k(Y_i))}\Omega^1_{\tau_i(k(Y_i))/k}$, and

$$P=\left(\sum_{i=1}^N K\otimes_{\sigma_i(k(x_i))}\Omega^1_{\sigma_i(k(x_i))/k}\right)\bigcap\left(\sum_{i=1}^N K\otimes_{\tau_i(k(Y_i))}\Omega^1_{\tau_i(k(Y_i))/k}\right).$$

   Due to the Lemma, $\dim P\geq rN-t$, so

$$\dim\left(\left(\sum_{i=1}^N K\otimes_{\sigma_i(k(x_i))}\Omega^1_{\sigma_i(k(x_i))/k}\right)+\left(\sum_{i=1}^N K\otimes_{\tau_i(k(Y_i))}\Omega^1_{\tau_i(k(Y_i))/k}\right)\right)\leq\sum_i d_i+N(d-r),$$

and therefore,

$$\mathrm{degtr}(\sigma_1(k(x_1))\ldots\sigma_N(k(x_N))\tau_1(k(Y_1))\ldots\tau_N(k(Y_N))/\tau_1(k(Y_1))\ldots\tau_N(k(Y_N)))\leq N(d-r)+t,$$

in other words, there are $\leq N(d-r)+t$ degrees of freedom in the choice of the field $\sigma_1(k(x_1))\ldots\sigma_N(k(x_N))\tau_1(k(Y_1))\ldots\tau_N(k(Y_N))$. Finally, since the number of the embeddings $\sigma_i(k(x_i))\hookrightarrow\sigma_1(k(x_1))\ldots\sigma_N(k(x_N))\tau_1(k(Y_1))\ldots\tau_N(k(Y_N))$ is countable, we are done. □

Corollary 2.2. *If the canonical sheaf Ω^d_X defines a birational map of X, then any "generic" in the above sense point $x\in X(K)$ is the only one in its rational equivalence class.* □

REMARKS. 1. If $\Gamma(X,\Omega^r_X)\neq 0$ is the non-trivial space of holomorphic forms of the maximal degree, $r>1$ is odd and M is big ($M>\dim\Gamma(X,\Omega^r_X)$) then for the M-th symmetric power $Y=\Sigma^M X$ of X we have $\Gamma(Y,\Omega^{rM}_Y)=0$, but Proposition 2.2 (2) still gives the same assertions for effective 0-cycles, when applied to Y. Namely, if $p:\Sigma^N Y\longrightarrow CH_0(X)$ is the natural map, then for any generic point $x\in\Sigma^N Y$ the set $p^{-1}(p(x))$ is a countable union of subvarieties of codimension $\geq(rM)N$ in $\Sigma^N Y$, since p factors through $\Sigma^{MN}X\longrightarrow CH_0(X)$.

2. One can easily construct an example of a variety with the canonical map to the point, where arbitrary set of rationally equivalent effective zero-cycles of degree N is Zariski-dense in $\Sigma^N X$. So the first part of the Proposition 2.2 is precise, and in the second part one cannot replace countable unions of subvarieties by finite ones.

Namely, let X be a smooth model of the quotient of an even-dimensional abelian variety A by the involution $x\longmapsto -x$. Clearly, $\mathrm{Alb}X=0$ and $\Gamma(X,\Omega^d_X)$ is a one-dimensional vector space. Due to a theorem of Roitman ([R2]), the group $CH_0(X)$ is torsion-free. On the other hand, any integer $n\neq 0$ induces an injection (in fact, an isomorphism) $n_*:CH_0(A)_{\mathbb{Q}}\longrightarrow CH_0(A)_{\mathbb{Q}}$. (For any non-zero z in $\mathrm{Ker}n_*$ one has $n^*n_*z=z*\sum_{n\zeta=0}[\zeta]=n^{2d}\cdot z+z*\sum_{n\zeta=0}([\zeta]-[0])=0\in CH_0(A)_{\mathbb{Q}}$, where $[a]*[b]:=[a+b]$. According to Bloch [B1], $z\in\left(CH_0(X)^0_{\mathbb{Q}}\right)^{*^i}$ but $z\notin\left(CH_0(X)^0_{\mathbb{Q}}\right)^{*^{i+1}}$ for some $i\geq 0$. So $n^{2d}\cdot z=z*\sum_{n\zeta=0}([\zeta]-[0])=0\in CH_0(A)_{\mathbb{Q}}$.) Therefore, images of a point of A and its translates by torsion elements of A in X are rationally equivalent. Adding N generic points in general position on X gives the example.

To construct odd-dimensional examples, one can use a smooth model X of the quotient of the cube of the elliptic curve $\mathbb{C}/\mathbb{Z}[(-1+\sqrt{-3})/2]$ by the automorphism of third order $(z_1,z_2,z_3)\longmapsto(-1+\sqrt{-3})/2\cdot(z_1,z_2,z_3)$.

3. When X is a general surface of degree ≥ 6 then rational equivalence classes of distinct points are distinct (see [V]).

4. When X is an abelian variety and $N\leq 3$ there are more precise bounds, see [AP].

3. Complex (2), Bloch's conjecture and divisibility problems

Bloch's conjecture asserts that for any smooth projective surface X over $\mathbb{C}$ with vanishing $H^2(X,\mathcal{O}_X)$ the group $CH_0(X)^0$ is generated by images of a finite number of Jacobians of curves on X.

As it was shown in Proposition 2.1, if for some embedding $\sigma:k\longrightarrow\mathbb{C}$ the group $CH_0(X\times_{k,\sigma}\mathbb{C})$ was generated by CH_0-groups of all divisors defined over k then in fact it was generated by CH_0-groups of a finite number of divisors. Therefore, the exactness of the complex (2) is a generalization of Bloch's conjecture.

Even more particularly, this implies that

- for any pair of k-varieties X and Y, and any q the group $CH^q(k(X)\otimes_k k(Y))$ is finitely generated and neglects base changes $k \longmapsto K$, and
- if the numerical equivalence coincides with the homological one modulo torsion then the $\mathbb{Q}$-algebras $CH_0(k(X)\otimes_k k(X))_{\mathbb{Q}}$ are semisimple, when non-trivial.

In case $q=1$ the exactness of the sequence (2) is easily verified:

Lemma 3.1. *Let $\mathfrak{X}$ and S be smooth projective varieties over $\overline{\mathbb{Q}}$ and $\lambda : \mathfrak{X} \longrightarrow S$ is a structure of S-scheme on $\mathfrak{X}$ such that generic fibre is smooth and absolutely irreducible. Suppose that K is an extension of the field $k=\overline{\mathbb{Q}}(S)$ and $X=\mathfrak{X}\times_S k$. Then the sequence*

$$\mathrm{Pic}^0\mathfrak{X}/\mathrm{Pic}^0 S \longrightarrow \mathrm{Pic}^0(X\times_k K) \longrightarrow H^2_{DR}(X\times_k K/\mathbb{Q}) \tag{8}$$

induced by the sheaf homomorphism $\mathcal{O}^\times_{X\times_k K} \stackrel{d\log}{\longrightarrow} \Omega^1_{X\times_k K/\mathbb{Q}}$, is exact. In particular,

$$\mathrm{Pic}^0(X\times_k K)/\mathrm{Pic}^0(X) \longrightarrow H^1(X,\mathcal{O}_X)\otimes_k H^1_{DR}(K/k) \tag{9}$$

is an embedding.

Proof. Without loss of generality, suppose that K is finitely generated over k and let T be a model of K over $\overline{\mathbb{Q}}$ with a morphism $f : T \longrightarrow S$. Consider a Cartesian square

$$\begin{tikzcd} & \mathfrak{X}\times_S T = \mathfrak{X}_T \arrow[dl] \arrow[dr] & \\ \mathfrak{X} \arrow[dr, "\lambda"'] & \diamond & T \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

By the definition of the Neron–Severi group, one has a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0\mathfrak{X}_T \arrow[r] & \mathrm{Pic}\mathfrak{X}_T \arrow[r] & NS(\mathfrak{X}_T) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Pic}^0 T \arrow[r] \arrow[u] & \mathrm{Pic} T \arrow[r] \arrow[u] & NS(T) \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

Therefore, the sequence

$$\mathrm{Pic}^0\mathfrak{X}_T/\mathrm{Pic}^0 T \longrightarrow \mathrm{Pic}\mathfrak{X}_T/\mathrm{Pic}T \longrightarrow NS(\mathfrak{X}_T)/NS(T) \longrightarrow 0$$

is exact.

The next step is an identifying of the group $\mathrm{Pic}^0\mathfrak{X}_T/\mathrm{Pic}^0 T$. Consider the Leray spectral sequence of the projection $\mathfrak{X}_T \longrightarrow T$: $E_2^{p,q}(T) = H^p(T, f^*R^q\lambda_*\mathbb{Q}) \Rightarrow H^{p+q}(\mathfrak{X}_T;\mathbb{Q})$.

Since the fibres of the map λ are connected, $E_2^{p,0}(T)$ is identified via pull-back f^* with the space $H^p(T;\mathbb{Q})$. Moreover, since any surjective morphism of smooth projective varieties induces injective pull-back on the cohomology, $E_2^{p,0}(T)$ is a subspace of $H^p(\mathfrak{X}_T;\mathbb{Q})$ and hence $E_2^{p,0}(T) = E_\infty^{p,0}(T)$ and $E_2^{0,1}(T) = E_\infty^{0,1}(T)$.

Note also, that $E_2^{0,p}(T) = f^*E_2^{0,p}(S)$. Therefore we get

$$E_\infty^{0,1}(T) = H^1(\mathfrak{X}_T;\mathbb{Q})/H^1(T;\mathbb{Q}) \cong E_\infty^{0,1}(S) = H^1(\mathfrak{X};\mathbb{Q})/H^1(S;\mathbb{Q}),$$

and thus $\mathrm{Pic}^0\mathfrak{X}_T/\mathrm{Pic}^0T = \mathrm{Pic}^0\mathfrak{X}/\mathrm{Pic}^0S$.

One can identify the group $\mathrm{Pic}\mathfrak{X}_T/\mathrm{Pic}T$ with the group $\mathrm{Pic}X_K$ to get an exact sequence

$$\mathrm{Pic}^0\mathfrak{X}/\mathrm{Pic}^0S \longrightarrow \mathrm{Pic}X_K \longrightarrow NS(\mathfrak{X}_T)/NS(T) \longrightarrow 0.$$

The rest is straightforward. □

Corollary 3.1. *If X is an abelian variety over k, $\alpha \in CH^q(X \times_k K)$ and a is a k-rational point of* Alb$X$*, then $t_a\alpha - \alpha \in F^1CH^q(X \times_k K)$, when $q = 0, 1, \dim X$. In other words, in these cases translations by elements of the group* Alb$X(k)$ *act trivially on $CH^q(k(X) \otimes_k K)$.*

Proof. Due to the Lemma, for any $a \in X(k)$ and any divisor D on $X \times_k K$ the difference $t_aD - D$ is in the subgroup $F^1CH^1(X \times_k K)$, and thus $t_aD_1 \cdots t_aD_q - D_1 \cdots D_q \in F^1CH^q(X \times_k K)$ for any divisors $D_1, \dots, D_q$. But as it is shown in [B1] p.216,

$$\mathrm{Pic}(X)^{\otimes^{g-1}} \otimes \mathrm{Pic}(X)^0 \twoheadrightarrow CH_0(A)^0,$$

so if $\sum_\alpha D_1^\alpha \cdots D_g^\alpha = [x] - [b] \in CH_0(X \times_k K)$ for some $b \in X(k)$ then

$$[x + a] - [b + a] - [x] + [a] = \sum_\alpha t_aD_1^\alpha \cdots t_aD_g^\alpha - \sum_\alpha D_1^\alpha \cdots D_g^\alpha \in F^1CH_0(X \times_k K)$$

due to the multiplicativity property of the filtration F (see the remark on p.6). □

Example (Bloch's conjecture for certain quotients of abelian varieties). Let A be an odd-dimensional abelian variety. Then there exist some divisors D_1 and D_2 on varieties Σ^2A and $A/\langle -1\rangle$, respectively, such that the inclusions induce surjections $CH_0(D_1) \twoheadrightarrow CH_0(\Sigma^2A)$ and $CH_0(D_2) \twoheadrightarrow CH_0(A/\langle -1\rangle)$.

Proof. We use an explicit decomposition of the diagonal in a g-dimensional abelian variety $\Delta \in CH^g(A \times_k A)$ into a sum of orthogonal projectors $\Delta = \sum_{s=0}^{2g} \pi_s$ from [K], where

$$\pi_s = \frac{(-1)^g}{(2g)!}\binom{2g}{s} l^s \circ {}^tl^{2g-s} = \frac{(-1)^g}{(2g)!}\binom{2g}{s} \mathrm{pr}_{A\times_kA*}(\mathrm{pr}^*_{A\times_k\widehat{A}}(l^s) \cdot \mathrm{pr}^*_{\widehat{A}\times_kA}({}^tl^{2g-s})),$$

and l is the first Chern class of the Poincaré line bundle on $A \times_k \widehat{A}$. Clearly, for $s > g$ the support of any representative of the class l^s cannot map surjectively onto A, and thus the support of some representative of the class π_s sits in a product $D_s \times_k A$ for some divisor D_s. In particular, π_s induces the trivial map $CH_0(A) \longrightarrow CH_0(A)$. Using the identity $\pi_s = {}^t\pi_{2g-s}$, we get that π_s maps $CH_0(A)$ to the image of $CH_0(D_s)$ for $s < g$. Let X be the quotient of an odd-dimensional abelian variety A by the involution $a \longmapsto -a$. Since $CH^*(X \times_k X)_{\mathbb{Q}} = CH^*(A \times_k A)_{\mathbb{Q}}^{(\mathbb{Z}/2\mathbb{Z})^2}$ and the elements $(1, 0)$ and $(0, 1)$ of $(\mathbb{Z}/2\mathbb{Z})^2$ change the sign of π_g, we see that all zero-cycles on X come from the divisor $\sum_{s=0}^{g-1} \overline{D_s}$ on X.

For the symmetric square of A, we note that all but a finite number of fibres of the summation map are isomorphic to X and the monodromy acts as a finite group. All 0-cycles on a fibre come from a divisor. Since almost all fibres are identified up to a finite action of the monodromy, all 0-cycles on the symmetric square of A come from a divisor. □

Remarks. 1. This implies in particular that for a sufficiently large field K the group $CH_0(k(A^N) \otimes_k K)$ is dominated by $\wedge^N CH_0(k(A) \otimes_k K)$.

Even more particularly, if E is an elliptic curve over k then from a commutative diagram

$$\begin{tikzcd} \wedge^N CH_0(k(E)\otimes_k K) \arrow[r, two heads] \arrow[d, hook, "\subset"] & CH_0(k(E^N)\otimes_k K) \arrow[d] \\ H^1(E,\mathcal{O}_E)^{\otimes N}\otimes \wedge^N H^1_{DR}(K/k) \arrow[r] & H^N(E^N,\mathcal{O}_{E^N})\otimes H^N_{DR}(K/k), \end{tikzcd}$$

if Y is a projective variety of dimension $\geq N$ such that $\wedge^N F^1 H^1_{DR}(Y/k) \longrightarrow H^N_{DR}(Y/k)$ is an injection, we get that $\wedge^N CH^1(k(E)\otimes_k k(Y)) \longrightarrow CH_0(k(E^N)\otimes_k k(Y))$ is an injection. The lower horizontal map is obviously not injective, even on the image of $\wedge^N CH_0(k(E)\otimes_k K)$ (if Y is a hyperplane section of E^N, embeddings $\sigma_i : k(E) \hookrightarrow K(Y)$'s correspond to the projections $\mathrm{pr}_i : Y \longrightarrow\!\!\!\!\rightarrow E$, and $\omega \in \Gamma(E,\Omega^1_E)$, then $\sigma_1(\omega),\dots,\sigma_N(\omega)$ generate an N-dimensional k-subspace in $\Omega^1_{k(Y)/k}$ but an $(N-1)$-dimensional $k(Y)$-subspace in $\Omega^1_{k(Y)/k}$).

2. Due to a theorem of Pirola, there are no hyperelliptic curves on a generic abelian variety of dimension ≥ 3 in characteristic zero. Therefore, there are rational curves neither on $A/\langle -1\rangle$ nor on the symmetric square of a generic abelian variety A.

3.1. **Divisibility problems.**

Proposition 3.1. *Suppose that the cohomological l-dimension of the field K is $< q$ and the class $c^N_q : CH^q(k(X)\otimes_k K) \longrightarrow H^q(X)/N^1H^q(X)\otimes H^q(K/k)$ is injective modulo torsion. Then the group $CH^q(k(X)\otimes_k K)$ is l-divisible modulo l-torsion.*

Proof. Let Y_i be a smooth projective variety over k with the field of rational functions $k(Y_i) \subseteq K$. If the class $c^N_q : CH^q(k(X)\otimes_k k(Y_i)) \longrightarrow H^q(X)/N^1H^q(X)\otimes H^q(Y_i)/N^1H^q(Y_i)$ is injective modulo torsion then we have an embedding

$$\varprojlim_n CH^q(k(X)\otimes_k k(Y_i))/l^n \longrightarrow H^q(X)/N^1H^q(X)\otimes \varprojlim_n (H^q(Y_i)/l^n)$$

modulo l-torsion.

Let (α_i) be an inductive system of non-trivial elements of $CH^q(k(X)\otimes_k k(Y_i))/l^n$ for a projective system of varieties Y_i as above with function fields approximating the field K. Then $\lim_i \alpha_i$ is a non-trivial element of the group $\varinjlim_i CH^q(k(X)\otimes_k k(Y_i))/l^n \hookrightarrow H^q(X)/N^1H^q(X)\otimes \varinjlim_i H^q(k(Y_i);\mathbb{Z}/l^n)$. But the latter group coincides with the group $H^q(X)/N^1H^q(X)\otimes H^q(\cup_i k(Y_i);\mathbb{Z}/l^n)$ and should be zero by the assumption on the cohomologiical l-dimension of the field K. $\square$

Remarks. 1. Under assumptions of the Proposition, if k is algebraically closed and K is finitely generated over k, then the group $CH^q(k(X)\otimes_k K)$ is trivial (Lemma 1.1).

2. In the case of algebraically closed field K this is obvious, since any Chow variety is define over the field k, so the divisible group $CH^q_{alg}(X\times_k K)$ surjects onto the groups $CH^q(X\times_k K)/CH^q(X) \longrightarrow\!\!\!\!\rightarrow CH^q(k(X)\otimes_k K)$.

3. If K is a p-adic field and $l \neq p$ is another prime, then the group $CH^q_{alg}(X\times_k K)$ is l-divisible modulo (finite) torsion, and, as above, we have the following sequence of surjections $CH^q_{alg}(X\times_k K) \longrightarrow\!\!\!\!\rightarrow CH^q(X\times_k K)/CH^q(X) \longrightarrow\!\!\!\!\rightarrow CH^q(k(X)\otimes_k K)$.

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