ZARISKI’S MULTIPLICITY QUESTION AND ALIGNED SINGULARITIES

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Abstract. We answer positively Zariski’s multiplicity question for special classes of non-isolated singularities.

Let \( f : (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0), (z_1, \ldots, z_n, t) \mapsto f(z_1, \ldots, z_n, t) = f_t(z_1, \ldots, z_n) \), with \( n \geq 3 \), be a germ (at the origin) of holomorphic function such that, for all \( t \) near 0, the germ \( f_t \) is reduced. Let \( \nu_{f_t} \) be the multiplicity of \( f_t \) at 0, that is, the number of points of intersection, near 0, of \( V_{f_t} := f_t^{-1}(0) \) with a generic (complex) line in \( \mathbb{C}^n \) passing arbitrarily close to 0 but not through 0. As we are assuming that \( f_t \) is reduced, \( \nu_{f_t} \) is also the order of \( f_t \) at 0, that is, the lowest degree in the power series expansion of \( f_t \) at 0. Let \( \mu_{f_t} \) be the Milnor number of \( f_t \) at 0. One says that \( (f_t) \) is topologically constant (respectively \( \mu \)-constant, equimultiple) if, for all \( t \) near 0, there is a germ of homeomorphism \( \varphi_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) such that \( \varphi(V_{f_t}) = V_{f_0} \) (respectively \( \mu_t = \mu_{f_0}, \nu_{f_t} = \nu_{f_0} \)). In the special case where \( (f_t) \) is a family of isolated singularities (i.e., when, for all \( t \) near 0, \( f_t \) has an isolated critical point at 0), if \( n \neq 3 \), then the topological constancy is equivalent to the \( \mu \)-constancy (cf. Lê [7], Teissier [16] and Lê–Ramanujam [8]).

In [21], Zariski asked the following question: if \( (f_t) \) is topologically constant, then is it equimultiple? More than thirty years later, the question is, in general, still unsettled (even for isolated hypersurface singularities). The answer is, nevertheless, known to be yes in several special cases: for example, for families of plane curve singularities (Zariski [22]), families of convenient Newton nondegenerate (isolated) singularities (Abderrahmane [1] and Saia–Tomazalla [15]), families of semiquasihomogeneous or quasihomogeneous isolated singularities\(^1\) (Greuel [4] and O’Shea [13]), families of isolated singularities of the form \( f_t(z) = a(z) + \theta(t) b(z) \), where \( a, b : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) and \( \theta : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \theta \neq 0 \), are germs of holomorphic functions (Greuel [4] and Trotman [19, 20]). For a detailed and more complete list, see the recent author’s survey article [3].

In this note, we concentrate our attention on families \( f = (f_t) \) of the following form:

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 f_t(z_1, \ldots, z_n) = g_t(z_1, \ldots, z_{n-1}) + z_n h_t(z_1, \ldots, z_n),
\]

where \( g : (\mathbb{C}^{n-1} \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0), (z_1, \ldots, z_{n-1}, t) \mapsto g(z_1, \ldots, z_{n-1}, t) = g_t(z_1, \ldots, z_{n-1}) \), and \( h : (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0), (z_1, \ldots, z_n, t) \mapsto h(z_1, \ldots, z_n, t) = h_t(z_1, \ldots, z_n) \), are germs of holomorphic functions such that, for all \( t \) near 0, the germ \( g_t \) (and \( f_t \)) is reduced.

In [5], Greuel–Pfister already considered families of this type and they proved the following result.

**Theorem 0.1** (Greuel–Pfister [5, Proposition 3.2]). Let \( f = (f_t) \) with \( f_t(z_1, \ldots, z_n) = g_t(z_1, \ldots, z_{n-1}) + z_n^2 h_t(z_1, \ldots, z_n) \) as above. Suppose that, for all \( t \) near 0, the germ \( f_t \) has an isolated critical point at 0 and the germ \( g_0 \) is semiquasihomogeneous (or the germ \( f_0 \) has an isolated critical at 0 and \( n = 3 \)). If \( (f_t) \) is topologically constant, then \( (g_t) \) is equimultiple. In

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\(^1\)In this case, it suffices to assume the semiquasihomogeneity or quasihomogeneity only for the germ \( f_0 \).
Theorem 0.3

Regarding this class of singularities, Massey proved the following theorem. Saia-Tomazella [15], Greuel [4], O’Shea [13] and Trotman [19, 20]. Note, nevertheless, that Theorem 0.1, together with the results of Zariski [22], Abderrahmane [1], Saia-Tomazella [15], Greuel [4], O’Shea [13] and Trotman [19, 20]. Note, nevertheless, that Theorem 0.2 is not an immediate consequence of Theorems 0.3 and 0.1 (cf. Remark 0.4).

Theorem 0.2. Let \( f = (f_1)_t \) with \( f_t(z_1, \ldots, z_n) = g_t(z_1, \ldots, z_{n-1}) + z_n^2 h_t(z_1, \ldots, z_n) \) as above. Assume that, for all \( t \) near 0, the germ \( f_t \) has an \( s \)-dimensional aligned singularity at 0. Also suppose that \( (f_1)_t \) is topologically constant. Let \( (t_k)_k \) be an infinite sequence of points in \( \mathbb{C} \) tending to 0. Assume that the coordinates \( z = (z_1, \ldots, z_n) \), or some circular permutation of them, form an aligning set of coordinates at 0 for \( f_0 \) and for \( f_{t_k} \), for all \( k \in \mathbb{N} \). Finally suppose that at least one of the following four conditions is satisfied:

1. For all \( t \) near 0, the germ \( g_t \) is convenient and has a nondegenerate Newton principal part with respect to the coordinates \( z' = (z_1, \ldots, z_{n-1}) \);
2. For all \( t \) near 0, the germ \( g_t \) is of the form \( g_t(z') = a(z') + \theta(t) b(z') \), where \( a, b: (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0) \) and \( \theta: (\mathbb{C}, 0) \to (\mathbb{C}, 0) \), \( \theta \neq 0 \), are germs of holomorphic functions;
3. \( g_0 \) is the germ of a semiquasihomogeneous polynomial with respect to \( z' \);
4. \( n = 3 \).

Then \( (g_t)_t \) is equimultiple. In particular, if, moreover, for all \( t \) near 0, the multiplicity at 0 of the germ \( g_t \) is less than or equal to the order at 0 of the (nonreduced) germ \( (z_1, \ldots, z_n) \mapsto z_n^2 h_t(z_1, \ldots, z_n) \), then \( (f_t)_t \) is equimultiple.

For the definition of aligned singularities and aligning sets of coordinates, see Massey [9]. For the basic material about Newton polyhedra, we refer to Kouchnirenko [6] and Oka [11, 12].

Aligned singularities were introduced by Massey in [9]. They generalize isolated singularities (obtained for \( s = 0 \)) and smooth one-dimensional singularities (in particular line singularities). Regarding this class of singularities, Massey proved the following reduction theorem.

Theorem 0.3 (Massey [9, Theorem 7.9]). The following are equivalent:

1. For all \( n \geq 4 \), the answer to Zariski’s multiplicity question is positive for families \( (f_t)_t \) of reduced analytic hypersurfaces with isolated singularities;
2. For all \( n \geq 4 \), there exists an integer \( s \) such that the answer to Zariski’s multiplicity question is positive for families \( (f_t)_t \) of reduced analytic hypersurfaces with \( s \)-dimensional aligned singularities (i.e., for all \( t \) near 0, \( f_t \) has an \( s \)-dimensional aligned singularity at 0);
3. For all \( n \geq 4 \), for all integer \( s \), the answer to Zariski’s multiplicity question is positive for families \( (f_t)_t \) of reduced analytic hypersurfaces with \( s \)-dimensional aligned singularities.

The proof of Theorem 0.2 is a combination of Massey’s proof of Theorem 0.3 and Greuel–Pfister’s proof of Theorem 0.1, together combined with the results of Zariski [22], Abderrahmane [1], Saia-Tomazella [15], Greuel [4], O’Shea [13] and Trotman [19, 20]. Note, nevertheless, that Theorem 0.2 is not an immediate consequence of Theorems 0.3 and 0.1 (cf. Remark 0.4).

Theorem 0.2 answers positively Zariski’s multiplicity question for special classes of high-dimensional singularities without any assumption on the topological constancy, that is, without any assumption on the homeomorphisms \( \varphi_t \). We recall that, under some additional hypotheses on the \( \varphi_t \)’s, positive answers to Zariski’s question for high-dimensional singularities already exist. For example, it is known that the multiplicity is an embedded \( C^1 \) invariant (cf. Ephraim [2] and Trotman [17, 18, 20]) and an embedded ‘right–left bilipschitz’ invariant (cf. Risler–Trotman [14]).
Let’s give an example where Theorem 0.2 applies. Set \( g_1(z_1, z_2) = z_1^2 + z_2^2 + (1-t) z_2^2 \) and 
\( h(t, z_1, z_2, z_3) = t z_2^2 \), so that \( f_1(z_1, z_2, z_3) = z_1^2 + z_2^2 + (1-t) z_1^2 + z_2^2 t z_2^2 \). For all \( t \) sufficiently close to 0, the singular locus of \( f_1 \) is just the \( z_3 \)-axis (so, \( f_1 \) has an 1-dimensional aligned singularity at 0). The coordinates \((z_1, z_2, z_3)\) are not aligning, but one checks easily that \((z_3, z_2, 2)\) are aligning for \((t, 2)\) too, and \((z_2, z_3)\) are not empty.

Theorem 0.2 (4) applies. Since \( z \) is defined by \( f_1 = 0 \), \( z \) is a topologically constant family of aligned singularities, the Lê numbers (cf. [9, Definition 1.11]) \( \lambda_{f_0, z}^i \) \((0 \leq i \leq n-1)\) of \( f_0 \) at 0 with respect to \( z \) are equal to the Lê numbers \( \lambda_{f_1, z}^i \) of \( f_1 \) at 0 with respect to \( z \), for all \( k \) large enough (cf. [9, Corollary 7.8]). Hence, by an inductive application of the Massey’s generalized Iomdine–Lê formula (cf. [9, Theorem 4.5 and Corollary 4.6]), for all integers \( j_1, \ldots, j_s \) such that \( 0 < j_1 < j_2 < \ldots < j_s \), the germs \( f_1^0 + \zeta^1 + \ldots + \zeta^s \) and \( f_1^0 + \zeta^1 + \ldots + \zeta^s \) have an isolated singularity at 0 and the same Milnor number at 0, provided \( k \) is large enough. In particular, by the upper semicontinuity of the Milnor number, this implies that, for all \( t \) sufficiently close to 0, the germ \( f_t + \zeta^1 + \ldots + \zeta^s \) has an isolated singularity at 0 and the same Milnor number, at 0, as \( f_0 + \zeta^1 + \ldots + \zeta^s \). In other words, the family \((f_t + \zeta^1 + \ldots + \zeta^s)_t \) is a \( \mu \)-constant family of isolated singularities. This implies, in particular, that \( g_t + \zeta^1 + \ldots + \zeta^s \), where, if \( 1 \leq p \leq s \), the term \( \zeta^p \) is omitted, has an isolated singularity at 0 for all small \( t \). Hence, as in [5, Proof of Proposition 3.2], by applying [5, Lemma 3.1] to the family \((f_t + \zeta^1 + \ldots + \zeta^s)_t \) with the hyperplane in \( \mathbb{C}^n \) defined by \( \zeta_t = 0 \), one gets that \( (g_t + \zeta^1 + \ldots + \zeta^s)_t \), where again, if \( 1 \leq p \leq s \), the term \( \zeta^p \) is omitted, is also a \( \mu \)-constant family of isolated singularities. Now, according to the case (1) or (3) that we consider, it follows from our hypotheses that, if the \( j_i \)'s are chosen sufficiently large, then for all \( t \) sufficiently close to 0, the germ \( g_t + \zeta^1 + \ldots + \zeta^s \) (\( \zeta^p \) omitted) is convenient and has a nondegenerate Newton principal part with respect to the coordinates \( \zeta_t \) (case (1)) or \( g_0 + \zeta^1 + \ldots + \zeta^s \) (\( \zeta^p \) omitted) is the germ of a semi-quasihomogeneous polynomial with respect to \( \zeta_t \) (case (3)). Since the \( j_i \)'s can be chosen arbitrarily large, Theorem 0.2 then follows from the results of Abderrahmane [1] and Saia–Tomazella [15] (case (1)), Greuel [4] and Trotman [19, 20] (case (2)), Greuel [4] and O’Shea [13] (case (3)), and Zariski [22] (case (4)).

Remark 0.4. If one replaces the word semi-quasihomogeneous by quasihomogeneous in Theorem 0.2 Part (3), the argument above does not work. Indeed, in this case, \( g_0 + \zeta^1 + \ldots + \zeta^s \) (\( \zeta^p \) omitted) is neither quasihomogeneous with an isolated singularity nor semi-quasihomogeneous, so that we cannot apply the result of Greuel [4] and O’Shea [13] (we recall that

2According to [9], since we are using the coordinates \((\zeta_1, \ldots, \zeta_n)\) for the germ \( f_t \), we use the coordinates \( \zeta = (\zeta_1, \ldots, \zeta_n, \zeta_{i+1}, \ldots, \zeta_{i+n}) \) for the germ \( f_t + \zeta^1 + \ldots + \zeta^s \).

3For the germ \( g_t \), we use the coordinates \( \zeta = (\zeta_1, \ldots, \zeta_n, \zeta_{i+1}, \ldots, \zeta_{i+n}) \), where \( \zeta_t \) is omitted. For the germ \( g_t + \zeta^1 + \ldots + \zeta^s \), where, if \( 1 \leq p \leq s \), the term \( \zeta^p \) is omitted, we use the coordinates \( \zeta = (\zeta_1, \zeta_{i+1}, \ldots, \zeta_{i+n}, \zeta_{i+1}, \ldots, \zeta_{i+n}) \), where \( \zeta_t \) is omitted.
a quasihomogeneous polynomial is not semiquasihomogeneous if it has a nonisolated critical point at 0). By contrast, one can replace semiquasihomogeneous by quasihomogeneous in Theorem 0.1. Indeed, the hypothesis for the $f_t$'s of having an isolated critical point at 0 automatically implies a similar property for the $g_t$'s and, consequently, if $g_0$ is quasihomogeneous, then it is necessarily semiquasihomogeneous too. This shows that Theorem 0.2 is not an immediate consequence of Theorems 0.3 and 0.1. Note that one can replace semiquasihomogeneous by quasihomogeneous with an isolated singularity in Theorem 0.2 Part (3)

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