On simple ideal hyperbolic Coxeter polytopes

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Introduction

Let $\mathbb{H}^n$ be the $n$-dimensional hyperbolic space and let $P$ be a simple polytope in $\mathbb{H}^n$. $P$ is called an ideal polytope if all vertices of $P$ belong to the boundary of $\mathbb{H}^n$. $P$ is called a Coxeter polytope if all dihedral angles of $P$ are submultiples of $\pi$.

There is no complete classification of hyperbolic Coxeter polytopes. In [6] Vinberg proved that there are no compact hyperbolic Coxeter polytopes in $\mathbb{H}^n$ when $n \geq 30$. Prokhorov [5] and Khovanskij [3] proved that there are no Coxeter polytopes of finite volume in $\mathbb{H}^n$ for $n \geq 996$. Examples of bounded Coxeter polytopes are known only for $n \leq 8$, and examples of finite volume non-compact Coxeter polytopes are known only for $n \leq 19$ [8] and $n = 21$ [1].

In this paper, we prove that no simple ideal Coxeter polytope exists in $\mathbb{H}^n$ when $n > 8$.

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1 Preliminaries

1.1 Coxeter diagrams

It is convenient to describe Coxeter polytopes in terms of Coxeter diagrams.

A Coxeter diagram is one-dimensional simplicial complex with weighted edges, where weights are either of the type $\cos \frac{\pi}{m}$ for some integer $m \geq 3$ or positive real numbers no less than one. We can draw edges of Coxeter diagram by the following way:

- if the weight equals $\cos \frac{\pi}{m}$ then the nodes are joined by either $(m - 2)$-fold edge or simple edge labeled by $m$;
- if the weight equals one then the nodes are joined by a bold edge;
- if the weight is greater than one then the nodes are joined by a dotted edge labeled by its weight.

A subdiagram of Coxeter diagram is a subcomplex that can be obtained by deleting several nodes and all edges that are incident to these nodes.

Let $\Sigma$ be a diagram with $d$ nodes $u_1, \ldots, u_d$. Define a symmetrical $d \times d$ matrix $Gr(\Sigma)$ by the following way: $g_{ii} = 1$; if two nodes $u_i$ and $u_j$ are adjacent then $g_{ij}$ equals negative weight of the edge $u_i u_j$; if two nodes $u_i$ and $u_j$ are not adjacent then $g_{ij}$ equals zero.
A Coxeter diagram \( \Sigma(P) \) of Coxeter polytope \( P \) is a Coxeter diagram whose matrix \( Gr(\Sigma) \) coincides with Gram matrix of \( P \). In other words, nodes of Coxeter diagram correspond to facets of \( P \). Two nodes are joined by either \((m-2)\)-fold edge or \( m \)-labeled edge if the corresponding dihedral angle equals \( \frac{\pi}{m} \). If the corresponding facets are parallel the nodes are joined by a bold edge, and if they diverge then the nodes are joined by a dotted edge.

By signature and rank of diagram \( \Sigma \) we mean the signature and the rank of the matrix \( Gr(\Sigma) \).

A Coxeter diagram \( \Sigma \) is called elliptic if the matrix \( Gr(\Sigma) \) is positively defined. A connected Coxeter diagram \( \Sigma \) is called parabolic if the matrix \( Gr(\Sigma) \) is degenerated, and any subdiagram of \( \Sigma \) is elliptic. Elliptic and connected parabolic diagrams are exactly Coxeter diagrams of spherical and Euclidean Coxeter simplices respectively. They were classified by Coxeter [2]. The complete list of elliptic and connected parabolic diagrams is represented in Table 1.

A non-connected diagram is called parabolic if it is a disjoint union of connected parabolic diagrams. A diagram is called indefinite if it contains at least one connected component that is neither elliptic nor parabolic.

Let \( f \) be a \( k \)-dimensional face of \( P \) (by abuse of notation we write \( f \) is a \( k \)-face of \( P \)). If \( P \) is a simple \( n \)-dimensional polytope then \( \alpha \) is an intersection of exactly \( n-k \) facets. Let \( f_1, \ldots, f_{n-k} \) be the facets containing \( f \) and let \( v_1, \ldots, v_{n-k} \) be the corresponding nodes of \( \Sigma(P) \). Let \( \Sigma_f \) be a subdiagram of \( \Sigma(P) \) with nodes \( v_1, \ldots, v_{n-k} \). We say that \( \Sigma_f \) is the diagram of the face \( f \).

The following properties of \( \Sigma(P) \) and \( \Sigma_f \) are proved in [7].

- [cor. of Th. 2.1] the signature of \( Gr(\Sigma(P)) \) equals \((n,1)\);
- [cor. of Th. 3.1] if a \( k \)-face \( f \) is not an ideal vertex of \( P \) (i.e. \( f \) is not a point at the boundary of \( \mathbb{H}^n \)), then \( \Sigma_f \) is an elliptic diagram of rank \( n-k \);
- [cor. of Th. 3.2] if \( f \) is an ideal vertex of \( P \) then \( \Sigma_f \) is a parabolic diagram of rank \( n-1 \); if \( f \) is a simple ideal vertex of \( P \) then \( \Sigma_f \) is connected;
- [cor. of Th. 3.1 and Th. 3.2] any elliptic subdiagram of \( \Sigma(P) \) corresponds to a face of \( P \); any parabolic subdiagram of \( \Sigma(P) \) is a subdiagram of the diagram of a unique ideal vertex of \( P \).

As a corollary, for simple ideal Coxeter polytope \( P \subset \mathbb{H}^n \) we obtain:

(i) Any two non-intersecting indefinite subdiagrams of \( \Sigma(P) \) are joined in \( \Sigma(P) \).

(ii) Any elliptic subdiagram of \( \Sigma(P) \) contains less than \( n \) nodes;

(iii) Any parabolic subdiagram of \( \Sigma(P) \) is connected and contains exactly \( n \) nodes;
Table 1: Connected elliptic and parabolic Coxeter diagrams are listed in left and right columns respectively.

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_n (n \geq 1) )</td>
<td>... ( \cdots ) ...</td>
</tr>
<tr>
<td>( \tilde{A}_1 )</td>
<td>( \tilde{A}_n (n \geq 2) )</td>
</tr>
<tr>
<td>( B_n = C_n ) (( n \geq 2 ))</td>
<td>( \tilde{B}_n (n \geq 3) )</td>
</tr>
<tr>
<td>( D_n (n \geq 4) )</td>
<td>... ( \cdots ) ...</td>
</tr>
<tr>
<td>( \tilde{D}_n (n \geq 4) )</td>
<td>( \tilde{D}_n (n \geq 4) )</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( \tilde{G}_2 )</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>( \tilde{F}_4 )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>( \tilde{E}_6 )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>( \tilde{E}_7 )</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>( \tilde{E}_8 )</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>( \tilde{H}_3 )</td>
</tr>
<tr>
<td>( H_4 )</td>
<td>( \tilde{H}_4 )</td>
</tr>
</tbody>
</table>

Note, that a connected parabolic diagram with more than 3 nodes contains neither bold nor \( k \)-fold edges for \( k > 2 \). Hence, a Coxeter diagram of simple ideal Coxeter polytope in \( BH^n \), \( n > 3 \), contains only simple edges, 2-fold edges and dotted edges.
Notation

Let \( P \) be an \( n \)-dimensional polytope. Denote by \( \alpha_i \), \( i = 0, 1, \ldots, n-1 \), the number of \( i \)-faces of \( P \). For a face \( f \) of \( P \), denote by \( \alpha_i^f \) the number of \( i \)-faces of \( f \) (e.g., \( \alpha_i = \alpha_i^P \)). Denote by

\[
\alpha_k^{(i)} = \frac{1}{\alpha_k} \sum_{\dim f = k} \alpha_i^f
\]

the average number of \( i \)-faces of a \( k \)-face of \( P \).

Proposition 1 (Nikulin [4]). For every simple convex bounded polytope \( P \) in \( \mathbb{R}^n \) for \( i < k \leq \lfloor n/2 \rfloor \) the following estimate holds:

\[
\alpha_k^{(i)} \leq \binom{n-i}{n-k} \frac{\left\lfloor n/2 \right\rfloor^i}{\left\lfloor k/2 \right\rfloor^i},
\]

Using this theorem for 2-faces (\( i = 0 \) and \( k = 2 \)), Vinberg proved that no compact Coxeter polytope exists in \( \mathbb{H}^n \), \( n \geq 30 \).

In [3], Khovanskij proved that Nikulin’s estimate holds for edge-simple polytopes (a polytope is called edge-simple if any edge is the intersection of exactly \( n-1 \) facets). This was used by Prokhorov [5] when he proved that no Coxeter polytope of finite volume exists in \( \mathbb{H}^n \) for \( n \geq 996 \).

In this paper, we study simple ideal hyperbolic Coxeter polytopes. Any hyperbolic Coxeter polytope of finite volume is edge-simple (see [3]). Thus, we can use Nikulin’s estimate. We consider the combinatorics of Coxeter diagrams of simple ideal hyperbolic Coxeter polytopes and prove that such a polytope has no triangular 2-faces and that the number of quadrilateral 2-faces of such a polytope is relatively small. This falls into a contradiction with Nikulin’s estimate in dimensions greater than 8.
2 Absence of triangular 2-faces 
and estimate for quadrilateral 2-faces.

Let $P$ be a simple ideal Coxeter polytope in $\mathbb{H}^n$ and let $V$ be a vertex of $P$. Since $P$ is simple, the vertex $V$ is contained in exactly $n$ edges $VV_i$, $i = 1, \ldots, n$. Denote by $v_i$ the node of $\Sigma V$ such that $\Sigma VV_i = \Sigma V\setminus \{v_i\}$. Denote by $u_i$ the node of $\Sigma(P)$ such that $\Sigma V_i = <u_i, \Sigma VV_i>$.

Now, starting from the diagram $\Sigma V$, we want to describe all possible diagrams $<v_i, u_i, \Sigma VV_i>$. For example, suppose that $\Sigma V = \tilde{A}_{n-1}$, $n \neq 3, 8, 9$. Then $\Sigma VV_i = \Sigma V\setminus v_i = A_{n-1}$. It is easy to see, that if $n \neq 3, 8, 9$ then $A_{n-1}$ is the only parabolic diagram with $n$ nodes containing a subdiagram $A_{n-1}$. Thus, $\Sigma V_i = A_{n-1}$. Note, that $|v_i, u_i| \neq 0$ and $|v_i, u_i| \neq 1$, otherwise $<v_i, u_i, \Sigma VV_i>$ does not satisfy property (iii). Hence, either $|v_i, u_i| = 2$ or $|v_i, u_i| = \infty$, and the subdiagram $<v_i, u_i, \Sigma VV_i>$ is one of two diagrams shown in Figure 1.

![Figure 1: Two possibilities for $<v_i, u_i, \Sigma VV_i>$, if $\Sigma V = \tilde{A}_{n-1}$, $n \neq 3, 8, 9.$](image)

Similarly, one can list all possible diagrams $<u_i, v_i, \Sigma VV_i>$ for any other type of $\Sigma V$. Recall that $\Sigma V$ is one of the diagrams shown in the right column of Table 1. A case-by-case check using properties (i)–(iii) shows the following:

**Lemma 1.** Suppose that $n > 5$. In the notation above $|v_i, u_i| \neq 0$. If $|v_i, u_i| = 1$ then, up to interchange of $v_i$ and $u_i$, the diagram $<v_i, u_i, \Sigma VV_i>$ coincides with one of the diagrams shown in Figure 2.

![Figure 2: Two possibilities for $<v_i, u_i, \Sigma VV_i>$ when $|v_i, u_i| = 1.$](image)

A node $v$ of a diagram $\Sigma$ is called a leaf of $\Sigma$ if $\Sigma$ contains exactly one node joined with $v$. 

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Lemma 2. Assume that $n > 3$. Then for the diagram $< v_i, u_i, \Sigma_{V,W}>$ the following property holds: if $v_i$ is a leaf of $\Sigma_V$ and $u_i$ is not a leaf of $\Sigma_W$ then $\Sigma_V = \overline{E}_k$, $\Sigma_W = \overline{A}_k$, where $k = 7$ or $8$. In this case $< v_i, u_i, \Sigma_{V,W}>$ is one of the diagrams shown in Figure 3.

Proof. Consider the subdiagram $\Sigma_{V,W}$. Since $\Sigma_V = < \Sigma_{V,W}, v_i>$ and $v_i$ is a leaf of $\Sigma_V$, $\Sigma_{V,W}$ is connected. Since $u_i$ is not a leaf of $\Sigma_W = < \Sigma_{V,W}, u_i>$, there are at least two edges joining $u_i$ with $\Sigma_{V,W}$. Hence, $\Sigma_W$ contains a cycle. Combined with (iii), this implies that $\Sigma_W = A_k$. Hence, $\Sigma_{V,W} = A_k$. The only parabolic diagrams with $k + 1$ nodes containing a subdiagram $A_k$ are $A_k$, $G_2$, $E_7$ and $E_8$.

Since $n > 3$ and $\Sigma_V$ has at least one leaf $v_i$, $\Sigma_V = \overline{E}_7$ or $\overline{E}_8$.

We are left to show that $[v_i, u_i] = 2$ or $[v_i, u_i] = \infty$. This follows from Lemma 1.

\[ \text{Figure 3: Possibilities for } < v_i, u_i, \Sigma_{V,W}> \text{ when } v_i \text{ is a leaf of } \Sigma_V \text{ and } u_i \text{ is not a leaf of } \Sigma_W. \]

Lemma 3. Let $P$ be a simple ideal Coxeter polytope in $\mathbb{H}^n$, $n > 5$. Then $P$ has no triangular 2-faces.

Proof. Suppose that $UVW$ is a triangular 2-face of $P$. Then there are exactly $n + 1$ facets of $P$ containing at least one of the points $U$, $V$ and $W$. The whole triangle $UVW$ is contained in exactly $n - 2$ of these facets. Since $P$ is simple, for each edge of $UVW$ there exists a unique facet containing the edge and not containing $UVW$. Denote these facets by $\bar{u}, \bar{v}$ and $\bar{w}$ for the edges $VW, UW$ and $UV$ respectively. Denote by $u$, $v$ and $w$ the nodes of $\Sigma(P)$ corresponding to $\bar{u}, \bar{v}$ and $\bar{w}$ respectively. Then $\Sigma_U = < v, w, \Sigma_{U,W}>$, $\Sigma_V = < u, w, \Sigma_{U,V}>$ and $\Sigma_W = < u, v, \Sigma_{U,V,W}>$ (see Figure 4a). In particular, (iii) implies that all these diagrams are parabolic.

Consider the edge of $\Sigma_W$ joining $u$ and $v$. By Lemma 1, either $[u, v] = 1$ or $[u, v] = 2$ or $[u, v] = \infty$.  

}\[ \text{Figure 3: Possibilities for } < v_i, u_i, \Sigma_{V,W}> \text{ when } v_i \text{ is a leaf of } \Sigma_V \text{ and } u_i \text{ is not a leaf of } \Sigma_W. \]
Suppose that \([u, v] = \infty\). Then \(\Sigma_W = \langle u, v, \Sigma_{UW} \rangle\) contains a dotted edge, in contradiction to the fact that \(\Sigma_W\) is parabolic. Thus, \([u, v] \neq \infty\) and, similarly, \([v, w] \neq \infty\) and \([u, w] \neq \infty\).

Suppose that \(u\) is a leaf of \(\Sigma_V\) and \(v\) is not a leaf of \(\Sigma_U\). Then Lemma 2 shows that \(\langle w, \Sigma_W \rangle = \langle u, v, w, \Sigma_{UW} \rangle\) is one of the diagrams shown in Figure 3. No of these diagrams contains a node \(w \neq u, v\), such that \(\langle u, v, \Sigma_{UW} \rangle\) is parabolic. Thus, no of these diagrams corresponds to a triangle, and we may assume that either both \(u\) and \(v\) are the leaves of \(\Sigma_V\) and \(\Sigma_U\) respectively or none of them is.

Suppose that \([u, v] = 2\). It follows from Table 1 and the assumption \(n > 5\) that either \(u\) or \(v\) is a leaf of \(\Sigma_W\). Without loss of generality we can assume that \(u\) is a leaf. Then it is easy to see that we have one of the diagrams shown in Figure 5. Consider the case shown in Figure 5a. Since \([u, w] \neq \infty\), the diagram \(\langle u, w, \Sigma_{UW} \rangle = \Sigma_V\) is elliptic, that is impossible by (ii). Consider the case shown in Figure 5b. If \([u, w] = 1\) then \(\langle u, w, \Sigma_{UW} \rangle = \Sigma_V\) is elliptic, that is impossible. If \([u, w] = 2\) then \(\langle u, v, w \rangle\) is a parabolic diagram with only three nodes in contradiction to (iii).

Suppose that \([u, v] = 1\). By Lemma 1, \(\Sigma_W = \langle u, v, \Sigma_{UW} \rangle\) coincides with one of the diagrams shown in Figure 2 (up to interchange of \(u\) and \(v\)). It is easy to see that \(\Sigma_{UW}\) contains no node \(w \neq u, v\) such that \(\langle u, v, \Sigma_{UW} \setminus w \rangle\) is a parabolic diagram. Note that \(\Sigma_{UV} = \langle w, \Sigma_{UW} \rangle\) and \(\langle u, v, \Sigma_{UW} \setminus w \rangle = \langle u, v, \Sigma_{UW} \rangle = \Sigma_W\). Thus, we have no parabolic diagram \(\Sigma_W\), so \([u, v] \neq 1\).

By Lemma 1, the case \([u, v] = 0\) is also impossible. There are no more possibilities for \([u, v]\). Hence, no diagram \(\Sigma_{UW}\) can be constructed, and \(P\) contains no triangular faces.

Figure 5: Possibilities for the case \([u, v] = 2\).
Note that an ideal Coxeter polytope in $\mathbb{H}^5$ may have a triangular 2-face. For example, the Coxeter diagram shown in Figure 6 determines a 5-dimensional ideal Coxeter simplex. All 2-faces of any simplex are triangles.

**Figure 6:** This diagram determines a 5-dimensional ideal Coxeter simplex.

**Lemma 4.** Let $V$ be a vertex of simple ideal Coxeter polytope $P$ in $\mathbb{H}^n$, $n > 9$. Then $V$ belongs to at most $n + 3$ quadrilateral 2-faces.

**Proof.** Let $q$ be a quadrilateral 2-face with vertices $V, V_i, V_j$ and $V_{ij}$. The 2-face $q$ belongs to $n - 2$ facets, each edge of $q$ belongs to $n - 1$ facets and each vertex belongs to $n$ facets. Denote by $\bar{v}_i, \bar{u}_i, \bar{v}_j$ and $\bar{u}_j$ the facets not containing $q$ and containing the edges $VV_j, V_iV_{ij}, VV_i$ and $V_jV_{ij}$ respectively (see Figure 4b). Denote by $v_i, u_i, v_j$ and $u_j$ the nodes of $\Sigma(P)$ corresponding to the facets $\bar{v}_i, \bar{u}_i, \bar{v}_j$ and $\bar{u}_j$ respectively.

Then $\Sigma_V = \langle v_i, v_j, \Sigma_q \rangle$, $\Sigma_{V_i} = \langle v_j, u_i, \Sigma_q \rangle$, $\Sigma_{V_j} = \langle v_i, u_j, \Sigma_q \rangle$, and $\Sigma_{V_{ij}} = \langle u_i, u_j, \Sigma_q \rangle$. See Figure 7 for an example of a quadrilateral.

**Figure 7:** Example of a quadrilateral

Suppose that $\Sigma_V = \tilde{A}_{n-1}$ and $v_i$ and $v_j$ are disjoint in $\Sigma_V$. Since $n > 8$, each of the vertices $V_i, V_j, V_{ij}$ are of the type $\tilde{A}_{n-1}$. Consider $\langle v_i, u_i, \Sigma_{V_{V_i}} \rangle$. 

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By (iii), either \(|v_i, u_i| = \infty\) or \(|v_i, u_i| = 2\) (cf. Figure 1). The same statement holds for \(|v_j, u_j|\). Since \(\Sigma_{V_{ij}}\) is a parabolic diagram \(\tilde{A}_{n-1}\), we have \(|u_i, u_j| = 0\) (see Figure 8). Then \(\Sigma(P)\) contains two disjoint indefinite subdiagrams \(v_i u_i w_i\) and \(v_j u_j w_j\). Thus, any quadrilateral containing \(V\) corresponds to a pair of neighbouring nodes of \(V\), and \(V\) belongs to at most \(n\) quadrilaterals.

![Figure 8](image)

**Figure 8:** \(v_i u_i w_i\) and \(v_j u_j w_j\) are disjoint indefinite subdiagrams. In this diagram, \(k_i, k_j = 2\) or \(\infty\).

From now on we assume that \(\Sigma_V \neq \tilde{A}_{n-1}\). Since \(n > 9\), \(\Sigma_V = \tilde{B}_{n-1}, \tilde{C}_{n-1}\) or \(\tilde{D}_{n-1}\). Define a distance \(\rho(u, w)\) between two nodes \(u\) and \(w\) of connected graph as the number of edges in the shortest path connecting \(u\) and \(w\).

Let \(x\) be a leaf of \(\Sigma_V\). Denote by \(\Sigma_{V}^{(5)}(x)\) a connected subdiagram of \(\Sigma_V\) spanned by five nodes closest to \(x\) in \(\Sigma_V\) (i.e., if \(v_k \in \Sigma_{V}^{(5)}(x)\) and \(v_l \notin \Sigma_{V}^{(5)}(x)\) then \(\rho(x, v_k) \leq \rho(x, v_l)\)). Note that for \(\Sigma_V = \tilde{B}_{n-1}, \tilde{C}_{n-1}\) and \(\tilde{D}_{n-1}\) when \(n \geq 9\) the diagram \(\Sigma_{V}^{(5)}(x)\) is well-defined for any leaf \(x\) of \(\Sigma_V\).

Denote by \(L(\Sigma_V)\) the set of leaves of \(\Sigma_V\). Define

\[
\Sigma_{V}^{(5)} = \bigcup_{x_i \in L(\Sigma_V)} \Sigma_{V}^{(5)}(x_i)
\]

(see Fig. 9). It is easy to see that if \(n > 10\) then \(\Sigma_{V}^{(5)}\) consists of two connected components.

![Figure 9](image)

**Figure 9:** Subdiagram \(\Sigma_{V}^{(5)}\) for \(\Sigma_V = \tilde{B}_{12}\).

Suppose that \(v_i\) and \(v_j\) do not belong to the same connected component of \(\Sigma_{V}^{(5)}\) (\(v_i\) or \(v_j\) may lie in \(\Sigma_V \setminus \Sigma_{V}^{(5)}\)). By the same reason as in the case \(\Sigma_V = \tilde{A}_{n-1}\), nodes \(v_i\) and \(v_j\) are neighbours in \(\Sigma_V\).

Suppose that \(v_i\) and \(v_j\) belong to the same connected component of \(\Sigma_{V}^{(5)}\). Suppose that \(v_i\) and \(v_j\) are disjoint. A straightforward check of possibilities with
use of properties (i)–(iii) shows that if $\Sigma_q$ is the diagram of a quadrilateral 2-face, then the connected component of $\Sigma_V^{(5)}$ is one of the following configurations (up to interchange of $v_i$ and $v_j$):

![Diagram of quadrilateral 2-faces]

Hence, the quadrilaterals containing $V$ are encoded either by one of $n - 1$ pair of neighbouring nodes of $\Sigma_V$ or by one of two pairs of nodes for each of two connected components of $\Sigma_V^{(5)}$. Thus, the number of quadrilaterals containing $A$ is less than or equal to $2 + 2 + (n - 1) = n + 3$.

\[\]

**Lemma 5.** Let $A$ be a vertex of a simple ideal Coxeter polytope $P$ in $\text{H}^9$. Then $A$ belongs to at most 15 quadrilateral 2-faces.

**Proof.** The existence of $\Sigma_V = \tilde{E}_8$ course a lot of possibilities for the diagram $<v, v_i, \Sigma_{Vv_i}>$. This leads to a large number of different diagrams $<v, v_i, \Sigma_q>$. To observe all these possibilities we use a case-by-case check organized as follows:

1. **Step 1.** We consider the cases $\Sigma_V = \tilde{A}_8, \tilde{B}_8, \tilde{C}_8, \tilde{D}_8$ and $\tilde{E}_8$ separately.

2. **Step 2.** For each node $v_i$, $i = 1, \ldots, 9$, of $\Sigma_V$ we list all possible diagrams $<v_i, u_i, \Sigma_{Vv_i}>$ such that $<u_i, \Sigma_{Vv_i}>$ is parabolic and $<v_i, u_i, \Sigma_{Vv_i}>$ satisfies properties (i)–(iii). We call such a diagram $<v_i, u_i, \Sigma_{Vv_i}>$ an edge-pattern. Clearly, any edge incident to $V$ corresponds to some edge-pattern $<v_i, u_i, \Sigma_{Vv_i}>$.

Some nodes $v_i$ of $\Sigma_V$ may admit several edge-patterns $<v_i, u_i, \Sigma_{Vv_i}>$ (up to 8 edge-patterns for one of the nodes of $\tilde{E}_8$). Denote the edge-patterns by $(v_i, u_i)_r$, $r = 1, \ldots, k_i$, where $k_i$ is the number of patterns for the node $v_i$ of $\Sigma_V$.

3. **Step 3.** For each edge-pattern $(v_i, u_i)_r$ we consider all edge-patterns $(v_j, u_j)_s$, $j \neq i$. We list all cases when $v_i, v_j, u_i, u_j$ correspond to the facets of some quadrilateral 2-face $q$ (where $\Sigma_q = \Sigma_V \setminus \{v_i, v_j\}$).

4. **Step 4.** For each node $v_i$, $1 \leq i \leq 9$, choose an edge-pattern $(v_i, u_i)_{r_i}$. Then compute the total number $Q(r_1, \ldots, r_9)$ of quadrilaterals determined by $(v_i, u_i)_{r_i}$ and $(v_j, u_j)_{r_j}$ for $1 \leq i < j \leq 9$.

5. **Step 5.** Denote by $Q(\Sigma_V)$ the maximal value of $Q(r_1, \ldots, r_9)$ for all $r_1, \ldots, r_9$. It turns out that

\[
Q(\tilde{A}_8) = 15, \\
Q(\tilde{B}_8) = 14, \\
Q(\tilde{C}_8) = 12,
\]
Thus, for any type of $\Sigma_V$ we obtain that $V$ belongs to at most 15 quadrilateral 2-facets.

Remark. At step 4 of the algorithm above one should check a huge number of possibilities (more than 15000 cases for $E_8$). This was done by computer.

3 Absence of simple ideal Coxeter polytopes in large dimensions.

Recall that $\alpha_i$ denotes the number of $i$-faces of a polytope $P$ and $\alpha_k^{(i)}$ denotes the average number of $i$-faces of $k$-face of $P$.

We will need the following lemma:

**Lemma 6.** Let $P$ be an $n$-dimensional simple polytope and let $l$ be the number of vertices of $P$. Then

$$\frac{l}{\alpha_2} = \frac{2}{n(n-1)} \alpha_2^{(1)}. \quad (1)$$

**Proof.** Denote by $m_i$ the number of $i$-angular 2-faces of $P$. Let us compute the total number $N$ of vertices of 2-faces. Clearly, $N = \sum_{i \geq 3} i \cdot m_i$. From the other hand, each pair of edges incident to one vertex of simple polytope determines a 2-face of the polytope. Thus, $N = \frac{l \cdot n(n-1)}{2}$, and we obtain the following equality

$$\frac{l \cdot n(n-1)}{2} = \sum_{i \geq 3} i \cdot m_i. \quad (2)$$

By definition,

$$\frac{\alpha_2^{(1)}}{\alpha_2} = \frac{\sum_{i \geq 3} i \cdot m_i}{\alpha_2}. \quad (3)$$

Combining (2) and (3), we obtain

$$\frac{l}{\alpha_2} = \frac{2}{n(n-1)} \frac{\sum_{i \geq 3} i \cdot m_i}{\alpha_2} = \frac{2}{n(n-1)} \alpha_2^{(1)}. \quad \square$$

**Theorem 1.** There is no simple ideal Coxeter polytope in $H^n$ for $n \geq 9$. 

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Proof. We use the notation from Lemma 6. Recall, that \( \alpha_2 = \sum_{i \geq 3} m_i \). By Lemma 3, \( m_3 = 0 \). Using (3), we have

\[
\alpha_2^{(1)} \geq \frac{1}{\alpha_2} (4m_4 + 5 \sum_{i \geq 3} m_i) = \frac{1}{\alpha_2} (5 \sum_{i \geq 4} m_i - m_4) = 5 - \frac{m_4}{\alpha_2}. \tag{4}
\]

Consider Nikulin’s estimate for \( \alpha_2^{(1)} \):

\[
\alpha_2^{(1)} < \left( \frac{n-1}{n-2} \left( \begin{array}{c} n/2 \\ 1 \end{array} \right) + \left( \begin{array}{c} (n+1)/2 \\ 1 \end{array} \right) \right) = \frac{n-1 + \varepsilon}{n-2 + \varepsilon}, \tag{5}
\]

where \( \varepsilon = 0 \) if \( n \) is even and \( \varepsilon = 1 \) if \( n \) is odd. Combining (4) with (5), we obtain

\[
5 - \frac{m_4}{\alpha_2} \leq \alpha_2^{(1)} < \frac{n-1 + \varepsilon}{n-2 + \varepsilon}. \tag{6}
\]

Denote by \( l \) the number of vertices of \( P \). Denote by \( N_4 \) the total number of vertices of quadrilateral 2-faces. Clearly, \( N_4 = 4m_4 \). By Lemmas 4 and 5 each of \( l \) vertices is incident to at most \( n + 6 \) quadrilaterals. Thus, \( N_4 \leq l(n + 6) \) and we have \( 4m_4 \leq l(n + 6) \). In view of (1) and (5), we have

\[
\frac{m_4}{\alpha_2} \leq \frac{1}{4} \left( \frac{l(n+6)}{n} \right) = \frac{n+6}{4n(n-1)} \frac{2}{\alpha_2^{(1)}} < \frac{n+6}{4(n-1)} \cdot \frac{4(n-1 + \varepsilon)}{(n-2 + \varepsilon)} = 2 \frac{n+6}{n(n-1)(n-2 + \varepsilon)}. \tag{7}
\]

Combining (6) and (7), we obtain

\[
5 - \frac{4(n-1 + \varepsilon)}{(n-2 + \varepsilon)} < \frac{m_4}{\alpha_2} < 2 \frac{n+6}{n(n-1)(n-2 + \varepsilon)}. \]

This implies

\[
(n-6 + \varepsilon)n(n-1) < 2(n+6)(n-1 + \varepsilon).
\]

This is equivalent to \( n^2 - 8n - 12 < 0 \) if \( n \) is even and to \( n^2 - 8n - 7 < 0 \) if \( n \) is odd. The first inequality has no solutions for \( n \geq 10 \), and the second one has no solutions for \( n \geq 9 \). So, the theorem is proved.

\[
\square
\]

References


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