Lagrangian 3-torus fibrations

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Abstract

We give a method to construct singular Lagrangian 3-torus fibrations over certain a priori given integral affine manifolds with singularities, which we call *simple*. The main result of this article is the proof that Mark Gross' [7] topological Calabi-Yau compactifications can be made into symplectic compactifications. As an example, we obtain a pair of compact symplectic 6-manifolds together with Lagrangian fibrations whose underlying affine structures are dual. The symplectic manifolds obtained are homeomorphic to a smooth quintic Calabi-Yau 3-fold and its mirror.

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1 Introduction.

A map $f : X \to B$ from a smooth symplectic manifold onto a smooth manifold is a Lagrangian fibration if the regular locus of fibres has half the dimension of X and the symplectic form restricts to zero there. The fibration is allowed to have singular fibres. In fact, interesting examples should in general include singular fibres. If the fibration map is smooth and proper, it is a well-known fact that the non-singular fibres are tori. Furthermore, away from the discriminant locus parametrizing the singular fibres, the base has the structure of an integral affine manifold. In other words, B has an atlas whose change of coordinates are integral affine linear transformations.

Lagrangian fibrations lie at the crossroads of integrable systems, toric symplectic geometry and more recently, Mirror Symmetry. For all three subjects, important issues are: the global topology of the fibration, the singularities of the fibres, the regularity of the fibration map and the affine structures induced on the base. In the recent years, integral affine geometry started to play a remarkably important role in Mirror Symmetry. The first evidence of this is given by Hitchin [20], who observed that the SYZ duality [32] can be interpreted as a Legendre transform between integral affine manifolds. Later, Kontsevich and Soibelman [22] and Gross and Wilson [14] proposed a landmark conjecture which, roughly speaking, says:

(1) Degenerating families of Calabi-Yau manifolds approaching large complex structure limits should collapse down to a singular integral affine S^n.

(2) Mirror families should be (re)constructed starting from the affine manifolds in (1).

The first part of this conjecture is referred to as the Gromov-Hausdorff collapse, while the second part is usually called the reconstruction problem [9]. We know that the Gromov-Hausdorff collapse does happen in dimension two [14]. More recently, Gross and Siebert [12, 13] develop a program to reconstruct the "complex side" of the mirror using Logarithmic geometry. Kontsevich and Soibelman [23] approach the complex reconstruction problem using non-Archimedean analytic spaces. The final explanation of Mirror Symmetry is likely to emerge from the work deriving from these two main streams.

On the "symplectic side" of the mirror, there is an analogous reconstruction problem. This paper is motivated by the following question. Can we construct symplectic manifolds starting from integral affine manifolds with singularities and obtain total spaces homeomorphic to mirror pairs of Calabi-Yau manifolds?

To answer this question we take Gross' Topological Mirror Symmetry [7] as a starting point. Gross developed a method to construct topological T^3 fibrations of 6-manifolds. This method consists, roughly, on the compactification of certain T^3 bundles by means of gluing suitable singular fibres. The discriminant locus in this case is a 3-valent graph with vertices labeled positive or negative. There are three types of singular fibres: *generic fibres*, *positive fibres* and *negative fibres*, mapping to either points on the edges, or positive or negative vertices of the graph, respectively. The names are given according to the Euler characteristic of the fibres which can be 0, $+1$ or -1 respectively[1]. Gross' compactification produces a class of fibrations that can be dualized. As an example of this construction, Gross obtained a pair of smooth manifolds with dual topological T^3 fibrations, the first one being homeomorphic to the quintic 3-fold and the second one homeomorphic to a mirror of the quintic.

[1] Gross uses a different convention: $(2,2)$, $(1,2)$ and $(2,1)$, for generic, positive and negative fibres, respectively

The main result of this paper is the proof that a compactification similar to that of Gross can be carried out in the symplectic category. The basic idea is the following. We start with an integral affine manifold with singularities $(B, \Delta, \mathscr{A})$ with 3-valent graph singular locus Δ. The affine structure on $B_0 = B - \Delta$ induces a family of maximal lattices $\Lambda \subseteq T^*B_0$, together with a symplectic manifold $X(B_0)$ and an exact sequence

$$0 \to \Lambda \to T^*B_0 \to X(B_0) \to 0.$$

This gives us a Lagrangian T^n bundle $f_0 : X(B_0) \to B_0$. When $\mathscr{A}$ is *simple* (cf. Definition 3.14), $X(B_0)$ can be compactified to a topological 6-manifold $X(B)$ using Gross method. To define a symplectic structure on $X(B)$, in other words, to achieve a symplectic compactification of $X(B_0)$, one needs *Lagrangian* models of generic, positive and negative singular fibres. The first two models have already been studied by the first author [1]. The construction of a Lagrangian negative model is much more delicate. An important part of this article is devoted to the construction of Lagrangian fibrations of negative type.

While the generic and positive models are given by smooth maps and have codimension two discriminant loci, our model for the negative fibration is piecewise smooth and has mixed codimension one and two discriminant: it is an "amoeba" whose three legs are pinched down to codimension 2 (cf. Figure 5). In fact it can be described as a perturbation of Gross' negative fibration, localized in a small neighborhood of the 'figure eight' (i.e. the singular locus of the negative fibre), which forces the singularities of the fibres to become isolated points and the discriminant locus to jump to codimension one near the vertex. The topology of the total space is unchanged by this perturbation. Joyce [21] had already conjectured that special Lagrangian fibrations should be in general piecewise smooth and should have codimension 1 discriminant locus. Over the codimension 1 part of the discriminant locus, our model has exactly the topology which Joyce proposed as the special Lagrangian version of Gross' negative fibre.

Our first attempt to construct a model of a Lagrangian negative fibration produces a fibration which fails to be smooth along a large codimension one subset, a whole plane containing the discriminant locus (cf. Example 5.8). This model is not suitable for the symplectic compactification. This is essentially due to the fact that piecewise smooth fibrations in general do not induce integral affine structures on the base. The affine structure induced by fibrations of this sort consists of two pieces separated by the codimension one wall. Piecewise smooth fibrations of this type are called *stitched* and have been studied in great detail by the authors [2, 3]. It turns out that the information on the lack of regularity of these fibrations can be encoded into certain invariants. This allows us to have good control on the regularity of stitched fibrations. In particular, we are able to modify Example 5.8 to a Lagrangian fibration which induces an integral affine structure on the complement of a closed 2-disc containing the codimension one component of the discriminant. Moreover, away from this 'bad disc', where the fibration fails to be smooth, the induced integral affine structure is simple.

Given a simple integral affine 3-manifold with singularities $(B, \Delta, \mathscr{A})$ a *localized thickening* of Δ is given by the data $(\Delta^{\blacklozenge}, \{D_{p^-}\}_{p^- \in \mathcal{N}})$ where:

(i) $\Delta^{\blacklozenge}$ is the closed subset obtained from Δ after replacing a neighborhood of each negative vertex with a shape of the type depicted in Figure 17 (an "amoeba" with thin legs).

(ii) $\mathcal{N}$ is the set of negative vertices and for each $p^- \in \mathcal{N}$, D_{p^-} is a disk containing the codimension 1 component of $\Delta^{\blacklozenge}$ around p^- (depicted as the gray area in Figure 17).

Given a localized thickening define

$$B_{\blacklozenge} = B - \left(\Delta \cup \bigcup_{p^- \in \mathcal{N}} D_{p^-} \right).$$

and denote by $\mathscr{A}_\blacklozenge$ the restriction of the affine structure on $B_\blacklozenge$

The main result of this paper is the following (cf. Theorem 8.2):

Theorem. Given a compact simple integral affine 3-manifold with singularities $(B, \Delta, \mathscr{A})$, all of whose negative vertices are straight. There is a localized thickening $(\Delta_\blacklozenge, \{D_{p^-}\}_{p^- \in \mathcal{N}})$ and a smooth, compact symplectic 6-manifold (X, ω) together with a piecewise smooth Lagrangian fibration $f : X \to B$ such that

(i) f is smooth except along $\bigcup_{p^- \in \mathcal{N}} f^{-1}(D_{p^-})$;

(ii) the discriminant locus of f is $\Delta_\blacklozenge$;

(iii) there is a commuting diagram

$$\begin{array}{ccc} X(B_\blacklozenge, \mathscr{A}_\blacklozenge) & \xrightarrow{\Psi} & X \\ {\scriptstyle f_0}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ B_\blacklozenge & \xrightarrow{\iota} & B \end{array}$$

where ψ is a symplectomorphism and ι the inclusion;

(iv) over a neighborhood of a positive vertex of $\Delta_\blacklozenge$ the fibration is positive, over a neighborhood of a point on an edge the fibration is generic-singular, over a neighborhood of D_{p^-} the fibration is Lagrangian negative.

As a corollary of Theorem 8.2 and Gross' topological compactification [7], when $(B, \Delta, \mathscr{A})$ is as in Example 3.17, the symplectic manifold obtained is homeomorphic to the quintic Calabi-Yau 3-fold. Applying the Legendre transform to Example 3.17 produces a compact simple integral affine manifold with singularities $(\check{B}, \check{\Delta}, \check{\mathscr{A}})$ [12]. The latter induces a bundle $X(\check{B}_0)$, dual to $X(B_0)$. By applying the Theorem we obtain a compact symplectic manifold $(\check{X}, \check{\omega})$ homeomorphic to Gross' topological compactification $X(\check{B}_0)$, therefore homeomorphic to a mirror of the quintic.

The affine structures we consider here satisfy a property called *simplicity.* Essentially, our notion of simplicity coincides with Gross and Siebert's simplicity in dimensions $n = 2$ and 3. Theorem 8.2 should produce pairs of compact symplectic manifolds fibering over Gross and Siebert's integral affine manifolds, therefore producing a vast number of examples of dual Lagrangian T^3 fibrations. For example, in [8], Gross shows that to the pairs of Calabi-Yau's constructed with the method of Batyrev and Borisov as complete intersections in dual Fano toric varieties, one can associate a pair of simple affine manifolds with singularities which, when compactified, give back a pair of manifolds homeomorphic to the two Calabi-Yau's. The latter statement is the content of [8]Theorem 0.1, which is proved in [11] by Gross and Siebert. Combining this with our result, we obtain a construction of symplectic manifolds fibred by Lagrangian tori, which are homeomorphic to the Batyrev and Borisov mirror pairs of Calabi-Yau manifolds. Also, another source of examples may come from the structures constructed in [16, 17, 18], provided they are simple.

We should mention at this point that Lagrangian T^3 fibrations of Calabi-Yau manifolds have been constructed before by Ruan [27, 29, 30]. Ruan's construction does not use integral affine geometry, rather, it depends on a gradient flow argument. In particular Ruan's construction depends on the embedding inside an ambient manifold. We suspect that Ruan's fibrations share many similarities with our symplectic compactifications but we haven't been able to verify this. It is not clear what kind of regularity Ruan's fibrations have, therefore whether they induce integral affine structures on the base. One interesting aspect of our method is that it makes explicit connection with the formulation of Mirror Symmetry in [22] and [13], where affine geometry is essential.

The main motivation of this paper is Mirror Symmetry but we expect interesting applications in symplectic topology to emerge from the results we present here. Our construction of Lagrangian fibrations has a flavor similar to the work on almost toric symplectic geometry of Leung and Symington [24]. A theory on almost toric 6-folds could emerge from the methods applied in this article. On the other hand, being our construction so explicitly connected to affine geometry, it is possible that the construction in Theorem 8.2 will eventually shed light onto the new methods in symplectic enumerative problems arising from tropical geometry.

The material of this paper is organized as follows. We start giving in §2 the description of Gross' compactification of topological T^n bundles with semi-stable monodromy. Here we explain how to modify Gross' negative fibration to a fibration with a localized thickening near the negative vertex. In §3 we introduce the integral affine manifolds we use in the rest of the paper. We formalize our notion of simplicity by means of standard models of affine manifolds with singularities with prescribed holonomy. Our notion of simplicity coincides with the one in [12] in dimension $n = 2$ and 3. Simplicity is, essentially, a condition which guarantees that the induced Lagrangian T^n bundles have semi-stable monodromy that can be compactified. We describe some examples of non-compact and compact simple integral affine manifolds with singularities. As an illustration of some of the methods we use, we show in Theorem 3.22 how, in dimension $n = 2$, one can produce symplectic manifolds diffeomorphic to K3 surfaces. In §3 we describe Lagrangian models of positive and generic fibrations and prove that they induce integral affine structures which are simple. These models can be used to produce semi-stable symplectic compactifications over simple affine manifolds *without* negative vertices (cf. Theorem 4.19). This is not enough, in general, to construct symplectic manifolds homeomorphic to Calabi-Yaus –such as a quintic and its mirror– as one should normally include negative vertices. In any event, given the existence of simple affine bases with positive vertices only (or without any vertices at all) Theorem 4.19 tells us how to construct a symplectic manifold together with a Lagrangian fibration over it. In this case, the Lagrangian fibrations obtained are everywhere smooth and the thickening of the discriminant is not necessary. There are explicit Examples of integral affine manifolds structures with no vertices [8] and Theorem 4.19 can be used to produce symplectic compactifications. In §5 we move on to piecewise smooth fibrations. We give concrete examples of piecewise smooth Lagrangian T^3 fibrations. In particular, in Example 5.8 we explicitly construct a Lagrangian version of the topological negative fibration with fat discriminant given in §2. This model is piecewise smooth over a large region. In §6 we review some of the techniques we developed in [2], which allow us to make certain non-smooth Lagrangian fibrations into smoother ones, such as the one in Example 5.8. The material of this section is rather technical and the reader may skip it in a first reading. In §7 we construct Lagrangian fibrations of negative type. These are local models whose discriminant is a localized thickening of a 3-valent negative vertex $p-$. The fibration is smooth away from a 2-disc D_{p-} containing the codimension 1 component of the discriminant. Away from D_{p-}, the affine structure is integral and simple. Finally, in §8 we prove Theorem 8.2.

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2 The topology.

In this section we review Mark Gross' Topological Mirror Symmetry [7], which is the starting point for the results of this paper. Gross developed a method to compactify certain T^n bundles over n-dimensional manifolds to obtain topological models of Calabi-Yau manifolds. We now outline how this method works. Along the way, we discuss how Gross' method can be modified to produce topological fibrations with mixed codimension one and two discriminant locus. We focus in dimension $n = 2$ and 3.

A *topological T^n fibration* $f : X \to B$ is a continuous, proper, surjective map between smooth manifolds, $\dim X = 2n$, $\dim B = n$, such that for a dense open set $B_0 \subseteq B$ and for all $b \in B_0$ the fibre $X_b = f^{-1}(b)$ is homeomorphic to an n-torus. We call the set $\Delta := B - B_0$ the *discriminant locus of f*. Sometimes we will denote a topological fibration by a triple $\mathcal{F} = (X, f, B)$. Notice that this notion of fibration differs from the usual differential geometric one in the sense that here $\mathcal{F}$ is allowed to have singular fibres over points in Δ. Allowing singular fibres is necessary if we aim at obtaining total spaces with interesting topology, such as Calabi-Yau manifolds other than complex tori. When X is a symplectic manifold, with symplectic form ω, a topological T^n-fibration is said to be Lagrangian if ω restricted to the smooth part of every fibre vanishes.

Definition 2.1. Let $\mathcal{F} = (X, f, B)$ and $\mathcal{F}' = (X', f', B')$ be a pair of topological fibrations with discriminant loci Δ and Δ' respectively. We define the following notions of *conjugacy* between $\mathcal{F}$ and $\mathcal{F}'$:

(i) We say that $\mathcal{F}$ is *conjugate* to $\mathcal{F}'$ if there exist a homeomorphism $\psi : X \to X'$ and a homeomorphism $\phi : B \to B'$ sending Δ to Δ' homeomorphically, such that $f' \circ \psi = \phi \circ f$. We shall say that $\mathcal{F}$ is (ψ, ϕ)*-conjugate* to $\mathcal{F}'$ whenever the specification is required.

(ii) If in addition X and X' are symplectic manifolds and the fibrations are Lagrangian, we will say that $\mathcal{F}$ is *symplectically conjugate* to $\mathcal{F}'$ if ψ is a C^∞ symplectomorphism and ϕ is a C^∞ diffeomorphism.

(iii) Given points $b \in \Delta$ and $b' \in \Delta'$, we shall say that $\mathcal{F}$ is *(symplectically) conjugate to $\mathcal{F}'$ over* Δ (or over b and b') if there are neighborhoods U and U' of Δ and Δ' (or of b and b') respectively, such that $(f^{-1}(U), f, U)$ is (symplectically) conjugate to $((f')^{-1}(U'), f', U')$.

Part (iii) can also be found in the literature as *semi-global (symplectic) equivalence* as it involves a fibred neighborhood of a fibre but not the total space. When $\mathcal{F}$ carries additional specified data, –e.g. a (Lagrangian) section or a choice of basis of $H_1(X, \mathbb{Z})$– one may also consider a slightly stronger version of (i)-(iii) which requires that the specified data is preserved, e.g. that ϕ sends the section of f to the section of f' and a basis of $H_1(X, \mathbb{Z})$ to a basis of $H_1(X', \mathbb{Z})$. Clearly all three notions define equivalence relations. The corresponding equivalence classes will be called *germs* of fibrations. Throughout this article we will often use conjugation to topologically or symplectically glue together fibred sets in order to obtain larger fibred sets and eventually produce compact (symplectic) manifolds.

Given a topological (or Lagrangian) fibration $\mathcal{F} = (X, f, B)$ and a subset $U \subset B$, we will often use the notation $\mathcal{F}|_U$ to denote the fibration $(f^{-1}(U), f, U)$ and we will refer to it as the restriction of $\mathcal{F}$ to U.

The topological fibrations considered by Gross have everywhere codimension two discriminant. For $n = 2$, Δ is a finite collection of points and the singular fibres are *nodal*. For $n = 3$, Δ is a connected trivalent graph with vertices labeled 'positive' or 'negative'. There are three types of singular fibres in this case: *generic-singular fibres*, i.e. the product of a nodal fibre with S^1; *positive fibres*, i.e. a 3-torus with a 2-cycle collapsed to a point; and *negative fibres*, singular along a 'figure eight'. For a more detailed description of these singular fibres we refer the reader to Examples 2.6, 2.7, 2.8 and 2.10 below or to [7] for further details.

In this article, we will allow Δ to jump dimension, i.e. Δ will include the region $\Delta_a \subseteq \Delta$, which may be regarded as a "fattening" of a graph near negative vertices. We also propose a new model with discriminant locus of type Δ_a (cf. Example 2.9) which is an alternative to Gross' negative fibration and, in some sense, it is a more generic version of it. The idea of using models with codimension one discriminant was first suggested by Joyce [21]§8, based on his knowledge of special Lagrangian singularities. Ruan's Lagrangian fibrations [28, 27, 29, 30] also have codimension one discriminant loci.

Consider the following three closed subsets of $\mathbb{R}^3$:

$$\mathscr{C}_e = \{x_1 = x_2 = 0\},$$

$$\mathscr{C}_d = \{x_1 = x_2 = 0,\, x_3 \le 0\} \cup \{x_1 = x_3 = 0,\, x_2 \le 0\} \cup \{x_1 = 0,\, x_2 = x_3 \ge 0\},$$

$$\mathscr{C}_a = \mathscr{C}_d \cup \left\{ x_1 = 0,\, x_2^2 + x_3^2 \le \frac{1}{2} \right\}.$$

Clearly $\mathscr{C}_d$ is a model of a neighborhood of a vertex in a three valent graph and $\mathscr{C}_a$ can be regarded as a fattening of $\mathscr{C}_d$ around the vertex. We also denote by D^3 the open unit ball in $\mathbb{R}^3$.

In this paper, we consider fibrations satisfying the following topological properties:

Assumption 2.2. Let $\mathcal{F} = (X, f, B)$ be a topological T^n fibration with discriminant locus $\Delta \subseteq B$ and fibre X_b over $b \in B$. We assume that $\mathcal{F}$ satisfies the following conditions:

1. for $n = 2$, Δ is a finite union of points and given a small neighborhood U of a point in Δ, the fibration $\mathcal{F}|_U$ is topologically conjugate to a nodal fibration (see Example 2.6);

2. for $n = 3$, there is a finite covering $\{U_i\}$ of Δ with open subsets of B such that one of the following three possibilities occur (see also Figure 1):

   (a) the pair $(U_i, U_i \cap \Delta)$ is homeomorphic to $(D^3, D^3 \cap \mathscr{C}_d)$ and $\mathcal{F}|_{U_i}$ is topologically conjugate to either a positive or a negative fibration (see Examples 2.10 and 2.8);

   (b) the pair $(U_i, U_i \cap \Delta)$ is homeomorphic to $(D^3, D^3 \cap \mathscr{C}_a)$ and $\mathcal{F}|_{U_i}$ is topologically conjugate to an alternative negative fibration (see Example 2.9);

   (c) the pair $(U_i, U_i \cap \Delta)$ is homeomorphic to $(D^3, D^3 \cap \mathscr{C}_e)$ and $\mathcal{F}|_{U_i}$ is topologically conjugate to a generic-singular fibration (see Example 2.7);

   We denote by Δ_d the set of points in Δ belonging to a U_i satisfying (a), which are the vertices of $U_i \cap \Delta$. We call these points vertices of Δ. We denote by Δ_a the union of the sets $U_i \cap \Delta$, where U_i satisfies (b); we can assume these sets to be pairwise disjoint. A point in Δ admitting open neighborhood U of B such that $(U, U \cap \Delta)$ is homeomorphic to $(D^3, D^3 \cap \mathscr{C}_e)$ is called an *edge point*. We denote by Δ_g the set of edge points.

We denote by Σ the locus formed by the singularities of all the fibres, therefore sometimes Σ will also be denoted by $\mathrm{Crit}(f)$; when f is smooth, $\mathrm{Crit}(f)$ will indeed coincide with the set of critical points of f. We insist, however, that f is not a priori required to be a smooth map. In fact, we will see that, near Δ_a, our fibrations are not smooth. Inspired by tropical geometry, we refer to a connected component of Δ_a as a *3-legged amoeba* (with thin ends). As we will see later when we will introduce affine structures, an important property of Δ_a is that it is locally planar, i.e. each connected component of Δ_a is contained, in some sense, in a 2-plane.

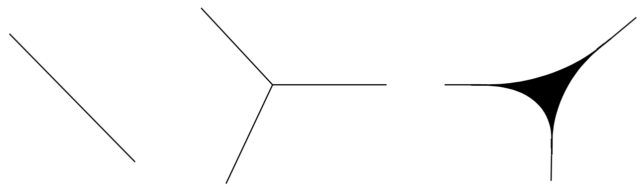


Figure 1: The three possibilities for $U_i \cap \Delta$, $n = 3$.

Definition 2.3. Let $f : X \to B$ be a topological T^n fibration and let $U \subset B$ be an open contractible neighborhood of $b \in \Delta$ such that $U \cap \Delta = \{b\}$, when $n = 2$; or else, when $n = 3$, such that U satisfies (a), (b) or (c) in point 2 of Assumption 2.2. Let X_{b_0} be a fibre over $b_0 \in U - \Delta$. Consider the monodromy representation

$$\mathcal{M}_b : \pi_1(U - \Delta, b_0) \to SL(H_1(X_{b_0}, \mathbb{Z})).$$

The image of $\mathcal{M}_b$ is called the *local monodromy group about* X_b (also denoted by $\mathcal{M}_b$).

Now we review the local models of these fibrations. For the details we refer the reader to [7]§2. The construction of the local models relies on the following:

Proposition 2.4. Let Y be a manifold of dimension $2n - 1$. Let $\Sigma \subseteq Y$ be an oriented submanifold of codimension three and let $Y' = Y - \Sigma$. Let $\pi' : X' \to Y'$ be a principal S^1-bundle over Y' with Chern class $c_1 = \pm 1$. For each triple (Y, Σ, π') there is a unique compactification $X = X' \cup \Sigma$ extending the topology of X', making X into a manifold and such that

$$\begin{array}{ccc} X' & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \hookrightarrow & Y \end{array}$$

commutes, with $\pi : X \to Y$ proper and $\pi|_\Sigma : \Sigma \to \Sigma$ the identity.

Remark 2.5. One can explicitly describe the above compactification as follows. For any point $p \in \Sigma$ there is a neighborhood $U \subset Y$ of p such that $U \cong \mathbb{R}^3 \times \mathbb{C}^{n-2}$ and $U \cap \Sigma$ can be identified with $\{0\} \times \mathbb{C}^{n-2}$. By unicity of π, there is a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\cong} & \mathbb{C}^2 \times \mathbb{C}^{n-2} \\ {\scriptstyle \pi}\Big\downarrow & & {\scriptstyle \bar{\pi}}\Big\downarrow \\ U & \xrightarrow{\cong} & \mathbb{R}^3 \times \mathbb{C}^{n-2} \end{array} \tag{1}$$

where $\bar{\pi}(z_1, z_2, \zeta) = (|z_1|^2 - |z_2|^2, z_1 z_2, \zeta)$, $\zeta \in \mathbb{C}^{n-2}$.

The constructions of topological T^n fibrations in this section are based on the following basic principle. One starts with a manifold $Y = B \times T^{n-1}$ with $\dim B = n$, a submanifold $\Sigma \subset Y$ and a map $\pi : X \to Y$ as in Proposition 2.4. The trivial T^{n-1} fibration $P : Y \to B$ can be lifted to a T^n fibration $f := P \circ \pi : X \to B$ with discriminant locus $\Delta := P(\Sigma)$. One can readily see that for $b \in \Delta$, the singularities of the fibre X_b occur along $\Sigma \cap P^{-1}(b)$. The set Σ –which is the locus of singular fibres of π– can be regarded as the locus where the vanishing cycles of the fibres of f collapse (cf. Figure 2).

Example 2.6 (Nodal fibration). This example is the topological model for the fibration over a point of Δ in the case $n = 2$. Let D be the unit disc in $\mathbb{C}$ and $D^* = D - \{0\}$. Let $f_0 : X_0 \to D^*$ be a T^2-bundle with monodromy generated by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We can use Proposition 2.4 to compactify X_0 as follows. The monodromy invariant cycle, $L \in$

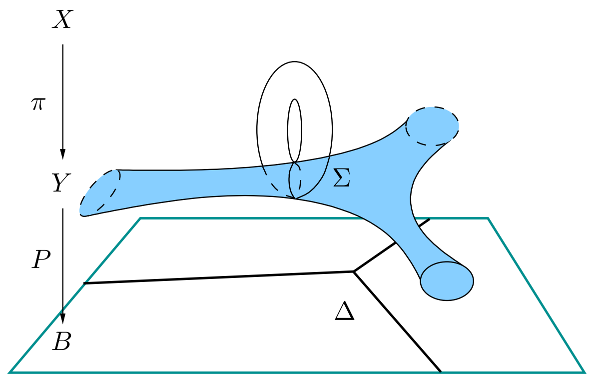


Figure 2: Negative fibration.

$H_1(f_0^{-1}(b), \mathbb{Z})$, induces a fibre preserving $T(L)$ action, with $T(L) = L \otimes \mathbb{R}/L$. The quotient modulo this action yields an S^1-bundle $\pi_0 : X_0 \to Y_0 = D^* \times S^1$. One can verify that π_0 extends to an S^1-bundle $\pi' : X' \to Y' = D \times S^1 - \{(0,p)\}$, where $p \in S^1$. Furthermore $c_1(\pi') = \pm 1$. Then Proposition 2.4 ensures that X' compactifies to a manifold $X = X' \cup \{pt\}$ and that there is a proper map $\pi : X \to Y = D \times S^1$ extending π'. Defining $P : Y \to D$ as the projection map, we obtain a fibration $f = P \circ \pi : X \to D$ extending f_0. The only singular fibre, $f^{-1}(0)$, is homeomorphic to $T^2 = S^1 \times S^1$ after $S^1 \times \{x\} \subset T^2$ is collapsed to x. We denote this fibre by I_1, following Kodaira's notation for singular fibres of elliptic fibrations. In Hamiltonian mechanics, a Lagrangian fibration with this topology is known as a *focus-focus fibration.*

Example 2.7 (Generic singular fibration). This example is the model for the fibration over a neighborhood of an edge point of Δ –in [7] this is called $(2,2)$ fibration. Let $B = D \times (0,1)$, where $D \subset \mathbb{C}$ is the unit disc, and let $Y = T^2 \times B$. Define $\Sigma \subset Y$ to be the cylinder sitting above $\{0\} \times (0,1) \subset B$ defined as follows. Let e_1, e_3 be a basis of $H_1(T^2, \mathbb{Z})$. Let $S^1 \subset T^2$ be a circle representing the homology class e_3. Define $\Sigma = S^1 \times \{0\} \times (0,1)$. Now let $\pi' : X' \to Y' := Y - \Sigma$ be an S^1-bundle with Chern class $c_1 = 1$. Then X' compactifies to a manifold $X = X' \cup \Sigma$ and there is a proper map $\pi : X \to Y$ extending π'. We can now define $f = P \circ \pi : X \to B$ where $P : Y \to B$ is the projection. Then it is clear f is a T^3 fibration with singular fibres homeomorphic to $I_1 \times S^1$ lying over $\Delta := \{0\} \times (0,1)$. If e_2 is an orbit of π, one can take e_1, e_2, e_3 as a basis of $H_1(X_b, \mathbb{Z})$, where X_b is a regular fibre. In this basis, e_2 and e_3 are monodromy invariant and a generator of the monodromy group of f about Δ is represented in this basis by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Example 2.8 (Negative fibration). This example is one of the two models over a neighborhood of a point in Δ_d –in [7] this is called $(2,1)$ fibration. Let $Y = T^2 \times B$ with B homeomorphic to a 3-ball. Let $\Delta \subset B$ be a cone over three distinct, non-collinear points. We write $\Delta = \{b_0\} \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$ where b_0 is the vertex of Δ and the Δ_i are the legs of Δ. Fix a basis e_2, e_3 for $H_1(T^2, \mathbb{Z})$. Define $\Sigma \subset T^2 \times B$ to be a pair of pants lying over Δ such that for $i = 1, 2, 3$, $\Sigma \cap (T^2 \times \Delta_i)$ is a leg of Σ which is the cylinder generated by $-e_3$, $-e_2$ and $e_2 + e_3$ respectively. These legs are glued together along a nodal curve or 'figure eight' lying over b_0. Now consider an S^1-bundle $\pi' : X' \to Y' = Y - \Sigma$ with Chern class $c_1 = 1$. This bundle compactifies to $\pi : X \to Y$. Now consider the projection map $P : Y \to B$. The composition $f = P \circ \pi$ is a proper map. The generic fibre of f is a 3-torus. For $b \in \Delta$ the fibre $f^{-1}(b)$ is singular along $P^{-1}(b) \cap \Sigma$, which is a circle when $b \in \Delta_i$, or the aforementioned figure eight when $b = b_0$. Thus the fibres over Δ_i are homeomorphic to $I_1 \times S^1$, whereas the central fibre, X_{b_0}, is singular along a nodal curve. A regular fibre can be regarded as the total space of an S^1-bundle over $P^{-1}(b)$. We can take as a basis of $H_1(X_b, \mathbb{Z})$, $e_1(b), e_2(b), e_3(b)$, where e_2 and e_3 are the 1-cycles in $P^{-1}(b) = T^2$ as before and e_1 is a fibre of the S^1-bundle. The cycle $e_1(b)$ vanishes as $b \to \Delta$. In this basis, the matrices generating the monodromy group corresponding to loops g_i about Δ_i with $g_1 g_2 g_3 = 1$, (cf. Figure 3) are

$$T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

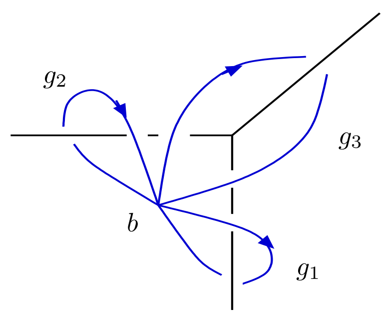


Figure 3: Loops g_1, g_2 and g_3, such that $g_1 g_2 g_3 = 1$.

Example 2.9 (Alternative negative fibration). This is the local model for a fibration over a neighborhood of a component of Δ_a. Consider Y and Σ as in Example 2.8. Now think of making a small perturbation of Σ just in a neighborhood of the "figure eight" –i.e. where the three cylinders forming Σ are joined together– and leaving the rest unchanged. A generic perturbation will be such that, near the fibre over b_0, Σ will intersect the fibres of $P: Y \to B$ in isolated points. Then $P(\Sigma)$ will have the shape of a 3-legged amoeba. One then constructs the bundle $\pi' : X' \to Y' = Y - \Sigma$ with Chern class $c_1 = 1$ and compactifies it to $\pi : X \to Y$. The total fibration is $f = P \circ \pi$.

We can give an explicit construction of a fibration of this type, following ideas in [6]§4. Consider $(\mathbb{C}^*)^2$ with the T^2 fibration $\mathrm{Log} : (v_1, v_2) \mapsto (\log|v_1|, \log|v_2|)$. Let $Y = \mathbb{R} \times (\mathbb{C}^*)^2$ and P be the fibration

$$P : (t, v) \to (t, \mathrm{Log}\, v),$$

where $t \in \mathbb{R}$ and $v = (v_1, v_2) \in (\mathbb{C}^*)^2$. Define a surface Σ' in $(\mathbb{C}^*)^2$ to be

$$\Sigma' = \{v_1 + v_2 + 1 = 0\},$$

and view it as a surface in $\{0\} \times (\mathbb{C}^*)^2 \subset Y$. Clearly $P(\Sigma')$ is $\{0\} \times \mathrm{Log}(\Sigma')$ and one can compute that it has the shape depicted in Figure 4. Images by Log of algebraic curves in $(\mathbb{C}^*)^2$ are known in the literature as amoebas, and this explains the name we gave to the components of Δ_a.

As a surface in $\mathbb{C}^2$, Σ' intersects $\{v_1 = 0\}$ in $q_1 = (0, 0, -1)$ and $\{v_2 = 0\}$ in $q_2 = (0, -1, 0)$. One can see that in a small neighborhood of q_1 one can twist Σ' slightly, so to make it coincide, in a smaller neighborhood, with $\{v_2 = -1\}$. Similarly one can twist Σ' near q_2, so to make it coincide with $\{v_1 = -1\}$. Finally, when $|v_1|$ and $|v_2|$ are both big, we can twist Σ' so to coincide with $\{v_1 + v_2 = 0\}$. Let Σ be this new twisted version of Σ'. A schematic description of these twistings is described in Figure 5, where Σ' is the light-colored diagonal line and Σ is the over-imposed twisted dark line. It is clear that $P(\Sigma) = \{0\} \times \mathrm{Log}(\Sigma)$ will have the shape of a 3-legged amoeba whose legs have

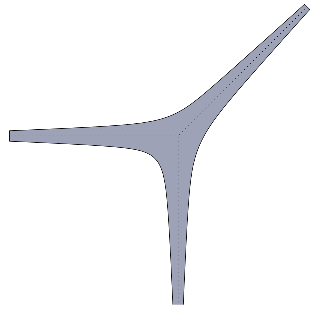


Figure 4: Amoeba of $v_1 + v_2 + 1 = 0$

been pinched to 1-dimensional segments toward the ends, as depicted in the right-hand side of Figure 5 (Mikhalkin [25] also defines a similar construction and calls this shape a localized amoeba). The bundle $\pi' : X' \to Y' = Y - \Sigma$ with Chern class $c_1 = 1$ and its compactification $\pi : X \to Y$ can again be constructed. The fibration is $f = P \circ \pi$ and $\Delta = P(\Sigma)$.

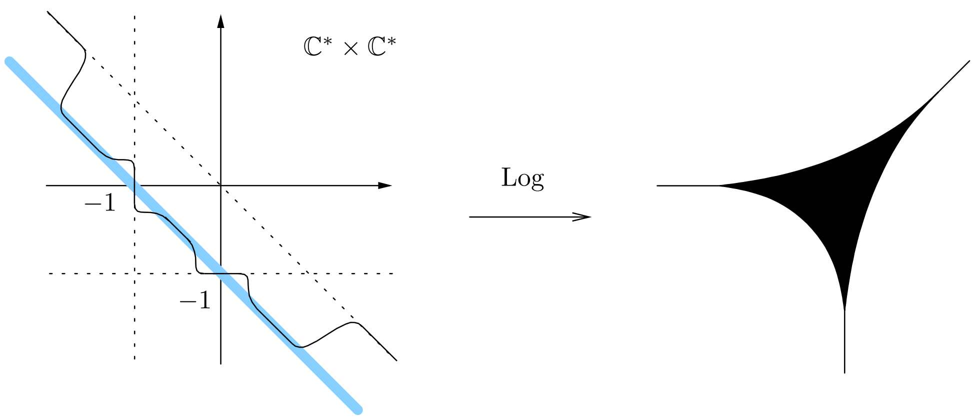


Figure 5: The twisted Σ gives and amoeba with thin legs.

We give a description of the fibration over the codimension 1 part of Δ. One can see that the fibres of Log over a point in the interior of the amoeba intersect Σ in two distinct points. These two points come together to a double point as the base point approaches the boundary of the amoeba. If p_1 and p_2 are two points on T^2 –which may coincide– then the singular fibres of f look like $S^1 \times T^2$ after $S^1 \times \{p_j\}$ is collapsed to a point. This behavior is topologically the same as the one conjectured by Joyce [21] for special Lagrangian T^3 fibrations. Moreover, the singularities of the fibres are modeled on those of an explicit example of a special Lagrangian fibration with non-compact fibres (cf. Joyce [21]§5).

In view of Proposition 2.4 and Remark 2.5, the total space X in this example is diffeomorphic to the one in Example 2.8, although the fibrations differ. In both cases the singularities of the fibres occur along the intersection of the critical surface Σ with the fibres of P. But the intersections happen in a different way. In Example 2.8 they occur either along circles, or along a figure eight. Here they occur along circles when the fibre is over a point in the codimension 2 part of Δ and as isolated points when the fibre is over a point in the codimension 1 part. As argued by Joyce, the isolated singularities are more generic in certain sense (cf. [21]§3). A schematic description of the fibration over the codimension 1 part of Δ is depicted in Figure 6. It can be compared with Figure 2.

We remark that over the codimension 2 part of Δ, the fibration has the same topology of the generic singular fibration of Example 2.7. It follows that the monodromy around the legs is same as the monodromy of Example 2.8, i.e. it is represented by the matrices (3).

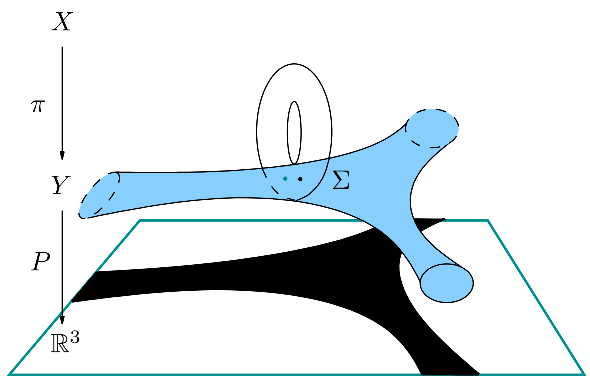


Figure 6: Negative fibration with amoeba-like discriminant.

Most of the effort in this paper is devoted to the construction of a fibration as in the previous example which is also *Lagrangian* with respect to a symplectic form on X. In the process we will also make more explicit the twistings which allow us to deform Σ' into Σ.

Observe that in the above examples there is a fibre-preserving S^1-action, induced by the S^1 bundle π'. One can use the same principle to construct T^2-invariant fibrations starting from suitable compactifications of T^2-bundles:

Example 2.10 (Positive fibration). This model is the other possible fibration over a neighborhood of a point in Δ_d –in [7] this is called $(1,2)$ fibration. Let $Y = S^1 \times B$ with B and $\Delta \subset B$ as in Example 2.8. Let $Y' = Y \setminus (\{p\} \times \Delta)$, where $p \in S^1$. Let $L \cong \mathbb{Z}^2$ and define $T(L) = L \otimes_{\mathbb{Z}} \mathbb{R}/L$. Now consider a principal $T(L)$-bundle $\pi' : X' \to Y'$. Under some mild assumptions on π' (cf. [7] Prop. 2.9), there is a unique manifold X with $X' \subset X$ extending the topology of X' and a proper extension $\pi : X \to Y$ of π'. The composition of π with the projection $Y \to B$ defines a topological T^3-fibration, $f : X \to B$. The fibre of f over $b \in B \setminus \Delta$ is T^3. The fibre over $b \in \Delta_i$ is homeomorphic to $S^1 \times I_1$, whereas the fibre over the vertex $b_0 \in \Delta$ is homeomorphic to $S^1 \times T^2/(\{point\} \times T^2)$. It is proved in [7] that the monodromy group of this model is generated, in some basis, by the inverse transpose of the matrices (3). The reader should not worry, at this point, for the lack of details in this description as we will give explicit Lagrangian models for this example later on.

Notice that the monodromy representation of the above models is *semi-stable*, in other words the monodromy matrices of $\mathcal{M}_b$ are unipotent. This terminology is imported from the classical theory of elliptic fibrations. The topological models described above may be regarded as 3-dimensional topological analogues to semi-stable singular elliptic fibres. We are now ready to state Gross' result. We refer the reader to [7]§2 for the details:

Theorem 2.11 (Gross). Let B be a 3-manifold and let $B_0 \subseteq B$ be a dense open set such that $\Delta := B - B_0$ is a trivalent graph, i.e. such that $\Delta = \Delta_d \cup \Delta_g$. Assume that the vertices of Δ are labeled, i.e. Δ_d decomposes as a union $\Delta_+ \cup \Delta_-$ of positive and negative vertices. Suppose there is a T^3 bundle $f_0 : X(B_0) \to B_0$ such that its local monodromy $\mathcal{M}_b$ is generated by

1. T as in (2), when $b \in \Delta_g$;

2. T_1, T_2, T_3 as in (3), when $b \in \Delta_-$;

3. $(T_1^t)^{-1}, (T_2^t)^{-1}, (T_3^t)^{-1}$, when $b \in \Delta_+$.

Then there is a T^3 fibration $f : X \to B$ and a commutative diagram

$$\begin{array}{ccc} X(B_0) & \hookrightarrow & X \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B. \end{array}$$

Over connected components of Δ_g, (X, f, B) is conjugate to the generic singular fibration, over points of Δ_+ it is conjugate to the positive fibration and over points of Δ_- to the negative fibration.

A topological manifold X obtained from $X(B_0)$ as in Theorem 2.11 is called a *topological semi-stable compactification*. Fibrations arising from semi-stable compactifications satisfy the so-called *topological simplicity* property (cf. [7]§2). This is intimately related to the *affine simplicity* of the subsequent sections. It is due to simplicity that Theorem 2.11 may be used to produce dual T^n fibrations of manifolds homeomorphic to mirror pairs of Calabi-Yau manifolds. In §3 we shall review Gross' construction of a T^3 bundle $X(B_0)$ which compactifies to a smooth manifold X homeomorphic to the quintic hypersurface in $\mathbb{P}^4$. The compactification of the dual bundle produces a manifold, $\check{X}$, which is homeomorphic to the mirror quintic. This construction gives evidence that the SYZ duality should indeed explain Mirror Symmetry.

Theorem 2.11 could be stated and proved, with little effort, replacing Δ_- with Δ_a, i.e. replacing a neighborhood of each negative vertex, with a 3-legged amoeba. Over connected components of Δ_a, the resulting fibration would then be conjugate to the alternative negative fibration of Example 2.9 but the topology of the total space remains the same. In fact we can do more: the main result of this paper is the proof that there exist *symplectic* semi-stable compactifications with respect to which the fibres are Lagrangian. The starting point for this compactifications will be the Lagrangian T^3 bundles obtained from affine 3-dimensional manifolds.

3 Affine manifolds and Lagrangian fibrations

Let us denote by $\mathrm{Aff}(\mathbb{R}) = \mathbb{R}^n \rtimes \mathrm{Gl}(n, \mathbb{R})$ the group of *affine linear* transformations, i.e. elements in $\mathrm{Aff}(\mathbb{R})$ are maps $A : \mathbb{R}^n \to \mathbb{R}^n$, $A(x) = L(x) + v$, where $L \in \mathrm{Gl}(n, \mathbb{R})$ and $v \in \mathbb{R}^n$. The subgroup of $\mathrm{Aff}(\mathbb{R})$ consisting of affine linear transformations with integral linear part will be denoted by:

$$\mathrm{Aff}_{\mathbb{R}}(\mathbb{Z}) = \mathbb{R}^n \rtimes \mathrm{Gl}(n, \mathbb{Z}).$$

Let us denote by

$$\mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^{n'}) = \mathbb{R}^{n'} \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n'})$$

and by

$$\mathrm{Aff}_{\mathbb{R}}(\mathbb{Z}^n, \mathbb{Z}^{n'}) = \mathbb{R}^{n'} \times \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^{n'}).$$

Definition 3.1. Let B be a topological n-dimensional manifold.

(i) An *affine manifold* is a pair $(B, \mathscr{A})$ where B is an n-dimensional manifold and $\mathscr{A}$ is a maximal atlas on B whose transition maps are $\mathrm{Aff}(\mathbb{R})$ transformations. We call $\mathscr{A}$ an *affine structure* on B.

(ii) An affine manifold $(B, \mathscr{A})$ is *integral* if the transition maps of the affine structure $\mathscr{A}$ are $\mathrm{Aff}_{\mathbb{R}}(\mathbb{Z})$ transformations. We call $\mathscr{A}$ an *integral affine structure* on B.

(iii) A continuous map $\alpha : B \to B'$ is *(integral) affine* if on each local coordinate chart, α is an element of $(\mathrm{Aff}_{\mathbb{R}}(\mathbb{Z}^n, \mathbb{Z}^{n'}))$ $\mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^{n'})$. Two (integral) affine manifolds B and B' are said to be *(integral) affine isomorphic* if there is an (integral) affine homeomorphism between them.

It is becoming standard to call an affine manifold as in (ii) *tropical manifold* [9]. Though convenient for various good reasons, this is not a well established terminology at the time this paper is being written, so we prefer to stick to definition (ii) instead. Our convention coincides with that in [23] and [16] and differs from [13]. Affine manifolds whose structure group is $\mathrm{Aff}(\mathbb{Z}) = \mathbb{Z}^n \rtimes \mathrm{Gl}(n, \mathbb{Z})$ will be denoted $\mathrm{Aff}(\mathbb{Z})$*-manifolds* (these are called integral affine in [13]).

Given an affine manifold $(B, \mathscr{A})$, consider a chart $(U, \phi) \in \mathscr{A}$ with affine coordinates $\phi = (u_1, \ldots, u_n)$. The cotangent bundle T^*_B admits a flat connection ∇ defined by

$$\nabla du_j = 0,$$

for all $j = 1, \ldots, n$ and all charts $(U, \phi) \in \mathscr{A}$. When $(B, \mathscr{A})$ is integral affine we can also define a maximal integral lattice $\Lambda \subset T^*_B$ by

$$\Lambda|_U = \mathrm{span}_{\mathbb{Z}}\langle du_1, \ldots, du_n\rangle$$

for all $(U, \phi) \in \mathscr{A}$. Therefore to every integral affine manifold $(B, \mathscr{A})$ we can associate the $2n$-dimensional manifold

$$X(B, \mathscr{A}) = T^*_B/\Lambda,$$

which together with the projection $f : X(B, \mathscr{A}) \to B$ forms a T^n fibre bundle. Also notice that the standard symplectic form ω on T^*_B descends to $X(B, \mathscr{A})$ and the fibres of f are Lagrangian.

The flat connection ∇ on T^*_B of an integral affine manifold $(B, \mathscr{A})$ has a holonomy representation $\rho^* : \pi_1(B, b) \to \mathrm{Gl}(n, \mathbb{Z})$ obtained by parallel transport along closed paths. A choice of basis of Λ_b is identified naturally with a choice of basis of $H_1(f^{-1}(b), \mathbb{Z})$. Under this identification, the holonomy representation ρ^* coincides with the monodromy representation of the bundle $X(B, \mathscr{A}) \to B$. More precisely, if $g \in \pi_1(B, b)$ and $\mathcal{M}_b(g)$ is the corresponding monodromy matrix, then $\mathcal{M}_b(g) = \rho^*(g)$. The integral affine manifold $(B, \mathscr{A})$ also induces a flat connection on T_B whose holonomy representation, ρ, is dual to ρ^*, i.e. the matrix $\rho(g)$ is the inverse transpose of $\rho^*(g)$. In what follows, unless otherwise stated, “holonomy representation” should be understood as the holonomy representation of the aforementioned flat connection on the *cotangent* bundle T^*_B.

It is well known that affine manifolds arise naturally from Lagrangian fibrations. This is the classical theory of action-angle coordinates in Hamiltonian mechanics.

Action-angle coordinates.

We review here some standard facts about Lagrangian fibrations which we will use in the next Sections. For details we refer to Duistermaat [4]. Assume we are given a $2n$-dimensional symplectic manifold X with symplectic form ω, a smooth n-dimensional manifold B and a proper smooth submersion $f : X \to B$ whose fibres are connected Lagrangian submanifolds. For every $b \in B$, denote by F_b the fibre of f at b.

Proposition 3.2 (Arnold-Liouville). In the above situation, for every $b \in B$, $T^*_b B$ acts transitively on F_b. In particular there exists a maximal sub-lattice Λ_b of $T^*_b B$ such that F_b is naturally diffeomorphic to $T^*_b B/\Lambda_b$, therefore F_b is an n-torus.

Proof. To every $\alpha \in T^*_b B$ we can associate a vector field v_α on F_b by

$$\iota_{v_\alpha}\omega = f^*\alpha.$$

Let ϕ_α^t be the flow of v_α with time $t \in \mathbb{R}$. Then we define the action θ_α of α on F_b by

$$\theta_\alpha(p) = \phi_\alpha^1(p),$$

where $p \in F_b$. One can check that such an action is well defined and transitive. Then, Λ_b defined as

$$\Lambda_b = \{\lambda \in T_b^*B \mid \theta_\lambda(p) = p, \text{ for all } p \in F_b\}$$

is a closed discrete subgroup of T_b^*B, i.e. a lattice. From the properness of F_b it follows that Λ_b is maximal (in particular homomorphic to $\mathbb{Z}^n$) and that F_b is diffeomorphic to T_b^*B/Λ_b. $\square$

We denote $\Lambda = \cup_{b\in B}\Lambda_b$. Given the presheaf on B defined by $U \mapsto H_1(f^{-1}(U), \mathbb{Z})$, the associated sheaf is a locally constant sheaf. We can identify it with Λ as follows. Let $U \subseteq B$ be a contractible open set. For every $b \in U$, $H_1(F_b, \mathbb{Z})$ can be naturally identified with $H_1(f^{-1}(U), \mathbb{Z})$. To every $\gamma \in H_1(f^{-1}(U), \mathbb{Z})$, we can associate a 1-form λ on U as follows. For every vector field v on U, if we denote by $\tilde{v}$ a lift, define

$$\lambda(v) = -\int_\gamma \iota_{\tilde{v}}\omega. \tag{4}$$

It turns out that this identifies the above sheaf with $\Lambda \subset T_B^*$. If $\gamma_1, \ldots, \gamma_n$ are a basis for $H_1(F_b, \mathbb{Z})$, then (4) gives us a $\mathbb{Z}$-basis $\lambda_1, \ldots, \lambda_n$ of Λ over a contractible open neighborhood U of b.

In particular, one can read the monodromy of $f : X \to B$ from the monodromy of Λ. We state now the fundamental theorem of smooth proper Lagrangian submersions:

Theorem 3.3 (Duistermaat). Given a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $H_1(F_b, \mathbb{Z})$, then the corresponding 1-forms $\lambda_1, \ldots, \lambda_n$ defined on a contractible open neighborhood U of b are closed and locally generate Λ. In particular, Λ is Lagrangian with respect to the standard symplectic structure in T_B^*. A choice of functions a_j such that $\lambda_j = da_j$ defines coordinates $a = (a_1, \ldots, a_n)$ called *action coordinates*. A covering $\{U_i\}$ of B by contractible open sets and a choice of action coordinates on each U_i defines an integral affine structure $\mathscr{A}$ on B. Moreover, if f has a Lagrangian section $\sigma : U \to X$ over an open set $U \subseteq B$, then there is a natural symplectomorphism

$$\Theta : T_U^*/\Lambda \to f^{-1}(U). \tag{5}$$

If σ is a global section then $X(B, \mathscr{A})$ is symplectically conjugate to X. If in addition the monodromy of Λ is trivial X is symplectically conjugate to $B \times T^n$. The map Θ is called the *period map* or *action-angle* coordinates map.

Proof. We just give a sketch of the proof. Using the Weinstein neighborhood theorem one can show that in a sufficiently small tubular neighborhood of a fibre F_b, the symplectic form is exact, i.e $\omega = -d\eta$ for some 1-form η. Notice that $\eta|_{F_b}$ is a closed 1-form. Define functions a_j on U by

$$a_j = \int_{\gamma_j} \eta.$$

One can show that

$$\lambda_j = da_j$$

and therefore λ_j is closed. It is clear that the coordinates $a = (a_1, ..., a_n)$ are well defined up to an integral affine transformation and therefore they define an integral affine structure on B inducing the lattice Λ in T_B^*. Finally, notice that given a section $\sigma : U \to X$ we have a covering map

$$\begin{array}{ccc} T_U^* & \to & f^{-1}(U) \\ \alpha & \mapsto & \theta_\alpha(\sigma(\pi(\alpha))) \end{array}$$

This map induces a diffeomorphism between T_U^*/Λ and $f^{-1}(U)$. One can check that in the case σ is Lagrangian this map is a symplectomorphism. For the proof of the last statement we refer the reader to [4]. $\square$

Corollary 3.4. Let $\mathcal{F} = (X, f, B)$ and $\mathcal{F}' = (X', f', B')$ be smooth proper Lagrangian fibrations inducing integral affine structures $\mathscr{A}$ and $\mathscr{A}'$ on B and B' respectively. Assume there exist Lagrangian sections σ and σ' of f and f' respectively. Then an integral affine isomorphism ϕ between B and B' induces a symplectic (ψ, ϕ)-conjugation between $\mathcal{F}$ and $\mathcal{F}'$ such that $\psi \circ \sigma = \sigma' \circ \phi$.

Proof. Let $\Lambda \subset T^*_B$ and $\Lambda' \subset T^*_{B'}$ be the lattices induced from the integral affine structures on B and B', respectively. From Theorem 3.3 it follows that X and X' are symplectomorphic to T^*_B/Λ and $T^*_{B'}/\Lambda'$, respectively. Given an integral affine isomorphism ϕ between B and B', clearly ϕ^* is a symplectomorphism between $T^*_{B'}$ and T^*_B inducing an isomorphism between Λ' and Λ. Therefore ϕ^* descends to a symplectomorphism $\tilde{\psi}$ between $T^*_{B'}/\Lambda'$ and T^*_B/Λ. Defining $\psi = \Theta' \circ (\tilde{\psi})^{-1} \circ \Theta^{-1}$ the claim follows. $\square$

The following is an easy but important consequence of Arnold-Liouville-Duistermaat theorem:

Corollary 3.5. Proper Lagrangian submersions do not have semi-global symplectic invariants. In other words, all such fibrations are symplectically conjugate to $U \times T^n$ when restricted to a small enough neighborhood U of a base point.

It is clear that *smoothness* of the fibration map plays a crucial role in the above result. Semi-global invariants do arise for certain piecewise C^∞ Lagrangian fibrations [2]. We say more about this in §6.

Affine manifolds with singularities.

When a Lagrangian fibration has singular fibres, its base is no longer an affine manifold but an affine manifold *with singularities.* These singularities can be a priori rather complicated. The topological properties described in §2 motivate the following:

Definition 3.6. An (integral) affine manifold with singularities is a triple $(B, \Delta, \mathscr{A})$, where B is a topological n-dimensional manifold, $\Delta \subset B$ a set which is locally a finite union of locally closed submanifolds of codimension at least 2 and $\mathscr{A}$ is an (integral) affine structure on $B_0 = B - \Delta$. A continuous map between (integral) affine manifolds with singularities

$$\alpha : B \to B'$$

is (integral) affine if $\alpha^{-1}(B_0') \cap B_0$ is dense in B and the restriction $\alpha_0 = \alpha|_{\alpha^{-1}(B_0')\cap B_0}$:

$$\alpha_0 : \alpha^{-1}(B_0') \cap B_0 \to B_0'$$

is an (integral) affine map. We say that α is an (integral) affine isomorphism if α is an homeomorphism and α_0 is an (integral) affine isomorphism of (integral) affine manifolds.

From now on we restrict to dimension $n = 2$ or 3. Let $(B, \Delta, \mathscr{A})$ be an affine manifold with singularities and let $(B_0, \mathscr{A})$ be the corresponding affine manifold. Let $X(B_0, \mathscr{A})$ be the Lagrangian T^n bundle over B_0 as introduced at the beginning of this section. We shall start imposing conditions on the singularities of the affine structure which, in particular, will imply that $X(B_0, \mathscr{A})$ is of the topological type described in §2, e.g. such that $X(B_0, \mathscr{A})$ will have semi-stable monodromy as in Theorem 2.11.

We start defining local models of integral affine manifolds with singularities. In dimension 2, the allowed behavior is described in the following:

Example 3.7 (The node). We define an affine structure with singularities on $B = \mathbb{R}^2$. Let $\Delta = \{0\}$ and let (x_1, x_2) be the standard coordinates on B. As the covering $\{U_i\}$ of $B_0 = \mathbb{R}^2 - \Delta$ we take the following two sets

$$U_1 = \mathbb{R}^2 - \{x_2 = 0 \text{ and } x_1 \geq 0\},$$

$$U_2 = \mathbb{R}^2 - \{x_2 = 0 \text{ and } x_1 \leq 0\}.$$

Denote by H^+ the set $\{x_2 > 0\}$ and by H^- the set $\{x_2 < 0\}$. Let T be the matrix

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{6}$$

The coordinate maps ϕ_1 and ϕ_2 on U_1 and U_2 are defined as follows

$$\begin{aligned} \phi_1 &= \mathrm{Id} \\ \phi_2 &= \begin{cases} \mathrm{Id} & \text{on } \bar{H}^+ \cap U_2, \\ (T^{-1})^t & \text{on } H^- \end{cases} \end{aligned}$$

The atlas $\mathscr{A} = \{U_i, \phi_i\}_{i=1,2}$ is clearly an affine structure on B_0. It is easy to check that given a point $b \in B_0$, we can chose a basis of $T^*_b B_0$ with respect to which the holonomy representation ρ^* sends the anti-clockwise oriented generator of $\pi_1(B_0)$ to the matrix T.

In dimension 3 we have the following models.

Example 3.8 (The edge). Let $I \subseteq \mathbb{R}$ be an open interval. Consider $B = \mathbb{R}^2 \times I$ and $\Delta = \{0\} \times I$. On $B_0 = (\mathbb{R}^2 - 0) \times I$ we take the product affine structure between the affine structure on $\mathbb{R}^2 - 0$ described in the previous example and the standard affine structure on I.

Example 3.9 (A variation). In the previous example the discriminant locus Δ was a straight line. We can slightly perturb Δ so that it becomes a smooth curve. More precisely, let $B = \mathbb{R}^2 \times I$ as before and consider a smooth function $\tau : I \to \mathbb{R}$. Let

$$\Delta_\tau = \{(\tau(s), 0, s), s \in I\} \subset B$$

and define a covering $\{U_i\}$ of $B_0 = B - \Delta_\tau$ to be

$$U_1 = (\mathbb{R}^2 \times I) - \{(x_1, 0, s) \mid x_1 \geq \tau(s)\},$$

$$U_2 = (\mathbb{R}^2 \times I) - \{(x_1, 0, s) \mid x_1 \leq \tau(s)\}.$$

Now let $H^+ = \{x_2 > 0\}$ and $H^- = \{x_2 < 0\}$. Take the following matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and define maps ϕ_j on U_j to be

$$\begin{aligned} \phi_1 &= \mathrm{Id} \\ \phi_2 &= \begin{cases} \mathrm{Id} & \text{on } \bar{H}^+ \cap U_2, \\ (T^{-1})^t & \text{on } H^-. \end{cases} \end{aligned}$$

Clearly $\mathscr{A} = \{U_i, \phi_i\}_{i=1,2}$ defines an affine structure on $B_0 = B - \Delta_\tau$. When $\tau = 0$, this example coincides with the previous one. Notice that the curve $(\tau(s), 0, s)$ is contained inside the 2-plane $\{x_2 = 0\}$, which can be viewed as an integral surface of the distribution spanned by the vectors in TB_0 which are invariant with respect to the holonomy representation ρ on TB_0. Two different curves give non-isomorphic singular affine structures, unless the curves can be taken one into the other via an integral affine transformation.

Example 3.10 (Positive vertex). Take $B = \mathbb{R} \times \mathbb{R}^2$, with coordinates (x_1, x_2, x_3) and identify $\mathbb{R}^2$ with $\{0\} \times \mathbb{R}^2$. Inside $\mathbb{R}^2$ consider the cone over three points:

$$\Delta = \{x_2 = 0,\, x_3 \leq 0\} \cup \{x_3 = 0,\, x_2 \leq 0\} \cup \{x_2 = x_3,\, x_3 \geq 0\}.$$

Now define closed sets in B

$$\begin{aligned} R &= \mathbb{R} \times \Delta, \\ R^+ &= \mathbb{R}_{\geq 0} \times \Delta, \\ R^- &= \mathbb{R}_{\leq 0} \times \Delta, \end{aligned}$$

and consider the following cover $\{U_i\}$ of $\mathbb{R}^3 - \Delta$:

$$\begin{aligned} U_1 &= \mathbb{R}^3 - R^+, \\ U_2 &= \mathbb{R}^3 - R^-. \end{aligned}$$

It is clear that $U_1 \cap U_2$ has the following three connected components

$$\begin{aligned} V_1 &= \{x_2 < 0,\ x_3 < 0\}, \\ V_2 &= \{x_2 > 0,\ x_2 > x_3\}, \\ V_3 &= \{x_3 > 0,\ x_3 > x_2\}. \end{aligned}$$

Take two matrices

$$T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

Now on U_1, U_2 we define coordinate maps ϕ_1, ϕ_2 as follows

$$\begin{aligned} \phi_1 &= \mathrm{Id}, \\ \phi_2 &= \begin{cases} \mathrm{Id} & \text{on } \bar{V}_1 \cap U_2, \\ T_1^{-1} & \text{on } \bar{V}_2 \cap U_2 \\ T_2 & \text{on } \bar{V}_3 \cap U_2 \end{cases} \end{aligned}$$

Again we see that $\mathscr{A} = \{U_i, \phi_i\}_{i=1,2}$ gives an affine structure on $B_0 = \mathbb{R}^3 - \Delta$. One can compute that given a point $b \in B_0$ and closed paths g_1, g_2 and g_3 as in Figure 3, we can choose a basis of $T_b^* B_0$ with respect to which the holonomy matrices satisfy $\rho^*(g_j) = (T_j^{-1})^t$ for $j = 1, 2, 3$.

Example 3.11 (A variation). In the previous example, Δ was a graph with three edges meeting in one vertex. All three edges were straight lines. In the spirit of Example 3.9 we can perturb each edge of Δ to a smooth curve starting at the vertex. Each straight edge of the previous example is contained in a 2-plane which is an integral plane of the distribution spanned by the vectors which are invariant with respect to the holonomy around that edge. For example, consider the edge $E_1 = \{x_1 = x_2 = 0, x_3 \leq 0\}$ of Δ. Then E_1 is contained inside the half plane, $P_1 = \{x_2 = 0, x_3 \leq 0\}$, whose tangent vectors are T_1 invariant, where $T_1 = \rho(g_1)$ is the holonomy of T_{B_0} with respect to E_1. An analogous thing happens with the other two edges. The union of all three half planes gives R. The new perturbed edges, E_j', must be curves inside the half planes P_j. More precisely, let τ be a function on Δ which is the restriction of a smooth function defined on an open neighborhood of Δ, such that $\tau(0) = 0$. If we let R be as in the previous example, define

$$\begin{aligned} \Delta_\tau &= \{(\tau(q), q) \in \mathbb{R} \times \Delta\} \\ R^+ &= \{(x_1, q) \in \mathbb{R} \times \Delta\ | x_1 \geq \tau(q)\} \\ R^- &= \{(x_1, q) \in \mathbb{R} \times \Delta\ | x_1 \leq \tau(q)\} \end{aligned}$$

Now charts $\mathscr{A} = \{U_i, \phi_i\}_{i=1,2}$ on $B - \Delta_\tau$ can be defined like in the previous example, but with these new definitions of R^+ and R^-. It is clear that $(B, \Delta_\tau, \mathscr{A})$ defines an affine manifold with singularities. Two different choices of functions τ define non-isomorphic integral affine manifolds with singularities, unless their graphs inside R can be mapped one to the other via an integral affine map.

Example 3.12 (Negative vertex). Let B and Δ be as in Example 3.10. Clearly, $\mathbb{R}^2 - \Delta$ has three connected components, which we denote C_1, C_2 and C_3. Let $\bar{C}_j = C_j \cup \partial C_j$. Viewing $\mathbb{R}^2$ embedded in B as $\{0\} \times \mathbb{R}^2$, consider the following three open subsets of B_0:

$$\begin{aligned} U_1 &= \mathbb{R}^3 - (\bar{C}_2 \cup \bar{C}_3), \\ U_2 &= \mathbb{R}^3 - (\bar{C}_1 \cup \bar{C}_3), \\ U_3 &= \mathbb{R}^3 - (\bar{C}_1 \cup \bar{C}_2). \end{aligned}$$

Let

$$\begin{aligned} V^+ &= \{x_1 > 0\}, \\ V^- &= \{x_1 < 0\}. \end{aligned}$$

Clearly $U_i \cap U_j = V^+ \cup V^-$ when $i \neq j$. If T_1 and T_2 are as in (7), define the following coordinate charts on U_1, U_2, U_3 respectively:

$$\begin{aligned} \phi_1 &= \mathrm{Id}, \\ \phi_2 &= \begin{cases} (T_1^{-1})^t & \text{on } \bar{V}^+ \cap U_2 \\ \mathrm{Id} & \text{on } \bar{V}^- \cap U_2 \end{cases} \\ \phi_3 &= \begin{cases} \mathrm{Id} & \text{on } \bar{V}^+ \cap U_3 \\ (T_2^{-1})^t & \text{on } \bar{V}^- \cap U_3 \end{cases} \end{aligned}$$

We can check that the affine structure defined by these charts is such that, for fixed $b \in B_0$, there exists a basis of $T_b^* B_0$ with respect to which the holonomy representation is such that $\rho^*(g_j) = T_j$, where g_j are as in Figure 3. In particular, the holonomy is given by the inverse transpose matrices of the holonomy in the previous example.

Example 3.13 (A variation). Again, we can perturb the above example by replacing the straight edges of Δ with smooth curves starting at the origin. This time these curves have to be contained inside $\{0\} \times \mathbb{R}^2$, which is the integral surface (containing Δ) of the distribution spanned by the ρ-holonomy invariant vectors in TB_0. The perturbed Δ, which we could denote Δ_τ, still separates $\mathbb{R}^2$ in three connected components C_1, C_2 and C_3. Then the definition of the affine structure carries through just like in the previous example and we denote it by $\mathscr{A}_\tau$.

We are now ready to give a definition of the specific affine structures with singularities which we will consider.

Definition 3.14. A 2-dimensional affine manifold with singularities $(B, \Delta, \mathscr{A})$ is said to be *simple* if Δ consists of a finite union of isolated points and a neighborhood of each $p \in \Delta$ is affine isomorphic to a neighborhood of $0 \in \mathbb{R}^2$ as in Example 3.7. We call $p \in \Delta$ a *node*. A 3-dimensional affine manifold with singularities $(B, \Delta, \mathscr{A})$ is *simple* if it satisfies:

(i) Δ is a trivalent graph;

(ii) a neighborhood of each vertex of Δ is affine isomorphic to a neighborhood of $0 \in \mathbb{R}^3$ in either Examples 3.10 or 3.11, in which case we call it a *positive vertex*; or to a neighborhood of $0 \in \mathbb{R}^3$ in either Examples 3.12 or 3.13, in which case we call it a *negative vertex*;

(iii) a neighborhood of each edge of the graph is affine isomorphic to a neighborhood of Δ in Example 3.8; or a neighborhood of Δ_τ in Example 3.9 for a suitable τ.

The following is direct consequence of the above definition and Theorem 2.11:

Corollary 3.15. Let $(B, \Delta, \mathscr{A})$ be a simple affine manifold with singularities and let $(B_0, \mathscr{A})$ be the underlying integral affine manifold. Then

$$f_0 : X(B_0, \mathscr{A}) \to B_0$$

is a T^n bundle with semi-stable monodromy as in Theorem 2.11. In particular, there is an $2n$-manifold X and a topological semi-stable compactification $X(B_0, \mathscr{A}) \hookrightarrow X$. Furthermore, the topological fibration $f : X \to B$ obtained is topologically simple.

Examples

Here we give some examples of affine manifolds with singularities and then we prove the 2-dimensional version of the main theorem of this article.

Example 3.16. In $\mathbb{R}^3$ consider the 3-dimensional simplex Ξ spanned by the points

$$P_0 = (-1, -1, -1), \quad P_1 = (3, -1, -1), \quad P_2 = (-1, 3, -1), \quad P_3 = (-1, -1, 3).$$

Let $B = \partial\Xi$. We explain how to construct a simple affine structure with singularities on B. Each edge ℓ_j of Ξ has 5 integral points (i.e. belonging to $\mathbb{Z}^n$), which divide ℓ_j into 4 segments. For each $j = 1, \ldots, 6$ denote by Δ^j_k, $k = 1, \ldots, 4$ the four barycenters of these four segments. We let

$$\Delta = \{\Delta^j_k; j = 1 \ldots 6 \text{ and } k = 1, \ldots, 4\}.$$

A covering of $B_0 = B - \Delta$ can be defined as follows. The first four open sets consist of the four open faces Σ_i, $i = 1 \ldots, 4$ with the affine coordinate maps ϕ_i induced by their affine embeddings in $\mathbb{R}^3$. Denote by I the set of integral points of B which lie on an edge. For every $Q \in I$ we can choose a small open set U_Q in B_0 such that $\{\Sigma_i\}_{i=1,\ldots,4} \cup \{U_Q\}_{Q \in I}$ is a covering of B_0. Let R_Q denote the 1-dimensional subspace of $\mathbb{R}^3$ generated by $Q \in I$. One can verify that if U_Q is small enough, the projection $\phi_Q : U_Q \to \mathbb{R}^3/R_Q$ is an homeomorphism. A computation shows that the atlas $\mathscr{A} = \{\Sigma_i, \phi_i\}_{i=1,\ldots,4} \cup \{U_Q, \phi_Q\}_{Q \in I}$ defines an affine structure on B_0 making $(B, \Delta, \mathscr{A})$ simple.

Example 3.17. This three dimensional example is taken from [10] §19.3. Let Ξ be the 4-simplex in $\mathbb{R}^3$ spanned by

$$P_0 = (-1, -1, -1, -1), \; P_1 = (4, -1, -1, -1), \; P_2 = (-1, 4, -1, -1),$$
$$P_3 = (-1, -1, 4, -1), \quad P_4 = (-1, -1, -1, 4).$$

Let $B = \partial\Xi$. Denote by Σ_j the open 3-face of B opposite to the point P_j and by F_{ij} the closed 2-face separating Σ_i and Σ_j. Each F_{ij} contains 21 integral points (including those on its boundary). These form the vertices of a triangulation of F_{ij} as in Figure 7. By joining the barycenter of each triangle with the barycenters of its sides we form a trivalent graph as in Figure 7. Define the set Δ to be the union of all such graphs in each 2-face. Denote by I the set of integral points of B. Just as in the previous example, we can form a covering of $B_0 = B - \Delta$ by taking the open 3-faces Σ_j and small open neighborhoods U_Q inside B_0 of $Q \in I$. A coordinate chart ϕ_i on Σ_i can be obtained from its affine embedding in $\mathbb{R}^4$. If we denote again by R_Q the linear space spanned by $Q \in I$, as a chart on U_Q we take the projection $\phi_Q : U_Q \to \mathbb{R}^4/R_Q$. A computation shows that this affine structure is simple. In fact the vertices of Δ which are contained in the interior of each 2-face are of negative type and those which are contained in the 1-faces are of positive type.

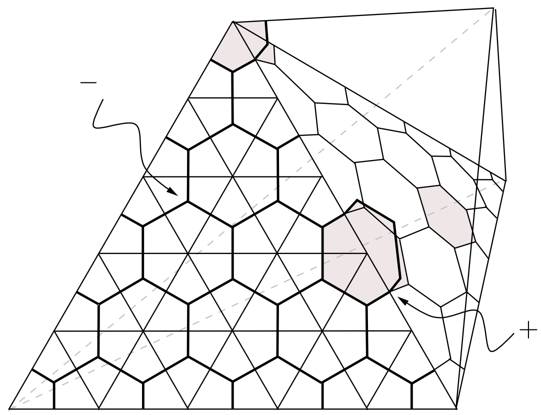


Figure 7: Affine S^3 with singularities.

Example 3.18 (A variation). In the previous example, all edges of Δ were straight lines, but one can perturb them in the sense of Examples 3.9, 3.11 and 3.13. In fact we can form a new Δ by keeping the vertices fixed and connecting them through smooth curves, which are small perturbations of the straight edges of the previous example. If these curves stay inside the 2-faces of B, then the affine structure on $B - \Delta$ can be defined just like above.

In some cases, such as in Examples 3.16 and 3.17 given an affine manifold with singularities, one can define a second affine structure $\check{\mathscr{A}}$ on B, via a discrete Legendre transform of $\mathscr{A}$ (cf. Gross and Siebert [12, 13]). Here we shall not give details about how this process works. Though it is important to mention that this method produces a second integral affine manifold with singularities, $(B, \check{\Delta}, \check{\mathscr{A}})$ which coincides topologically with $(B, \Delta, \mathscr{A})$ but with holonomy representation $\check{\rho}$ dual to ρ. In dimension 3 this means, in particular, that the positive vertices of Δ become negative vertices of $\check{\Delta}$ and vice-versa.

These examples of singular affine manifolds are very important. The bundles associated to them satisfy the hypothesis of Theorem 2.11 so they can be used to produce topological semi-stable compactifications which are homeomorphic to well known examples of Calabi-Yau manifolds:

Theorem 3.19 (Gross [7]). Let $(B, \Delta, \mathscr{A})$ be the integral affine manifold with singularities described in Example 3.17 and let

$$(B, \Delta, \mathscr{A}) \to (B, \check{\Delta}, \check{\mathscr{A}})$$

be its Legendre transform. Let $X(B_0, \mathscr{A}) \hookrightarrow X$ and $X(B_0, \check{\mathscr{A}}) \hookrightarrow \check{X}$ be the corresponding topological semi-stable compactifications. Then X is homeomorphic to the quintic hypersurface and $\check{X}$ is homeomorphic to its mirror.

Later in this article we show that there are *symplectic* semi-stable compactifications recovering the quintic and its mirror. These compactifications rely deeply on the existence of suitable local models of Lagrangian fibrations with singular fibres. The construction of such models is a highly delicate issue.

The focus-focus fibration

In dimension 2 it is much easier to produce symplectic semi-stable compactifications. Now we will show how Example 3.16 gives rise to a symplectic semi-stable compactification diffeomorphic to a K3 surface. This will require a local model of Lagrangian T^2 fibration with a semi-stable singular fibre, such as the one in the following:

Example 3.20. Let $X = \mathbb{C}^2 - \{z_1 z_2 + 1 = 0\}$ and let ω be the restriction to X of the standard symplectic form on $\mathbb{C}^2$. One can easily check that the following map $f : X \to \mathbb{R}^2$ is a Lagrangian fibration:

$$f(z_1, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log |z_1 z_2 + 1| \right). \tag{8}$$

The only singular fibre is $f^{-1}(0)$, which has the topology of a I_1 fibre. It follows that this fibration is conjugate to the topological fibration in Example 2.6.

Lagrangian fibrations with semi-stable singular fibres, e.g., conjugate to the fibration in Example 2.6, are called *focus-focus fibrations*. They have been studied extensively in Hamiltonian Mechanics [4], [34] –where they got their name– and more recently in symplectic topology [24], [33] and Mirror Symmetry [14].

Let $\arg : \mathbb{C}^* \to \mathbb{R}$ be the multi-valued function $\rho e^{i\theta} \mapsto \theta$. Denote by $D \subseteq \mathbb{C}$ the unit open disk and let $D^* = D - \{0\}$. Let $\mathcal{F} = (X, \omega, f, D)$ be a focus-focus fibration. It has been shown [33] that there are coordinates $b = (b_1, b_2)$ on $\mathbb{R}^2$, with values in D, a smooth function $q : D \to \mathbb{R}$ such that $q(0) = 0$ and a choice of generators of $H_1(f^{-1}(b), \mathbb{Z})$ with respect to which the periods λ_1 and λ_2 of $\mathcal{F}$ can be written as

$$\begin{aligned} \lambda_1 &= -\log |b| \; db_1 + \arg b \; db_2 + dq \\ \lambda_2 &= 2\pi \, db_2. \end{aligned}$$

Clearly λ_1 is multi-valued and blows up as $b \to 0$. The lattice

$$\Lambda = \operatorname{span}_{\mathbb{Z}} \langle \lambda_1, \lambda_2 \rangle$$

has monodromy given by T as in (6). We now describe the affine structure induced on D^*. Consider the two open subsets

$$\begin{aligned} U_1 &= D - \{\operatorname{Im} b = 0 \text{ and} \operatorname{Re} b \geq 0\}, \\ U_2 &= D - \{\operatorname{Im} b = 0 \text{ and} \operatorname{Re} b \leq 0\}. \end{aligned}$$

On U_1 we chose the branch of arg with values in $(0, 2\pi)$ and we denote it by $\arg_1$. On U_2 we chose the branch with values in $(-\pi, \pi)$ which we denote by $\arg_2$. Clearly on $U_1 \cap U_2$ we have $\arg_1 = \arg_2 + 2\pi$. A computation shows that the maps $\psi_j : U_j \to \mathbb{R}^2$ given by

$$\psi_j(b) = (-b_1 \log |b| + b_1 + q(b) + b_2 \arg_j b, \; 2\pi b_2),$$

with $q(0) = 0$, are a choice of affine coordinates associated to λ_1 and λ_2.

It is easy to check that the map ψ_1 (or ψ_2) extends continuously to D. Call $\alpha : D \to \mathbb{R}^2$ the extended map. On a sufficiently small neighborhood $V \subseteq D$ of 0, the map α is a homeomorphism of V onto $\alpha(V)$. The reader may verify that $0 \in V$ is a node with respect to the affine structure given by $\{U_j, \psi_j\}$. In other words, the map α restricted to $V^* = V - \{0\}$ is an affine isomorphism between V^* and the affine manifold $\alpha(V^*)$ whose affine structure is the restriction of the one in Example 3.7. The affine structure with singularities on D induced by a focus-focus fibration is therefore simple. In particular, the affine structure induced by Example 3.20 is simple.

Remark 3.21. Germs of focus-focus fibrations –with respect to symplectic conjugation– are classified by formal power series in two variables $\mathbb{R}[\![x, y]\!]$ with vanishing constant term [33]. Such series correspond to the Taylor coefficients of functions $q \in C^\infty(D)$ as above evaluated at $0 \in \mathbb{R}^2$. This means that there is an infinite number of *different* germs of focus-focus fibrations, all inducing simple affine manifolds with singularities, i.e. inducing *the same* singular affine structure on the base. In §4 we will see that a similar phenomenon happens in higher dimensions.

The K3 surface.

Theorem 3.22. Let $(B, \Delta, \mathscr{A})$ be the affine manifold with singularities in Example 3.16 and let $X(B_0, \mathscr{A})$ be the associated T^2 bundle with symplectic structure ω_0 and projection f_0 induced by the standard ones in $T^*_{B_0}$. There exists a compact symplectic manifold (X, ω), a Lagrangian fibration $f : X \to B$ and an embedding $\iota : X(B_0, \mathscr{A}) \to X$ such that $\iota^*\omega = \omega_0$ and $f \circ \iota = f_0$. Moreover X is diffeomorphic to a smooth $K3$ surface.

Proof. Let $f_V : X_V \to V$ be a focus-focus fibration over a small open neighborhood V of its node $0 \in V$. Let $V^* = V - \{0\}$ and denote by $(V^*, \mathscr{A}_V)$ the integral affine manifold induced by f_V. Let $X(V^*, \mathscr{A}_V)$ be the associated Lagrangian T^2 bundle over V^*. It can be shown that f_V has a Lagrangian section $s : V \to X_V$ such that $s(V) \cap \mathrm{Crit}(f_V) = \varnothing$. Then from Theorem 3.3 it follows that $f_V^{-1}(V^*) \subset X_V$ is symplectically conjugate to $X(V^*, \mathscr{A}_V)$.

Now let $P \in \Delta$ and let $U \subset B$ be a small neighborhood of P. Denote by $U^* = U - P$ and by $X(U^*, \mathscr{A})$ the Lagrangian T^2 bundle over U^* given by the restriction of $X(B_0, \mathscr{A})$ to U^*. Recall that both U and V are simple affine manifold with singularities. Then, after taking U and V small enough, there exists an integral affine isomorphism $V^* \cong U^*$. From Corollary 3.4, the latter isomorphism induces is a symplectic conjugation,

$$f_V^{-1}(V^*) \cong X(V^*, \mathscr{A}_V) \cong X(U^*, \mathscr{A}),$$

which can be used to symplectically glue X_V to $X(B_0)$. Define (X, ω) to be the symplectic manifold obtained after applying this gluing over all points $P \in \Delta$ and $f : X \to B$ the resulting fibration. It is clear that (X, ω) is a semi-stable compactification of $(X(B_0, \mathscr{A}), \omega_0)$ such that $\iota^*\omega = \omega_0$. It is easy to check that (X, ω, f, B) is topologically conjugate to a simply connected elliptic fibration with 24 singular fibres of type I_1. It follows that X is diffeomorphic to a K3 surface. □

Corollary 3.23. In view of Remark 3.21, given $(B, \Delta, \mathscr{A})$ as in Example 3.16, a compactification $X(B_0, \mathscr{A}) \hookrightarrow (X, \omega)$ as above is uniquely determined up to symplectic conjugation by a choice of 24 formal power series in two variables:

$$\mathfrak{q}_1, \ldots, \mathfrak{q}_{24} \in \mathbb{R}[\![x, y]\!]$$

corresponding to germs of focus-focus fibrations $\mathcal{F}_1, \ldots \mathcal{F}_{24}$. In particular, there are infinitely many Lagrangian fibrations of a symplectic K3 surface, fibering over $(B, \Delta, \mathscr{A})$, which are all topologically conjugate but not *symplectically* conjugate.

The space $\mathbb{R}[[x, y]]$ being contractible, implies that every two focus-focus fibrations can be connected with a path in $\mathbb{R}[[x, y]]$. The standard Moser's argument implies that the corresponding total spaces are symplectomorphic. Similarly, any two symplectic structures obtained using Theorem 3.22 can be connected with a path in $\mathbb{R}[[x, y]]^{24}$. Moser's argument implies that all such manifolds are symplectomorphic.

Following an alternative approach, Zung obtained a Lagrangian fibration of a symplectic 4-manifold which is also diffeomorphic to a K3 surface (cf. [35]Example 4.19). Leung and Symington [24] use affine geometry as starting point to construct and classify –up to diffeomorphism– the so-called *almost toric symplectic 4-manifolds.* The fibration we obtained in Theorem 3.22 coincides with one of the list in [24].

Other ways of constructing affine manifolds with singularities have been proposed by Gross and Siebert [12, 13], Hasse and Zharkov [16, 17, 18]. In [8], Gross finds a combinatorial method to obtain simple affine manifolds with singularities out of the geometry of the polytopes which Batyrev and Borisov use to construct pairs of Calabi-Yau varieties as complete intersections inside Fano toric varieties. From Theorem 0.1 of [8] (proved by Gross and Siebert in [11]) it follows that these affine manifolds give rise to topological semi-stable compactifications homeomorphic to the two Batyrev-Borisov's Calabi-Yau varieties. We shall see in this paper that similar compactifications can be carried out in the symplectic category.

4 Positive and generic-singular fibrations.

We describe some of the local models needed to produce symplectic compactifications. These models may be regarded as 3-dimensional analogues to focus-focus fibrations. The arguments given here can be generalized to dimension $n > 3$. All fibrations in this Section are given by smooth maps.

Definition 4.1. Let $\mathcal{F} = (X, \omega, f, B)$ be a Lagrangian fibration.

(i) A *Lagrangian generic-singular fibration* is a smooth Lagrangian fibration $\mathcal{F}$, with non-degenerate singularities (in the sense of [26]) which is conjugate to a topological T^3 fibration of generic type (cf. Example 2.7).

(ii) A *Lagrangian positive fibration* is a Lagrangian fibration $\mathcal{F}$ which is conjugate to a topological T^3 fibration of positive type (cf. Example 2.10).

The non-degeneracy condition implies that the singularity is of rank-1 focus-focus type, such singularities are normalized [26].

Examples

We start giving examples of non-proper Lagrangian fibrations describing the singular behavior of (i) and (ii) near $\mathrm{Crit}(f)$. Let $D^k \subseteq \mathbb{R}^k$ be the standard open ball.

Example 4.2. Consider $\mathbb{R}^4$ with standard coordinates (x_1, x_2, y_1, y_2) and let $D^4 \subseteq \mathbb{R}^4$. Let $D^1 \times S^1$ have coordinates (r, θ). Define $V = D^4 \times D^1 \times S^1$ with the standard symplectic structure and $F(x_i, y_i, r, \theta) = (b_1, b_2, b_3)$ where

$$b_1 = x_1y_1 + x_2y_2, \quad b_2 = x_1y_2 - x_2y_1, \quad b_3 = r_3. \tag{9}$$

The reader may verify that $\mu = (b_2, b_3)$ is the moment map of a Hamiltonian action of T^2 and that F is a T^2 invariant Lagrangian fibration of V over $D^2 \times D^1$. The singular fibres are homeomorphic to $\mathbb{R} \times S^1 \times S^1$ after $\{p\} \times S^1 \times S^1$ is collapsed to $\{p\} \times S^1$.

Example 4.3. Consider $\mathbb{C}^3$ with canonical coordinates z_1, z_2, z_3. Define $F(z) = (b_1, b_2, b_3)$, where

$$b_1 = \operatorname{Im} z_1z_2z_3, \quad b_2 = |z_1|^2 - |z_2|^2, \quad b_3 = |z_1|^2 - |z_3|^2. \tag{10}$$

Here $\mu(z_1, z_2, z_3) = (b_2, b_3)$ is the moment map of a T^2-action, furthermore the above functions Poisson commute, so the fibres of F are Lagrangian. The critical locus of F is $\mathrm{Crit}(F) = \bigcup_{ij}\{z_i = z_j = 0\}$ and its discriminant locus is $\Delta = \{b_1 = 0, b_2 = b_3 \geq 0\} \cup \{b_1 = b_2 = 0, b_3 \leq 0\} \cup \{b_1 = b_3 = 0, b_2 \leq 0\}$, i.e. a cone over three points with vertex at $0 \in \mathbb{R}^3$. The regular fibres are homeomorphic to $\mathbb{R} \times T^2$. The singular fibre over $0 \in \Delta$ is homeomorphic to $\mathbb{R} \times T^2$ after $\{p\} \times T^2$ is collapsed to $p \in \mathbb{R}$. All the other singular fibres are homeomorphic to $\mathbb{R} \times T^2$ after a two cycle $\{p\} \times T^2 \subset \mathbb{R} \times T^2$ is collapsed to a circle. This is one of the examples of special Lagrangian fibrations by Harvey and Lawson [19].

Now we give explicit examples of Lagrangian positive and generic-singular fibrations.

Example 4.4. Let $X = \mathbb{C}^3 - \{1 + z_1z_2z_3 = 0\}$ with canonical coordinates z_1, z_2, z_3 and the standard symplectic structure. Consider the T^2-action on X given by $(z_1, z_2, z_3) \mapsto (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{-i(\theta_1+\theta_2)}z_3)$. We obtain $f : X \to \mathbb{R}^3$ given by $f = (f_1, f_2, f_3)$ where

$$f_1 = \log|1 + z_1z_2z_3|, \quad f_2 = |z_1|^2 - |z_2|^2, \quad f_3 = |z_1|^2 - |z_3|^2.$$

It is straightforward to check that the above functions Poisson commute, hence the fibres of f are Lagrangian. It follows that f is modeled on Example 4.3 near $\mathrm{Crit}(f)$. In particular, the discriminant locus is a cone over three points which coincides with the one in Example 4.3. This example has the topology of a positive fibration.

Example 4.5. Let $X' = \mathbb{C}^2 - \{z_1 z_2 - 1 = 0\}$ and let $X = X' \times \mathbb{C}^*$ with the standard symplectic structure. Define $f : X \to \mathbb{R}^3$ by $f = (f_1, f_2, f_3)$ where

$$f_1 = \tfrac{|z_1|^2 - |z_2|^2}{2}, \quad f_2 = \log|z_3|, \quad f_3 = \log|z_1 z_2 - 1|.$$

Again, these functions Poisson commute, hence f is Lagrangian. The singular fibres of f are lying over $\Delta = \{(0, r, 0) \mid r \in \mathbb{R}\}$. The reader may verify that the above gives a generic-singular fibration.

The reader should be aware that the above are just examples of Lagrangian positive and generic-singular fibrations. In fact, there are infinitely many germs of such fibrations [1].

The affine structures.

Now we describe the integral affine structures induced by the above models by giving their period lattices explicitly. For the details we refer the reader to [1]. Fibrations with generic-singular fibres can be normalized near $\mathrm{Crit}(f)$ according to the following:

Theorem 4.6. Let $\mathcal{F} = (X, \omega, f, B)$ be a generic-singular fibration. Assume that $\Sigma = \mathrm{Crit}(f)$ is non-degenerate. Then there is a T^2 invariant neighborhood $U \subseteq X$ of Σ and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Psi"] \arrow[d, "f|_U"'] & D^4 \times D^1 \times S^1 \arrow[d, "F"] \\ B \arrow[r, "\psi"] & D^2 \times D^1 \end{tikzcd} \tag{11}$$

where coordinates (x, y) on D^4 and (r, θ) on $D^1 \times S^1$ define standard symplectic coordinates, the map Ψ is a symplectomorphism, ψ is a diffeomorphism sending Δ to $\{0\} \times D^1$ and F is given by (9). Furthermore Ψ can be taken to be T^{n-1} equivariant.

The above is a corollary of a result due to Miranda and Zung [26]; we refer the reader to [1]§3 for the details.

Remark 4.7. For convenience we shall assume that $B = f(U)$ where U is as in Theorem 4.6. We can think of the above normalization as providing U with canonical coordinates and $B \cong D^2 \times D^1$ with coordinates b_1, b_2, b_3 such that the Hamiltonian vector fields of $b_i \circ f|_U$ are linear. This linearization will be used to compute the action coordinates explicitly. This is crucial to understand the singularities of the affine structure in the base.

Proposition 4.8. Let $\mathcal{F} = (X, \omega, f, B)$ be any generic-singular fibration and $F_{\bar{b}} = f^{-1}(\bar{b})$ a smooth fibre. There is a basis of $H_1(F_{\bar{b}}, \mathbb{Z})$ whose corresponding basis $\lambda_1, \lambda_2, \lambda_3$ of the period lattice Λ of $\mathcal{F}$, in the coordinates $b = (b_1, b_2, b_3)$ on $B \cong D^2 \times D^1$ given by Theorem 4.6, can be written as

$$\lambda_1 = \lambda_0 + dH, \qquad \lambda_2 = 2\pi db_2, \qquad \lambda_3 = db_3, \tag{12}$$

where $H \in C^\infty(B)$ is such that $H(0) = 0$ and $\lambda_0 = -\log|b_1 + ib_2| db_1 + \mathrm{Arg}(b_1 + ib_2) db_2$. The monodromy of Λ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

Proof. The proof is the same as in [1] Proposition 3.10. Let $s = b_1 + \sqrt{-1} b_2$ and $r_3 = b_3$. Roughly speaking, one considers the maps given by $\sigma_1(s, r) = (\bar{s}/\epsilon, r, \theta_0)$ and $\sigma_2(s, r) = (\epsilon, s/\epsilon, r, \theta_0)$ for small $\epsilon > 0$ and $\theta_0 \in S^1$ fixed; these define sections of $f|_U = F$ disjoint from $\mathrm{Crit}(F)$, where F is as in (9). The Hamiltonian vector fields η_i of F_i extend to $X \setminus U$. One can define a basis γ of $H_1(F_{\bar{b}}, \mathbb{Z})$ in terms of suitable composition of the integral curves of η_i. The period λ_1 is obtained by integrating along the path γ_1 starting at $\sigma_1(s, r)$, passing through $\sigma_2(s, r)$ and going back to $\sigma_1(s, r)$. The contribution of $\gamma_1 \cap U$ to the period λ_1 is λ_0, whereas the contribution of $\gamma_1 \cap X \setminus U$ is dH. The remaining periods can be computed integrating along classes in $H_1(F_{\bar{b}}, \mathbb{Z})$ represented by integral curves of η_2 and η_3, respectively. □

As in the 2-dimensional focus-focus fibration, one can choose suitable branches of λ_0 and define action coordinates on these branches. One can easily verify that this defines a simple singular affine structure on B. We have:

Corollary 4.9. A generic-singular fibration $\mathcal{F} = (X, \omega, f, B)$ induces a simple affine structure with singularities on B.

Proof. Consider the coordinates (b_1, b_2, b_3) on $B = D^2 \times D^1$ and the period lattice as in Proposition 4.8. With respect to these coordinates $\Delta = \{b_1 = b_2 = 0\}$. Define open subsets of $B_0 = B - \Delta$:

$$\begin{aligned} V_1 &= B - \{(b_1, 0, b_3) \mid b_1 > 0\}, \\ V_2 &= B - \{(b_1, 0, b_3) \mid b_1 < 0\}. \end{aligned}$$

On V_j the action coordinates have the form

$$A_j(b_1, b_2, b_3) = (\psi_j(b_1, b_2) + H(b_1, b_2, b_3), 2\pi b_2, b_3),$$

where ψ_j is a choice of primitive of λ_0. Then $\mathscr{A} = \{U_j, A_j\}$ gives the integral affine structure on B_0. As in the focus-focus case, for either $j = 1, 2$, the map A_j extends to a homeomorphism, $A : B \to A(B) \subseteq \mathbb{R}^2 \times \mathbb{R}$ such that $A(0) = 0$. It is easy to show that, if $\tau(t) = H(0, 0, t)$, then A is an isomorphism between $(B, \Delta, \mathscr{A})$ and a neighborhood of Δ_τ in the affine manifold with singularities of Example 3.9. □

The case of Lagrangian fibrations of positive type is analogous. Positive fibrations are locally modeled on the fibration in Example 4.3 in a neighborhood of its critical locus. One can use this local description to compute the periods. We have (cf. [1]Theorem 4.19):

Proposition 4.10. Let $\mathcal{F} = (X, \omega, f, B)$ be a Lagrangian fibration of positive type and $F_{\bar{b}} = f^{-1}(\bar{b})$ a smooth fibre. Then there is a basis of $H_1(F_{\bar{b}}, \mathbb{Z})$ and local coordinates (b_1, b_2, b_3) on B around $\bar{b}$, such that the corresponding period 1-forms are:

$$\lambda_1 = \lambda_0 + dH, \quad \lambda_2 = 2\pi db_2, \quad \lambda_3 = 2\pi db_3 \tag{14}$$

where H is a smooth function on B such that $H(0) = 0$ and λ_0 is multi-valued 1-form blowing up at $\Delta \subset B$, where

$$\Delta = \{b_1 = 0, b_2 = b_3 \geq 0\} \cup \{b_1 = b_2 = 0, b_3 \leq 0\} \cup \{b_1 = b_3 = 0, b_2 \leq 0\}.$$

In the basis $\lambda_1, \lambda_2, \lambda_3$ of Λ and for suitable generators of $\pi_1(B - \Delta)$ satisfying $g_1 g_2 g_3 = I$ (cf. Figure 3), the monodromy representation of $\mathcal{F}$ is generated by the matrices:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We now prove that the affine structure on the base of a positive fibration is simple.

Proposition 4.11. A Lagrangian fibration $\mathcal{F} = (X, \omega, f, B)$ of positive type induces on B the structure of a simple affine manifold with singularities with positive vertex.

Proof. Let (b_1, b_2, b_3) be the coordinates on B and $\Delta \subseteq B$ as in Proposition 4.10. To avoid cumbersome notation let us assume $B = \mathbb{R} \times \mathbb{R}^2$. We may identify $\mathbb{R}^2$ with $\{0\} \times \mathbb{R}^2$. Then $\Delta \subset \mathbb{R}^2$. Let $\lambda_1, \lambda_2, \lambda_3$ be the periods of $\mathcal{F}$ as in (14). We want to show that the affine structure on $B - \Delta$ induced by $\mathcal{F}$ is isomorphic to the one given in Examples 3.10 or 3.11. To do this we will consider the locally defined map $A = (A_1, A_2, A_3)$, where each A_j is a suitable branch of a primitive of λ_j such that $A_j(0) = 0$. First we will show that –perhaps after replacing B by a smaller neighborhood of 0– the map A extends to a homeomorphism $A : B \to A(B) \subseteq \mathbb{R}^3$. Let

$$\begin{aligned} R &= \mathbb{R} \times \Delta \\ R^+ &= \mathbb{R}_{\geq 0} \times \Delta \\ R^- &= \mathbb{R}_{\leq 0} \times \Delta \end{aligned}$$

and take the open cover $\{U_1, U_2\}$ of $B - \Delta$ where

$$\begin{aligned} U_1 &= B - R^+, \\ U_2 &= B - R^-. \end{aligned} \tag{15}$$

On U_1 we can choose an affine coordinates map given by

$$A(b_1, b_2, b_3) = (\psi_1(b_1, b_2, b_3), 2\pi b_2, 2\pi b_3),$$

where ψ_1 is a primitive of λ_1. Clearly $A(R^-) \subset R$. We now show that A extends continuously to B. The key observation is that the symplectic form ω is exact in a neighborhood of the singular fibre over the vertex of Δ. This is straightforward in the case of Example 4.4, where ω is the standard symplectic form on $\mathbb{C}^3$ but it is also true in general. So assume $\omega = d\eta$ for some 1-form η. Now let us fix a basis $e = (e_1, e_2, e_3)$ of $H^1(f^{-1}(U_1), \mathbb{Z})$, corresponding to the periods λ_1, λ_2 and λ_3 respectively. Recall that action coordinates can be computed by

$$A(b) = \left(-\int_{e_1(b)} \eta, \ -\int_{e_2(b)} \eta, \ -\int_{e_3(b)} \eta \right),$$

where $e_j(b)$ is a 1-cycle, contained in $f^{-1}(b)$, representing e_j. We prove first that A, as a map, extends continuously to $B - \Delta$. Notice that e_2 and e_3 are monodromy invariant, so we may assume that $e_2(b)$ and $e_3(b)$ are well defined for all $b \in B - \Delta$ and that

$$-\int_{e_j(b)} \eta = 2\pi b_j, \tag{16}$$

for $j = 2, 3$. In particular, A_2 and A_3 are defined on B. Let us study

$$\psi_1(b) = -\int_{e_1(b)} \eta.$$

Suppose that $\psi_1(\bar{b}) = 0$ for a fixed point $\bar{b} \in U_1$. Given another point $b \in U_1$ let $\Gamma : [0, 1] \to U_1$ be a path such that $\Gamma(0) = \bar{b}$ and $\Gamma(1) = b$. Consider the cylinder S inside $f^{-1}(U_1)$ spanned by the cycles $e_1(\Gamma(t))$. Then one can see that

$$\psi_1(b) = \int_S \omega. \tag{17}$$

We may use (17) to define $\psi_1(b)$ for $b \in R^+ - \Delta$. Since $B - \Delta$ is not simply connected, this expression of ψ_1 is well defined provided that it is independent of the chosen path Γ. Suppose that Γ_1 and Γ_2 are two different paths from $\bar{b}$ to b such that $\Gamma_1 - \Gamma_2$ is not homotopically trivial in $B - \Delta$, then we have to show that if S_1 and S_2 are the corresponding cylinders, then

$$\int_{S_1 - S_2} \omega = 0.$$

Denote by $e_1^+(b)$ and $e_1^-(b)$ those boundary components of S_1 and S_2 respectively, which lie on top of b (the endpoint of both Γ_1 and Γ_2). Then

$$\partial(S_1 - S_2) = e_1^+(b) - e_1^-(b),$$

and

$$\int_{S_1 - S_2} \omega = \int_{e_1^+(b) - e_1^-(b)} \eta.$$

Because of monodromy, $e_1^+(b)$ and $e_1^-(b)$ may not coincide and it is not obvious that the above integral vanishes. Nevertheless, we know that $b \in R^+$ and there are three cases: if $b = (b_1, b_2, b_3)$ then either $b_2 = 0$, $b_3 = 0$ or $b_2 = b_3$. Let us look at that the latter case. With respect to the basis $e = (e_1, e_2, e_3)$ as above, the monodromy matrices T_1, T_2 and T_3 corresponding respectively to generators g_1, g_2 and g_3 of $\pi_1(B - \Delta)$ as depicted in Figure 3 are those given in Proposition 4.10.

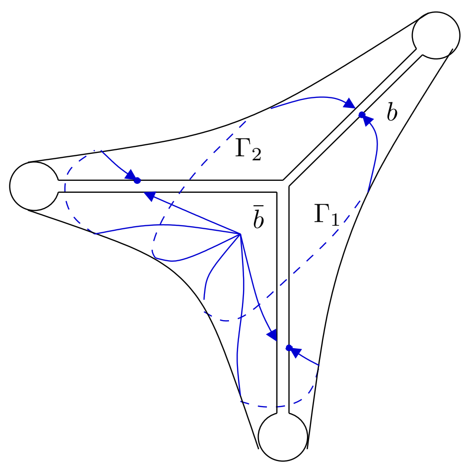


Figure 8: The cut pair of pants are wrapping around Δ and give a schematic picture for $U_1 = B - R^+$, the cut represents R^+. Here $b \in R^+$ and Γ_1 and Γ_2 are two possible paths from $\bar{b}$ to b.

Let $\bar{b}$, b, Γ_1 and Γ_2 be given as in Figure 8, then one can see that $\Gamma_1 - \Gamma_2 = g_1^{-1} g_2^{-1}$. This implies that

$$e_1^+(b) = e_1^-(b) - e_2(b) + e_3(b)$$

and therefore that

$$\int_{e_1^+(b) - e_1^-(b)} \eta = \int_{-e_2(b) + e_3(b)} \eta = 2\pi(b_2 - b_3) = 0,$$

where in the second equality we have used (16). Similarly one treats the cases $b_2 = 0$ or $b_3 = 0$ using monodromy matrices T_1 and T_2 respectively. This shows that ψ_1 extends continuously to $B - \Delta$. It can be easily seen that it also extends continuously to points in Δ. In fact one can use (17) as a definition of $\psi_1(b)$ when $b \in \Delta$. This makes sense since the cycles $e_1(b)$ spanning S can be extended as cycles on singular fibres when $b \in \Delta$, e.g. when $b = 0$, $e_1(0)$ is a homologically non trivial closed curve passing through the singularity of $f^{-1}(0)$, in particular $e_1(0)$ is the generator of $H_1(f^{-1}(0), \mathbb{Z}) = \mathbb{Z}$.

We argue that A is injective onto its image, at least when restricted to a smaller neighborhood of $b = 0$. This would imply that A is a homeomorphism. Clearly, A is injective if and only if for fixed values of b_2 and b_3, the function $\psi_1(\,\cdot\,, b_2, b_3)$ is injective in a neighborhood of $b = 0$. Since $d\psi_1 = \lambda_1$, this holds if the coefficient of db_1 in λ_1 is never zero in a neighborhood of $b = 0$. In fact, it was shown in §4 of [1] that this coefficient blows up to infinity as $b \to 0$, in particular it never vanishes.

One can easily check that A defines an isomorphism between the affine structure with singularities induced on B by the fibration $\mathcal{F}$ and the one described in Example 3.11, where $\tau : \Delta \to \mathbb{R}$ is given by $\tau = \psi_1|_\Delta$. We only need to verify that τ is smooth. In fact, it turns out that $\tau = H|_\Delta$ where H is the smooth function in (14); this follows from the computation of λ_0 given in [1]§4. Consider the fibration $F : \mathbb{C}^3 \to \mathbb{R}^3$ of Example 4.3. This is the local model for the singularity of a positive fibration. Consider two sections σ_- and σ_+ of F, disjoint from $\mathrm{Crit}(F)$ and such that for every $b \in \Delta$, $\sigma_-(b)$ and $\sigma_+(b)$ lie on distinct connected components of the smooth part of the fibre over b. For every $b \in \mathbb{R}^3$ consider a curve $\gamma(b)$ contained $F^{-1}(b)$ joining $\sigma_-(b)$ to $\sigma_+(b)$ and define the function

$$a_0(b) = -\int_{\gamma(b)} \eta.$$

Then $\lambda_0 = da_0$. Clearly a_0 can be continuously defined on $\mathbb{R}^3$. Using the fact that F satisfies $F(-z_1, z_2, z_3) = (-b_1, b_2, b_3)$, where $F(z_1, z_2, z_3) = (b_1, b_2, b_3)$, one can show that a_0 satisfies $a_0(-b_1, b_2, b_3) = -a_0(b_1, b_2, b_3)$ and therefore that $a_0|_\Delta = 0$. This proves that $\tau = H|_\Delta$. □

Gluing over the discriminant locus

Given a simple affine manifold with singularities, we show how to symplectically glue singular fibres of positive or generic type to the associated T^3 bundle. This gives us a (partial) symplectic compactification over positive and generic points of the singular locus.

Consider a cylinder $D^2 \times I$ inside $\mathbb{R}^2 \times \mathbb{R}$, where I is an open interval, and let $\Delta = \{0\} \times I$. Let H be a smooth real-valued function on $D^2 \times I$. The *germ* of H along Δ, denoted H_Δ, is the Taylor expansion series of H along Δ. This is a formal power series in two variables whose coefficients are smooth functions on I.

Remark 4.12. For any given formal power series in two variables $h = \sum h_{ij} x_1^i x_2^j$ whose coefficients are smooth functions $h_{ij} = h_{ij}(r)$ on I, there is a function H on $D^2 \times I$ whose germ along Δ is h. An analogous statement in the case of a formal power series in one variable with real coefficients is standard (cf. [31] Exercise 13, page 384). It is an exercise to check that it is also true in two variables with coefficients depending on a parameter.

Recall that the generators of the period lattice of a generic-singular fibration may be written as $\lambda_1 = \lambda_0 + dH$, $\lambda_2 = 2\pi db_2$ and $\lambda_3 = db_3$, where (b_1, b_2, b_3) are coordinates in in $D^2 \times I$, λ_0 as (12) and H a smooth function. One can prove the following (cf. [1]):

Theorem 4.13. For any smooth function H over $B = D^2 \times I$, there is a generic-singular fibration $\mathcal{F}_H = (X, \omega, f, B)$ whose period lattice is generated by 1-forms as in (12). Furthermore, two generic-singular fibrations $\mathcal{F}_H$ and $\mathcal{F}_{H'}$ are symplectically conjugate in a neighborhood of Δ if and only if $H_\Delta = H'_\Delta$.

We call H_Δ the *invariant* of the fibration $\mathcal{F}_H$. We proved in Corollary 4.9 that the affine base of a generic-singular fibration is always simple, isomorphic to Example 3.9. Furthermore, the shape of its discriminant locus (in affine coordinates), as well as the isomorphism class of its singular affine base is determined by the function $\tau(r) = H(0, 0, r)$ which is the restriction of H to Δ. In other words, by the zero order term of the germ H_Δ. In the special case when the zero order term of H_Δ vanishes, the base is affine isomorphic to the product of an affine disc with a node times the standard affine interval, in this case we call the associated fibration $\mathcal{F}_H$ *straight*, in all other cases we call it *twisted*.

Lemma 4.14. Given any function $\tau \in C^\infty(\Delta)$ on an edge $\Delta \subset D^2 \times I$ with $\tau(0) = 0$, there is a generic-singular fibration whose base is locally affine isomorphic to the affine manifold with singularities $(\mathbb{R}^2 \times I, \Delta_\tau, \mathscr{A})$ of Example 3.9.

Proof. In view of Remark 4.12, we can certainly find a smooth function H on $D^2 \times I$ such that $H|_\Delta = \tau$. We can then form $\mathcal{F}_H$ using Theorem 4.13. □

Analogously, positive fibrations are also classified by germs H_Δ, where in this case $\Delta \subset D^3$ is a trivalent vertex and H a smooth function on D^3 as in Proposition 4.10; for the details we refer to [1]. Given a positive fibration, Proposition 4.11 tells us that its base is locally isomorphic to $(\mathbb{R}^3, \Delta_\tau, \mathscr{A})$ as in Example 3.11. A particular case is when $\tau = 0$ which gives a straight vertex. More generally we showed (cf. proof of Proposition 4.11) that $\tau = H|_\Delta$. In particular, we have:

Lemma 4.15. Given any function $\tau \in C^\infty(\Delta)$ on a trivalent vertex $\Delta \subset D^3 \subset \mathbb{R}^3$ with $\tau(0) = 0$, there is a positive fibration whose base is locally affine isomorphic to the affine manifold with singularities $(\mathbb{R}^3, \Delta_\tau, \mathscr{A})$ of Example 3.11.

We stress that the constructions described in Lemmas 4.14 and 4.15 only involve the zero order term of H_Δ, which is enough for determining the affine structure. From [1] it follows that we have many possible choices of H_Δ giving the same affine structure:

Corollary 4.16. Given a prescribed affine manifold with singularities $(B, \Delta_\tau, \mathscr{A})$ either as in Example 3.9 in the generic case or as in Example 3.11 in the positive case, there are infinitely many non symplectically conjugate germs of Lagrangian fibrations whose bases are locally affine isomorphic to $(B, \Delta_\tau, \mathscr{A})$.

Observe that the above result holds also in the case when $\tau \equiv 0$, i.e. when the discriminant is completely straight. Exploiting the flexibility given by Lemmas 4.14 and 4.15, we can show that we can always locally compactify a torus bundle given by simple affine manifolds with singularities near a positive or generic point of the discriminant locus:

Proposition 4.17. Let $(B, \Delta, \mathscr{A})$ be a given simple affine 3-manifold with singularities. Then we have the following

(i) if $J \subseteq \Delta_g$ is an edge of Δ, then there is a generic-singular fibration $\mathcal{F}$, with affine base $(B', \Delta', \mathscr{A}')$ and neighborhood $U \subseteq B$ of J such that there exists an integral affine isomorphism $(B', \Delta', \mathscr{A}') \cong (U, J, \mathscr{A})$ inducing a symplectic conjugation $X(B'_0, \mathscr{A}') \cong X(U - J, \mathscr{A})$;

(ii) if $p \in \Delta_d$ is a positive vertex of Δ, then there is positive fibration $\mathcal{F}$ with base $(B', \Delta', \mathscr{A}')$ and a neighborhood $U \subseteq B$ of p such that there exists an integral affine isomorphism $(B', \Delta', \mathscr{A}') \cong (U, U \cap \Delta, \mathscr{A})$ inducing a symplectic conjugation $X(B'_0, \mathscr{A}') \cong X(U - (U \cap \Delta), \mathscr{A})$.

Moreover, using the symplectic conjugations in (i) and (ii), we can symplectically glue the germ of $\mathcal{F}$ into $X(B_0, \mathscr{A})$.

Proof. It is just a matter of applying Lemmas 4.14 and 4.15 to find suitable $\mathcal{F}$. Since both positive and generic singular fibrations have a Lagrangian section, the result follows from Corollary 3.4. □

Gluing legs

While for the gluing in Proposition 4.17 it is sufficient to consider the zero order term of H_Δ, to glue two singular Lagrangian fibrations $\mathcal{F}$ and $\mathcal{F}'$ along their legs one should take into account *all* terms. This is essentially due to the fact that, gluing legs also involves gluing them along their singular fibres. We will see that Theorem 4.13 also takes care of this.

Suppose we are given a simple affine 3-manifold with singularities $(B, \Delta, \mathscr{A})$ and two points p and p' of Δ connected by an edge J (p and p' may be generic, positive or negative points). Let us assume that we have glued to $X(B_0, \mathscr{A})$ the germs of singular Lagrangian fibrations $\mathcal{F}$ and $\mathcal{F}'$ fibering over disjoint neighborhoods V and V' of p and p' respectively (e.g. using Proposition 4.17, if p and p' are positive or generic). We do not consider only the case when p and p' are either positive of generic, since we want the arguments here to hold also for negative points onto which we can glue fibrations like the ones in §7. We only assume here that $\mathcal{F}$ and $\mathcal{F}'$ have legs with generic-singular fibres on their ends and these ends are connected by J. We now explain how to glue to $X(B_0, \mathscr{A})$ a generic singular fibration along J in such a way that this gluing is made compatible with the gluing of $\mathcal{F}$ and $\mathcal{F}'$.

We can assume that there are disjoint neighborhoods U and U' of the ends of J, as in Figure 9, and generic-singular fibrations $\mathcal{L} = \mathcal{F}|_U$ and $\mathcal{L}' = \mathcal{F}'|_{U'}$ over U and U'. Let H_Δ and H'_Δ be, respectively, the invariants of $\mathcal{L}$ and $\mathcal{L}'$ as in Theorem 4.13.

Since J is an edge of Δ, there is a neighborhood W of J, with $W \cap \Delta = J$, such that (W, J) is (locally) affine isomorphic to $(D^2 \times I, \Delta_\tau)$ as in Example 3.9. Without loss of generality, we can assume $I = (-1, 1)$ and that there exists $\delta \in (0, 1)$ such that $U \cong D^2 \times (-1, -\delta)$ and $U' \cong D^2 \times (\delta, 1)$. Denote $I_{-\delta} = (-1, -\delta)$ and $I_\delta = (\delta, 1)$. Clearly, we can interpret H_Δ and H'_Δ as formal power series along $I_{-\delta}$ and I_δ respectively. By the arguments of the previous section, we must have that the zero order terms of H_Δ and H'_Δ coincide with $\tau|_{I_{-\delta}}$ and $\tau|_{I_\delta}$ respectively.

It is now clear that we can choose a formal power series $\tilde{H}_\Delta$ along I such that

(a) the zero order term of $\tilde{H}_\Delta$ is τ;

(b) $\tilde{H}_\Delta$ coincides with H_Δ and H'_Δ along $I_{-\delta}$ and I_δ respectively.

This can be done using cut-off functions. For this purpose it may be necessary to shrink $I_{-\delta}$ and I_δ by taking a slightly bigger δ.

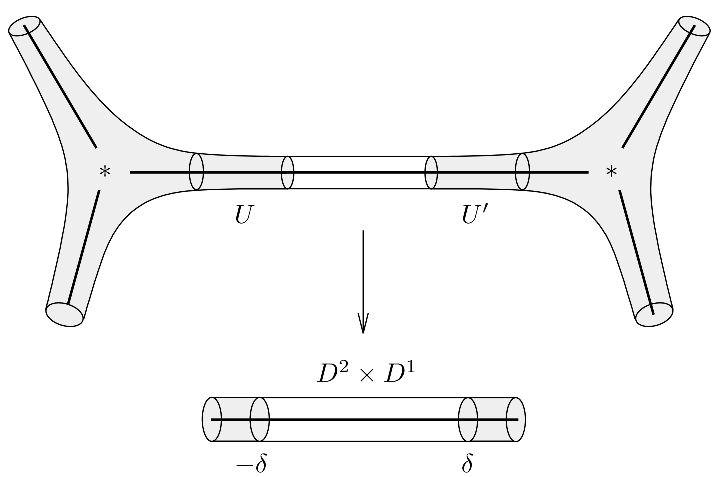


Figure 9: The gluing of two legs along their ends. The asterisk represents components of the discriminant of $\mathcal{F}$ and $\mathcal{F}'$, which can be of either positive or negative type (or void).

We can now apply Remark 4.12 and the first part of Theorem 4.13 to find the germ of a generic-singular Lagrangian fibration $\tilde{\mathcal{L}}$ fibering over W whose invariant is $\tilde{H}_\Delta$. The second part of Theorem 4.13 and condition (b) above imply that $\tilde{\mathcal{L}}|_U \cong \mathcal{L}$ and $\tilde{\mathcal{L}}|_{U'} \cong \mathcal{L}$, moreover condition (a) implies that $\tilde{\mathcal{L}}$ can be glued to $X(B_0, \mathscr{A})$ along J. It is clear that the symplectic conjugations $\tilde{\mathcal{L}}|_U \cong \mathcal{L}$ and $\tilde{\mathcal{L}}|_{U'} \cong \mathcal{L}'$ coincide with the map gluing $\tilde{\mathcal{L}}$ to $X(B_0, \mathscr{A})$.

We have proved:

Proposition 4.18. Let $(B, \Delta, \mathscr{A})$ be a simple affine 3-manifold with singularities and let $p, p' \in \Delta$ be points connected by an edge J. Suppose there are disjoint neighborhoods V and V' of p and p' respectively and a neighborhood W of J, with $W \cap \Delta = J$, such that the following conditions hold

(i) if $\tilde{B} = B_0 \cup (V \cup V')$, there exists a Lagrangian fibration $\mathcal{F} = (X, \omega, f, \tilde{B})$ and a commuting diagram

$$\begin{array}{ccc} X(B_0, \mathscr{A}) & \xrightarrow{\Psi} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ B_0 & \xrightarrow{\iota} & \tilde{B} \end{array}$$

where Ψ is a symplectomorphism and ι the inclusion.

(ii) $\mathcal{F}|_{W \cap V}$ and $\mathcal{F}|_{W \cap V'}$ are generic-singular fibrations.

Then, if we let $\tilde{B}' = \tilde{B} \cup W$, there exists a Lagrangian fibration $\mathcal{F}' = (X', \omega', f', \tilde{B}')$ and a commuting diagram

$$\begin{array}{ccc} X(B_0, \mathscr{A}) & \xrightarrow{\Psi'} & X' \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f'} \\ B_0 & \xrightarrow{\iota} & \tilde{B}' \end{array}$$

where Ψ' is also a symplectomorphism.

The upshot of the results of this Section is that: 1) we can construct local models of generic and positive singular fibres; 2) we know how to glue them onto any given simple affine manifold with generic and positive singularities; 3) these gluings can be made compatible over common intersections. In fact, we can show:

Theorem 4.19. Let $(B, \Delta, \mathscr{A})$ be a compact simple integral affine 3-manifold with singularities without negative vertices. Then there is a compact smooth symplectic 6-manifold (X, ω) and a C^∞ Lagrangian fibration $f : X \to B$ with discriminant locus Δ, which is a semi-stable compactification of the T^3 bundle $X(B_0, \mathscr{A}) \to B_0$.

The proof is an application of the above preparation results. Using Proposition 4.17 we can first glue in the positive vertices, then using Proposition 4.18 we glue in the generic-singular fibres over the edges. Theorem 4.19 is a particular case of our more general result we shall prove in §8, where we also include negative fibrations. We emphasize that the fibration obtained in Theorem 4.19 is *smooth*. This will not happen if Δ includes negative vertices. In that case, the resulting fibration will be *piecewise* smooth only.

As a further remark we point out that Theorem 4.19 can be generalized to dimension $n \geq 3$, since there are natural generalizations of generic and positive singularities and the analysis of their affine structures carries through as in the $n = 3$ case. Our notion of simplicity can also be generalized to higher dimensions, though for $n > 3$ it may no longer coincide with the notion of simplicity in the sense of Gross and Siebert [13].

5 Piecewise smooth fibrations

It is now commonly accepted that to produce Lagrangian fibrations of the type described in §2 one should also allow *piecewise smooth* fibrations (cf. [6], [21], [29]). Here we present a simple way to produce local models of piecewise smooth Lagrangian fibrations. We suspect that models of the sort presented here are also implicit in Ruan's fibrations but we have been unable to verify this. Our method is inspired by ideas of Gross [6], Goldstein [5] and Joyce [21].

Fibrations with torus symmetry.

Let (X, ω) be a symplectic $2n$-manifold and let $\mu : (X, \omega) \to \mathfrak{t}^*$ be the moment map of a Hamiltonian T^k-action. Let $t \in \mu(X)$ and let $\pi_t : \mu^{-1}(t) \to X_t$ be the projection modulo the T^k action. When t is a regular value of μ, X_t is a smooth manifold and the symplectic form ω descends to a symplectic form ω_t on X_t. When t is a critical value of μ, X_t may be a singular space and ω_t will be only defined on the smooth part of X_t. The space (X_t, ω_t) is the Marsden-Weinstein reduced space at t.

Remark 5.1. We shall denote by

$$\omega_{\mathbb{C}^m} = \frac{i}{2} \sum_k dz_k \wedge d\overline{z}_k$$

the standard symplectic structure on $\mathbb{C}^m$ and ω_0 will denote the reduced symplectic form of the reduced space (X_t, ω_t) at time $t = 0$.

Goldstein [5] and Gross [6] used reduced spaces to construct T^k-invariant (special) Lagrangian fibrations. The following is a particular case of [6]Thm. 1.2:

Proposition 5.2. Let T^k act effectively on X, $k \leq n-1$. Suppose that there is a continuous map $G : X \to M$ to an $(n-k)$-dimensional manifold M such that $G(T \cdot x) = G(x)$ for all $T \in T^k$. Suppose that for t in a dense subset of $\mu(X)$ the induced maps $G_t : X_t \to M$ have fibres that are Lagrangian with respect to ω_t. Then $f : X \to \mu(X) \times M$ given by:

$$f = (\mu, G) \tag{18}$$

defines a T^k-invariant Lagrangian fibration.

When the T^k-action has fixed points, the construction of Proposition 5.2 will produce fibrations with interesting singular fibres. We will give some explicit examples shortly.

Remark 5.3. In the extremal case when $k = n-1$, constructing Lagrangian fibrations using Proposition 5.2 is very easy. In this situation, the reduced spaces X_t are two dimensional and every map $G_t : X_t \to \mathbb{R}$ with 1-dimensional level sets defines a Lagrangian fibration on X_t. In particular, any T^{n-1}-invariant continuous map $G : X \to \mathbb{R}$ which, on each X_t, descends to a map G_t with 1-dimensional level sets can be used to construct Lagrangian fibrations. We will make much use of this fact later on.

The reduced geometry.

Consider the following S^1 action on $\mathbb{C}^n$, with $n \geq 2$:

$$e^{i\theta}(z_1, z_2, z_3, \ldots, z_n) = (e^{i\theta} z_1, e^{-i\theta} z_2, z_3, \ldots, z_n). \tag{19}$$

This action is Hamiltonian with respect to $\omega_{\mathbb{C}^n}$. Clearly it is singular along the $2(n-2)$ dimensional symplectic submanifold $\mathrm{Crit}(\mu) = \{z_1 = z_2 = 0\}$. The moment map is:

$$\mu(z_1, \ldots, z_n) = \frac{|z_1|^2 - |z_2|^2}{2}. \tag{20}$$

The only critical value of μ is $t = 0$ and $\mathrm{Crit}(\mu) \subset \mu^{-1}(0)$.

Now consider the map $\bar{\pi}$ as in Remark 2.5. Recall that $\bar{\pi}$ is given by

$$\begin{array}{rcl} \bar{\pi} : \mathbb{C}^n & \to & \mathbb{R} \times \mathbb{C}^{n-1} \\ (z_1, \ldots, z_n) & \mapsto & (\mu, z_1 z_2, z_3, \ldots, z_n). \end{array} \tag{21}$$

When restricted to $\mathbb{C}^n - \mathrm{Crit}(\mu)$, the above is an S^1-bundle onto $(\mathbb{R}\times\mathbb{C}^{n-1}) - \bar{\pi}(\mathrm{Crit}(\mu))$ with Chern class $c_1 = 1$. Let π_t be the restriction to $\mu^{-1}(t)$ of the map

$$(z_1, \ldots, z_n) \mapsto (z_1 z_2, z_3, \ldots, z_n). \tag{22}$$

Then π_t can be used to identify the reduced space $\mu^{-1}(t)/S^1$ with $\mathbb{C}^{n-1}$. Under this identification, i.e. letting the coordinates $u_1 = z_1 z_2$ and $u_j = z_{j+1}$ when $2 \leq j \leq n-1$, the reduced symplectic form ω_t can be written as:

$$\omega_t = \frac{i}{2} \left(\frac{1}{2\sqrt{t^2 + |u_1|^2}} du_1 \wedge d\overline{u}_1 + \sum_{j=2}^{n-1} du_j \wedge d\overline{u}_j \right). \tag{23}$$

Clearly, away from $t = 0$, the reduced spaces are smooth manifolds.

On the other hand, at $t = 0$ the reduced form ω_0 blows up along the hyperplane

$$\Sigma := \pi_0(\mathrm{Crit}(\mu)) = \{u_1 = 0\},$$

so the reduced space $(\mathbb{C}^{n-1}, \omega_0)$ is singular. However, it was observed by Guillemin and Sternberg in [15], that it can be smoothed out, i.e. it can be identified with $(\mathbb{C}^{n-1}, \omega_{\mathbb{C}^{n-1}})$. Indeed, the identification is given by the following

$$\Gamma_0 : (u_1, u_2, \ldots, u_{n-1}) \mapsto \left(\frac{u_1}{\sqrt{|u_1|}}, u_2, \ldots, u_{n-1} \right). \tag{24}$$

The map Γ_0 is continuous, smooth away from $u_1 = 0$ and such that $\Gamma_0^* \omega_{\mathbb{C}^{n-1}} = \omega_0$. One can do more: one can identify all the reduced spaces with $(\mathbb{C}^{n-1}, \omega_{\mathbb{C}^{n-1}})$ at once. Consider the map

$$\Gamma_t : (u_1, u_2, \ldots, u_{n-1}) \mapsto \left(\frac{u_1}{\sqrt{|t| + \sqrt{t^2 + |u_1|^2}}}, u_2, \ldots, u_{n-1} \right). \tag{25}$$

One can verify that Γ_t is a symplectomorphism between $(\mathbb{C}^{n-1}, \omega_t)$ and the standard symplectic space $\mathbb{C}^{n-1}$. However, this identification has the problem that, although continuous and smooth for fixed $t \in \mathbb{R}$, it is not smooth in t when $t = 0$. In fact one can show that it cannot be otherwise.

A construction

We now illustrate a general method to construct piecewise smooth Lagrangian fibrations using Proposition 5.2 and the observations about the reduced geometry with respect to the S^1 action as in (19).

Let $\mathrm{Log} : (\mathbb{C}^*)^{n-1} \to \mathbb{R}^{n-1}$ be the map defined by

$$\mathrm{Log}(v_1, \ldots, v_{n-1}) = (\log|v_1|, \ldots, \log|v_{n-1}|). \tag{26}$$

Clearly, the above map is a Lagrangian fibration with respect to the restriction of $\omega_{\mathbb{C}^{n-1}}$ to $(\mathbb{C}^*)^{n-1}$. Moreover, it defines a trivial T^{n-1}-bundle over $\mathbb{R}^{n-1}$. Let the map

$$\Phi : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$$

be a smooth symplectomorphism of the standard $\mathbb{C}^{n-1}$. Let X_t be the open and dense subsets of $(\mathbb{C}^{n-1}, \omega_t)$ defined by

$$X_t = \Gamma_t^{-1} \circ \Phi^{-1}\left((\mathbb{C}^*)^{n-1}\right).$$

Denote, with slight abuse of notation,

$$\Sigma := \{u_1 = 0\} \cap X_0.$$

Then examples of maps $G_t : X_t \to \mathbb{R}^{n-1}$ as in Proposition 5.2 can be defined by

$$G_t = \operatorname{Log} \circ \Phi \circ \Gamma_t.$$

This clearly makes sense also when $t = 0$. It is also clear that, for all fixed $t \in \mathbb{R}$, G_t is a Lagrangian fibration with respect to the reduced symplectic form (23). We summarize this in the following:

Proposition 5.4. Let Φ, X_t and G_t be as defined above. Let Q be the map given by

$$Q(t, u_1, \ldots, u_{n-1}) = (t, G_t(u_1, \ldots, u_{n-1})). \tag{27}$$

Then Q is defined on the dense open subset $Y \subseteq \mathbb{R} \times \mathbb{C}^{n-1}$ defined by

$$Y = \{(t, u_1, \ldots, u_{n-1}) \in \mathbb{R} \times \mathbb{C}^{n-1} \mid (u_1, \ldots, u_{n-1}) \in X_t\}.$$

Letting $\bar{\pi}$ be as in (21) and

$$X = (\bar{\pi})^{-1}(Y)$$

with the standard symplectic form induced from $\mathbb{C}^n$, the map $f : X \to \mathbb{R}^n$ given by

$$f = Q \circ \bar{\pi}$$

is a piecewise smooth Lagrangian fibration of X which fails to be smooth on the $(2n-1)$-dimensional subspace $\mu^{-1}(0) \cap X$.

It is clear that all the singular fibres of f must lie in $\mu^{-1}(0) \cap X$. In fact, the singular fibres are all the lifts of fibres of G_0 in X_0 which intersect Σ. The topology of the singularity depends on the topology of this intersection. The discriminant locus of the fibration is therefore the set $\Delta \subset \mathbb{R}^n$ given by

$$\Delta = \{0\} \times (\operatorname{Log} \circ \Phi \circ \Gamma_0(\Sigma)).$$

Given a point $b = (0, b_1, \ldots, b_{n-1}) \in \Delta$, the fibre $f^{-1}(b)$ looks like $S^1 \times G_0^{-1}(b_1, \ldots, b_{n-1})$ after the circles over all points in $G_0^{-1}(b_1, \ldots, b_{n-1}) \cap \Sigma$ have been collapsed to points (cf. Figure 6).

Examples

In the following examples we use the above construction with $n = 2$ or 3. Define the piecewise smooth map $\gamma : \mathbb{C}^2 \to \mathbb{C}$ by

$$\gamma(z_1, z_2) = \begin{cases} \frac{z_1 z_2}{|z_1|}, & \text{when } \mu(z_1, z_2) \geq 0 \\ \\ \frac{z_1 z_2}{|z_2|}, & \text{when } \mu(z_1, z_2) < 0. \end{cases} \tag{28}$$

If π_t is the restriction of the map (22) to $\mu^{-1}(t)$, then one can easily see that for all $(z_1, z_2, z_3) \in \mu^{-1}(t)$, the map $\Gamma_t \circ \pi_t$ is given by

$$\Gamma_t \circ \pi_t : (z_1, z_2, z_3) \mapsto (\gamma(z_1 z_2), z_3).$$

From Proposition 5.4, we see that $\Gamma_t \circ \pi_t$ can be twisted by a symplectomorphism Φ. The topology of the resulting fibration depends on how we choose Φ.

Example 5.5 (The amoeba). Take as a symplectomorphism Φ the linear map

$$\Phi(u_1, u_2) = \frac{1}{\sqrt{2}} \left(u_1 - u_2, u_1 + u_2 - \sqrt{2} \right). \tag{29}$$

Then the fibration resulting from Proposition 5.4 can be written explicitly in the coordinates of the total space. We obtain:

$$f(z_1, z_2, z_3) = \left(\frac{1}{2} \left(|z_1|^2 - |z_2|^2 \right), \log \frac{1}{\sqrt{2}} |\gamma - z_3|, \log \frac{1}{\sqrt{2}} \left| \gamma + z_3 - \sqrt{2} \right| \right), \tag{30}$$

where γ is as in (28). It is not difficult to see that $\Phi \circ \Gamma_0$ sends Σ to the surface in $(\mathbb{C}^*)^2$ given by

$$\Sigma' = \{v_1 + v_2 + 1 = 0\},$$

which is, topologically, a pair of pants. Then the discriminant locus is

$$\Delta = \{0\} \times \mathrm{Log}(\Sigma'),$$

which has the shape in Figure 4. This example is topologically conjugate to the one in Example 2.9, before the surface Σ' has been twisted. For the discussion of the topology of the fibres in this example we refer to Example 2.9.

In dimension $n = 2$ we have the following:

Example 5.6 (Stitched focus-focus). Using Proposition 5.4 we can obtain the following piecewise smooth fibration:

$$f(z_1, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log |\gamma(z_1, z_2) + 1| \right). \tag{31}$$

It is clearly well defined on $X = \{(z_1, z_2) \in \mathbb{C}^2 \mid \gamma(z_1, z_2) + 1 \neq 0\}$. Observe that f is topologically conjugate to a focus-focus fibration, hence to Example 2.6. The only singular fibre is $f^{-1}(0)$ and it is a pinched torus. The fibration fails to be smooth on $\mu^{-1}(0)$. This example consists of the union of two smooth Lagrangian fibrations meeting along the "stitch", $\mu^{-1}(0)$. We study this kind of piecewise smoothness in detail in §6.

Notice that in this example we are in the extremal case of Proposition 5.2, i.e. the reduced spaces are 2-dimensional and Remark 5.3 applies. In particular, the second component of f in (31) could be replaced by any T^2 invariant function G, i.e. depending on $t = \frac{1}{2} \left(|z_1|^2 - |z_2|^2 \right)$ and $u_1 = z_1 z_2$, subject to the condition that all maps G_t have 1-dimensional level sets. Using this idea it is easy to construct everywhere smooth fibrations, such as the one in Example 3.20 where $G(t, u_1) = \log |u_1 + 1|$. Of course the topology of the resulting fibration depends on the topology of the maps G_t.

We have an analogous model in dimension $n = 3$:

Example 5.7 (The leg). Consider the following affine symplectomorphism of $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$

$$\Phi : (u_1, u_2) \mapsto (-u_2, u_1 - 1). \tag{32}$$

The surface Σ is sent by $\Phi \circ \Gamma_0$ to $\Sigma' = \{v_2 + 1 = 0\}$. The amoeba of Σ' is just a straight line. The resulting fibration f is

$$f(z_1, z_2, z_3) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log |z_3|, \log |\gamma(z_1, z_2) - 1| \right). \tag{33}$$

The discriminant locus is $\{0\} \times \mathbb{R} \times \{0\} \subset \mathbb{R}^3$, a horizontal line in the plane $\{0\} \times \mathbb{R}^2$. The fibration is a piecewise smooth version of the generic singular fibration in Example 4.5. Notice that this fibration is invariant under the Hamiltonian T^2-action

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2, z_3) = (e^{i\theta_1} z_1,\, e^{-i\theta_1} z_2,\, e^{2i\theta_2} z_3), \tag{34}$$

whose moment map is

$$(z_1, z_2, z_3) \mapsto \left(\frac{|z_1|^2 - |z_2|^2}{2}, \, |z_3|^2 \right).$$

There are other choices of symplectomorphisms Φ giving piecewise smooth generic fibrations. Although not very different from the previous one, we will write other two for convenience, since we will need them in the next example. The first one is

$$\Phi : (u_1, u_2) \mapsto (u_1 - 1, u_2 - \sqrt{2}). \tag{35}$$

It gives the fibration

$$f(z_1, z_2, z_3) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \, \log|\gamma(z_1, z_2) - 1|, \, \log\left|z_3 - \sqrt{2}\right| \right), \tag{36}$$

whose discriminant locus is the vertical line $\{0\} \times \{0\} \times \mathbb{R} \subset \mathbb{R}^3$. Also in this case it is clearly invariant under a T^2 action. The last choice of Φ is

$$\Phi : (u_1, u_2) \mapsto \frac{1}{\sqrt{2}}(u_1 - u_2, \, u_1 + u_2), \tag{37}$$

giving

$$f(z_1, z_2, z_3) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \, \log|\gamma(z_1, z_2) - z_3|, \, \log|\gamma(z_1, z_2) + z_3| \right), \tag{38}$$

whose discriminant is the slope $+1$ diagonal through zero in $\{0\} \times \mathbb{R}^2$. The T^2 action in this case is given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2, z_3) = (e^{i(\theta_2+\theta_1)} z_1, \, e^{i(\theta_2-\theta_1)} z_2, \, e^{2i\theta_2} z_3), \tag{39}$$

whose moment map is

$$(z_1, z_2, z_3) \mapsto \left(\frac{|z_1|^2 - |z_2|^2}{2}, \, \frac{|z_1|^2 + |z_2|^2}{2} + |z_3|^2 \right).$$

In the above examples, the reduced spaces are all 2-dimensional. Using Remark 5.3 we can construct variations of (33) by replacing the last component of (33) with any function depending on $t = \frac{|z_1|^2 - |z_2|^2}{2}$, $s = |z_3|^2$ and $u_1 = z_1 z_2$, subject to the condition that all the maps G_t have 1-dimensional level sets. A choice providing an example of a smooth fibration is given by $G = \log|u_1 - 1|$, which gives us Example 4.5. One can do more. In fact, one can take a function G which gives an interpolation between the piecewise smooth fibration in (33) and the smooth one in Example 4.5. This can be done by taking G depending also on s, such that G is equal to $\log|\gamma(z_1, z_2) - 1|$ when s is big and equal to $\log|u_1 - 1|$ when s is small. We will say more about this later on, as this idea is useful in an important step of the main construction of the paper.

Example 5.8 (The amoeba with thin legs). We now construct an example which interpolates Example 5.5 and 5.7. Consider the smooth function:

$$H_0 = \frac{\pi}{4} \operatorname{Im}(u_1 \overline{u}_2)$$

and let η_{H_0} be the Hamiltonian vector field associated to H_0. If Φ_s is the flow generated by η_{H_0}, then the Hamiltonian symplectomorphism associated to H_0 is defined to be $\Phi_{H_0} = \Phi_1$. One computes that in our case

$$\Phi_{H_0} : (u_1, u_2) \mapsto \frac{1}{\sqrt{2}}(u_1 - u_2, u_1 + u_2).$$

It maps $\{u_1 = 0\}$ to $\{v_1 + v_2 = 0\}$. We now want a symplectomorphism which acts like Φ_{H_0} in a small ball centered at the origin and like the identity outside a slightly bigger ball. So choose a cut-off function $k : \mathbb{R}_{\geq 0} \to [0,1]$ such that, for some $\epsilon > 0$,

$$k(t) = \begin{cases} 1 & \text{when } 0 < t \leq \epsilon; \\ 0 & \text{when } t \geq 2\epsilon \end{cases} \tag{40}$$

and define the Hamiltonian

$$H = k(|u_1|^2 + |u_2|^2)H_0.$$

The Hamiltonian symplectomorphism Φ_H associated to H satisfies

$$\Phi_H(u_1, u_2) = \begin{cases} \mathrm{Id}_{\mathbb{C}^2}, & \text{when } |u_1|^2 + |u_2|^2 \geq 2\epsilon; \\ \\ \frac{1}{\sqrt{2}}(u_1 - u_2, u_1 + u_2), & \text{when } |u_1|^2 + |u_2|^2 \leq \epsilon. \end{cases}$$

Now let Ψ be the affine symplectomorphism

$$\Psi : (v_1, v_2) \mapsto \frac{1}{\sqrt{2}}(v_1 - v_2, v_1 + v_2 - \sqrt{2}).$$

and finally, define $\Phi = \Psi \circ \Phi_H$. It is clear that

$$\Phi(u_1, u_2) = \begin{cases} \Psi, & \text{when } |u_1|^2 + |u_2|^2 \geq 2\epsilon; \\ \\ (-u_2, u_1 - 1), & \text{when } |u_1|^2 + |u_2|^2 \leq \epsilon. \end{cases}$$

Notice that Φ acts like in (32) on the ball of radius $\sqrt{\epsilon}$ around the origin, i.e. in a neighborhood of the surface $\{u_1 = 0\}$, and like in (29) outside a larger ball. We use this Φ to construct a fibration f using Proposition 5.4. One can then see that $\Phi \circ \Gamma_0$ sends Σ to a surface Σ' such that $\mathrm{Log}(\Sigma') \subset \mathbb{R}^2$ is a 3-legged amoeba with the end of the horizontal leg pinched down to a straight line. The discriminant locus of f is then $\Delta = \{0\} \times \mathrm{Log}(\Sigma') \subset \mathbb{R}^3$. Of course, f fails to be smooth on the slice $\mu^{-1}(0)$. Using the same method we can twist Σ suitably and obtain a fibrations having discriminant locus an amoeba with three thin legs (cf. Figure 5). For example, to pinch the diagonal leg to a thin line, choose a smooth function H_0 generating the Hamiltonian symplectomorphism

$$(u_1, u_2) \to \left(u_1 + \frac{1}{\sqrt{2}}, u_2 + \frac{1}{\sqrt{2}}\right).$$

Cut H_0 off with a function ρ which vanishes when $|u_2|^2 \leq M/2$, for some big M, and is equal to 1 when $|u_2|^2 \geq M$. This produces a Hamiltonian H. Now one proceeds as before. With an almost identical procedure one pinches down the vertical leg. The final choice of symplectomorphism Φ pinching down all three legs simultaneously may look like:

$$\Phi(u_1, u_2) = \begin{cases} (-u_2, u_1 - 1), & \text{when } |u_1|^2 + |u_2|^2 \leq \epsilon; \\ \\ (u_1 - 1, u_2 - \sqrt{2}), & \text{when } |u_1|^2 + |u_2 - \sqrt{2}|^2 \leq \epsilon; \\ \\ \frac{1}{\sqrt{2}}(u_1 - u_2,\, u_1 + u_2), & \text{when } |u_2|^2 \geq M; \\ \\ \Psi, & \text{everywhere else.} \end{cases} \tag{41}$$

It is clear that this piecewise smooth example is topologically conjugate to the one in Example 2.9. Here we have made explicit the twistings described there. In §7 we will show that this fibration can be modified so that it is actually smooth towards the ends of the three legs. For this we will develop further the smoothing method sketched at the end of Example 5.7. Also in §7, we will show that this fibration can be modified so that it is smooth away from a neighborhood homeomorphic to a 2-disk containing the codimension 1 part of its discriminant.

The next result states existence of Lagrangian sections of the fibrations in the previous examples.

Proposition 5.9. The fibrations in Example 5.5 and 5.8 have smooth Lagrangian sections which do not intersect the critical surface $\mathrm{Crit}(f)$.

Proof. Consider the symplectomorphism Φ from Example 5.5. The reduced fibration at time $t = 0$, i.e. the map $G_0 = \mathrm{Log} \circ \Phi \circ \Gamma_0$, has many Lagrangian sections, since the Log fibration has many. In particular we can choose one which does not intersect $\Sigma = \mathrm{Crit}(f)$, this follows for example by observing that the following Lagrangian section of the Log fibration

$$(x_1, x_2) \mapsto (ie^{x_1}, e^{x_2}) \tag{42}$$

does not intersect the surface $\Sigma' = \{v_1 + v_2 + 1\}$. It is easy to see that a section which does not intersect Σ can be lifted to $\mu^{-1}(0)$. The image of this lift is a coisotropic 2 dimensional submanifold of X. Applying the coisotropic embedding theorem, we can extend this submanifold to a Lagrangian submanifold along a direction which is transversal to $\mu^{-1}(0)$, e.g. along $i\eta$, where η is the Hamiltonian vector field of the S^1 action. This submanifold is then the image of a section of the fibration in Example 5.5.

In the case of Φ from Example 5.8, $\Phi(\Sigma)$ is a small perturbation of Σ' as above. One can see that the section in (42) also avoids $\Phi(\Sigma)$. Then the argument follows as before. □

We notice that "smooth section" in the above statement means a section whose image is a smooth, manifold. In fact there is no obvious notion of what a smooth map from the base is, since there is no notion of smooth coordinates.

In view of Proposition 5.4, the fibrations of Examples 5.5 and 5.8 are all given by piecewise C^∞ maps. More precisely, away from Σ, they are the union of two honest C^∞ fibrations meeting and coinciding along $\mu^{-1}(0)$. A similar phenomenon occurs in special Lagrangian geometry [21]. This kind of piecewise smoothness deserves careful attention and we study it in the next Section.

6 Stitched fibrations

In [3] we proposed to extend the classical theory of action-angle coordinates to a particular type of piecewise smooth fibrations, which we called *stitched fibrations*. Here we review how this theory was further developed in [2] and extend some of those techniques to fibrations which are not proper. For details and complete proofs we refer the reader to [2]. The material in this section is primarily technical but necessary to understand the lack of regularity of the fibrations in §5. The techniques here are useful, in particular, for the construction of Lagrangian fibrations of negative type §7.

Definition 6.1. Let (X, ω) be a smooth $2n$-dimensional symplectic manifold. Suppose there is a free Hamiltonian S^1 action on X with moment map $\mu : X \to \mathbb{R}$. Let $X^+ = \{\mu \geq 0\}$ and $X^- = \{\mu \leq 0\}$. Given a smooth $(n-1)$-dimensional manifold M, a map $f : X \to \mathbb{R} \times M$ is said to be a *stitched Lagrangian fibration* if there is a continuous S^1 invariant function $G : X \to M$, such that the following holds:

(i) Let $G^\pm = G|_{X^\pm}$. Then G^+ and G^- are restrictions of C^∞ maps on X;

(ii) f can be written as $f = (\mu, G)$ and f restricted to $X^\pm$ is a proper submersion with connected Lagrangian fibres.

We call $Z = \mu^{-1}(0)$ the *seam* and $\Gamma = f(Z) \subseteq \{0\} \times M$ the *wall*. We denote $f^{\pm} = f|_{X^{\pm}}$.

Notice that we do not require f to be onto $\mathbb{R} \times M$, so we denote $B = f(X)$ and $B^{\pm} = f(X^{\pm})$. In general, a stitched fibration will only be piecewise C^{∞}, however all its fibres are smooth Lagrangian tori. Observe also that $f^{\pm}$ is the restriction of a C^{∞} map, it is not a priori required to extend to a smooth Lagrangian fibration beyond $X^{\pm}$. Throughout this section we will always assume (unless otherwise stated) that the pair (B, Γ) is diffeomorphic to the pair (D^n, D^{n-1}), where $D^k \subset \mathbb{R}^k$ is an open unit ball centered at the origin and $\mathbb{R}^{n-1}$ is embedded in $\mathbb{R}^n$. Later on we will consider more general bases –e.g. non-simply-connected– when we speak about monodromy.

We now review some the examples given in §5:

Example 6.2 (Stitched focus-focus, revisited). Consider the piecewise smooth fibration in Example 5.6. One can easily see that the restriction of f to $X - f^{-1}(0)$ is a stitched Lagrangian fibration.

Analogously, the piecewise smooth fibration in Example 5.7 gives rise to a stitched fibration when restricted to the complement of the union of the singular fibres. There is another important example in dimension three:

Example 6.3 (The amoeba, revisited). Consider the fibration in Example 5.5. When restricted to $X - f^{-1}(\Delta)$, f defines a stitched Lagrangian fibration. The seam is $Z = \mu^{-1}(0) - f^{-1}(\Delta)$, notice that in this case Z has three connected components.

To understand the geometry of stitched fibrations in a neighborhood of a point on the wall, it is convenient to allow a more general set of coordinates than just the smooth ones.

Definition 6.4. A set of coordinates on $B \subseteq \mathbb{R} \times M$, given by a map $\phi : B \to \mathbb{R}^n$, is said to be *admissible* if the components of $\phi = (\phi_1, \dots, \phi_n)$ satisfy the following properties:

(i) ϕ_1 is the restriction to B of the projection map $\mathbb{R} \times M \to \mathbb{R}$;

(ii) for $j = 2, \dots, n$ the restrictions of ϕ_j to B^+ and B^- are locally restrictions of smooth functions on B.

Essentially, admissible coordinates are those such that $\phi \circ f$ is again stitched. Let $f : X \to B$ be a stitched Lagrangian fibration and let ϕ be a set of admissible coordinates. For $j = 2, \dots, n$, $f_j^{\pm} = \phi_j \circ f|_{X^{\pm}}$ is the restriction of a C^{∞} function on X to $X^{\pm}$ and we can write $f = (\mu, f_2^{\pm}, \dots, f_n^{\pm})$. Let η_1 and $\eta_j^{\pm}$ be the Hamiltonian vector fields of μ and $f_j^{\pm}$ respectively. In order to measure how far f is from being smooth, it makes sense to compare η_j^+ and η_j^- in the only place where they exist simultaneously, i.e. along Z. In fact it is not difficult to show that there are S^1 invariant functions a_j on Z such that

$$(\eta_j^+ - \eta_j^-)|_Z = a_j \, \eta_1|_Z. \tag{43}$$

Clearly, when $\phi \circ f$ is smooth $a_2 = \cdots = a_n = 0$.

It is convenient to interpret the S^1 invariant functions $(a_2, \dots, a_n)$ in (43) as follows. First observe that the seam of a stitched fibration is an S^1-bundle $p : Z \to \bar{Z} := Z/S^1$ such that:

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "f|_Z"'] & & \bar{Z} \arrow[dl, "\bar{f}"] \\ & \Gamma & \end{tikzcd}$$

where $\bar{Z}$ has the reduced symplectic form and $\bar{f}$ is the reduced Lagrangian fibration over the wall Γ. We also have the vertical $(n-1)$-plane distribution:

$$\mathfrak{L} = \ker \bar{f}_* \subset T\bar{Z}$$

tangent to the fibres of $\bar{f}$. Clearly, a choice of coordinates around $b \in \Gamma$ induces a frame $\bar{\eta} = (\bar{\eta}_2, \dots, \bar{\eta}_n)$ of $\mathfrak{L}$, where $\bar{\eta}_j = p_* \eta_j^+ = p_* \eta_j^-$. Define ℓ_1 to be the section of $\mathfrak{L}^*$ such that:

$$\ell_1(\bar{\eta}_j) = a_j.$$

It is not difficult to see (and we prove it in [2]) that ℓ_1 is *fibrewise closed*, i.e. when restricted to the fibres of $\bar{f}$, ℓ_1 is a closed 1-form. One can prove that a different choice of coordinates around $b \in \Gamma$ induces a frame $\bar{\eta}'$ and a section ℓ_1' such that $\ell_1' - \ell_1 = \delta$, where δ is fibrewise constant, i.e. the Lie derivative $\mathcal{L}_{\bar{\eta}_j} \delta = 0$ for all $j = 2, \dots, n$ (cf. [2]Proposition 4.2). As a corollary, if there is a change of coordinates in the base which makes a stitched fibration smooth, then ℓ_1 is fibrewise constant. The invariant ℓ_1 is a first order measure of how much f fails to be smooth along Z. Of course one also needs to consider "higher order terms" to fully understand the behavior of a stitched fibration near the seam.

In the smooth case, action-angle coordinates defined over B depend on a choice of a basis of $H_1(X, \mathbb{Z})$. In the case of stitched fibrations it is convenient to generalize this idea as follows. We choose a pair of bases $\gamma^{\pm} = (\gamma_1, \gamma_2^{\pm}, \dots, \gamma_n^{\pm})$ of $H_1(X, \mathbb{Z})$ such that

(a) γ_1 is represented by an orbit of the S^1 action,

(b) $\gamma_j^+ = \gamma_j^- + m_j \gamma_1$, for some $m_2, \dots, m_n \in \mathbb{Z}$.

Condition (b) simply means that $p_* \gamma^+ = p_* \gamma^-$ under the map $p_* : H_1(X, \mathbb{Z}) \to H_1(X/S^1, \mathbb{Z})$. Such a choice of bases will be useful to understand fibrations over non simply connected bases where monodromy may occur. The following proposition generalizes the notion of action angle coordinates on the base.

Proposition 6.5. Let $f : X \to B$ be a stitched fibration and let $\gamma^{\pm}$ be bases of $H_1(X, \mathbb{Z})$ satisfying the above conditions. Then the restrictions of $\gamma^{\pm}$ to $H_1(X^{\pm}, \mathbb{Z})$ induce embeddings,

$$\Lambda^{\pm} \hookrightarrow T^*_{B^{\pm}}.$$

Let $\alpha^{\pm} : B^{\pm} \to \mathbb{R}^n$ be the corresponding action coordinates satisfying $\alpha^{\pm}(b) = 0$ for some $b \in \Gamma$. Then the map

$$\alpha = \begin{cases} \alpha^+ & \text{on } B^+ \\ \alpha^- & \text{on } B^- \end{cases}$$

is an admissible change of coordinates. If $b_1, \dots b_n$ denote the action coordinates on B given by α, then $\{db_1, \dots db_n\}$ is a basis of Λ^+ and Λ^-. Furthermore, the reduced space $\bar{Z}$ can be identified with $T^*\Gamma / \langle db_2, \dots, db_n \rangle_{\mathbb{Z}}$ and the reduced fibration $\bar{f}$ can be identified with the standard projection $\bar{\pi}$. Moreover ℓ_1 satisfies

$$\int_{[db_j]} \ell_1 = m_j, \quad j = 2, \dots, n \tag{44}$$

where $[db_j] \in H_1(\bar{Z}, \mathbb{Z})$ is the class represented by db_j.

Proof. The first statements follow from the results in §3. For the proof of the last statement we refer the reader to [2]§4. □

Recall that to establish the existence of action-angle coordinates, in the classical case, one chooses a smooth Lagrangian section. In the stitched case we choose a continuous section $\sigma : B \to X$ such that $\sigma|_{B^{\pm}}$ are the restrictions of smooth maps and $\sigma(B)$ is a smooth Lagrangian submanifold. Such sections always exist locally, for example the one constructed in Proposition 5.9 is a section of this type. We denote a stitched fibration $f : X \to B$ together with a choice of basis γ of $H_1(X, \mathbb{Z})$ and a section σ as above by $\mathcal{F} = (X, B, f, \gamma, \sigma)$.

Definition 6.6. Two stitched fibrations $\mathcal{F} = (X, B, f, \gamma, \sigma)$ and $\mathcal{F}' = (X', B', f', \gamma', \sigma')$, with seams Z and Z' respectively are *symplectically conjugate* if there are neighborhoods $W \subseteq B$ of $\Gamma := f(Z)$ and $W' \subseteq B'$ of $\Gamma' := f'(Z')$ such that $\mathcal{F}|_W$ and $\mathcal{F}'|_{W'}$ are (ψ, ϕ)-conjugate, where ψ is an S^1 equivariant C^∞ symplectomorphism sending Z' to Z and ϕ is a C^∞ diffeomorphism such that $\psi \circ \sigma' = \sigma \circ \phi$ and $\psi_* \gamma' = \gamma$. The set of equivalence classes under this relation will be called *germs of stitched fibrations*.

Notice that in the above definition we are allowed to shrink to a smaller neighborhood of Γ but not to a smaller Γ. So germs are meant to be defined around Γ and not around a point. In [2] we classified stitched Lagrangian fibrations up to symplectic conjugation in terms of certain invariants. We review this classification here.

First we illustrate a basic construction of stitched fibrations.

Example 6.7 (Normal forms). Let $(b_1, \ldots, b_n)$ be the standard coordinates on $\mathbb{R}^n$. Let (U, Γ) be a pair of subsets of $\mathbb{R}^n$ diffeomorphic to (D^n, D^{n-1}) and $\Gamma = U \cap \{b_1 = 0\}$. Define $U^+ = U \cap \{b_1 \geq 0\}$ and $U^- = U \cap \{b_1 \leq 0\}$. Consider the lattice $\Lambda = \operatorname{span}\langle db_1, \ldots, db_n \rangle_{\mathbb{Z}}$ and form the symplectic manifold T^*U/Λ. Denote by π the standard projection onto U. Let $Z = \pi^{-1}(\Gamma)$ and $\bar{Z} = Z/S^1$, where the S^1 action is the one generated by db_1. Suppose there is an open neighborhood $V \subseteq T^*U/\Lambda$ of Z and a map $u : V \to \mathbb{R}^n$ which is a proper, smooth, S^1-invariant Lagrangian submersion with components $(u_1, \ldots, u_n)$ such that $u|_Z = \pi$ and $u_1 = b_1$. Now define the following subsets of T^*U/Λ,

$$Y^+ := \pi^{-1}(U^+), \quad Y := Y^+ \cup V, \quad Y^- := Y \cap \pi^{-1}(U^-)$$

and define the map $f_u : Y \to \mathbb{R}^n$ by

$$f_u = \begin{cases} u & \text{on } Y^-, \\ \pi & \text{on } Y^+. \end{cases} \tag{45}$$

Clearly $f_u : Y \to \mathbb{R}^n$ is a stitched fibration. Denote $B_u := f_u(Y)$. The zero section σ_0 of π is, perhaps after a change of coordinates in the base, a section of f_u. Let γ_0 be the basis of $H_1(Y, \mathbb{Z})$ induced by Λ. We call the stitched fibration $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$ a *normal form*.

Now suppose $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$ is as above and let $(b, y) = (b_1, \ldots, b_n, y_1, \ldots, y_n)$ be canonical coordinates on T^*B_u so that y gives coordinates on the fibre $T^*_b B_u$. Let W be a neighborhood of Γ inside $u(V)$. If $r \in \mathbb{R}$ is a parameter, for any $b = (0, b_2, \ldots, b_n) \in \Gamma$, let (r, b) denote the point $(r, b_2, \ldots, b_n) \in \mathbb{R}^n$. Given $(r, b) \in W$, denote by $L_{r,b}$ the fibre $u^{-1}((r, b))$. For every fibre $F_b \subset Z$ of π, consider the symplectomorphism

$$(y_1, \ldots, y_n, \sum_{k=1}^{n} x_k dy_k) \mapsto (x_1, b_2 + x_2, \ldots, b_n + x_n, y_1, \ldots, y_n), \tag{46}$$

between a neighborhood of the zero section of T^*F_b and a neighborhood of F_b in V. If W is sufficiently small, for every $(r, b) \in W$, the Lagrangian submanifold $L_{r,b}$ will be the image of the graph of a closed 1-form on F_b. Due to the S^1 invariance of u and the fact that $u_1 = b_1$, this 1-form has to be of the type

$$r dy_1 + \ell(r, b),$$

where $\ell(r, b)$ is the pull back to F_b of a closed one form on $\bar{F}_b$. Denote by $\ell(r)$ the smooth one parameter family of sections of $\mathfrak{L}^*$ such that $\ell(r)|_{\bar{F}_b} = \ell(r, b)$. The condition $u|_Z = \pi$ implies that $\ell(0, b) = 0$. Furthermore, the N-th order Taylor series expansion of $\ell(r)$ in the parameter r can be written as

$$\ell(r) = \sum_{k=1}^{N} \ell_k \, r^k + o(r^N), \tag{47}$$

where the ℓ_k's are fibrewise closed sections of $\mathfrak{L}^*$.

Definition 6.8. With the above notation, we define

(i) $\mathscr{L}_Z$ the set of sequences $\ell = \{\ell_k\}_{k\in\mathbb{N}}$ such that ℓ_k is a fibrewise closed section of $\mathfrak{L}^*$;

ii) $\mathscr{U}_Z$ the set of pairs (V, u) where $V \subseteq T^*U/\Lambda$ is a neighborhood of Z and $u : V \to \mathbb{R}^n$ is a proper, smooth, S^1-invariant Lagrangian submersion with components $(u_1, \ldots, u_n)$ such that $u|_Z = \pi$ and $u_1 = b_1$.

As above, to a given $(V, u) \in \mathscr{U}_Z$ we can associate a unique sequence $\ell \in \mathscr{L}_Z$. Conversely, in [2]§5 we showed that for any given sequence $\ell \in \mathscr{L}_Z$ there is some $(V, u) \in \mathscr{U}_Z$, therefore a normal form, associated to it. Clearly, this (V, u) is not unique.

In [2] we proved that stitched fibrations are normalized according to the following:

Proposition 6.9. Every stitched fibration $\mathcal{F} = (X, B, f, \sigma, \gamma)$ is symplectically conjugate to a normal form $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$

Proof. Let Z be the seam of $\mathcal{F}$, ω_{red} the reduced symplectic form on $\bar{Z}$ and $\bar{f} : \bar{Z} \to \Gamma$ the reduced fibration. Using the coisotropic embedding theorem we can assume w.l.o.g. that $X = \mathbb{R} \times S^1 \times \bar{Z}$ with symplectic form $\omega = \omega_{red} + ds \wedge dt$, where (t, s) are coordinates on $\mathbb{R} \times S^1$ and the projection onto $\mathbb{R}$ is the moment map μ. On X, we can define an "auxiliary" smooth Lagrangian fibration given by

$$\tilde{\pi}(t, s, p) = (t, \bar{f}(p)).$$

Fix a basis γ of $H_1(X, \mathbb{Z}) \cong H_1(S^1 \times \bar{Z}, \mathbb{Z})$ and a smooth Lagrangian section of $\tilde{\pi}$. The action-angle coordinates of $\tilde{\pi}$ with respect to γ and σ induce a C^∞ symplectomorphism

$$T^*U/\Lambda \cong X \tag{48}$$

for some open neighborhood U of $0 \in \mathbb{R}^n$ with action coordinates $(b_1, \ldots, b_n)$. The angle coordinates are $(y_1, \ldots, y_n)$. In these coordinates $Z = \{b_1 = 0\}$ and $\Gamma = U \cap \{b_1 = 0\}$. While f becomes:

$$f = \begin{cases} u^+ & \text{on } X^+; \\ u^- & \text{on } X^-, \end{cases} \tag{49}$$

where $u^\pm$ correspond to $f^\pm$. It follows that $u^+|_Z = u^-|_Z = \pi|_Z$.

One can show that u^+ can be extended as a smooth proper Lagrangian fibration a little bit beyond X^+, i.e. we can find a smooth proper Lagrangian fibration $\tilde{u}^+$ defined on a set $X^+ \cup V$, where V is some open neighborhood of Z, such that $\tilde{u}^+|_{X^+} = u^+$. For the details of this extension see [2], Proposition 6.3. To put f in normal form, we consider the action-angle coordinates associated to $\tilde{u}^+$ with section σ and basis γ of $H_1(X, \mathbb{Z})$ as above. In these coordinates, $X^+ \cup V$ becomes T^*U/Λ and $\tilde{u}^+$ becomes the projection π. Again in action-angle coordinates of $\tilde{u}^+$, a Lagrangian extension $\tilde{u}^-$ of u^-, becomes $(W, u) \in \mathscr{U}_Z$ for some $W \subseteq T^*U/\Lambda$ and some Lagrangian fibration u. Then we simply define $Y^+ = T^*U^+/\Lambda$, $Y = Y^+ \cup W$, $Y^- = Y \cap \pi^{-1}(U^-)$ and

$$f_u = \begin{cases} u & \text{on } Y^-, \\ \pi & \text{on } Y^+. \end{cases} \tag{50}$$

□

When $\mathcal{F}$ is smooth, its normal form is $\mathcal{F}_\pi$. This is Arnold-Liouville theorem (cf. Corollary 3.5). Given a stitched Lagrangian fibration $\mathcal{F} = (X, B, f, \sigma, \gamma)$ with normal form $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$, we respectively denote by Z_{nor} and Γ_{nor} the seam and the wall of $\mathcal{F}_u$ and by $\bar{Z}_{\text{nor}}$ the S^1 reduction of Z_{nor}.

Definition 6.10. Let $\mathcal{F} = (X, B, f, \sigma, \gamma)$ be a stitched fibration with normal form $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$. Let $\ell \in \mathscr{L}_{\bar{Z}_{\text{nor}}}$ be the unique sequence determined by $(V, u) \in \mathscr{U}_{Z_{\text{nor}}}$ defining $\mathcal{F}_u$. We call $\text{inv}(\mathcal{F}) := (\bar{Z}_{\text{nor}}, \ell)$ the *invariants* of $\mathcal{F}$. We say that the invariants of $\mathcal{F}$ vanish if for all $k \in \mathbb{N}$, $\ell_k \equiv 0$ when restricted to the reduced fibres of $\mathcal{F}_u$. We say that the invariants of $\mathcal{F}$ are fibrewise constant if all the ℓ_k's are fibrewise constant.

We prove in [2]Corollary 6.9 that $\text{inv}(\mathcal{F})$ is independent on the choice of normal form.

We will now see that every specified data $(\bar{Z}_{\text{nor}}, \ell)$, with ℓ_1 satisfying an integrality condition can be realized as the invariants of a stitched fibration. Notice that $\bar{Z}_{\text{nor}}$ is uniquely determined by Γ as $\bar{Z}_{\text{nor}} = T^*\Gamma/\bar{\Lambda}$, where $\bar{\Lambda} = \text{span}\langle db_2, \ldots, db_n\rangle_{\mathbb{Z}}$. We have

Theorem 6.11. Given any pair (U, Γ_{nor}) of subsets of $\mathbb{R}^n$, diffeomorphic to (D^n, D^{n-1}) and with $\Gamma_{\text{nor}} = U \cap \{b_1 = 0\}$, a sequence $\ell = \{\ell_k\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_{\text{nor}}}$ and integers $m_2, \ldots, m_n$ such that

$$\int_{[db_j]} \ell_1 = m_j, \quad \text{for all } j = 2, \ldots, n, \tag{51}$$

there exists a smooth symplectic manifold (X, ω) and a stitched Lagrangian fibration $f : X \to U$ satisfying the following properties:

(i) the coordinates $(b_1, \ldots, b_n)$ on U are action coordinates of f with $\mu = f^*b_1$ the moment map of the S^1 action;

(ii) the periods $\{db_1, \ldots, db_n\}$, restricted to $U^\pm$ correspond to bases $\gamma^\pm = \{\gamma_1, \gamma_2^\pm, \ldots, \gamma_n^\pm\}$ of $H_1(X, \mathbb{Z})$ satisfying conditions (a) and (b) prior to Proposition 6.5;

(iii) there is a Lagrangian section σ of f, such that $(\bar{Z}_{\text{nor}}, \ell)$ are the invariants of $(X, f, U, \sigma, \gamma^+)$.

Proof. We refer the reader to [2]Theorem 6.12 for the details. Roughly, one starts with U^+ and U^- regarded as disjoint sets. These give two disjoint pieces $X^\pm = T^*U^\pm/\Lambda^\pm$, where $\Lambda^\pm = \langle db_1, \ldots, db_n\rangle_{\mathbb{Z}}$. Let $Z^\pm = \partial X^\pm$. On X^+ we have Hamiltonian vector fields $\eta_1 = \partial_{b_1}$ and $\eta_j^+ = \partial_{b_j}$ for $j = 2, \ldots, n$. We can also define vector fields on Z^+:

$$\eta_j^- = \eta_j^+ - a_j\eta_1$$

where $(a_2, \ldots, a_n)$ are the coefficients of ℓ_1. One can (topologically) glue X^+ and X^- using a map $Q : Z^- \to Z^+$ defined in terms of the $\mathbb{R}^n$ action induced by the flows of η_j^-. Intuitively, Q identifies the fibres inside each of the two halves Z^- and Z^+ after the fibres inside Z^- have been twisted by iteratively flowing in the direction of $\eta_1, \eta_2^-, \ldots, \eta_n^-$. The integrality condition (51) guarantees that (ii) is satisfied. One can extend Q to give a *smooth* symplectomorphism $\tilde{Q}$ between open neighborhoods of $Z^\pm$. For this one needs to consider invariants ℓ_k, for $k > 1$. The choice of $\tilde{Q}$ is determined by $\{\ell_k\}$. This gluing gives a smooth symplectic manifold (X, ω) and a stitched fibration $f : X \to U$, which by construction is such that $\text{inv}(\mathcal{F}) = (\bar{Z}_{\text{nor}}, \ell)$. ∎

We also have the following (cf. [2]Theorem 6.11):

Theorem 6.12. Let $\mathcal{F}$ and $\mathcal{F}'$ be stitched fibrations. Then,

(i) two stitched fibrations $\mathcal{F}$ and $\mathcal{F}'$ are conjugate if and only if $\text{inv}(\mathcal{F}) = \text{inv}(\mathcal{F}')$;

(ii) $\mathcal{F}$ is smooth if and only if $\text{inv}(\mathcal{F})$ vanish;

(iii) $\mathcal{F}$ becomes smooth after an admissible change of coordinates on the base if and only if $\text{inv}(\mathcal{F})$ are fibrewise constant.

In other words, the set of germs of stitched fibrations is classified by the pairs $(\bar{Z}_{\mathrm{nor}}, \ell)$. We say that a fibration is *fake stitched* if it becomes smooth after an admissible change of coordinates on the base. One interesting consequence of Theorem 6.11, which we will exploit later on, is that from a given set of invariants we can form another one for example by summing to the sequence ℓ another sequence or by multiplying elements ℓ_k by pull backs of smooth functions on the base. The new invariants give rise to new stitched fibrations.

Example 6.13. Consider a smooth proper Lagrangian fibration $f: X \to B$, with $B = \mathbb{R} \times M$ and $f = (\mu, G)$, where μ is the moment map of a free S^1 action and G is S^1 invariant. Assuming B is contractible and having chosen bases $\gamma^{\pm}$ of $H_1(X, \mathbb{Z})$ as in (a) and (b) above, on B we can apply the admissible change of coordinates α as in Proposition 6.5. Clearly $f' = \alpha \circ f$ is (tautologically) a fake stitched fibration. Given a Lagrangian section σ of f', it easy to see that the normal form for $(X, f'(B), f', \gamma^+, \sigma)$ is of the type $(Y, U, f_u, \gamma_0, \sigma_0)$ where $Y = T^*U/\Lambda$ and

$$u(y_1, \ldots, y_n, b_1, \ldots, b_n) = (b_1, b_2 - m_2 b_1, \ldots, b_n - m_n b_1),$$

i.e. the projection composed with a linear change of coordinates. In this case the only non-zero invariant is ℓ_1 which is given by

$$\ell_1 = \sum_j m_j dy_j.$$

Clearly ℓ_1 is fibrewise constant.

Monodromy

We now study stitched fibrations defined over non simply connected bases. In this case, the underlying topological T^n bundle may have monodromy. When $\mathcal{F}$ is smooth, monodromy can be read from the holonomy of the affine structure on the base. This is no longer true for stitched fibrations in general. This is the case, for instance, of Example 5.5; in fact, in [3]Proposition 7 (cf. also Remark 5) we gave explicit evidence of this. We show now that monodromy can alternatively be detected from the behavior of the first order invariant ℓ_1. We restrict to some specific examples with unipotent monodromy.

Example 6.14. Let $U \subset \mathbb{R}^2$ be an open annulus in $\mathbb{R}^2$ centered at the origin. As usual denote $U^+ = U \cap \{b_1 \geq 0\}$, $U^- = U \cap \{b_1 \leq 0\}$ and $\Gamma = U^+ \cap U^-$. This time Γ is disconnected. We let $\Gamma_u = \Gamma \cap \{b_2 \geq 0\}$ and $\Gamma_d = \Gamma \cap \{b_2 \leq 0\}$ be the upper and lower parts of Γ respectively. Now let $f: X \to \mathbb{R}^2$ be a stitched Lagrangian fibration such that $f(X) = U$. Observe that the seam Z has two connected components: $Z_u = f^{-1}(\Gamma_u)$ and $Z_d = f^{-1}(\Gamma_d)$. Denote by $\bar{Z}_u$ and $\bar{Z}_d$ the respective S^1 quotients, i.e. the connected components of $\bar{Z}$. Let $b \in \Gamma_u$ and choose as generator of $\pi_1(U, b)$ an anti-clock-wise oriented curve starting at b and going once around 0. Suppose that with respect to a basis $\{\gamma_1, \gamma_2\}$ of $H_1(F_b, \mathbb{Z})$ the monodromy is

$$\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}, \tag{52}$$

for some integer $m \neq 0$. In this case we must have that γ_1 is represented by the orbits of the S^1 action. As usual let $X^{\pm} = f^{-1}(U^{\pm})$. Since $U - \Gamma_d$ is contractible we can think of $\{\gamma_1, \gamma_2\}$ as a basis of $H_1(f^{-1}(U - \Gamma_d), \mathbb{Z})$. Consider the diagrams:

$$\begin{tikzcd} & H_1(X^+, \mathbb{Z}) \arrow[dr] & \\ H_1(f^{-1}(U - \Gamma_d), \mathbb{Z}) \arrow[ur] \arrow[rr, "j_+"] & & H_1(f^{-1}(U - \Gamma_u), \mathbb{Z}) \end{tikzcd}$$

or

$$
\begin{tikzcd}
H_1(f^{-1}(U-\Gamma_d),\mathbb{Z}) \arrow[rr, "j_-"] \arrow[dr] & & H_1(f^{-1}(U-\Gamma_u),\mathbb{Z}) \\
& H_1(X^-,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

induced by inclusions and restrictions. The map j_+ identifies $\{\gamma_1,\gamma_2\}$ with a basis $\{\gamma_1,\gamma_2^+\}$ of $H_1(f^{-1}(U-\Gamma_u),\mathbb{Z})$, whereas j_- with a basis $\{\gamma_1,\gamma_2^-\}$. Notice that monodromy is given by $j_+^{-1}\circ j_-$. Therefore we must have $\gamma_2^+ = m\gamma_1+\gamma_2^-$. Hence $\{\gamma_1,\gamma_2^+\}$ and $\{\gamma_1,\gamma_2^-\}$ satisfy conditions (a) and (b) in the previous section. Applying Proposition 6.5 to f restricted to $f^{-1}(U-\Gamma_u)$ we can consider the action coordinates map α constructed by taking action coordinates with respect to $\{\gamma_1,\gamma_2^+\}$ on U^+ and with respect to $\{\gamma_1,\gamma_2^-\}$ on U^-. Denote by (b_1^d,b_2^d) such coordinates. Similarly on $U-\Gamma_d$ we can consider action angle coordinates with respect to the basis $\{\gamma_1,\gamma_2\}$. Denote by (b_1^u,b_2^u) these coordinates. In particular we have the identifications

$$\bar{Z}_d = T^*\Gamma_d \,/\, \langle db_2^d\rangle_{\mathbb{Z}}$$

and

$$\bar{Z}_u = T^*\Gamma_u \,/\, \langle db_2^u\rangle_{\mathbb{Z}}.$$

With respect to this choice of coordinates we can compute the first order invariants of f, ℓ_1^u and ℓ_1^d on $\bar{Z}_u$ and $\bar{Z}_d$, respectively. Then (44) should hold, therefore we obtain

$$\int_{[db_2^u]} \ell_1^u = 0 \quad\text{and}\quad \int_{[db_2^d]} \ell_1^d = m.$$

This tells us that monodromy can be read from a jump in cohomology class of the first order invariant associated to action coordinates.

Using the methods of Theorem 6.11 we can also construct stitched Lagrangian fibrations with prescribed monodromy and invariants. In fact we have

Theorem 6.15. Let $U\subset\mathbb{R}^2$ be an annulus as above with coordinates (b_1,b_2). Let $\bar{Z}_d = T^*\Gamma_d\,/\,\langle db_2\rangle_{\mathbb{Z}}$ and $\bar{Z}_u = T^*\Gamma_u\,/\,\langle db_2\rangle_{\mathbb{Z}}$ with projections $\bar{\pi}^d$ and $\bar{\pi}^u$ and bundles $\mathfrak{L}_d = \ker\bar{\pi}^d_*$ and $\mathfrak{L}_u = \ker\bar{\pi}^u_*$ respectively. Given an integer m and sequences $\ell^d = \{\ell_k^d\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_d}$ and $\ell^u = \{\ell_k^u\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_u}$ such that

$$\int_{[db_2]} \ell_1^u = 0 \quad\text{and}\quad \int_{[db_2]} \ell_1^d = m,$$

there exists a smooth symplectic manifold (X,ω) and a stitched Lagrangian fibration $f:X\to U$ having monodromy (52) with respect to some basis $\gamma=\{\gamma_1,\gamma_2\}$ of $H_1(f^{-1}(U-\Gamma_d),\mathbb{Z})$ and satisfying the following properties:

(i) the coordinates (b_1,b_2) are action coordinates of f with moment map f^*b_1;

(ii) the periods $\{db_1,db_2\}$, restricted to $U^{\pm}$ correspond to the basis $\{\gamma_1,\gamma_2\}$;

(iii) there is a Lagrangian section σ of f, such that $(\bar{Z}_u,\ell^u)$ and $(\bar{Z}_d,\,\ell^d)$ are the invariants of $(f^{-1}(U-\Gamma_d),\ f,\ U-\Gamma_d,\ \sigma,\ \gamma)$ and $(f^{-1}(U-\Gamma_u),\ f,\ U-\Gamma_u,\ \sigma,\ j_+(\gamma))$ respectively.

The fibration (X,f,U) satisfying the above properties is unique up to fibre preserving symplectomorphism.

Proof. This is just a repetition of the arguments in Theorem 6.11 for each component of $\Gamma=\Gamma_d\cup\Gamma_u$. We leave the details as an exercise. $\square$

Remark 6.16. Notice that the stitched fibrations discussed in Example 6.14 are more general than the ones constructed in Theorem 6.15. We illustrate this with an example. Let U^- and U^+ be two "half annuli" of the same width but of different radii (as depicted in Figure 10). If $b^\pm = (b_1^\pm, b_2^\pm)$ denote coordinates on $U^\pm$ and we let $\Lambda^\pm = \langle db_1^\pm, db_2^\pm \rangle_{\mathbb{Z}}$, then we can glue together $X^+ = T^*U^+/\Lambda^+$ and $X^- = T^*U^-/\Lambda^-$ after choosing suitable invariants and applying the usual method of Theorem 6.11. We first glue the lower boundaries of X^+ and X^- and then the upper boundaries, (as indicated by the arrows in Figure 10). This produces a stitched fibration of the type discussed in Example 6.14, in fact we would obtain a total space X which fibres over a base obtained as the result of the gluing of the two half annuli, which is clearly diffeomorphic to an annulus. The fibration is not of the type constructed in Theorem 6.15. There are two main differences between the two constructions. In the examples from Theorem 6.15 action coordinates extend continuously to the whole annulus and the symplectic form on the total space is exact. These two facts do not hold in the example just described, in fact if the symplectic form were exact then the action coordinates would extend continuously to the whole annulus (to show this one can use an argument similar to the one used in Proposition 4.11).

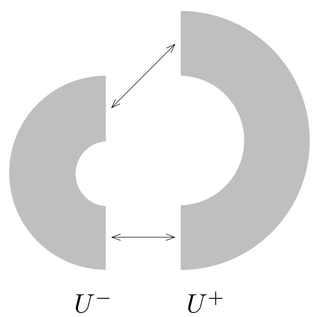


Figure 10: Gluing half annuli with different radii.

Example 6.17. An example of a stitched Lagrangian fibration constructed using Theorem 6.15 is the following. We can choose the elements of the sequence ℓ^u to be all zero, while the elements of the sequence ℓ^d to be all zero except ℓ_1^d which we define to be

$$\ell_1^d = m\, dy_2.$$

It is clear that the resulting fibration is only fake stitched, in fact the invariants are fibrewise constant. One can also see that, in the case $m = 1$ and $U = \mathbb{R}^2 - \{0\}$, the fibration is symplectically conjugate to $(X, \alpha \circ f)$, where (X, f) is a smooth focus-focus fibration (where the singular fibre has been removed) and α is the action coordinates map (see the discussion after Example 3.20 and Example 6.13). In particular this fibration induces an affine structure on the base which is simple.

We now discuss a three dimensional example.

Example 6.18. In $\mathbb{R}^3$ consider the 3-valent graph

$$\Delta = \{(0,0,-t),\ t \geq 0\} \cup \{(0,-t,0),\ t \geq 0\} \cup \{(0,t,t),\ t \geq 0\}$$

and let D be a tubular neighborhood of Δ. Take $U = \mathbb{R}^3 - D$ and assume we have a stitched Lagrangian fibration $f : X \to \mathbb{R}^3$ such that $U = f(X)$ and the seam is $Z = f^{-1}(\{b_1 = 0\} \cap U)$. Again we let $U^+ = U \cap \{b_1 \geq 0\}$, $U^- = U \cap \{b_1 \leq 0\}$ and $\Gamma = U^+ \cap U^-$. Also let $X^\pm = f^{-1}(U^\pm)$. This time Γ (hence Z) has three connected components

$$\begin{aligned} \Gamma_c &= \{(0,t,s),\ t,s < 0\} \cap U, \\ \Gamma_d &= \{(0,t,s),\ t > 0, s < t\} \cap U, \\ \Gamma_e &= \{(0,t,s),\ s > 0, t < s\} \cap U. \end{aligned}$$

Also denote by Z_c, Z_d and Z_e the corresponding connected components of Z and by $\bar{Z}_c$, $\bar{Z}_d$ and $\bar{Z}_e$ their S^1 quotients.

Fix $b \in \Gamma_c$ and suppose that there is a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(F_b, \mathbb{Z})$ and generators g_1, g_2, g_3 of $\pi_1(U, b)$, satisfying $g_1 g_2 g_3 = 1$, with respect to which the monodromy transformations are

$$\mathcal{M}_b(g_1) = T_1 = \begin{pmatrix} 1 & -m_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}_b(g_2) = T_2 = \begin{pmatrix} 1 & 0 & -m_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{53}$$

and $\mathcal{M}_b(g_3) = T_3 = T_2^{-1} T_1^{-1}$, for non zero integers m_1 and m_2. We have that γ_1 is represented by the orbits of the S^1 action, since it is the only monodromy invariant cycle. Now, since $U - (\Gamma_d \cup \Gamma_e)$ is contractible, $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of $H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z})$. Consider the diagrams:

$$\begin{tikzcd} & H_1(X^+, \mathbb{Z}) \arrow[dr] & \\ H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z}) \arrow[ur] \arrow[rr, "j_+"] & & H_1(f^{-1}(U - (\Gamma_c \cup \Gamma_d)), \mathbb{Z}) \end{tikzcd}$$

or

$$\begin{tikzcd} H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z}) \arrow[dr] \arrow[rr, "j_-"] & & H_1(f^{-1}(U - (\Gamma_c \cup \Gamma_d)), \mathbb{Z}) \\ & H_1(X^-, \mathbb{Z}) \arrow[ur] & \end{tikzcd}$$

induced by inclusions and restrictions. The map j_+ identifies $\{\gamma_1, \gamma_2, \gamma_3\}$ with a basis of $H_1(f^{-1}(U - (\Gamma_c \cup \Gamma_d)), \mathbb{Z})$, which we call $\{\gamma_1, \gamma_2^+, \gamma_3^+\}$, while j_- identifies it with another basis, which we call $\{\gamma_1, \gamma_2^-, \gamma_3^-\}$. Notice that the monodromy map $\mathcal{M}_b(g_2) = j_+^{-1} \circ j_-$. We must have

$$\begin{cases} \gamma_2^+ = \gamma_2^-, \\ \gamma_3^+ = m_2 \gamma_1 + \gamma_3^-. \end{cases} \tag{54}$$

Applying Proposition 6.5 to f restricted to $f^{-1}(U - (\Gamma_c \cup \Gamma_d))$, we can consider the action coordinates map α on $U - (\Gamma_c \cup \Gamma_d)$ computed with respect to $\{\gamma_1, \gamma_2^+, \gamma_3^+\}$ on U^+ and with respect to $\{\gamma_1, \gamma_2^-, \gamma_3^-\}$ on U^-. Let us denote these coordinates by (b_1^e, b_2^e, b_3^e). Similarly we can consider action coordinates on $U - (\Gamma_d \cup \Gamma_e)$ with respect to the basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z})$. We denote them by (b_1^c, b_2^c, b_3^c). We have the identifications

$$\bar{Z}_e = T^* \Gamma_e \, / \, \langle db_2^e, db_3^e \rangle_{\mathbb{Z}}$$

and

$$\bar{Z}_c = T^* \Gamma_c \, / \, \langle db_2^c, db_3^c \rangle_{\mathbb{Z}}.$$

With respect to these coordinates we can compute the first order invariants ℓ_1^e and ℓ_1^c on $\bar{Z}_e$ and $\bar{Z}_c$ respectively. From Proposition 6.5 and identities (54) applied to ℓ_1^c and ℓ_1^e we obtain

$$\int_{[db_2^c]} \ell_1^c = \int_{[db_3^c]} \ell_1^c = 0$$

and

$$\int_{[db_2^e]} \ell_1^e = 0 \ \text{ and } \ \int_{[db_3^e]} \ell_1^e = m_2.$$

Similarly we construct the first order invariant ℓ_1^d on $\bar{Z}_d$. It will satisfy

$$\int_{[db_2^d]} \ell_1^d = m_1 \ \text{ and } \ \int_{[db_3^d]} \ell_1^d = 0.$$

Again, monodromy is understood in terms of the difference in the cohomology class of the first order invariant. Example 5.5 is a special case of this situation, where $m_1 = m_2 = 1$.

Conversely, we can construct stitched fibrations like the one in previous example by specifying gluing data and applying Theorem 6.11. In fact we can prove

Theorem 6.19. Let $U \subset \mathbb{R}^3$, Γ_c, Γ_d and Γ_e be as in Example 6.18 and let (b_1, b_2, b_3) be coordinates on U. Define $\bar{Z}_c = T^*\Gamma_c \,/\, \langle db_2, db_3 \rangle_{\mathbb{Z}}$, $\bar{Z}_d = T^*\Gamma_d \,/\, \langle db_2, db_3 \rangle_{\mathbb{Z}}$ and $\bar{Z}_e = T^*\Gamma_e \,/\, \langle db_2, db_3 \rangle_{\mathbb{Z}}$ with projections $\bar{\pi}^c$, $\bar{\pi}^d$, $\bar{\pi}^e$ and bundles $\mathfrak{L}_c = \ker \bar{\pi}^c_*$, $\mathfrak{L}_d = \ker \bar{\pi}^d_*$, $\mathfrak{L}_e = \ker \bar{\pi}^e_*$. Suppose we are given integers m_1, m_2 and sequences $\ell^c = \{\ell^c_k\}_{k \in \mathbb{N}} \in \mathscr{L}_{\bar{Z}_c}$, $\ell^d = \{\ell^d_k\}_{k \in \mathbb{N}} \in \mathscr{L}_{\bar{Z}_d}$ and $\ell^e = \{\ell^e_k\}_{k \in \mathbb{N}} \in \mathscr{L}_{\bar{Z}_e}$ satisfying

$$
\begin{aligned}
\int_{[db_2]} \ell^c_1 &= \int_{[db_3]} \ell^c_1 = 0, \\
\int_{[db_2]} \ell^e_1 &= 0 \quad \text{and} \quad \int_{[db_3]} \ell^e_1 = m_2, \\
\int_{[db_2]} \ell^d_1 &= m_1 \quad \text{and} \quad \int_{[db_3]} \ell^d_1 = 0.
\end{aligned}
\tag{55}
$$

Then there exists a smooth symplectic manifold (X, ω) and a stitched Lagrangian fibration $f : X \to U$ having the same monodromy of Example 6.18 with respect to some basis $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z})$ and satisfying the following properties:

(i) the coordinates (b_1, b_2, b_3) are action coordinates of f with moment map f^*b_1;

(ii) the periods $\{db_1, db_2, db_3\}$, restricted to $U^{\pm}$ correspond to the basis γ;

(iii) there is a Lagrangian section σ of f, such that $(\bar{Z}_c, \ell^c)$, $(\bar{Z}_d,\, \ell^d)$ and $(\bar{Z}_e,\, \ell^e)$ are respectively the invariants of:

$$(f|_{U-(\Gamma_d \cup \Gamma_e)}, \sigma,\, \gamma),\ \ (f|_{U-(\Gamma_c \cup \Gamma_e)}, \sigma,\, j_+(\gamma)) \text{ and } (f|_{U-(\Gamma_c \cup \Gamma_d)}, \sigma,\, j_+(\gamma)).$$

The fibration (X, f, U) satisfying the above properties is unique up to fibre preserving symplectomorphism.

Remark 6.20. Also in this case (cf. Remark 6.16) we notice that fibrations of the type discussed in Example 6.18 are more general than the ones constructed using Theorem 6.19. To show this one can use higher dimensional versions of the fibration in Remark 6.16, with discontinuous action coordinates. We leave the details to the reader.

Example 6.21. A simple example of stitched Lagrangian fibration which can be constructed using Theorem 6.19 is as follows. Define the sequence ℓ^c to be identically zero and choose the terms of ℓ^d and ℓ^e to be zero except the first order ones, which we define to be

$$\ell^d_1 = m_1\, dy_2 \quad \text{and} \quad \ell^e_1 = m_2\, dy_3.$$

Clearly ℓ^c_1, ℓ^d_1 and ℓ^e_1 satisfy the integral conditions of Theorem 6.19, moreover they are fibrewise constant, therefore they define fake stitched fibrations. Since the fibration is smooth after a change of coordinates on the base, it induces an affine structure on the base. One can easily see that in the case $m_1 = -1$ and $m_2 = 1$ and $U = \mathbb{R}^3 - \Delta$, this affine structure is simple and affine isomorphic to a negative vertex of Example 3.12. Notice that we could also replace Δ with Δ_τ and obtain an affine structure which is isomorphic to the one in Example 3.13.

Non-proper stitched fibrations

This section is rather technical and the methods introduced will only be used in the proof of Lemma 7.6, therefore the reader may skip it on first reading. Here we study some special cases of piecewise smooth fibrations with non compact fibres. The results extend the ones concerning proper maps. For this reason and for sake of brevity we shall only give full proofs when the arguments do not follow directly from the previous case.

Let X be a smooth symplectic 6-manifold together with a smooth Hamiltonian S^1 action with moment map $\mu : X \to \mathbb{R}$. Assume μ has exactly one critical value $0 \in \mathbb{R}$ and a codimension four submanifold $\Sigma = \operatorname{Crit}\mu$. Let M be a smooth 2-dimensional manifold and let $B \subseteq \mathbb{R} \times M$ be a contractible open neighborhood of a point $(0, m) \in \mathbb{R} \times M$. Let $\Gamma = B \cap (\{0\} \times M)$. As usual we define $Z = \mu^{-1}(0)$ and $\bar{Z}$ the S^1 quotient of Z and $X^+ = \{\mu \geq 0\}$, $X^- = \{\mu \leq 0\}$.

We consider fibrations satisfying the following:

Assumption 6.22. The map $f : X \to B$ is a topological T^3 fibration with discriminant locus $\Delta \subset \Gamma$ such that $f(\Sigma) = \Delta$ satisfying

(a) (X, ω, f, B) is topologically conjugate to a generic singular fibration.

(b) There is a continuous S^1 invariant map $G : X \to M$ such that

  (i) if $G^{\pm} = G|_{X^{\pm}}$ then G^+ and G^- are restrictions of C^∞ maps on X;

  (ii) f can be written as $f = (\mu, G)$ and f restricted to $X^{\pm}$ is a proper map with connected Lagrangian fibres.

(c) There is a connected, S^1 invariant, open neighborhood $\mathfrak{U} \subseteq X$ of Σ such that $f(\mathfrak{U}) = B$ and such that $f_{\mathfrak{U}} = f|_{\mathfrak{U}}$ is a C^∞ map with non degenerate singular points.

We can think of B as $D^2 \times I$ with $\Delta = \{0\} \times I$. Clearly, the restriction of f to $X - f^{-1}(\Delta)$ is a stitched fibration in the sense of the previous sections. Example 5.7, as well as the legs of Example 5.8 satisfy conditions (a) and (b). Furthermore, one can deform such examples near Σ to produce fibrations which, in addition, satisfy condition (c) (cf. Lemma 7.4).

Let $\mathfrak{U}' \subset \mathfrak{U}$ be a smaller open set satisfying condition (c) (maybe after shrinking B). If we remove $\mathfrak{U}'$ we obtain a topologically trivial compact cylinder fibration

$$f|_{X - \mathfrak{U}'} : X - \mathfrak{U}' \to B \tag{56}$$

which fails to be smooth along a subset of $Z - (\mathfrak{U}' \cap Z)$. Notice though that the fibration is actually smooth toward the ends of each cylindrical fibre.

Let $X^\circ = X - \overline{\mathfrak{U}'}$ with symplectic structure $\omega^\circ = \omega|_{X^\circ}$. The restriction $f^\circ = f|_{X^\circ}$ defines a piecewise smooth open cylinder fibration

$$f^\circ : X^\circ \to B. \tag{57}$$

We denote $F^\circ(b)$ the cylindrical fibre of f° over $b \in B$. On the other hand, the smooth part $f_{\mathfrak{U}}$ of f defines an integrable Hamiltonian system with non-degenerate singularities which can be normalized as in Theorem 4.6. This normalization defines smooth coordinates (b_1, b_2, b_3) on the base.

Denote by $X^\# = X - \Sigma$ and by $f^\# : X^\# \to B$ the restriction of f to $X^\#$. Let $(f^\#)^\pm$ be the restriction of $f^\pm$ to $(X^\#)^\pm = X^\# \cap X^\pm$ and let $Z^\# = Z - \Sigma$ and $\bar{Z}^\#$ the corresponding reduced space with reduced symplectic structure ω_{red} on $\bar{Z}^\#$.

Proposition 6.23. Let $f : X \to B$ be a fibration satisfying Assumption 6.22 and let $F_{\bar{b}} = f^{-1}(\bar{b})$ be a smooth fibre. There is a basis $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ of $H_1(F_{\bar{b}}, \mathbb{Z})$ and coordinates (b_1, b_2, b_3) on B with respect to which the periods of $f^{\pm} : X^{\pm} \to B^{\pm}$ can be written

$$\begin{aligned} \lambda_1^{\pm} &= 2\pi db_1, \\ \lambda_2^{\pm} &= dH^{\pm} + \lambda_0, \\ \lambda_3^{\pm} &= db_3, \end{aligned}$$

where $\lambda_0 = \arg(b_1 + ib_2)db_1 + \log|b_1 + ib_2|db_2$ and $H^{\pm} \in C^{\infty}(B^{\pm})$. Moreover, there is a fibre preserving symplectomorphism

$$\Theta^{\pm} : T^*B^{\pm}/\Lambda_{H^{\pm}} \to (X^{\#})^{\pm} \tag{58}$$

where $\Lambda_{H^{\pm}}$ is the integral lattice generated by $\lambda_1^{\pm}, \lambda_2^{\pm}, \lambda_3^{\pm}$.

Proof. We take as coordinates (b_1, b_2, b_3) on B the ones given by the normalization of the singularity in Theorem 4.6. Then the proof goes essentially as in Proposition 4.8. As in the smooth case, one can define γ as being represented by an 3-tuple of sections $b \mapsto (\gamma_1(b), \gamma_2(b), \gamma_3(b))$, each one given by certain composition of Hamiltonian flows. In this case, however, $b \mapsto \gamma_2(b)$ does not vary smoothly but piecewise smoothly, failing to be smooth along Γ. The contribution of the path $\gamma_2 \cap \mathfrak{U}$ to the periods $\lambda_2^{\pm}$ is λ_0. On the other hand, the contribution of $\gamma_2 \cap X - \mathfrak{U}$ is $dH^{\pm}$. In contrast, the other two periods can be computed along paths entirely contained in $\mathfrak{U}$ which implies that they are smoothly defined on B. □

We will from now on denote $\lambda_1^{\pm}$ and $\lambda_3^{\pm}$ simply by λ_1 and λ_3 respectively.

Remark 6.24. Notice that in the above we can assume $H^+|_{\Gamma} = H^-|_{\Gamma}$, therefore we can define $\bar{\Lambda}_H = \Lambda_{H^+} \mod db_1 = \Lambda_{H^-} \mod db_1$. Via the identification in the above Proposition, the space $\bar{Z}^{\#}$ corresponds to $T^*\Gamma/\bar{\Lambda}_H$ and $\bar{f}^{\#} : \bar{Z}^{\#} \to \Gamma$ becomes the projection $\bar{\pi}^{\#}$.

We now introduce a standard model for fibrations satisfying Assumption 6.22.

Example 6.25 (Normal form of cylindrical type). Let (U, Γ) be a pair of subsets of $\mathbb{R}^2 \times \mathbb{R}$ diffeomorphic to $(D^2 \times D^1, D^1 \times D^1)$ with $\Gamma = U \cap \{b_1 = 0\}$. Let $\Delta = \{b_1 = b_2 = 0\}$. Given $H \in C^{\infty}(U)$ denote by H_{Δ} the germ of H along Δ. Consider the integral lattice Λ_H in T^*U generated by:

$$\begin{aligned} \lambda_1 &= 2\pi db_1, \\ \lambda_2 &= dH + \arg(b_1 + ib_2)db_1 + \log|b_1 + ib_2|db_2, \\ \lambda_3 &= db_3. \end{aligned} \tag{59}$$

Let (y_1, y_2, y_3) denote the locally defined vertical coordinates on T^*U, which it is convenient to think of as Λ_H-periodic coordinates. For fixed positive $L \in \mathbb{R}$ consider the following subset of T^*U:

$$C_L = \{|y_2| < L\} \tag{60}$$

and denote $C_L(b) = T_b^*U \cap C_L$. If U is a sufficiently small neighborhood of Δ, we can assume that for every $b \in U$, $2L < |\log|b| + \partial_{b_2}H|$. Therefore the projection $T_b^*U \to T_b^*U/\Lambda_H$ maps $C_L(b)$ to a cylinder which closes up in the y_1 and y_3 direction but not in the y_2 direction. So let us think of $C_L(b)$ as this cylinder and define $J_L^{\circ} = \bigsqcup_{b \in U} C_L(b)$, which is an open subset of T_b^*U/Λ_H. The projection π restricts to an open cylinder fibration:

$$\pi^{\circ} : J_L^{\circ} \to U.$$

Clearly there is an S^1 action on J_L° induced by λ_1, whose moment map is b_1. Let $Z_L^{\circ} = (\pi^{\circ})^{-1}(\Gamma)$ and let $\bar{Z}_L^{\circ}$ be the corresponding S^1 reduced space. Let $\bar{\pi}^{\circ} : \bar{Z}_L^{\circ} \to \Gamma$ be the reduced fibration. We denote the fibre of $\bar{\pi}^{\circ}$ by $\bar{C}_L(b)$.

For $L' < L$, construct $J^\circ_{L'}$, which is a cylinder fibration with shorter cylinders, and define its closure $K_{L'} = \overline{J^\circ_{L'}}$. Define the open set $E_{L,L'} = J^\circ_L - K_{L'}$, which we can think of as the union of the ends of the cylinders. Suppose now that we have an open neighborhood V of Z°_L and a smooth S^1 invariant Lagrangian submersion $u : V \to \mathbb{R}^3$ with cylindrical fibres satisfying: $u|_{Z^\circ_L} = \pi^\circ$, $u|_{E_{L,L'}} = \pi^\circ$ and $u_1 = b_1$. Then we can define $Y^+_L = (\pi^\circ)^{-1}(U^+)$, $Y_L = Y^+_L \cup V$, $Y^-_L = Y_L \cap (\pi^\circ)^{-1}(U^-)$ and the piecewise smooth function $f^\circ_u : Y_L \to B_u \subseteq \mathbb{R}^n$ to be the map

$$f^\circ_u = \begin{cases} \pi^\circ & \text{on } Y^+_L, \\ u & \text{on } Y^-_L. \end{cases} \tag{61}$$

Clearly, if we think of Y_L as playing the role of X°, $f^\circ_u : Y_L \to B_u$ is a Lagrangian fibration of type (57). Notice that the fibres of f°_u coincide with the fibres of π° inside $E_{L,L'}$, in particular f°_u is smooth restricted to $E_{L,L'}$. In some sense, the fibres of f_u are straight towards their ends (cf. Figure 11).

We now compactify by adding the singularities. Let $J^\#_H = T^*U/\Lambda_H$ and let $\pi^\# : J^\#_H \to U$ be the Lagrangian fibration induced by the standard projection on T^*U. Clearly J°_L and therefore Y_L are open subsets of $J^\#_H$. When $b \in \Delta$, the fibre $C(b) = (\pi^\#)^{-1}(b)$ is an open cylinder, with ends at $+\infty$ and $-\infty$ in the y_2-direction, otherwise $C(b)$ is a torus. From the results in [1], $J^\#_H$ can be compactified to a symplectic manifold X by adding the singularity at the ends of the cylinders $C(b)$ when $b \in \Delta$. The fibration $\pi^\#$ extends to a smooth fibration $f_H : X \to U$ of generic-singular type. The open subset $J^\#_H - K_{L'}$ extends to an open neighborhood E of the singular set Σ. The fibres of f°_u coincide with the fibres of f_H toward their ends and therefore f°_u may be extended to make it coincide with f_H on E. More precisely, define $\mathfrak{U} = f_H^{-1}(B_u) \cap E$ and $Y = Y_L \cup \mathfrak{U}$. Now we can define

$$f_{u,H} = \begin{cases} f_H & \text{on } \mathfrak{U}, \\ f^\circ_u & \text{on } Y_L. \end{cases} \tag{62}$$

Clearly $f_{u,H} : Y \to B_u$ is a well defined Lagrangian fibration satisfying Assumption 6.22. The zero section σ_0 of π° is, perhaps after a change of coordinates in the base, a section of f_u. If $F_{\bar{b}}$ is a smooth fibre of $f_{u,H}$, with $\bar{b} \in U^+$, let γ_0 be the basis of $H_1(F_{\bar{b}}, \mathbb{Z})$ determined by $\lambda_1, \lambda_2, \lambda_3$. We call $\mathcal{F}_{u,H} = (Y, f_{H,u}, \sigma_0, \gamma_0)$ a *normal form of cylindrical type.*

The set $Y_L \subset J^\#_H$ can be visualized in Figure 11 as the square with open top and bottom. The straight light-colored lines are the fibres of $\pi^\#$ and the fibres of $f^\circ_u : Y_L \to B_u$ are depicted as dark lines. The upper and lower rectangular regions represent the components of $E_{L,L'}$.

Given the above construction we denote $Z^\#_H = (\pi^\#)^{-1}(\Gamma)$ and by $\bar{Z}^\#_H$ its S^1 quotient. Notice that if we let $\bar{\Lambda}_H = \Lambda_H \mod db_1$, then $\bar{Z}^\#_H = T^*\Gamma/\bar{\Lambda}_H$. If $\bar{\pi}^\#$ is the projection, let $\mathfrak{L} = \ker \bar{\pi}^\#_*$. We can assume u is a well defined map in a neighborhood of $Z^\#_H$ which coincides with the projection outside a neighborhood of Z°_L, therefore we can associate to the pair (V, u) a sequence $\ell = \{\ell_k\}_{k \in N}$ of fibrewise closed section of $\mathfrak{L}^*$, just as we did in the proper case. We can easily see that the sequence ℓ must vanish outside $\bar{Z}^\circ_L$, in particular each ℓ_k, when restricted to a fibre, has compact support contained in the cylinder $\bar{C}_L(b)$. With respect to the proper case, in this situation we have an additional piece of data, i.e. the smooth function H.

The following is analogous to Definition 6.8:

Definition 6.26. With the above notation,

i) Let $\mathscr{L}_{\bar{Z}^\#_H}$ the set of sequences of fibrewise closed sections of $\mathfrak{L}^*$ which vanish outside $\bar{Z}^\circ_L$ for some positive L such that $2L < |\log|b| + \partial_{b_2} H|$ for every $b \in \Gamma$.

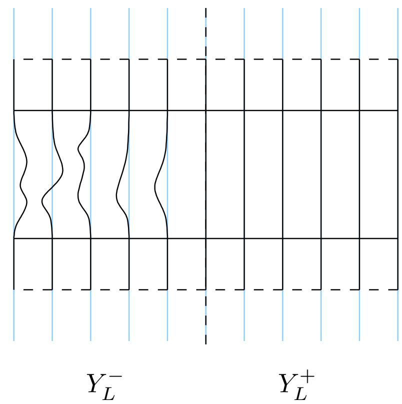


Figure 11: Normal form of cylindrical type.

ii) Let $\mathscr{U}_{\tilde{Z}_H^\#}$ be the set of pairs (V, u) where, for some positive L and L' satisfying $2L' < 2L < |\log|b| + \partial_{b_2} H|$, V is a neighborhood of Z_L° and $u : V \to \mathbb{R}^n$ is a smooth, S^1-invariant Lagrangian submersion, with cylindrical fibres, with components (u_1, u_2, u_3) such that $u|_{Z_L^\circ} = \pi^\circ$, $u|_{E_{L,L'}} = \pi^\circ$ and $u_1 = b_1$.

iii) Let $\mathscr{H}_\Delta$ be the set of germs H_Δ of smooth functions H defined on neighborhoods of Δ.

We define the invariants of a normal form of cylindrical type $\mathcal{F}_{u,H}$ to be:

$$\mathrm{inv}(\mathcal{F}_{u,H}) = (Z_H^\#, \ell, H_\Delta).$$

A little explanation is necessary to see in which sense these are invariants.

Remark 6.27. Suppose we are given two normal forms of cylindrical type $\mathcal{F}_{u,H}$ and $\mathcal{F}_{u',H'}$. From the results in [1] (cf. also Theorem 4.13), a necessary condition for f_H and $f_{H'}$ to be symplectically conjugate is that $H_\Delta = H'_\Delta$, so suppose this holds. This gives a symplectomorphism, which we denote by $\Phi_{H,H'}$, between the total spaces X and X' of the two fibrations which conjugates (X, f_H, B) and $(X', f_{H'}, B')$. By pulling back (V', u') via this symplectomorphism and computing the Taylor series, we obtain a sequence of fibrewise closed sections of $\mathfrak{L}^*$ which we call $\Phi_{H,H'} \cdot \ell'$. Using the same arguments as in the proof of Theorem 6.12 (cf.[2], Theorem 6.11), we can then show that $\mathcal{F}_{u,H}$ and $\mathcal{F}_{u',H'}$ are symplectically conjugate if and only if $\Phi_{H,H'} \cdot \ell' = \ell$. In particular, when $H = H'$, they are symplectically conjugate if and only if $\ell = \ell'$.

For the classification of fibrations satisfying Assumption 6.22, it is useful to have the following result.

Proposition 6.28. Let $f : X \to B$ be a Lagrangian fibration satisfying Assumption 6.22. Given a smooth fibre $F_{\bar{b}}$ of f there is a basis γ of $H_1(F_{\bar{b}}, \mathbb{Z})$ and a section σ of f, such that $\mathcal{F} = (X, f, B, \sigma, \gamma)$ is symplectically conjugate to a normal form of cylindrical type $\mathcal{F}_{u,H}$.

Proof. One uses the same arguments as in the proof of Proposition 6.9. Suppose there is an extension of $f^+ : X^+ \to B^+$ to a smooth Lagrangian fibration $\tilde{f}^+$ defined on a neighborhood $W \subseteq X$ of Z such that $\tilde{f}^+|_{\mathfrak{U}} = f|_{\mathfrak{U}}$. Then one may compute the period lattice of $\tilde{f}^+$; this gives a smooth function H extending the function H^+ in Proposition 6.23. Assuming that also f^- has been extended to $\tilde{f}^-$ so that $\tilde{f}^-|_{\mathfrak{U}} = f|_{\mathfrak{U}}$, one may verify that the period map $\Theta^+ : T^*U/\Lambda_H \to W^\#$ gives the required equivalence between $\mathcal{F}$ and $\mathcal{F}_{u,H}$ where $u = \tilde{f}^- \circ \Theta^+$.

To extend f^+, notice that $f_{\mathfrak{U}} = f|_{\mathfrak{U}}$ is smooth so, tautologically, $f_{\mathfrak{U}}$ is an extension of f^+ to $\mathfrak{U}$. It remains to extend f^+ away from $\mathfrak{U}$. Let $\mathfrak{U}' \subset \mathfrak{U}$ and define $f^\circ : X^\circ \to B$ as in (57). Denote $Z^\circ = Z \cap X^\circ$ and by $\bar{Z}^\circ$ its S^1 quotient with $\bar{f}^\circ : \bar{Z}^\circ \to \Gamma$ the reduced fibration. Then $\bar{f}^\circ$ is a smooth Lagrangian cylinder fibration.

The coisotropic neighborhood theorem allows us to identify a neighborhood of Z° inside X° with a neighborhood V of $\{0\} \times S^1 \times \bar{Z}^\circ$ inside $\mathbb{R} \times S^1 \times \bar{Z}^\circ$ (t will denote the $\mathbb{R}$ coordinate). Moreover, since $\bar{Z}^\#$ can be identified with $T^*\Gamma/\bar{\Lambda}_H$ (see Remark 6.24), $\bar{Z}^\circ$ can be identified with a subset of $T^*\Gamma/\bar{\Lambda}_H$ of the type $\bar{Z}^\circ_L$ for some positive L (see Example 6.25). The pullback of f° under these identifications gives a piecewise smooth Lagrangian fibration on $V \subset \mathbb{R} \times S^1 \times \bar{Z}^\circ_L$

$$g = \begin{cases} u^+ & \text{on } V^+; \\ u^- & \text{on } V^- \end{cases} \tag{63}$$

where $V^+ = V \cap \{t \geq 0\}$, $V^- = V \cap \{t \leq 0\}$ and $u^\pm$ is the restriction to $V^\pm$ of a C^∞ map. The set $Z^\circ \cap \mathfrak{U}$ where f° is smooth, corresponds (under the above identifications) to the interior of $Z^\circ_L - Z^\circ_{L'}$ which we denote $C_{L,L'}$, where $L' < L$. Notice that the map g above is then smooth along $C_{L,L'}$, in particular the Taylor expansions in t of u^+ and u^- coincide along $C_{L,L'}$. With the same arguments used in the proper case one can show that $u^\pm$ can be smoothly extended to a Lagrangian fibration $\tilde{u}^\pm$ beyond $V^\pm$ (cf. Proposition 6.9 above, or [2] Proposition 6.3 for more details). In fact with a little more care one can do this so that along $\mathbb{R} \times C_{L,L'}$, where an extension already exists, namely g itself, we have $\tilde{u}^\pm|_{\mathbb{R}\times C_{L,L'}} = g|_{\mathbb{R}\times C_{L,L'}}$. The map $\tilde{u}^+$ gives the required extension $\tilde{f}^+$ of f^+, where the last observation guarantees that $\tilde{f}^+|_{\mathfrak{U}} = f|_{\mathfrak{U}}$. □

From the above result, it follows that to every Lagrangian fibration $\mathcal{F}$ satisfying Assumption 6.22 we can assign the invariants of a normal form for $\mathcal{F}$, i.e. a triple $(Z^\#_H, \ell, H_\Delta)$. Notice that two normal forms $\mathcal{F}_{u,H}$ and $\mathcal{F}_{u',H'}$ for the same fibration $\mathcal{F}$ must be related in the way described in Remark 6.27. It is worth stating this in the following:

Theorem 6.29. Two germs of fibrations $\mathcal{F}$ and $\mathcal{F}'$ satisfying Assumption 6.22 are symplectically conjugate if and only if their invariants are related in the way described in Remark 6.27.

We also have:

Proposition 6.30. Given $H_\Delta \in \mathscr{H}_\Delta$, there is a function H defined on a neighborhood of Γ whose germ is H_Δ, such that for every $\ell \in \mathscr{L}_{\bar{Z}^\#_H}$, there is a normal form of cylindrical type whose invariants are $(Z^\#_H, \ell, H_\Delta)$.

The results in this section extend those in [1] to stitched fibrations with generic singularities (satisfying Assumption 6.22). The arguments here can also be carried through in the stitched focus-focus case, the positive case and their higher dimensional analogues.

7 Lagrangian negative fibrations

The purpose of this section is two-fold. We first use the analysis in §6 to refine the piecewise smooth fibrations constructed in §5. Subsequently, we study the affine structures associated to the resulting fibrations.

Recall that we defined a negative vertex to be an integral affine manifold with singularities modeled on Example 3.13.

Definition 7.1. Let (X, ω) be a 6-dimensional symplectic manifold and $B \subseteq \mathbb{R}^3$ an open subset. Let $f : X \to B$ be a piecewise smooth Lagrangian fibration. $\mathcal{F} = (X, \omega, f, B)$ is called a *Lagrangian negative fibration* if it satisfies the following properties:

(i) $\mathcal{F}$ is topologically conjugate to the alternative negative fibration of Example 2.9.

(ii) there exists a submanifold with boundary $D \subset B$, homeomorphic to a closed disc in $\mathbb{R}^2$, such that $\Delta \cap (B - D)$ consists of three one dimensional disjoint segments (the legs of Δ) and f is smooth when restricted to $X - f^{-1}(D)$;

(iii) let $B_0 = B - (D \cup \Delta)$, $X_0 = f^{-1}(B_0)$ and $f_0 = f|_{X_0}$. Let $(B_0, \mathscr{A})$ be the integral affine manifold induced by the Lagrangian T^3 bundle $\mathcal{F}_0 = (X_0, f_0, B_0)$. For some choice of model of negative vertex $(\mathbb{R}^3, \Delta_\tau, \mathscr{A}_\tau)$ as given in Example 3.13, there exist an open neighborhood $U \subseteq \mathbb{R}^3$ of $0 \in \mathbb{R}^3$, a submanifold with boundary $D' \subset U$ homeomorphic to a closed disc in $\mathbb{R}^2$, satisfying $0 \in D' \subset \{x_1 = 0\} \subset \mathbb{R}^3$ and an integral affine isomorphism

$$(B_0, \mathscr{A}) \cong (U - (D' \cup \Delta_\tau), \mathscr{A}_\tau).$$

Corollary 3.4 directly implies the following:

Proposition 7.2. Let $\mathcal{F}$ be a Lagrangian negative fibration. With the notation as in Definition 7.1, let $U_0 = U - (D' \cup \Delta_\tau)$ and $X(U_0, \mathscr{A}_\tau)$ be the associated Lagrangian torus bundle. Then, if $\mathcal{F}$ has a smooth Lagrangian section, $\mathcal{F}_0$ is symplectically conjugate to $X(U_0, \mathscr{A}_\tau)$.

The main result of this section is

Theorem 7.3. There exists a symplectic manifold (X, ω) and a map $f : X \to B$ such that (X, ω, f, B) is a Lagrangian negative fibration.

The starting point aiming at the proof of Theorem 7.3 will be the Lagrangian fibration described in Example 5.8, which satisfies Definition 7.1(i). The proof will consist essentially of three steps. First we modify Example 5.8 so to obtain a fibration which is smooth towards the ends of the 1-dimensional legs (Smoothing I and II). In the second step (Smoothing III) we use the invariants of stitched Lagrangian fibrations to modify the fibration once more so that it satisfies property (ii). Finally we show that these modifications have been done in a way that also (iii) holds.

Smoothing I

Let us consider the fibration as in Example 5.8 with its discriminant locus Δ. Recall that this fibration is constructed using Proposition 5.4, by taking as symplectomorphism Φ the one described by (41). For positive $M \in \mathbb{R}$ let us define

$$\Delta_{h,M} = \Delta \cap \{b_2 \leq -M\}, \quad \Delta_{v,M} = \Delta \cap \{b_3 \leq -M\}, \quad \Delta_{d,M} = \Delta \cap \{b_2, b_3 \geq M\}. \quad (64)$$

When M is sufficiently big, $\Delta_{h,M}$, $\Delta_{v,M}$ and $\Delta_{d,M}$ are 1-dimensional. In fact, they are the ends of the horizontal, vertical and diagonal legs of Δ respectively. Now let $\Sigma_{h,M}$, $\Sigma_{v,M}$ and $\Sigma_{d,M}$ be the parts of the critical surface Σ which are mapped to $\Delta_{h,M}$, $\Delta_{v,M}$ and $\Delta_{d,M}$ respectively.

We have the following

Lemma 7.4. The piecewise smooth Lagrangian fibration $\mathcal{F} = (X, \omega, f, B)$ in Example 5.8 can be perturbed, without changing its topology, so that, for sufficiently big M, it becomes smooth on small neighborhoods $N_{h,M}$, $N_{v,M}$ and $N_{d,M}$ of $\Sigma_{h,M}$, $\Sigma_{v,M}$ and $\Sigma_{d,M}$ respectively.

Proof. From the way f is defined in Example 5.8, we can assume

$$\Sigma_{h,M} = \{t = 0, \ u_1 = 0, \ |u_2|^2 < \epsilon/4\},$$

where ϵ is as in (41) and $M = \log(\sqrt{\epsilon}/2)$. For any $\tau > 0$ denote open sets

$$N^\tau = \{(t, u_1, u_2) \mid \max(|u_1|, |u_2|^2) < \tau\}.$$

From now on we assume f is restricted to $N^{\epsilon/2}$. As one can easily see from the construction, the map G_t defining f, restricted to $N^{\epsilon/2}$ is

$$G_t(u_1, u_2) = \left(\log|u_2|, \log\left| \frac{u_1}{\sqrt{|t| + \sqrt{t^2 + |u_1|^2}}} - 1 \right| \right). \tag{65}$$

This is the map that we want to perturb, but just on a smaller neighborhood. We do it applying the idea already anticipated at the end of Example 5.7. In fact we notice that G_t is invariant with respect to the S^1 action

$$e^{i\theta}(u_1, u_2) = (u_1, e^{2i\theta}u_2),$$

which is also Hamiltonian with respect to the reduced symplectic form ω_t given in (23). The moment map is

$$(u_1, u_2) \mapsto |u_2|^2.$$

So, if g is a real function depending only on u_1, t and $s = |u_2|^2$, then

$$(u_1, u_2) \mapsto (\log|u_2|,\ g(u_1, t, |u_2|^2))$$

is a Lagrangian fibration with respect to ω_t, provided the level sets of $u_1 \mapsto g(u_1, t, s)$ are one dimensional submanifolds for every s and t. For example, consider a real non-negative function ρ defined on $\mathbb{R}^3$ such that, for every fixed $(t, s) \in \mathbb{R}^2$, the map

$$u \mapsto \frac{u}{\rho(|u|^2, t, s)} \tag{66}$$

is a local homeomorphism of a neighborhood of $u = 0$, then $g = \log|\frac{u}{\rho} - 1|$ defines a Lagrangian fibration (at least in a neighborhood of 0). In particular

$$\rho_0(r, t) = \sqrt{|t| + \sqrt{t^2 + r}},$$

with $(r, t) \in \mathbb{R}^2$ gives the map G_t in (65), but it is not smooth. It is easy to see that if ρ is smooth on $\mathbb{R}^3$ and satisfies

$$\rho > \rho_0 \tag{67}$$

then the map (66) is an orientation preserving diffeomorphism (at least near $u = 0$). So let us choose a smooth ρ_1, defined on $\mathbb{R}^2$ and satisfying $\rho_1 > \rho_0$, and let

$$g_j = \log\left| \frac{u_1}{\rho_j(|u_1|^2, t)} - 1 \right|,$$

for $j = 0, 1$. We wish to find a g which interpolates between g_0 and g_1. More precisely, we want g to be equal to g_0 outside $N^{3\epsilon/8}$ and to g_1 on some smaller open neighborhood of $\Sigma_{h,M}$. Clearly $(u_1, u_2) \in N^{3\epsilon/8}$ if and only if $(|u_1|^2, |u_2|^2)$ is in the rectangle

$$S_0 = [-9\epsilon^2/64, 9\epsilon^2/64] \times [-3\epsilon/8, 3\epsilon/8].$$

Now let S_1 be a closed neighborhood of 0 in $\mathbb{R}^2$ which is contained in the interior of S_0, e.g. a smaller rectangle. Taking a $\sigma \in C^\infty(\mathbb{R}^2)$, which is 0 outside S_0 and 1 on S_1, let us define

$$\rho(r, t, s) = (1 - \sigma(r, s))\rho_0(r, t) + \sigma(r, s)\rho_1(r, t),$$

so that ρ is equal to ρ_0 outside S_0 and it is equal to ρ_1 on S_1. Clearly $\rho > \rho_0$. We leave it to the reader to check that choices can be made so that with this ρ, (66) is indeed a homeomorphism. Now define

$$g = \log \left| \frac{u_1}{\rho(|u_1|^2, t, |u_2|^2)} - 1 \right|.$$

Clearly g is equal to g_0 outside $N^{3\epsilon/8}$ and to g_1 on

$$N_{h,M} = \{(|u_1|^2, |u_2|^2) \in S_1\}$$

which, with a suitable choice of S_1, is a neighborhood of $\Sigma_{h,M}$. Moreover $u \mapsto g(u, t, s)$ has 1-dimensional level sets. We can therefore replace the second component of G_t in (65) with g and redefine

$$G_t(u_1, u_2) = (\log |u_2|, g),$$

which is smooth on $N_{h,M}$. This proves the lemma for $\Sigma_{h,M}$. A schematic picture of this smoothing is described in Figure 12. The vertical lines represent fibres of f over the horizontal leg. The base of the fibration is represented by the horizontal line on the bottom of the picture; the bold segment on the right represents the region where the codimension one part of Δ begins. The shaded region represents the locus where f is not smooth. The dashed region is $N_{h,M}$.

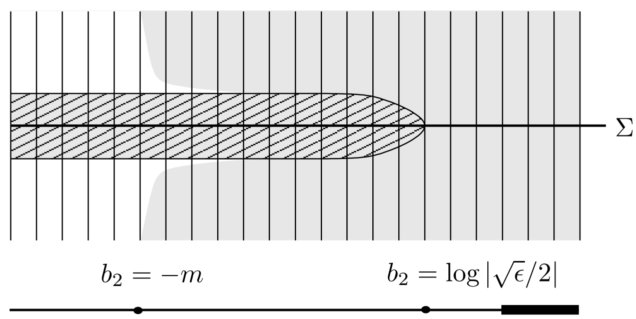


Figure 12: Horizontal leg. The dashed region is $N_{h,M}$ as in Lemma 7.4. After Smoothing II there will be a full fibred neighborhood (white region) where the fibration is smooth.

The case of the vertical leg is done in the same way. At first sight it is not so obvious that also the diagonal leg can be treated in the same way. So let us give some explanation. When $|u_2|^2 \geq M$, the map G_t becomes

$$G_t(u_1, u_2) = \left(\log \left| \frac{u_1}{\rho_0(|u_1|^2, t)} - u_2 \right|, \log \left| \frac{u_1}{\rho_0(|u_1|^2, t)} + u_2 \right| \right). \tag{68}$$

The first observation is that this map is invariant under the S^1-action

$$e^{i\theta}(u_1, u_2) = (e^{i\theta} u_1, e^{i\theta} u_2). \tag{69}$$

After the following change of coordinates on the base

$$(x_1, x_2) \mapsto \left(\frac{e^{2x_1} + e^{2x_2}}{2}, x_1 - x_2 \right)$$

this becomes

$$G_t(u_1, u_2) = \left(\frac{\sqrt{t^2 + |u_1|^2 + |u_2|^2}}{2} - \frac{|t|}{2}, \log \frac{|u_1/\rho_0 - u_2|}{|u_1/\rho_0 + u_2|} \right). \tag{70}$$

One can check that for every fixed $t \in \mathbb{R}$ the map

$$(u_1, u_2) \mapsto \frac{\sqrt{t^2 + |u_1|^2} + |u_2|^2}{2},$$

is the moment map of the S^1-action (69), with respect to the reduced symplectic form ω_t. Moreover, if one replaces $u_1 = z_1 z_2$, $u_2 = z_3$ and $t = \frac{|z_1|^2 - |z_2|^2}{2}$, then the above map becomes

$$\nu : (z_1, z_2, z_3) \mapsto \frac{|z_1|^2 + |z_2|^2}{4} + \frac{|z_3|^2}{2}$$

which is a smooth map on the total space. Let us denote

$$s = \frac{\sqrt{t^2 + |u_1|^2} + |u_2|^2}{2}.$$

The second component of (70) can be rewritten as

$$g_0(u_1, u_2) = \log \left| \frac{2u_1/\rho_0}{u_1/\rho_0 + u_2} - 1 \right|.$$

We can now apply the same strategy we used in the case of the horizontal leg. We observe that we could replace this g_0 with any other S^1-invariant function g. In particular we could replace ρ_0, which is S^1-invariant, with another smooth S^1-invariant ρ_1. As before, we then interpolate ρ_0 and ρ_1 with a cut off function σ depending on $|u_1|^2$ and s. We avoid writing the details here, as they just follow the same argument as before.

In the end we obtain that, in a small neighborhood of $\Sigma_{d,M}$, G_t can be written as:

$$G_t = \left(s - \frac{|t|}{2}, \log \left| \frac{2u_1/\rho_1}{u_1/\rho_1 + u_2} - 1 \right| \right),$$

where now the second component is smooth. The first component is not quite smooth yet. We saw that s is smooth when lifted to the total space, but $|t|$ isn't. The total fibration becomes of the type

$$f(z_1, z_2, z_3) = \left(\mu, \nu - \frac{|\mu|}{2}, g(z_1 z_2, z_3, \mu, \nu) \right),$$

where g is smooth. We see that after a change of coordinates on the base of the type

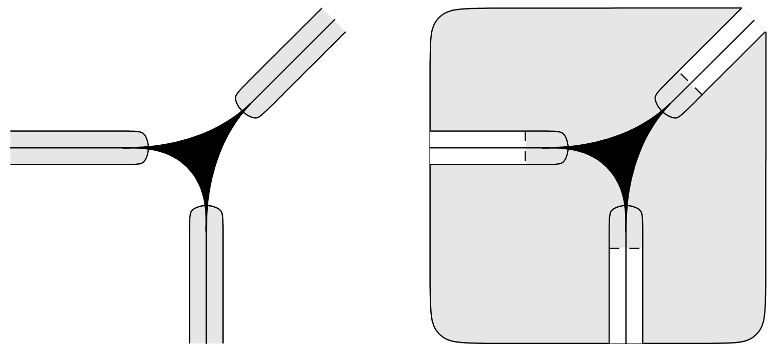
$$(b_1, b_2, b_3) \mapsto \left(b_1, b_2 + \frac{|b_1|}{2}, b_3 \right) \tag{71}$$

this fibration becomes

$$f(z_1, z_2, z_3) = (\mu, \nu, g(z_1 z_2, z_3, \mu, \nu)),$$

which is smooth. One can find a global change of coordinates on the base which acts like (71) only in a neighborhood of the end of the diagonal leg and is the identity elsewhere. This ends the proof of the Lemma. □

Remark 7.5. Notice that the new perturbed fibration of Lemma 7.4 has a Lagrangian section. In fact one can easily see that the section of the fibration in Example 5.8 survives the smoothing above, since it is far from the critical surface Σ.



(a) (b)

Figure 13: Smoothing over the legs.

Smoothing II

Lemma 7.4 gives us a piecewise smooth fibration $\mathcal{F}$, topologically conjugate to the one in Example 5.8 but smooth along $N_{h,M}$, $N_{v,M}$ and $N_{d,M}$. The latter are sets mapping down onto open neighborhoods $B_{h,M}$, $B_{v,M}$ and $B_{d,M}$ of the legs as depicted in Figure 13 (a). Given a positive $m \in \mathbb{R}$, let us denote by $B_{h,m}$, $B_{v,m}$ and $B_{d,m}$ neighborhoods of $\Delta_{h,m}$, $\Delta_{v,m}$ and $\Delta_{d,m}$ and for brevity let us define $\mathcal{F}_{h,m} = \mathcal{F}|_{B_{h,m}}$, $\mathcal{F}_{v,m} = \mathcal{F}|_{B_{v,m}}$ and $\mathcal{F}_{d,m} = \mathcal{F}|_{B_{d,m}}$. Clearly when M is as in Lemma 7.4, $\mathcal{F}_{h,M}$, $\mathcal{F}_{v,M}$ and $\mathcal{F}_{d,M}$ satisfy Assumption 6.22.

Our goal now is to use the results on non-proper stitched fibrations in Section 6 to perturb $\mathcal{F}$ so that for some $m > M$ and neighborhoods $B_{h,m}$, $B_{v,m}$ and $B_{d,m}$, the fibrations $\mathcal{F}_{h,m}$, $\mathcal{F}_{v,m}$ and $\mathcal{F}_{d,m}$ are smooth. This will produce a fibration whose base is depicted in Figure 13 (b). Over the white rectangular regions the fibration is completely smooth but on the shaded region it is still piecewise smooth. The result is the following:

Lemma 7.6. Let $\mathcal{F}$ denote the fibration obtained in Lemma 7.4. Given a positive real number $m > M$, there exists a perturbation $\tilde{\mathcal{F}}$ of $\mathcal{F}$ (perhaps defined over a smaller neighborhood of the plane $\{b_1 = 0\}$), such that

(i) $\tilde{\mathcal{F}}$ is topologically conjugate to $\mathcal{F}$;

(ii) there are open neighborhoods $B_{h,m}$, $B_{v,m}$ and $B_{d,m}$ of $\Delta_{h,m}$, $\Delta_{v,m}$ and $\Delta_{d,m}$ respectively so that the fibrations $\tilde{\mathcal{F}}_{h,m}$, $\tilde{\mathcal{F}}_{v,m}$ and $\tilde{\mathcal{F}}_{d,m}$ are smooth.

Proof. Consider one of the fibrations $\mathcal{F}_{h,M}$, $\mathcal{F}_{v,M}$ or $\mathcal{F}_{d,M}$ as above (whenever necessary, we allow ourselves to restrict to smaller neighborhoods of $\Delta_{h,M}$, $\Delta_{v,M}$ or $\Delta_{d,M}$). To keep the notation simple we temporarily drop the subindices and denote it by $\mathcal{F}$.

Since $\mathcal{F}$ satisfies Assumption 6.22, it follows from Proposition 6.28 that we can associate to $\mathcal{F}$ a normal form of cylindrical type $\mathcal{F}_{u,H}$ together with its invariants given by a triple $(Z^{\#}_{H}, \ell, H_{\Delta})$ which, in view of Theorem 6.29, uniquely determine $\mathcal{F}$ as a germ around $\Gamma = B \cap \{b_1 = 0\}$. By slight abuse of notation we will denote by the same letter Γ both $B \cap \{b_1 = 0\}$ and $B_u \cap \{b_1 = 0\}$, where B_u is the base of $\mathcal{F}_{u,H}$. For the duration of this proof H will remain unchanged, so we drop the subindex H and denote $\mathcal{F}_u := \mathcal{F}_{u,H}$ for short.

The proof consists in suitably deforming the sequence ℓ. Let $\bar{A} \subset \Gamma$ and $\bar{A}' \subset \bar{A}$ be (planar) regions as depicted in Figure 14. Given a cut-off function $\rho \in C^{\infty}(\Gamma)$ such that ρ is 1 on $\Gamma - \bar{A}$ and 0 on $\bar{A}'$, define a new (fibrewise closed) sequence $\tilde{\ell}$ whose

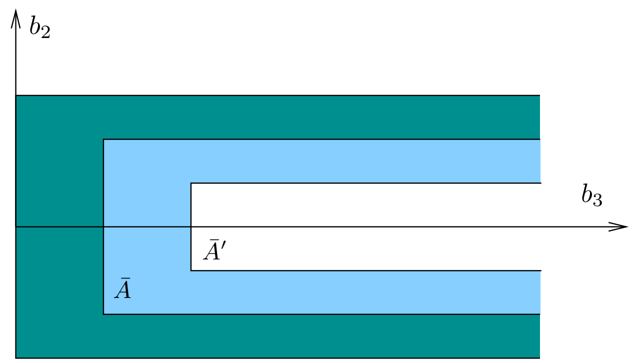


Figure 14: Γ (or $\Gamma_{h,M}$).

elements are $\tilde{\ell}_k = (\rho \circ \bar{\pi}^{\#})\, \ell_k$ for each $k \in \mathbb{N}$. We obtain a triple $(Z_H^{\#}, \tilde{\ell}, H_{\Delta})$, such that $\ell|_{(\bar{\pi}^{\#})^{-1}(\Gamma - \bar{A})} = \tilde{\ell}|_{(\bar{\pi}^{\#})^{-1}(\Gamma - \bar{A})}$ and $\tilde{\ell}|_{(\bar{\pi}^{\#})^{-1}(\bar{A}')} = 0$.

In view of Proposition 6.30, $(Z_H^{\#}, \tilde{\ell}, H_{\Delta})$ gives rise to a normal form of cylindrical type $\mathcal{F}_{\tilde{u}}$ defined over a neighborhood of Γ. By construction and by Theorem 6.29, $\mathcal{F}_u$ and $\mathcal{F}_{\tilde{u}}$ define the same germ around $\Gamma - \bar{A}$, i.e. there are open neighborhoods U and $\tilde{U}$ of $\Gamma - \bar{A}$ (satisfying $U \cap \{b_1 = 0\} = \tilde{U} \cap \{b_1 = 0\} = \Gamma - \bar{A}$) such that $\mathcal{F}_u|_U$ and $\mathcal{F}_{\tilde{u}}|_{\tilde{U}}$ are symplectically conjugate. Moreover $\mathcal{F}_{\tilde{u}}$ is smooth when restricted to any open neighborhood A' of $\bar{A}'$ such that $A' \cap \{b_1 = 0\} = \bar{A}'$. Now recall that $\mathcal{F}_u$ is symplectically conjugate to $\mathcal{F}$, so we have that $\mathcal{F}_{\tilde{u}}|_{\tilde{U}}$ is symplectically conjugate $\mathcal{F}|_U$.

Let us summarize the result using our original notation for the horizontal leg. For $\Gamma_{h,M} = B_{h,M} \cap \{b_1 = 0\}$, we have found sets $\bar{A}' \subset \bar{A} \subset \Gamma_{h,M}$ (as in Figure 14) and a normal form of cylindrical type $\mathcal{F}_{\tilde{u}}$, defined over a neighborhood of $\Gamma_{h,M}$, smooth over $\bar{A}'$ and such that $\mathcal{F}_{\tilde{u}}|_{\tilde{U}}$ is symplectically conjugate to $\mathcal{F}_{h,M}|_U$, where U and $\tilde{U}$ are neighborhoods of $\Gamma_{h,M} - \bar{A}$ (satisfying $U \cap \{b_1 = 0\} = \tilde{U} \cap \{b_1 = 0\} = \Gamma_{h,M} - \bar{A}$).

If we go back denoting by $\mathcal{F}$ the fibration of Lemma 7.4, we can form a new fibration $\tilde{\mathcal{F}}$ in the following way. Let $\mathcal{F}' = \mathcal{F}|_{\mathbb{R}^3 - (\mathbb{R} \times \bar{A})}$ and symplectically glue $\mathcal{F}_{\tilde{u}}$ to $\mathcal{F}'$ using the conjugation between $\mathcal{F}_{\tilde{u}}|_{\tilde{U}}$ and $\mathcal{F}'|_U = \mathcal{F}_{h,M}|_U$. The fibration $\tilde{\mathcal{F}}$ is the result of this gluing. Notice that $\tilde{\mathcal{F}}$, due to the properties of $\mathcal{F}_{\tilde{u}}$, is such that for some $m > M$ (depending on $\bar{A}'$) and a suitable neighborhood of $B_{h,m}$ of $\Delta_{h,m}$, the restriction $\tilde{\mathcal{F}}_{h,m}$ is smooth. Notice that $\bar{A}'$ can be chosen so that the latter holds for any $m > M$.

The above method applied to all legs, produces the required result. $\square$

The idea of deforming the sequence ℓ by multiplying it by a cut-off function on the base will be used again in the subsection Smoothing III. This is actually the main application of the results on stitched fibrations in this paper.

Remark 7.7. We observe that the Lagrangian section of Example 5.8 survives also this second smoothing.

The normal form

Consider the Lagrangian fibration $\mathcal{F}$ produced in Lemma 7.6. If we let $U = \mathbb{R}^3 - \Delta$, then $\mathcal{F}|_U$ is a stitched T^3 fibration whose seam consists of three disjoint components. It is clear that $\mathcal{F}|_U$ is a fibration of the type described in Example 6.18. The goal of this section is to show that $\mathcal{F}|_U$ is in fact symplectically conjugate to a fibration which can be constructed with Theorem 6.19, maybe after restricting the latter to a smaller neighborhood of the vertex of Δ (see Remarks 6.16 and 6.20). Essentially, we need to show that the action coordinates, a priori defined only on a contractible open set, extend continuously to $\mathbb{R}^3$. We need the following

Lemma 7.8. Let (X, ω) be the total space of the fibration produced in Lemma 7.6. Then ω is exact on X.

Proof. Recall that the fibration produced in Lemma 7.6 is a perturbation of the one in Example 5.8, whose total space is an open set of $\mathbb{C}^3$ with standard symplectic form, which is exact. One can see that the successive perturbations of this fibration have not modified the cohomology class of ω. $\square$

To describe the fibration $\mathcal{F}$ we use the same notation of Example 6.18. Given $\bar{b} \in \Gamma_c$, there exists a basis $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(F_{\bar{b}}, \mathbb{Z})$ with respect to which monodromy is generated by the matrices in (53) with $m_1 = m_2 = 1$. We can compute the action coordinates $\alpha : U - (\Gamma_e \cup \Gamma_d) \to \mathbb{R}^3$ with respect to γ, normalized so that $\alpha(\bar{b}) = (0, 0, 0)$ (cf. Proposition 6.5). From Lemma 7.8, there exists a primitive η of ω, such that for every $b = (b_1, b_2, b_3) \in U - (\Gamma_e \cup \Gamma_d)$ we have

$$\alpha(b) = \left(-\int_{\gamma_1(b)} \eta, \ -\int_{\gamma_2(b)} \eta, \ -\int_{\gamma_3(b)} \eta \right),$$

where $\gamma_j(b)$ is a cycle in F_b representing γ_j. Clearly α is well defined and continuous on $U - (\Gamma_d \cup \Gamma_e)$. Actually, we have:

Lemma 7.9. The action coordinates map α extends continuously to $\mathbb{R}^3$.

Proof. We apply a similar argument to the one used in the case of the positive fibre (see Proposition 4.11). Clearly, since γ_1 is represented by the orbits of the S^1 action

$$-\int_{\gamma_1(b)} \eta = b_1,$$

which is continuous. We now prove that, for $j = 2, 3$

$$\alpha_j(b) = -\int_{\gamma_j(b)} \eta \tag{72}$$

extends continuously to points in Γ_d or in Γ_e. As we did in Proposition 4.11, we can think of $\alpha_j(b)$ as

$$\alpha_j(b) = \int_S \omega,$$

where S is a surface spanned by the cycles $\gamma_j(b')$ as b' moves along a curve joining $\bar{b}$ and b. Suppose $b \in \Gamma_e$ (or Γ_d), then we need to show that $\alpha_j(b)$ is independent of the curve from $\bar{b}$ to b, or equivalently that

$$\int_{S_1 - S_2} \omega = 0,$$

where S_1 and S_2 are the surfaces corresponding to two different paths from $\bar{b}$ to b. The boundary $\partial(S_1 - S_2)$ is determined by monodromy. It is easy to see that $\partial(S_1 - S_2)$ is a multiple of $\gamma_1(b)$, therefore for some integer k we have

$$\int_{S_1 - S_2} \omega = -\int_{\partial(S_1 - S_2)} \eta = k \int_{\gamma_1(b)} \eta = 0,$$

where the last equality follows from the fact that $b \in \Gamma_d$ or Γ_e. To show that α extends continuously also to points of Δ we can argue that (72) makes sense also over singular fibres, since both η and $\gamma_j(b)$ are well defined when $b \in \Delta$. $\square$

We also have:

Lemma 7.10. The map $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism onto its image.

Proof. Since $\alpha_1(b) = b_1$, it is enough to show that, if for fixed $t \in \mathbb{R}$ we let $U_t = \{b_1 = t\}$, then $\alpha_t = \alpha|_{U_t}$ is a bijection onto its image. If λ_2 and λ_3 are the periods of the fibration corresponding to γ_2 and γ_3, then α_t is computed by taking primitives of $\lambda_2|_{U_t}$ and $\lambda_3|_{U_t}$. If we let X_t denote the symplectic reduction of X at t and $G_t : X_t \to \mathbb{R}^2$ the reduced fibration, then it is not difficult to see that $\lambda_2|_{U_t}$ and $\lambda_3|_{U_t}$ are in fact periods of G_t (cf. [3]Lemma 5.9). Now the conclusion follows by simply observing that G_t is a proper Lagrangian submersion, i.e. an integrable system. The argument works also when $t = 0$.

An explicit computation of the periods was done in [3]Proposition 5.10 for the fibration in Example 5.8. There we found that

$$\begin{aligned} \lambda_2 &= \beta_1\, db_1 - e^{2b_2} db_2, \\ \lambda_3 &= \beta_2\, db_1 - e^{2b_3} db_3, \end{aligned} \tag{73}$$

where β_1 and β_2 are functions depending only on b_1. The periods of the perturbed fibration obtained in Lemma 7.6 will have this same expression away from where the perturbation took place (i.e. away from the white region in Figure 15), for example in a neighborhood of the codimension 1 part of Δ. It is easy to see from this expression of the periods that α extends continuously to Δ and that it is a bijection. □

Corollary 7.11. Let $\mathcal{F}$ be the fibration constructed in Lemma 7.6 and let $U = \mathbb{R}^3 - \Delta$. The stitched fibration $\mathcal{F}|_U$ is symplectically conjugate to a fibration constructed in Theorem 6.19.

Proof. The fibrations constructed in Theorem 6.19 have smooth Lagrangian sections and the action coordinates extend continuously to the whole base. Since $\mathcal{F}|_U$ also has a Lagrangian section (cf. Remarks 7.7) and the action coordinates extend continuously to the whole base, the statement easily follows from the results on stitched fibrations such as the existence of a normal form. The latter is found extending the maps f^+ and f^- beyond all connected components of the seam and then using the Lagrangian section to normalize with the period map.

□

Smoothing III

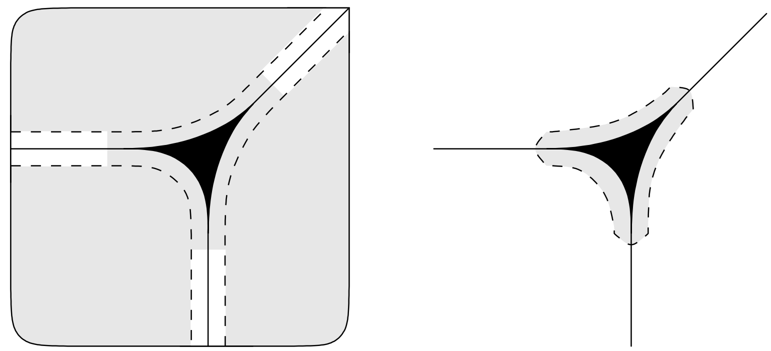
Now we show that the fibration in Example 5.8 can be perturbed to make it smooth on an even larger region. We consider the fibration $\mathcal{F}$ obtained in Lemma 7.6 whose base is depicted in Figure 15 (a). Over the white region complete smoothness was achieved. In the previous section we saw that over $U = \mathbb{R}^3 - \Delta$ the fibration is (symplectically conjugate to) a stitched Lagrangian fibration which can be constructed as in Theorem 6.19. In this section we want to deform the invariants over each connected component of the seam so to achieve smoothness beyond the (planar) gray region in Figure 15 (b).

Lemma 7.12. Let $\mathcal{F}$ be the fibration obtained in Lemma 7.6. There is a perturbation $\tilde{\mathcal{F}}$ of $\mathcal{F}$ such that:

(i) $\tilde{\mathcal{F}}$ is topologically conjugate to $\mathcal{F}$;

(ii) there exists a submanifold with boundary $D \subset B$, homeomorphic to a closed disc in $\mathbb{R}^2$, with $\Delta \cap (B - D)$ consisting of three disjoint segments, such that $\tilde{\mathcal{F}}|_{\mathbb{R}^3 - D}$ is a smooth Lagrangian fibration.

Proof. The proof follows the same lines of Lemma 7.6. Assume that $\mathcal{F}|_{\mathbb{R}^3 - \Delta}$ has been constructed with Theorem 6.19. In particular the wall Γ consists of the union of three disjoint sets, denoted Γ_c, Γ_d and Γ_e. The corresponding components of the seam are



(a) (b)

Figure 15: Smoothing away from the legs.

$Z_c = f^{-1}(\Gamma_c)$, $Z_d = f^{-1}(\Gamma_c)$ and $Z_e = f^{-1}(\Gamma_e)$ with corresponding quotients denoted by $\bar{Z}_c$, $\bar{Z}_d$ and $\bar{Z}_e$. The invariants of $\mathcal{F}|_{\mathbb{R}^3-\Delta}$ are given by sequences ℓ^c, ℓ^d and ℓ^e. In particular the first order invariants satisfy the integral conditions (55) with $m_1 = -1$ and $m_2 = 1$.

Over the same wall Γ and seam Z, we could define another triple of invariants as follows. Define $(\ell^c)'$ to be the zero sequence, while $(\ell^d)'$ and $(\ell^e)'$ to be sequences whose only non-zero terms are the first order ones, which we define to be

$$(\ell^d_1)' = -dy_2 \quad \text{and} \quad (\ell^e_1)' = dy_3.$$

As we saw in Example 6.21, these choices of invariants give rise to a fake stitched fibration $\mathcal{F}'$ which is topologically conjugate to $\mathcal{F}|_{\mathbb{R}^3-\Delta}$.

Using Theorem 6.19 we now construct a new stitched fibration with the same wall Γ and seam Z as $\mathcal{F}|_{\mathbb{R}^3-\Delta}$, but whose invariants interpolate between those of $\mathcal{F}'$ and those of $\mathcal{F}|_{\mathbb{R}^3-\Delta}$. Let A' be a small tubular neighborhood of Δ and denote $\bar{A}' = A' \cap \{b_1 = 0\}$. Assume that $\bar{A}'$ is entirely contained in the region in Figure 15 (a) delimited by the dotted lines. In particular we want the ends of $\bar{A}'$ to be contained in the white region where $\mathcal{F}$ is smooth. Let $A \subset A'$ be a smaller open neighborhood of Δ and denote $\bar{A} = A \cap \{b_1 = 0\}$. Let $\rho \in C^\infty(\Gamma)$ be a cut-off function which is 1 on $\bar{A}$ and 0 on $\Gamma - \bar{A}'$. Define $\tilde{\ell}^c_k = (1-\rho)(\ell^c_k)' + \rho\, \ell^c_k$ and similarly define $\tilde{\ell}^d_k$ and $\tilde{\ell}^e_k$. It follows from Theorem 6.19 that the sequences $\tilde{\ell}_c$, $\tilde{\ell}_d$ and $\tilde{\ell}_e$ give rise to a stitched Lagrangian fibration $\tilde{\mathcal{F}}^o$ which is topologically conjugate to $\mathcal{F}|_{\mathbb{R}^3-\Delta}$. Moreover $\tilde{\mathcal{F}}^o|_{A-\Delta}$ and $\mathcal{F}|_{A-\Delta}$ are symplectically conjugate so we can glue $\mathcal{F}|_A$ to $\tilde{\mathcal{F}}^o|_{A-\Delta}$ along $\mathcal{F}|_{A-\Delta}$. This produces a piecewise smooth Lagrangian fibration $\tilde{\mathcal{F}}$ which is topologically conjugate to $\mathcal{F}$, moreover the chosen invariants guarantee that after a change of coordinates on the base $\tilde{\mathcal{F}}$ satisfies the smoothness condition (ii). $\square$

The fibration $\tilde{\mathcal{F}}$ obtained via Lemma 7.12 clearly satisfies properties (i) and (ii) of Definition 7.1, but finally we can also give

Proof of Theorem 7.3. It only remains to show that $\tilde{\mathcal{F}}$ satisfies property (iii) of Definition 7.1, but this immediately follows from the construction. In fact, $\tilde{\mathcal{F}}|_{\mathbb{R}^3-A'}$ coincides with the fibration described in Example 6.21 restricted to a suitable neighborhood of the vertex. We observed that the latter fibration induces an affine structure on the base which is affine isomorphic to a negative vertex of Example 3.12 (or of Example 3.13). This concludes the proof. $\square$

8 The compactification.

The main theorem

Finally, having completed the construction of the negative fibration, in this last section we prove the main result of the article. In order to give a correct statement of the theorem, we need first to make a few observations.

We start with a compact simple integral affine 3-manifold with singularities $(B, \Delta, \mathscr{A})$. The goal is to symplectically compactify the torus bundle $X(B_0, \mathscr{A})$ by gluing to it singular fibres. We have already seen in Section 4, Proposition 4.17 how the gluing of positive or generic singular fibres is quite straightforward. In the case of negative vertices we have seen that our construction gives a fibration whose discriminant locus contains components of type Δ_a, i.e. of codimension 1. For this reason around negative points one needs to replace Δ with a slightly perturbed discriminant locus containing components of type Δ_a.

Let us consider the fibration of Example 5.8. The periods of this fibration were computed in [3] and they are given by formulas (73). Let us consider the corresponding primitives (action coordinates) restricted to the plane $\{b_1 = 0\}$, which is the plane where the discriminant locus lies. We can easily see that the action coordinates map α transforms the amoeba with thin legs into a slightly different shape, depicted in in Figure 16. This shape does not change much after we have done the smoothings of Lemmas 7.4, 7.6 and 7.12, what may happen is that the codimension 2 part –i.e. the legs– may become slightly curved. Nevertheless, it is not difficult to see that we can prolong the legs of a negative fibration so that they become straight toward their ends. This can be done by gluing suitable generic-singular Lagrangian fibrations using the methods of Proposition 4.18.

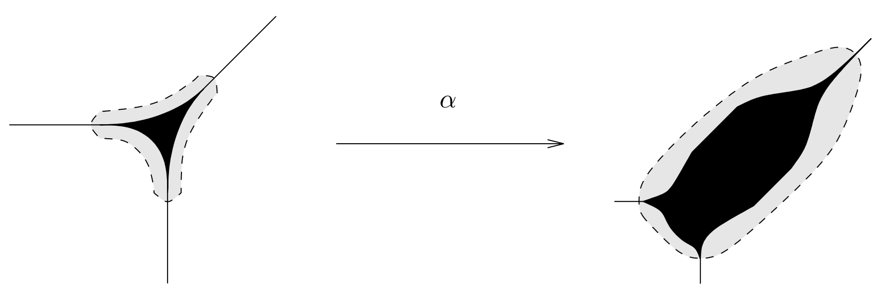


Figure 16: The affine image of the amoeba with thin legs

Definition 8.1. Given a simple integral affine 3-manifold with singularities $(B, \Delta, \mathscr{A})$, all of whose negative vertices are straight (i.e. locally affine isomorphic to Example 3.12), a *localized thickening* of Δ is given by the data $(\Delta^{\blacklozenge}, \{D_{p^-}\}_{p^- \in \mathcal{N}})$ where:

(i) $\Delta^{\blacklozenge}$ is the closed subset obtained from Δ after replacing a neighborhood of each negative vertex with a shape of the type depicted in Figure 17. This replacement takes place in the plane corresponding to $\{x_1 = 0\}$ of the local model, Example 3.12.

(ii) $\mathcal{N}$ is the set of negative vertices and for each $p^- \in \mathcal{N}$, D_{p^-} is a submanifold of B, homeomorphic to a disk and containing the codimension 1 component of $\Delta^{\blacklozenge}$ around the negative vertex p^-. Moreover, D_{p^-} is contained in the plane $\{x_1 = 0\}$. We depict D_{p^-} as the gray area in Figure 17.

The requirement that all negative vertices are straight is only to avoid unnecessary complications. Given a localized thickening of Δ, define

$$B_{\blacklozenge} = B - \left(\Delta \cup \bigcup_{p^- \in \mathcal{N}} D_{p^-} \right).$$

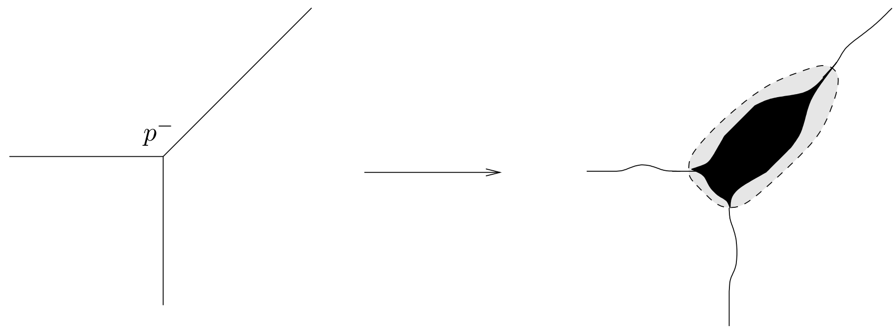


Figure 17: A localized thickening of a negative vertex.

Clearly, the integral affine structure $\mathscr{A}$ on $B - \Delta$ restricts to an integral affine structure on $B_{\blacklozenge}$ which we denote by $\mathscr{A}_{\blacklozenge}$, therefore we can form the torus bundle $X(B_{\blacklozenge}, \mathscr{A}_{\blacklozenge})$.

Now we can state and prove the theorem:

Theorem 8.2. Given a compact simple integral affine 3-manifold with singularities $(B, \Delta, \mathscr{A})$, all of whose negative vertices are straight (i.e. locally isomorphic to Example 3.12), there is a localized thickening $(\Delta_{\blacklozenge}, \{D_{p^-}\}_{p^- \in \mathcal{N}})$ of Δ and a smooth, compact symplectic 6-manifold (X, ω) together with a piecewise smooth Lagrangian fibration $f : X \to B$ such that

(i) f is smooth except along $\bigcup_{p^- \in \mathcal{N}} f^{-1}(D_{p^-})$;

(ii) the discriminant locus of f is $\Delta_{\blacklozenge}$;

(iii) there is a commuting diagram

$$\begin{array}{ccc} X(B_{\blacklozenge}, \mathscr{A}_{\blacklozenge}) & \xrightarrow{\Psi} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ B_{\blacklozenge} & \xrightarrow{\iota} & B \end{array}$$

where ψ is a symplectomorphism and ι the inclusion;

(iv) over a neighborhood of a positive vertex of $\Delta_{\blacklozenge}$ the fibration is positive, over a neighborhood of a point on an edge the fibration is generic-singular, over a neighborhood of D_{p^-} the fibration is Lagrangian negative.

Proof. The proof is quite simple. First we glue positive fibrations over sufficiently small neighborhoods of positive vertices of Δ using Proposition 4.17. Now given a negative vertex $p^- \in \mathcal{N}$, we have that a neighborhood of p^- is affine isomorphic to a neighborhood U of zero in the local model Example 3.12. Consider a negative Lagrangian fibration $\mathcal{F}^- = (X^-, \omega^-, f^-, B^-)$ (cf. Definition 7.1), which we have constructed in Theorem 7.3. The discriminant locus Δ^- of f^- has the shape of an amoeba with thin legs and there is a disc D containing the codimension 1 part of Δ^- such that f^- is smooth except at points of $(f^-)^{-1}(D)$ (cf. part (i) and (ii) of Definition 7.1). Moreover we may assume that $B^- - (\Delta^- \cup D)$ is affine isomorphic to $(U' - (D' \cup \Delta_\tau), \mathscr{A}_\tau)$, where U' is a neighborhood of 0 in the affine manifold with singularities of Example 3.13 and $D' \subset \{x_1 = 0\}$ contains 0 and is homeomorphic to a disc (cf. point (iii) of Definition 7.1). It may happen that U' is too big for us to glue the Lagrangian negative fibration as it is. However, if we replace ω^- with $\epsilon\,\omega^-$ for a sufficiently small $\epsilon > 0$, this has the effect of scaling the affine coordinates on the base by a factor of ϵ (i.e. of making the amoeba as small as we please). Therefore we may assume that $U' \subset U$. Moreover, we may also assume that the legs of Δ^- (in affine coordinates) are straight towards their ends, i.e. they coincide with the legs of Δ outside an open subset U'' such that $D' \subset \bar{U}'' \subset U'$. The localized thickening $\Delta_\blacklozenge$ of Δ around p^- consists in replacing $U' \cup \Delta$ with Δ^- and defining $D_{p^-} = D'$. The affine structure $\mathscr{A}_\blacklozenge$ is inherited from $\mathscr{A}$. This can be done at every negative vertex p^-. Tautologically, we have that $X(U' - (D_{p^-} \cup \Delta_\blacklozenge), \mathscr{A}_\blacklozenge)$ is symplectically conjugate to $(f^-)^{-1}(B^- - (\Delta^- \cup D))$ and therefore we can glue X^- to $X(B_\blacklozenge, \mathscr{A}_\blacklozenge)$.

Finally, now that singular fibres have been glued on top of all vertices, it only remains to glue generic-singular fibres along the edges. This can be easily done by applying directly Proposition 4.18, notice in fact that Lagrangian negative fibrations are smooth and generic-singular towards the ends of the legs. $\square$

We remark that the manifolds we obtain with this theorem are diffeomorphic to Gross' semi-stable compactifications of Theorem 2.11. Also, as a corollary of this construction we have

Corollary 8.3. A smooth quintic X in $\mathbb{P}^4$ has a symplectic form ω with a piecewise smooth Lagrangian fibration $f : X \to S^3$.

Proof. If we apply Theorem 8.2 to Example 3.17 we obtain a symplectic manifold X with a piecewise smooth Lagrangian fibration $f : X \to S^3$. By Gross' Theorem 3.19, X is homeomorphic to a non-singular quintic. $\square$

We do not know whether the symplectic manifold (X, ω) obtained in this corollary is actually symplectomorphic to a quintic with a Kähler form, although we conjecture it is.

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