Einstein-Weyl structures on complex manifolds

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Abstract
A Hermitian Einstein-Weyl manifold is a complex manifold admitting a Ricci-flat Kähler covering $\tilde{M}$, with the deck transform acting on $\tilde{M}$ by homotheties. We show that a Hermitian Einstein-Weyl structure on a compact complex manifold is unique, if it exists. This result is a conformal analogue of Calabi’s theorem stating the uniqueness of Calabi-Yau metrics in a given Kähler class.

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1 Introduction
E. Calabi ([C]) has shown that a compact manifold of Kähler type with vanishing first Chern class can admit at most one Kähler-Einstein metric in a given Kähler class (see [B] for details and implications of this extremely influential work).

In this note, we generalize this result to conformal setting. Recall that a locally conformally Kähler (LCK) manifold is a complex manifold admitting a Kähler covering $\tilde{M}$, with the deck transform acting on $\tilde{M}$ by holomorphic homotheties. If $\tilde{M}$ is, in addition, Ricci-flat, $\tilde{M}$ is called Hermitian Einstein-Weyl, or locally conformally Kähler Einstein-Weyl.\(^1\)

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\(^1\)Normally, one defines Hermitian Einstein-Weyl differently, and then this definition becomes a theorem; see Claim 3.2.
Since the deck transform group acts on $\widetilde{M}$ conformally, the LCK-structure defines a conformal class of Hermitian metrics on $M$. A metric in this class is called an LCK-metric. In the literature, the distinction between “LCK-metrics” and “LCK-structures” is often ignored.

We give an introduction to LCK-geometry in Section 2, and explain the properties of Einstein-Weyl structures in Section 3.

**Theorem 1.1.** Let $(M, J)$ be a compact complex manifold. Then it admits at most one Einstein-Weyl locally conformally Kähler structure, up to a constant multiplier.

We prove Theorem 1.1 in Section 4.

**Remark 1.2.** For a Calabi-Yau manifold, the metric is uniquely determined by the complex structure and the Kähler class in cohomology. In a conformal setting, the Einstein-Weyl LCK-structure is defined uniquely. This happens because a relevant cohomology group is $H^2(M, L)$, where $L$ is the weight bundle of the conformal structure (see Definition 2.2). It is easy to show that all cohomology of the local system $L$ vanish, cf. [L, Remark 6.4].

The compatibility between a complex structure and a Weyl structure naturally leads to the LCK-condition. This was observed by I. Vaisman (see also [PPS]). Moreover, as shown by P. Gauduchon ([G]), a compact Einstein-Weyl locally conformally Kähler manifold is necessarily Vaisman (see Theorem 3.4). Then Theorem 1.1 is translated into the uniqueness of an Einstein-Weyl Vaisman metric on a given compact complex manifold.

The Vaisman manifolds are intimately related to Sasakian geometry (see e.g. [OV1]). Given a Sasakian manifold $X$, the product $S^1 \times X$ has a natural Vaisman structure. Conversely, any Vaisman manifold admits a canonical Riemannian submersion to $S^1$, with fibers which are isometric and equipped with a natural Sasakian structure.

Under this correspondence, the Einstein-Weyl Vaisman manifolds correspond to Sasaki-Einstein manifolds. The Sasaki-Einstein manifolds recently became a focus of much research, due to a number of new and unexpected examples constructed by string physicists (see [MSY], [CLPP], [GMSW1], [GMSW2], and the references therein). For a physicist, Sasaki-Einstein manifolds are interesting because of AdS/CFT correspondence in string theory. From the mathematical point of view, these examples are as mysterious as the Mirror Symmetry conjecture 15 years ago.
The Sasakian manifolds, being transverse Kähler\textsuperscript{2}, can be studied by the means of algebraic geometry. One might hope to obtain and study the Sasaki-Einstein metrics by the same kind of procedures as used to study the Kähler-Einstein metrics in algebraic geometry. However, this analogy is not perfect. In particular, it is possible to show that the Sasaki-Einstein structures on CR-manifolds are not unique. We shall address this problem in a forthcoming paper.

One may hope to approach the classification of Sasaki-Einstein structures using the Einstein-Weyl geometry.

\section{Vaisman manifolds}

We first review the necessary notions of locally conformally Kähler geometry. See [DO], [OV1], [OV2], [OV3], [Ve] for details and examples.

Let $(M, J, g)$ be a complex Hermitian manifold of complex dimension $n$. Denote by $\omega$ its fundamental two-form $\omega(X, Y) = g(X, JY)$.

\begin{definition}
A Hermitian metric $g$ on $(M, J)$ is \textbf{locally conformally Kähler} (LCK for short) if \[d\omega = \theta \wedge \omega.\]
for a closed 1-form $\theta$.
\end{definition}

Clearly, for any function $f : M \to \mathbb{R}^0$, $f\omega$ is also an LCK-metric. A conformal class of LCK-metrics is called an \textbf{LCK-structure}.

The form $\theta$ is called \textbf{the Lee form of the LCK-metric}, and the dual vector field $\theta^\sharp$ is called \textbf{the Lee field}.

The one-form $\theta$ can be interpreted as a (flat) connection one-form in the bundle of densities of weight 1, usually denoted $L$. This is the real line bundle associated to the representation

$A \mapsto |\det(A)\|^{1/2n},\ A \in \text{GL}(2n, \mathbb{R})$

\begin{definition}
The bundle $L$, equipped with a connection $\nabla_0 + \theta$, is called \textbf{the weight bundle of a locally conformally Kähler structure}. One
\end{definition}

\textsuperscript{2}This viewpoint was systematically developed in the work of C.P. Boyer, K. Galicki and collaborators. See e.g. [BG].
could consider the form $\omega$ as a closed, positive $(1,1)$-form, taking values in $L^2$.

**Remark 2.3.** Passing to a covering, we may assume that the flat bundle $L$ is trivial. Then $\omega$ can be considered as a closed, positive $(1,1)$-form taking values in a trivial vector bundle, that is, a Kähler form. Therefore, any LCK-manifold admits a covering $\tilde{M}$ which is Kähler. The deck transform acts on $\tilde{M}$ by homotheties. This property can be used as a definition of LCK-structures (see Section 1).

**Definition 2.4.** A Vaisman manifold is an LCK manifold whose Lee form is parallel with respect to the Levi-Civita connection of $g$.

**Definition 2.5.** Let $(\mathcal{C}, g, \omega)$ be a Kähler manifold. Assume that $\rho$ is a free, proper action of $\mathbb{R}^{>0}$ on $\mathcal{C}$, and $g$ and $\omega$ are homogeneous of weight 2:

$$\text{Lie}_v \omega = 2\omega, \quad \text{Lie}_v g = 2g,$$

where $v$ is the tangent vector field of $\rho$. The quotient $\mathcal{C}/\rho$ is called a Sasakian manifold. If $N = \mathcal{C}/\rho$ is given, $\mathcal{C}$ is called the Kähler cone of $N$. As a Riemannian manifold, $\mathcal{C}$ is identified with the Riemannian cone of $(N, g_N)$, $\mathcal{C}(N) = (N \times \mathbb{R}^{>0}, t^2 g_N + dt^2)$.

The Sasakian manifolds are discussed in [BG], in great detail.

The following characterization of compact Vaisman manifolds is known (see [OV1]):

**Remark 2.6.** A compact complex manifold $(M, J)$ is Vaisman if it admits a Kähler covering $(\tilde{M}, J, h) \to (M, J)$ such that:

- The monodromy group $\Gamma \cong \mathbb{Z}$ acts on $\tilde{M}$ by holomorphic homotheties with respect to $h$ (this means that $(M, J)$ is equipped with an LCK-structure).

- $(\tilde{M}, J, h)$ is isomorphic to a Kähler cone over a compact Sasakian manifold $S$. Moreover, there exists a Sasakian automorphism $\varphi$ and a positive number $q > 1$ such that $\Gamma$ is isomorphic to the cyclic group generated by $(x, t) \mapsto (\varphi(x), qt)$. 


Remark 2.7. In these assumptions, denote by $\theta^2$ the vector field $t \frac{d}{dt}$ on $\tilde{M} = (S \times \mathbb{R}^{>0}, g_st^2 + dt^2)$. Chose the metric $g = g_S + dt^2$ on $M = \tilde{M}/\Gamma$. Clearly, $\theta^2$ descends to a Lee field on $M$, denoted by the same letter. Then $J(\theta^2)$ is tangent to the fibers of the natural projection $\tilde{M} \longrightarrow \mathbb{R}^{>0}$, hence belongs to $TS$. This vector field is called the Reeb field of the Sasakian manifold $S$. Clearly, the orbits of $J(\theta^2)$ on $\tilde{M}$ are precompact (contained in a compact set).

Remark 2.8. It will be important for us to note that the Kähler metric $h$ on the covering $\tilde{M} = C(S) = S \times \mathbb{R}^{>0}$ has a global Kähler potential $\psi$, which is expressed as $\psi(x, t) = t^2$. The metric $\psi^{-1} \cdot h$ projects on $N$ into the LCK metric $g$. Moreover, $\psi = |\theta|^2$, the norm being taken with respect to the lift of $g$.

On a Vaisman manifold, the Lee field $\theta^2$ is Killing, parallel and holomorphic. One easily proves that $L_{\theta^2} \omega = 2\omega$.

Recall from [To] the notion of transverse geometry:

Definition 2.9. Consider a manifold endowed with a foliation $F$ with tangent bundle $F$ and normal bundle $Q$. A differential, or Riemannian, form $\alpha$ on $X$ is basic (or transverse) if $X|\alpha = 0$ and $\text{Lie}_X \alpha = 0$ for every $X \in F$. A transverse geometry of $F$ is a geometry defined locally on the leaf space of $F$. A Kähler transverse structure on $(M, F)$ is a complex Hermitian structure on $Q$ defined by a pair $g_F, \omega_F$ of transverse forms, in such a way that the induced almost complex structure defined locally on the leaf space $M/F$ is integrable and Kähler.

Example 2.10: Let $(M, J, \omega)$ be a Vaisman manifold, $\theta^2$ its Lee field. Consider the holomorphic foliation $F$, generated by $\theta^2$ and $J\theta^2$. The form $\omega - \theta \wedge J\theta$ is transverse Kähler. Hence the Vaisman manifolds provide examples of transverse Kähler foliations ([Va], [Ts1]). Similarly, a Sasakian manifold has a transverse Kähler geometry associated to the foliation generated by the Reeb field.

A compact complex manifold of Vaisman type can have many Vaisman structures, still the Lee field is unique up to homothety:

Proposition 2.11. If $g_1$, $g_2$ are Vaisman metrics on the same compact
manifold $(M, J)$, then $\theta_1^i = c \theta_2^i$, for some real constant $c$.

**Proof.** The result was proven by Tsukada in [Ts2]. Here we include an alternative proof. Recall from [Ve] that for a Vaisman structure $(g, J)$, the two-form

$$\eta := \omega - \theta \wedge J\theta$$

is exact and positive, with the null-space generated by $\langle \theta^2, J\theta^2 \rangle$. It is the transverse Kähler form of $(M, \mathcal{F})$ (see Example 2.10). Let $g_1, g_2$ be Vaisman metrics, $\omega_1, \omega_2$ the corresponding Hermitian forms, $\theta_i$ and $\theta^i$ the corresponding Lee forms and Lee fields. Consider the $(1,1)$-forms $\eta_1, \eta_2$, defined as above,

$$\eta_i := \omega_i - \theta_i \wedge J\theta_i.$$

Unless their null-spaces coincide, the sum $\eta_1 + \eta_2$ is strictly positive. Then

$$\int_M (\eta_1 + \eta_2) \dim M > 0.$$

This is impossible, because $\eta_i$ are exact. We obtained that the 2-dimensional bundles generated by $\theta_i^2, J\theta_i^2$ are equal:

$$\langle \theta_1^2, J\theta_1^2 \rangle = \langle \theta_2^2, J\theta_2^2 \rangle$$

This implies that $\theta_1^2$, considered as a vector in $T^{1,0}(M)$, is proportional to $\theta_2^2$ over $\mathbb{C}$.

$$\theta_1^2 = a \theta_2^2 + b J\theta_2^2, \quad a, b \in \mathbb{R}. \quad (2.1)$$

Since $\theta_i^2$ is holomorphic, the proportionality coefficient is constant.

To finish the proof of Proposition 2.11, it remains to show that this proportionality coefficient is real. Here we use Remark 2.7: the orbits of $J\theta_1^2$ should be pre-compact. From (2.1) we obtain

$$J\theta_1^2 = a J\theta_2^2 - b \theta_2^2.$$ 

But $a J\theta_2^2 - b \theta_2^2$ acts on the metric by a homothety, with a coefficient which is proportional to $e^{-b}$. Therefore, an orbit of this vector field is contained in a compact set if and only if $b = 0$.

**Remark 2.12.** Let $L_C = L \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the weight bundle of the Vaisman manifold $(M, J, g)$. The Lee form then is the connection form of the standard Hermitian connection in $L_C$, and one can prove (see [Ve]) that its curvature can be identified with the above form $\eta = \omega - \theta \wedge J\theta$, hence it is exact.
3 Einstein-Weyl LCK manifolds

Einstein-Weyl structures are defined and studied for their own, see e.g. [CP]. Here we specialize the definitions to LCK structures.

The Levi-Civita connection $\nabla^g$ of $g$ is not the best tool to study the conformal properties of an LCK manifold. Instead, the **Weyl connection** defined by

$$\nabla = \nabla^g - \frac{1}{2}\{\theta \otimes Id + Id \otimes \theta + g \otimes \theta^2\}$$

is torsion-free and satisfies $\nabla g = \theta \otimes g$.

The Ricci tensor of the Weyl connection is not symmetric. Hence, to obtain the analogue of the Einstein condition one gives:

**Definition 3.1.** An LCK-manifold is **Einstein-Weyl** if the symmetric part of the Ricci tensor of the Weyl connection is proportional to the metric. An Einstein-Weyl LCK-manifold is also called **Hermitian Einstein-Weyl**.

Let $\nabla$ be a Weyl connection on an LCK-manifold. One can see that $\nabla$ is the covariant derivative associated to the connection one-form $\theta$ in the weight bundle $L$. Since $\theta$ is closed, we can take a covering $\tilde{M}$ of $M$, with $\theta = df$, for some function $f$ on $\tilde{M}$. The Weyl connection becomes the Levi-Civita connection for the metric $e^{-f}g$ on $\tilde{M}$. Since $\nabla(e^{-f}g) = \nabla(J) = 0$, $e^{-f}g$ is a Kähler metric. This way one obtains a Kähler covering of an LCK-manifold, starting from a Weyl connection. The converse construction is also clear: The Levi-Civita connection on a Kähler covering $\tilde{M}$ of an LCK-manifold $M$ is independent from homotheties, hence descends to $M$, and satisfies the conditions for Weyl connection.

This gives the following claim.

**Claim 3.2.** Let $\nabla$ be a Weyl connection on a complex Hermitian manifold. Then $\nabla$ satisfies the Einstein-Weyl condition if and only if $\nabla$ is Ricci-flat on the Kähler covering of $M$.

\[\blacksquare\]

**Remark 3.3.** Claim 3.2 also follows from Proposition 3.5 (below). Indeed, a trivialization of the weight bundle $L_C$ induces a trivialization of canonical class $K = L_C^{-n}$.
From a deep result of Gauduchon in [G], it follows that:

**Theorem 3.4.** Let \((M, J, g)\) be a compact Einstein-Weyl LCK manifold. Then the Ricci tensor of the Weyl connection vanishes identically and the Lee form is parallel. In particular, \((M, J, g)\) is Vaisman.

From Theorem 3.4, we obtain that all Kähler coverings of an Einstein-Weyl LCK-manifold are Ricci-flat. This property can be used as a definition of Einstein-Weyl LCK-manifolds.

The locally conformally Kähler Einstein-Weyl structures can be expressed in terms of the complexified weight bundle.

**Proposition 3.5.** ([Ve, Proposition 5.6]) Let \(M\) be an Einstein-Weyl LCK-manifold, \(K\) its canonical class, \(L_{\mathbb{C}}\) its weight bundle. Consider \(K, L_{\mathbb{C}}\) as Hermitian holomorphic bundles, with the metrics induced from \(M\). Then \(L_{\mathbb{C}}^n \cong K^{-1}\).

Let \((M, J, g)\) be an Einstein-Weyl Vaisman LCK-manifold, and \(\tilde{M}\) its Kähler covering, which trivialises \(L\). From Proposition 3.5, it is clear that \(\tilde{M}\) has trivial canonical class. Let \(\Omega\) be a section of canonical class of \(\tilde{M}\) which is equivariant under the monodromy action. Such a section is unique up to a constant. Indeed, if \(\Omega_1, \Omega_2\) are two equivariant sections of canonical class, the quotient \(\frac{\Omega_1}{\Omega_2}\) is a holomorphic function on \(\tilde{M}\) which is invariant under monodromy, hence descends to a global holomorphic function on \(M\). Therefore \(\frac{\Omega_1}{\Omega_2} = \text{const.}\) Rescaling \(\Omega\) such that \(|\Omega| = 1\), we obtain

\[
\Omega \wedge \overline{\Omega} = \frac{1}{n! 2^n} \omega^n,
\]

where \(n = \dim_{\mathbb{C}} M\). In particular, given two Einstein-Weyl structures \(\omega_1\) and \(\omega_2\), we always have \(\omega_1^n = \lambda \omega_2^n\), where \(\lambda\) is a positive constant. After rescaling, we may also assume that

\[
\det \omega_1 = \det \omega_2, \quad (3.1)
\]

where \(\det \omega_i = \omega_i^n\), \(n = \dim_{\mathbb{C}} M\).
4 Uniqueness of Einstein-Weyl structures

In this Section, we prove Theorem 1.1. Clearly, Theorem 1.1 follows from (3.1) combined with the following proposition.

Proposition 4.1. Let \((M, J)\) be a compact complex manifold admitting two Vaisman metrics \(\omega_1\) and \(\omega_2\), such that \(\det \omega_1 = \det \omega_2\). Then \(\omega_1 = \omega_2\).

Proof: We start with the following claim, which is implied by Tsukada’s theorem (Proposition 2.11).

Claim 4.2. In these assumptions, denote the corresponding Lee fields by \(\theta_i^\alpha, i = 1, 2\). Then

\[ \theta_1^\alpha = \theta_2^\alpha. \]

Proof: By Proposition 2.11, \(\theta_1^\alpha = c \theta_2^\alpha\). Denote by \(\tilde{\omega}_i\) the Kähler forms on \(\tilde{M}\) corresponding to \(\omega_i\). By construction, \(\text{Lie}_{\theta_i^\alpha} \omega_i = 2\omega_i\), where \(\text{Lie}\) denotes the Lie derivative. Therefore,

\[ \text{Lie}_{\theta_i^\alpha} \omega_i^n = 2n\omega_i^n. \]

Using (3.1), we obtain that

\[ 2n\omega_1^n = \text{Lie}_{\theta_1^\alpha} \omega_1^n = c \text{Lie}_{\theta_2^\alpha} \omega_1^n = 2nc\omega_1^n \]

Therefore, \(c = 1\). We proved Claim 4.2. ■

Return to the proof of Proposition 4.1. Consider a form

\[ \eta_i := \omega_i - \theta_i \wedge J\theta_i. \]

(4.1)

This is a positive, exact \((1, 1)\)-form on \(M\), which can be interpreted as a curvature of the weight bundle (see the proof of Proposition 2.11). First of all, we deduce from \(\eta_1 = \eta_2\) the statement of Proposition 4.1.

Lemma 4.3. In the assumptions of Proposition 4.1, assume that \(\eta_1 = \eta_2\), where \(\eta_i\) are \((1, 1)\)-forms defined in (4.1). Then \(\omega_1 = \omega_2\).

Proof. As follows from (4.1), to prove \(\omega_1 = \omega_2\) it suffices to show \(\theta_1 = \theta_2\). Let \(\tilde{M}\) be the Kähler \(Z\)-covering of \(M\), which is a cone over a compact
Sasakian manifold, and \( \varphi_1, \varphi_2 \) the corresponding Kähler potentials, obtained as in Remark 2.8. It is easy to see that \( \theta_i = d \log \varphi_i \) and \( \eta_i = d^c \theta_i \) ([Ve]). Therefore,

\[
\eta_1 - \eta_2 = d^c d \log \left( \frac{\varphi_1}{\varphi_2} \right) \tag{4.2}
\]

The functions \( \varphi_i \) are automorphic under the deck transform action on \( \tilde{M} \), with the same factors of monodromy. Therefore, their quotient \( \frac{\varphi_1}{\varphi_2} \) is well defined on \( M \). By (4.2), \( 0 = \eta_1 - \eta_2 = d^c d \log \left( \frac{\varphi_1}{\varphi_2} \right) \), hence \( \psi := \log \frac{\varphi_1}{\varphi_2} \) is pluriharmonic on a compact complex manifold \( M \). Therefore \( \psi \) is constant. This gives \( \theta_1 - \theta_2 = d\psi = 0 \). Lemma 4.3 is proven. ■

Return to the proof of Proposition 4.1. Note that \( \eta_i \) are transverse Kähler forms. Since

\[
\det \eta_i = (\theta^2 \wedge J \theta^2) \det \omega_i,
\]

it follows that

\[
\det \eta_1 = \det \eta_2.
\]

Let \( \rho \) be a transverse form, defined as \( \rho = \sum_{k+l=n-2} \eta_1^k \wedge \eta_2^l \). Then

\[
(\eta_1 - \eta_2) \wedge \rho = 0. \tag{4.3}
\]

As \( \eta_i \) are both positive, \( \rho \) is strictly positive, transversal \((n - 2, n - 2)\)-form. It is well known that on a complex manifold \( X \), any positive \((\dim X - 1, \dim X - 1)\)-form is an \((\dim X - 1)\)-st power of a Hermitian form. Therefore, there exists a transverse form \( \alpha \) such that \( \rho = \alpha^{n-2} \). Then (4.3) gives

\[
(\eta_1 - \eta_2) \wedge \alpha^{n-2} = 0.
\]

From (4.2), we obtain

\[
\eta_1 - \eta_2 = dd^c \psi,
\]

where \( \psi := \log \left( \frac{\varphi_1}{\varphi_2} \right) \) is a smooth, transversal function on \( M \).

We now associate to \( \alpha \) a second-order differential operator \( \mathcal{D} \) acting on transverse \( C^\infty \) functions, which is defined as follows. For any transverse function \( f \), \( dd^c f \wedge \alpha^{n-2} \) is a transverse top \((n - 1, n - 1)\) form, and hence there exists a unique transverse function \( g \) such that \( dd^c f \wedge \alpha^{n-2} = g \cdot \alpha^{n-1} \). We define

\[
\mathcal{D}(f) = g, \quad \text{where} \quad dd^c f \wedge \alpha^{n-2} = g \cdot \alpha^{n-1}.
\]
In other words,

\[ D(f) = \frac{dd^c f \wedge \alpha^{n-2}}{\alpha^{n-1}}. \]

From the definition, we have \( D(\psi) = 0 \). Obviously \( D \) has positive symbol on the ring of transverse functions, identified locally with functions on a space of leaves of \( \mathcal{F} \).\(^1\) This allows us to apply the generalized maximum principle:

**Proposition 4.4.** ([PW]) Let \( D \) be a second order differential operator on \( \mathbb{R}^n \) with positive symbol, satisfying \( D(const.) = 0 \), and let \( f \in \ker D \) be a function in its kernel. Assume that \( f \) has a local maximum. Then \( f \) is constant.

Return to the proof of Theorem 1.1. Recall that from (4.2), we have

\[ \eta_1 - \eta_2 = d^c d \log(\psi), \quad \psi \in \ker D. \]

To show that \( \eta_1 = \eta_2 \) it is enough to prove that the kernel of \( D \) contains only constant functions. As follows from the generalized maximum principle, a function in \( \ker D \) which has a local maximum is necessarily constant. Since \( M \) is compact, any continuous function on \( M \) must have a maximum. Therefore, \( \psi \in \ker D \) is constant, and \( \eta_1 - \eta_2 = dd^c \psi = 0 \). The proof of Theorem 1.1 is finished.

All locally conformal Kähler structures underlying a locally conformally hyperkähler structure on a compact hypercomplex manifold are necessarily Einstein-Weyl. This gives

**Corollary 4.5.** Let \((M, I_1, I_2, I_3)\) be a compact hypercomplex manifold. Then it can admit at most one locally conformally hyperkähler structure.

\[ \blacklozenge \]

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\(^{1}\)In fact, the symbol of \( D \) is equal to the symmetric, positive definite 2-form associated with \( \alpha \).
References


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