AN EXACT GEOMETRIC MASS FORMULA

CHIA-FU YU

Abstract. We show an exact geometric mass formula for superspecial points in the reduction of any quaternionic Shimura variety modulo at a good prime \( p \).

1. Introduction

Let \( p \) be a rational prime number. Let \( B \) be a totally indefinite quaternion algebra over a totally real field \( F \) of degree \( d \), together with a positive involution \(*\). Assume that \( p \) is unramified in \( B \). Let \( O_B \) be a maximal order stable under the involution \(*\). Let \((V, \psi)\) be a non-degenerate \( \mathbb{Q} \)-valued skew-Hermitian (left) \( B \)-module with dimension \( 2g \) over \( \mathbb{Q} \). Put \( m := \frac{d}{2d} \), a positive integer. A polarized abelian \( O_B \)-variety \( \mathbb{A} = (A, \lambda, \iota) \) is a polarized abelian variety \((A, \lambda)\) together with a ring monomorphism \( \iota : O_B \rightarrow \text{End}(A) \) such that \( \lambda \circ \iota(b^*) = \iota(b)^* \circ \lambda \) for all \( b \in O_B \). Let \( k \) be an algebraically closed field of characteristic \( p \). An abelian variety over \( k \) is said to be superspecial if it is isomorphic to a product of supersingular elliptic curves. Denote by \( \Lambda_g^B \) the set of isomorphism classes of \( g \)-dimensional superspecial principally polarized abelian \( O_B \)-varieties over \( k \). Define the mass of \( \Lambda_g^B \) to be

\[
\text{Mass}(\Lambda_g^B) := \sum_{\mathbb{A} \in \Lambda_g^B} \frac{1}{|\text{Aut}(A, \lambda, \iota)|}.
\]

The mass \( \text{Mass}(\Lambda_g^B) \) is studied in Ekedahl [1] (Ekedahl’s result relies on an explicit volume computation in Hashimoto-Ibukiyama [4, Proposition 9, p. 568]) in the special case \( B = M_2(\mathbb{Q}) \). He proved

**Theorem 1.1** (Ekedahl, Hashimoto-Ibukiyama). One has

\[
\text{Mass}(\Lambda_g^B) = \frac{(-1)^{g(g+1)/2}}{2g} \prod_{i=1}^{g} \zeta(1 - 2i) \cdot \prod_{i=1}^{g} p^i - (-1)^i,
\]

where \( \Lambda_g \) is the set of isomorphism classes of \( g \)-dimensional superspecial principally polarized abelian varieties over \( k \) and \( \zeta(s) \) is the Riemann zeta function.

Let \( B_{p, \infty} \) be the quaternion algebra over \( \mathbb{Q} \) ramified exactly at \( \{p, \infty\} \). Let \( B' \) be the quaternion algebra over \( F \) such that \( \text{inv}_v(B') = \text{inv}_v(B_{p, \infty} \otimes \mathbb{Q} B) \) for all \( v \). Let \( \Delta' \) be the discriminant of \( B' \) over \( F \).

In this paper we prove

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The maximal order of $B$ if denoted by $\mathcal{O}$. Theorem 1.2.
One has
\[
\text{Mass}(\Lambda^B_g) = \frac{(-1)^{dm(m+1)/2}}{2\pi d} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta'} N(v)^i + (-1)^i \prod_{v|\mathfrak{p}, v|\Delta'} N(v)^i + 1 \right\},
\]
where $\zeta_F(s)$ is the Dedekind zeta function.

Let $N \geq 3$ be a prime-to-$p$ positive integer. Choose a primitive $n$-th root of unity $\zeta_N \in \mathbb{Q} \subset \mathbb{C}$ and fix an embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$. Let $M$ be the moduli space over $\mathbb{F}_p$ of $g$-dimensional principally polarized abelian $O_B$-varieties with a symplectic $O_B$-linear level-$N$ structure w.r.t. $\zeta_N$. Let $L_0$ be a self-dual $O_B$-lattice of $V$ with respect to $\psi$. Let $G_1$ be the automorphism group scheme over $\mathbb{Z}$ associated to the pair $(L_0, \psi)$. As an immediate consequence of Theorem 1.2, we get
Theorem 1.3. The moduli space $M$ has
\[
|G_1(\mathbb{Z}/N\mathbb{Z})| \frac{(-1)^{dm(m+1)/2}}{2\pi d} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta'} N(v)^i + (-1)^i \prod_{v|\mathfrak{p}, v|\Delta'} N(v)^i + 1 \right\}
\]
superspecial points.

We divide the proof of Theorem 1.2 into 4 parts; each part is treated in one section. The first part is to express the weighted sum in terms of an arithmetic mass; this is done in the author’s recent work [8]. The second part is to compute the mass associated to a quaternion unitary group and a standard open compact subgroup; this is done by Shimura [7] (re-obtained by Gan and J.-K. Yu [3, 11.2, p. 522]) using the theory of Bruhat-Tits Buildings). The third part is to compare the derived arithmetic mass in Section 1 with “the” standard mass in Section 2. This reduces the problem to computing a local index at $p$. The last part uses Dieudonné theory to compute this local index. A crucial step is choosing a good basis for the superspecial Dieudonné module concerned; this makes the computation easier.

Notation. $\mathbb{A}$ denotes the Hamilton quaternion algebra over $\mathbb{R}$. $A_f$ denotes the finite adele ring of $\mathbb{Q}$ and $\hat{\mathbb{A}} = \prod_p \mathbb{A}_p$. For a number field $F$ and a finite place $v$, denote by $O_F$ the ring of integers, $F_v$ the completion of $F$ at $v$, $e_v$ the ramification index for $F/\mathbb{Q}$, $\kappa_v$ the residue field, $f_v := [\kappa_v : \mathbb{F}_p]$ and $q_v := N(v) = |\kappa_v|$. For an $O_F$-module $A$, write $A_v$ for $A \otimes_{O_F} O_{F_v}$. For a scheme $X$ over $\text{Spec } A$ and an $A$-algebra $B$, write $X_B$ for $X \times_{\text{Spec } A} \text{Spec } B$. For a linear algebraic group $G$ over $\mathbb{Q}$ and an open compact subgroup $U$ of $G(A_f)$, denote by $\text{DS}(G, U)$ the double coset space $G(\mathbb{Q}) \backslash G(A_f)/U$, and write $\text{Mass}(G, U) := \sum_{\Gamma_i} |\Gamma_i|^{-1}$ if $G$ is $\mathbb{R}$-anisotropic, where $\Gamma_i := G(\mathbb{Q}) \cap c_i U c_i^{-1}$ and $c_1, \ldots, c_h$ are complete representatives for $\text{DS}(G, U)$. For a central simple algebra $B$ over $F$, write $\Delta(B/F)$ for the discriminant of $B$ over $F$. If $B$ is a central division algebra over a non-archimedean local field $F_v$, denote by $O_B$ the maximal order of $B$, $m(B)$ the maximal ideal and $\kappa(B)$ the residue field. $\mathbb{Q}_p$ denotes the unramified extension of $\mathbb{Q}_p$ of degree $n$ and write $\mathbb{Z}_p^n := O_{\mathbb{Q}_p^n}$.

2. Simple mass formulas

Let $B$ be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with a positive involution $*$, and $O_B$ be an order of $B$ stable under *. Let $k$ be any field.
To any polarized abelian $O_B$-varieties $\mathcal{A} = (A, \lambda, \iota)$ over $k$, we associate a pair $(G_x, U_x)$, where $G_x$ is the group scheme over $\mathbb{Z}$ representing the functor

$$R \mapsto \{ h \in (\text{End}_O(A_k) \otimes R)^\times \mid h'h = 1 \},$$

where $h \mapsto h'$ is the Rashot involution, and $U_x$ is the open compact subgroup $G_x(\mathbb{Z})$. For any prime $\ell$, we write $A_\ell(\ell)$ for the associated $\ell$-divisible group with additional structures $(A[\ell^\infty], \lambda, \iota_\ell)$, where $\lambda$ is the induced quasi-polarization from $A[\ell^\infty]$ to $A'[\ell^\infty] = A[\ell^\infty]'$ (the Serre dual), and $\iota_\ell: O_B \otimes \mathbb{Z}_\ell \rightarrow \text{End}(A[\ell^\infty])$ the induced ring monomorphism. For any two objects $A_0$ and $A_2$ over $k$, denote by $\text{Q-isom}_k(A_0, A_2)$ the set of $O_B$-linear quasi-isogenies $\varphi: A_1 \rightarrow A_2$ over $k$ such that $\varphi^*\lambda_2 = \lambda_1$, and $\text{Isom}_k(A_0(\ell), A_2(\ell))$ the set of $O_B \otimes \mathbb{Z}_\ell$-linear isomorphisms $\varphi: A_1[\ell^\infty] \rightarrow A_2[\ell^\infty]$ over $k$ such that $\varphi^*\lambda_2 = \lambda_1$.

Let $x := A_0 = (A_0, \lambda_0, \iota_0)$ be a fixed polarized abelian $O_B$-variety over $k$. Denote by $A_x(k)$ the set of isomorphisms classes of polarized abelian $O_B$-varieties $\mathcal{A}$ over $k$ such that

$$\text{(I)}: \text{Isom}_k(A_0, A_\ell(\ell)) \neq \emptyset \text{ for all primes } \ell.$$

Let $\Lambda'_x(k) \subseteq A_x(k)$ be the subset consisting of objects such that

$$\text{(Q)}: \text{Q-isom}_k(A_0, A) \neq \emptyset.$$

Let $\ker^1(Q, G_x)$ denote the kernel of the local-global map $H^1(Q, G_x) \rightarrow \prod_v H^1(Q_v, G_x)$.

**Theorem 2.1.** ([8, Theorem 2.3]) Suppose that $k$ is a field of finite type over its prime field.

1. There is a natural bijection $\Lambda'_x(k) \cong \text{DS}(G_x, U_x)$. Consequently, $\Lambda'_x(k)$ is finite.
2. One has $\text{Mass}(\Lambda'_x(k)) = \text{Mass}(G_x, U_x)$.

**Theorem 2.2.** ([8, Theorem 4.6 and Remark 4.7]) Notation as above. If $k \supset \mathbb{F}_p$ is algebraically closed and $A_0$ is supersingular, then $\text{Mass}(\Lambda'_x(k)) = \text{Mass}(G_x, U_x)$ and $\text{Mass}(\Lambda_x(k)) = |\ker^1(Q, G_x)| \cdot \text{Mass}(G_x, U_x)$.

**Remark 2.3.** The statement of Theorem 2.2 is valid for basic abelian $O_B$-varieties in the sense of Kottwitz (see [6] for the definition). The present form is enough for our purpose.

3. **AN EXACT GEOMETRIC MASS FORMULA**

Let $D$ be a totally definite quaternion division algebra over a totally real field $F$ of degree $d$. Let (bar) $d \mapsto \bar{d}$ denote the canonical involution. Let $(V', \varphi)$ be a $D$-valued totally definite quaternion Hermitian $D$-module of rank $m$. Let $G^e$ denote the unitary group attached to $\varphi$. This is a reductive group over $F$ and is regarded as a group over $\mathbb{Q}$ via the Weil restriction of scalars from $F$ to $\mathbb{Q}$. Choose a maximal order $O_D$ of $D$ stable under the canonical involution $\bar{\cdot}$. Let $L$ be an $O_D$-lattice in $V'$ which is maximal among the lattices on which $\varphi$ takes its values in $O_B$. Let $U_0$ be the open compact subgroup of $G^e(\mathbb{A}_f)$ which stabilizes the adelic lattice $L \otimes \mathbb{Z}$. The following is deduced from a mass formula of Shimura [7] (also see Gan - J.-K. Yu [3, 11.2, p. 522]). This form is more applicable to prove Theorem 1.2.
Theorem 3.1 (Shimura). One has
\begin{equation}
\text{Mass}(G^\varphi, U_0) = \frac{(-1)^{dm(m+1)/2}}{2^{md}} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta(O/F)} N(v)^i + (-1)^i \right\}.
\end{equation}

\begin{equation}
\text{Mass}(G^\varphi, U_0) = |D_F|^{m^2} \prod_{i=1}^{m} D_F^{1/2} \left[ (2i-1)!(2\pi)^{-2i} \right] \zeta_F(2i) \prod_{v|\Delta(D/F)} N(v)^i + (-1)^i,
\end{equation}
where $D_F$ is the discriminant of $F$ over $\mathbb{Q}$. Using the functional equation for $\zeta_F(s)$, we deduce (3.1) from (3.2).

4. Global comparison

Keep the notation as in Section 1. Fix a $g$-dimensional superspecial principally polarized abelian $O_B$-variety $x = (A_0, \lambda_0, t_0)$ over $k$. Define $\Lambda_x := \Lambda_x(k)$ as in Section 2. Let $(G_x, U_x)$ be the pair associated to $x$.

Lemma 4.1. Any two self-dual $O_B \otimes \mathbb{Z}_p$-lattices of $(V_{\psi}, \psi)$ are isomorphic.

Proof. The proof is elementary and omitted.

Lemma 4.2. One has (1) $\Lambda_x = \Lambda^B_x \; \ker^1(\mathbb{Q}, G_x) = \{1\}$.

Proof. (1) The inclusion $\Lambda_x \subset \Lambda^B_x$ is clear. We show the other direction. Let $A \in \Lambda^B_x$. It follows from Lemma 4.1 that the condition $(I_\ell)$ is satisfied for primes $\ell \neq p$. Let $M$ be the covariant Dieudonné module of $A$. One chooses an isomorphism $O_{B,p} \simeq M_2(O_{F,p})$ so that $* : (a_{ij}) \mapsto (a_{ij})^t$. Using the Morita equivalence, it suffices to show that any two superspecial principally quasi-polarized Dieudonné modules with compatible $O_{F,p}$-action are isomorphic. This follows from Theorem 5.1.

(2) Since $G_x$ is semi-simple and simply connected (as it is an inner form of $\text{Res}_{F/\mathbb{Q}} \text{Sp}_{2m,F}$), the Hasse principle for $G_x$ holds.

4.1. We compute that

(i) $G_x(\mathbb{R}) = \{ \tilde{h} \in M_m(\mathbb{H})^d | \tilde{h}^t \tilde{h} = 1 \}$,

(ii) for $\ell \neq p$, we have $G_x(\mathbb{Q}_\ell) = \prod_{v|\ell} G_{x,v}$ and $U_{x,\ell} = \prod_{v|\ell} U_{x,v}$, where

\begin{equation}
G_{x,v} = \begin{cases}
\text{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta(B/F), \\
\{ h \in M_m(B_v) | \tilde{h}^t h = 1 \}, & \text{otherwise},
\end{cases}
\end{equation}

\begin{equation}
U_{x,v} = \begin{cases}
\text{Sp}_{2m}(O_{F,v}), & \text{if } v \nmid \Delta(B/F), \\
\{ h \in M_m(O_{B_v}) | \tilde{h}^t h = 1 \}, & \text{otherwise},
\end{cases}
\end{equation}

(iii) $G_x(\mathbb{Q}_p) = \prod_{v|p} G_{x,v}$, where

\begin{equation}
G_{x,v} = \begin{cases}
\text{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta', \\
\{ h \in M_m(B'_{v}) | \tilde{h}^t h = 1 \}, & \text{otherwise}.
\end{cases}
\end{equation}

Take $D = B'$ and $V' = D_m^{\varphi}$ with $\varphi(x, y) = \sum x_i y_i$, and take $L = O_D^{\varphi}$. We compute that

(i) $G^\varphi(\mathbb{R}) = \{ h \in M_m(\mathbb{H})^d | \tilde{h}^t h = 1 \}$,
(ii)' for any \( \ell \), we have \( G_x(\mathbb{Q}_\ell) = \prod_{v \mid \ell} G^c_v \) and \( U_{0, \ell} = \prod_{v \mid \ell} U_{0, v} \), where

\[
G^c_v = \begin{cases} \text{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta', \\
\{ h \in M_m(B'_v) \mid \tilde{h}h = 1 \}, & \text{otherwise,}
\end{cases}
\]

\[
U_{0, v} = \begin{cases} \text{Sp}_{2m}(O_{F_v}), & \text{if } v \nmid \Delta', \\
\{ h \in M_m(O_{B'_v}) \mid \tilde{h}h = 1 \}, & \text{otherwise.}
\end{cases}
\]

(4.3)

For \( \ell \neq p \) and \( v \mid \ell \), one has \( B_v = B'_v \) and that \( v \nmid \Delta(B/F) \) if and only if \( v \nmid \Delta' \). It follows from computation above that \( G_x, \mathbb{Q}_\ell \simeq G^c_v \) and \( G_x, \mathbb{Q}_\ell \simeq G^c_v \) for all \( \ell \). Since the Hasse principle holds for the adjoint group \( G^x, \mathbb{Q}_\ell \), we get \( G_x \simeq G^c \) over \( \mathbb{Q} \). We fix an isomorphism and write \( G_x = G^c \). For \( \ell \neq p \) and \( v \mid \ell \), the subgroups \( U_{0, v} \) and \( U_{x, v} \) are conjugate, and hence they have the same local volume.

4.2. Applying Theorem 2.2 in our setting (Section 1) and using Lemma 4.2, we get \( \text{Mass}(\mathbb{A}^B_g) = \text{Mass}(G_x, U_x) \). Using the result in Subsection 4.1, we get

\[
\text{Mass}(\mathbb{A}^B_g) = \text{Mass}(G^c, U_0) \cdot \mu(U_{0, p}/U_{x, p}),
\]

(4.4)

where \( \mu(U_{0, p}/U_{x, p}) = [U_{x, p} : U_{0, p} \cap U_{x, p}]^{-1} [U_{0, p} : U_{0, p} \cap U_{x, p}] \).

5. Local index \( \mu(U_{0, p}/U_{x, p}) \)

Let \( (M', \langle , \rangle', \iota') \) be the covariant Dieudonné module associated to the point \( x = (A_0, \lambda_0, \iota_0) \) in the previous section. Choose an isomorphism \( O_B \otimes \mathbb{Z}_p \simeq M_2(O_F \otimes \mathbb{Z}_p) \) so that \( \ast \) becomes the transpose. Let \( M := eM', \langle , \rangle := \langle , \rangle'|_M \) and \( \iota := \iota'|_{O_F} \),

where \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) in \( M_2(O_F \otimes \mathbb{Z}_p) \). The triple \( (M, \langle , \rangle, \iota) \) is a superspecial principally quasi-polarized Dieudonné module with compatible \( O_F \otimes \mathbb{Z}_p \)-action of rank \( g = 2dm \). Let \( M = \oplus_{v \mid p} M_v \) be the decomposition with respect to the decomposition \( O_F \otimes \mathbb{Z}_p = \oplus_{v \mid p} O_v \); here we write \( O_v \) for \( O_{F_v} \). By the Morita equivalence, we have

\[
U_{x, p} = \text{Aut}_{DM, O_B}(M', \langle , \rangle') = \text{Aut}_{DM, O_F}(M, \langle , \rangle) = \prod_{v \mid p} U_{x, v},
\]

(5.1)

where \( U_{x, v} := \text{Aut}_{DM, O_v}(M_v, \langle , \rangle) \).

Let \( W := W(k) \) be ring of Witt vectors over \( k \) and \( \sigma \) the absolute Frobenius map on \( W \). Let \( \mathcal{J} := \text{Hom}(O_v, W) \) be the set of embeddings; write \( \mathcal{J} = \{ \sigma_i \}_{i \in \mathbb{Z}/f_v \mathbb{Z}} \) so that \( \sigma \sigma_i = \sigma_{i+1} \) for all \( i \). We identify \( \mathbb{Z}/f_v \mathbb{Z} \) with \( \mathcal{J} \) through \( i \mapsto \sigma_i \). Decompose \( M_v = \oplus_{i \in \mathbb{Z}/f_v \mathbb{Z}} M^i_v \) into \( \sigma_i \)-isotypic components \( M^i_v \). One has \((1)\) each component \( M^i_v \) is a free \( W \)-module of rank \( 2m \), which is self-dual with respect to the pairing \( \langle , \rangle \), \((2)\) \( \langle M^i_v, M^j_v \rangle = 0 \) if \( i \neq j \), and \((3)\) the operations \( F \) and \( V \) shift by degree 1 and degree -1, respectively.

**Theorem 5.1.** Let \( (M_v, \langle , \rangle, \iota) \) be as above. There is a symplectic basis \( \{ X^i_j, Y^i_j \}_{i=1, \ldots, m} \) for \( M^i_v \) such that

(i) \( Y^i_j \in VM_{v+1}^i \),

(ii) \( FX^i_j = -Y^{i+1}_j \) and \( FY^i_j = pX^{i+1}_j \),

for all \( i \in \mathbb{Z}/f_v \mathbb{Z} \) and all \( j \).
Proof. We write $f$, $M$ and $q$ for $f_v$, $M_v$ and $q_v$, respectively. Suppose that $f = 2c$ is even. Let $N := \{x \in M \mid F^ex = (-1)^cV^ex\}$. Since $M$ is superspecial, we have $(\ast) F^2N = pN$, $\overline{N} \otimes_{\mathbb{Q}_v} W \simeq M$ and $N = \oplus N^i$. Since $VN^T$ is isotropic with respect to $(\cdot, \cdot)$ in $N/pN$, we can choose a symplectic basis $\{X^i_j, Y^i_j\}_{j=1, \ldots, m}$ for $N^0$ such that $Y^0_j \in VN^1$ for all $j$. Define $X^i_j$ and $Y^i_j$ recursively for $j = 1, \ldots, i$:

\begin{equation}
X^i_{j+1} = p^{-1}FY^i_j, \quad Y^i_{j+1} = -FX^i_j.
\end{equation}

One has $X^i_{j+2} = \frac{1}{p}F^2X^i_j$ and $Y^i_{j+2} = \frac{1}{p}F^2Y^i_j$; hence

\begin{equation}
X^i_j = (-1)^cF^{2c}X^0_j = X^0_j, \quad Y^i_j = (-1)^cF^{2c}Y^0_j = Y^0_j,
\end{equation}

for all $j$. It is easy to see that $\{X^i_j, Y^i_j\}_{j=1, \ldots, m}$ forms a symplectic basis for $N^i$. Suppose that $f = 2c + 1$ is odd. Let $N := \{x \in M \mid F^2fx + pf(x) = 0\}$. We construct a symplectic basis $\{X^i_j, Y^i_j\}_{j=1, \ldots, m}$ for $N^0$ with the properties: $X^0_j \notin VN^1$, $Y^0_j \in VN^1$ and $Y^0_j = (-1)^c1p^{-c}F^2FX^0_j$ for all $j$. We can choose $X^0_j \in N^0 \setminus VN^1$ so that $\langle X^0_j, (-1)^c1p^{-c}F^2FX^0_j \rangle \in \mathbb{Q}_v$. This follows from the fact that the form $(x, y) := \langle x, p^{-c}F^2fy \rangle \mod p$ is a non-degenerate Hermitian form on $N^0/VN^1$. Set $Y^0_j = (-1)^c1p^{-c}FFX^0_j$ and let $\mu := \langle X^0_j, Y^0_j \rangle$. From $\langle FFX^0_j, FFX^0_j \rangle = \langle (-1)^c1p^{-c}FFX^0_j, (-1)^c1p^{-c}FFX^0_j \rangle$, we get $\mu \in \mathbb{Q}_v$. Since $\mathbb{Q}_v/\mathbb{Q}_q$ is unramified, replacing $X^0_j$ by a suitable $\lambda X^0_j$, we get $\langle X^0_j, Y^0_j \rangle = 1$. Do the same construction for the complement of the submodule $< X^0_j, Y^0_j >$ and use induction; we exhibit such a basis for $N^0$.

Define $X^i_j$ and $Y^i_j$ recursively for $i = 1, \ldots, f$ as (5.2). We verify again that $X^i_j = X^0_j$ and $Y^i_j = Y^0_j$. It follows from the relation (5.2) that $\{X^i_j, Y^i_j\}_{j=1, \ldots, m}$ forms a symplectic basis for $N^i$ for all $i$. This completes the proof. ■

Proposition 5.2. Notation as above.

1. If $f_v$ is even, then

\begin{equation}
U_{x,v} = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}_{2m}(\mathbb{Z}_{q^v}) \mid B \equiv 0 \mod p \right\}.
\end{equation}

2. If $f_v$ is odd, then

\begin{equation}
U_{x,v} \simeq \{ h \in M_m(O_{\mathcal{B}^v}) \mid \hat{h}h = 1 \}.
\end{equation}

Proof. Let $\phi \in U_{x,v}$. Choose a symplectic basis $B$ for $M_v$ as in Theorem 5.1. Since $\phi$ commutes with the $O_F$-action, we have $\phi = (\phi_i)$, where $\phi_i \in \text{Aut}(M_v, \langle \cdot, \cdot \rangle)$. Write $\phi_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \text{Sp}_{2m}(W)$ using the basis $B$. Since the map $F$ is injective, $\phi_0$ determines the remaining $\phi_i$. From $\phi F^2 = F^2 \phi$, we have $\phi_{i+2} = \phi^{(2)}_i$ (as matrices). Here we write $\phi^{(n)}_i$ for $\phi^{(n)}_i$. From $\phi F = F \phi$ we get $A^{(1)}_i = D_{i+1}$, $B^{(1)}_i = -pC_{i+1}$, $C^{(1)}_i = -B_{i+1}$ and $D^{(1)}_i = A_{i+1}$.

1. If $f_v$ is even, then $A_0, B_0, C_0, D_0 \in \mathbb{Z}_{q^v}$ and $B_0 \equiv 0 \mod p$. This shows (5.3).

2. Suppose $f_v$ is odd. From $\phi_0^{(f_v+1)} = \phi_1$ we get $A^{(f_v)}_0 = D_0$, $B^{(f_v)}_0 = -pC_0$, $pC^{(f_v)}_0 = -B_0$, $D^{(f_v)}_0 = A_0$. Hence

\begin{equation}
U_{x,v} = \left\{ \left( \begin{array}{cc} A & -pC^T \\ C & A^T \end{array} \right) \in \text{Sp}_{2m}(\mathbb{Z}_{q^v}) \right\}.
\end{equation}
where \( \tau \) is the involution of \( \mathbb{Q}_{q^2} \) over \( \mathbb{Q}_q \). Note that \( O_B' = \mathbb{Z}_{q^2} \langle \Pi \rangle \) with \( \Pi^2 = -p \) and \( \Pi a = a^7 \Pi \) for all \( a \in \mathbb{Z}_{q^2} \). The map \( A + C \Pi \mapsto \begin{pmatrix} A & -pC' \\ C & A' \end{pmatrix} \) gives rise to an isomorphism (5.4). This proves the proposition. ■

Let \( (V_0 = \mathbb{F}_q^{2m}, \psi_0) \) be a standard symplectic space. Let \( P \) be the stabilizer of the standard maximal isotropic subspace \( \mathbb{F}_q \langle e_1, \ldots, e_m \rangle \).

**Lemma 5.3.** \( |Sp_{2m}(\mathbb{F}_q)/P| = \prod_{i=1}^{m}(q^i + 1) \).

**Proof.** We have a natural bijection between the group \( Sp_{2m}(\mathbb{F}_q) \) and the set \( B(m) \) of ordered symplectic bases \( \{v_1, \ldots, v_{2m}\} \) for \( V_0 \). The first vector \( v_1 \) has \( q^{2m} - 1 \) choices. The first companion vector \( v_{m+1} \) has \( q^{2m-1} \) choices as it does not lie in the hyperplane \( v_1^\perp \) and we require \( \psi_0(v_1, v_{m+1}) = 1 \). The remaining ordered symplectic basis can be chosen from the complement \( \mathbb{F}_q \langle v_1, v_{m+1} \rangle^\perp \). Therefore, we have proved the recursive formula \( |Sp_{2m}(\mathbb{F}_q)| = (q^{2m} - 1)q^{2m-1}|Sp_{2m-2}(\mathbb{F}_q)| \). From this, we get

\[
|Sp_{2m}(\mathbb{F}_q)| = q^{m^2} \prod_{i=1}^{m}(q^{2i} - 1).
\]

We have

\[
P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : AD^t = I_m, \ BA^t = AB^t \right\}.
\]

This yields

\[
|P| = q^{2m^2} |GL_m(\mathbb{F}_q)| = q^{m^2} \prod_{i=1}^{m}(q^i - 1).
\]

From (5.5) and (5.6), we prove the lemma. ■

By Proposition 5.2 and Lemma 5.3, we get

**Theorem 5.4.** One has

\[
\mu(U_{0,p}/U_{x,p}) = \prod_{v|p} \mu(U_{0,v}/U_{x,v}) = \prod_{v|p, v^\perp \Delta} \prod_{i=1}^{m}(q_v^i + 1).
\]

Plugging the formula (5.7) in the formula (4.4), we get the formula (1.3). The proof of Theorem 1.2 is complete.

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**References**


Institute of Mathematics, Academia Sinica, 128 Academia Rd. Sec. 2, Nankang, Taipei, Taiwan, and NCTS (Taipei Office)

E-mail address: chiafu@math.sinica.edu.tw

Max-Planck-Institut für Mathematik, Vivatsgasse 7, Bonn, 53111, Germany