On Ramanujan relations between Eisenstein series

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Abstract

The Ramanujan relations between Eisenstein series can be interpreted as an ordinary differential equation in a parameter space of a family of elliptic curves which is inverse to the Gauss-Manin connection of the corresponding period map constructed by elliptic integrals of first and second kind. In this article we consider a slight modification of elliptic integrals by allowing non-algebraic integrants and we get in a natural way generalizations of Ramanujan relations between Eisenstein series.

1 Introduction

In the inverse of period map of the classical two parameter Weierstrass family of elliptic curves there appaers Eisenstein series of weight 4 and 6, and the Schwarz triangle function with triangular parameters $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$, where $p, q, r \in \mathbb{N}$, is the inverse of an automorphic function for the triangle group with signature $\langle p, q, r \rangle$. In all these examples the period maps of differential forms of the first kind are considered and considering periods of differential forms of the second kind, one gets differential automorphic functions which are solutions to certain ordinary differential equations (see [11]). Looking in this way, it is not necessary to define (differential) automorphic functions by functional equations which they satify with respect to a Kleinian group, but as functions which are solutions to certain ordinary differential equations. To explain better this idea, let us state the main results of this paper for an example:

Theorem 1. Consider the multi-valued function

\begin{equation}
\text{pm} : \mathbb{C}^3 \setminus \{(t_1, t_2, t_3) \in \mathbb{C}^3 \mid 27t_3^2 - 4t_2^3 = 0\} \rightarrow \text{SL}(2, \mathbb{C})
\end{equation}

\begin{equation}
t \mapsto \left(\int_{\delta_1} \frac{dx}{y}, \int_{\delta_1} \frac{dx}{y}, \int_{\delta_2} \frac{dx}{y}, \int_{\delta_2} \frac{dx}{y}\right),
\end{equation}

where

\begin{equation}
y = \gamma^{\frac{1}{3}} (27t_3^2 - 4t_2^3)^{\frac{1}{3}} (x - t_1)^3 - t_2(x - t_1) - t_3^a
\end{equation}

and $\delta_1$ and $-\delta_2$ are two straight paths in the $x$-plane connecting one root of $y$ to the other two roots. Here $\gamma$ is a complex number depending only on $a, b$ and $c$, and it is taken so that the image of $\text{pm}$ is in $\text{SL}(2, \mathbb{C})$.

1. For $a \neq \frac{2}{3}$, $\text{pm}$ is a local biholomorphism and its local inverse restricted to $\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$, namely $(g_{1,a}(z), g_{2,a}(z), g_{3,a}(z))$, where $z$ is in some small open set $U$ in $\mathbb{C}$, satisfies
the system of ordinary differential equations:

\[
\begin{align*}
\dot{t}_1 &= t_1^2 + \frac{3a-1}{9a-6} t_2 \\
\dot{t}_2 &= 4t_1t_2 + \frac{3}{5a-2} t_3 \\
\dot{t}_3 &= 6t_1t_3 + \frac{2a}{9a-6} t_2^2,
\end{align*}
\]

where \(\dot{\cdot}\) is the derivation with respect to \(z\).

2. The integrals \(\int_\delta \frac{x^2}{y} \, dx\), where \(\delta\) is a path connecting two roots of \(y\), are constant along the solutions of (3).

3. \(g_{k,a}\)'s with respect to the group

\[
\Gamma := \langle M_1, M_2 \rangle \subset \text{SL}(2, \mathbb{C}),
\]

\[
M_1 := \frac{i}{e^{2\pi i a}} \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & 1 \end{pmatrix},
M_2 := \frac{i}{e^{2\pi i a}} \begin{pmatrix} 1 & e^{2\pi i a} \\ 0 & -e^{2\pi i a} \end{pmatrix},
\]

have the following automorphic properties: for every \(A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma\) and \(z \in U\) such that \(cz + d \neq 0\) there exists an analytic continuation of \(g_{k,a}, k = 1, 2, 3\) along a path which connects \(z\) to \(Az\) such that

\[
(cz + d)^{-2k} g_{k,a}(Az) = g_{k,a}(z), \quad k = 2, 3,
\]

\[
(cz + d)^{-2k} g_{k,a}(Az) = g_{k,a}(z) + c(cz + d)^{-1}.
\]

One can show that

\[
g_{k,a} = a_k \left(1 + (-1)^k \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i zn}\right),
\]

\(k = 1, 2, 3, \quad z \in \mathbb{H},\)

is the Eisenstein series of weight \(2k\), where \(\mathbb{H}\) is the upper half plane, \(B_k\) is the \(k\)-th Bernoulli number (\(B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{12}, \ldots\)), \(\sigma_i(n) := \sum_{d|n} d^i\) and

\[
(a_1, a_2, a_3) = \left(\frac{2\pi i}{12}, 3\left(\frac{2\pi i}{12}\right)^2, 2\left(\frac{2\pi i}{12}\right)^3\right),
\]

(see [11]). In the case \(a = \frac{1}{2}\) the system of ordinary differential equations (3) is known as the Ramanujan relations between \(g_{k,a}, k = 1, 2, 3\) because he had observed that in this case the series (7) satisfy the differential equation (3) (see for instance [8]). I do not know any explicit expressions like (7) for an arbitrary \(a \in \mathbb{C}\). Throughout the text we will consider the family (9) which is a generalization of (2) and it has the advantage that it contains full Gauss hypergeometric functions.

The text is organized in the following way: In \(\S 2\) we consider a more general family of transcendent curves. In \(\S 3\) and \(\S 4\) we fix up the paths of integration and calculate the monodromies. In \(\S 5\) we calculate the derivation of the period map. The calculation is similar to the calculation of Gauss-Manin connections in the algebraic context. In \(\S 6\) we calculate the determinant of the period map and according to this calculation in \(\S 7\) we redefine the period map. In \(\S 8\) we calculate the inverse of the period map and obtain Ramanujan type relations. \(\S 9\) is devoted the action of an algebraic group. Finally in \(\S 10\) we discuss the automorphic properties of the functions which appear in the inverse of the period map.

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2 Families of transcendent curves

For \( a, b, c \in \mathbb{C} \) fixed, we consider the following family of transcendent curves:

\[
E_{t,a,b,c} = E_t : y = f(x),
\]

\[
f(x) := t_0^a(x - t_1)^b(x - t_2)^c(x - t_3)^d.
\]

Here \( t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4 \) is a parameter. The discriminant of \( E_t \) is defined to be

\[
\Delta = \Delta(t) := t_0^2(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2
\]

and we will work with regular parameters, i.e.

\[
t \in T := \{ t \in \mathbb{C}^4 \mid \Delta(t) \neq 0 \}.
\]

The parameter \( t_0 \) is just for simplifying the calculations related to the Gauss-Manin connection of the family (see §5). If \( a, b \) and \( c \) are rational numbers then \( E_t \)'s are algebraic curves. In this case one can use algebro-geometric methods in order to study the periods of \( E_t \), see for instance [14]. In general, \( E_t \) is a solution of the following logarithmic differential equation

\[
\frac{dy}{y} = \frac{adx}{x - t_1} + \frac{bdx}{x - t_2} + \frac{cdx}{x - t_3}.
\]

In order to prove Theorem 1 we also consider the family

\[
\tilde{E}_t : y = f(x), \quad f(x) = \tilde{t}_0^a((x - \tilde{t}_1)^3 - \tilde{t}_2(x - \tilde{t}_1) - \tilde{t}_3)^a.
\]

In the case \( a = b = c \) there is a canonical map from the parameter space of the first curve to the parameter space of the second curve:

\[
\tilde{t}_0 = t_0, \quad \tilde{t}_1 = \frac{t_1 + t_2 + t_3}{3}, \quad \tilde{t}_2 = (\tilde{t}_1 - t_1)(\tilde{t}_1 - t_2) + (\tilde{t}_1 - t_2)(\tilde{t}_1 - t_3) + (\tilde{t}_1 - t_3)(\tilde{t}_1 - t_3), \quad \tilde{t}_3 = (\tilde{t}_1 - t_1)(\tilde{t}_1 - t_2)(\tilde{t}_1 - t_3).
\]

For simplicity we will also use \( t \) instead of \( \tilde{t} \); being clear parameters of which family we are talking about.

3 Paths of integration and Pochhammer cycles

We distinguish three, not necessarily closed, paths in \( E_t \). In the \( x \)-plane let \( \delta_i, \ i = 1, 2, 3 \) be the straight path connecting \( t_{i+1} \) to \( t_{i-1} \), \( i = 1, 2, 3 \) (by definition \( t_4 := t_1 \) and \( t_0 = t_3 \)).

There are many paths in \( E_t \) which are mapped to \( \delta_i \) under the projection on \( x \). We choose one of them and call it \( \delta_i \). For the case in which \( \text{Re}(a), \text{Re}(b), \text{Re}(c) < 0 \) the paths \( \delta_i \)'s and \( \tilde{\delta}_i \)'s are depicted in Figure 1. We can make our choices so that \( \delta_1 + \delta_2 + \delta_3 \) is a limit of a closed and homotopic-to-zero path in \( E_t \). For instance, we can take the path \( \tilde{\delta}_i \)'s in such a way that the triangle formed by them has almost zero area. Now, we have the integral

\[
\int_{\delta} \frac{p(x)dx}{y} = \int_{\delta} \frac{p(x)dx}{f(x)}, \quad p \in \mathbb{C}[x],
\]

where \( \delta \) is one of the paths explained above. By a linear change in the variable \( x \) such integrals can be written in terms of the Gauss hypergeometric function (see [7]).
Another way to study the integrals (11) is by using Pochhammer cycles. For simplicity we explain it for the pairs \((t_1, t_2)\). The Pochhammer cycle associated to the points \(t_1, t_2 \in \mathbb{C}\) and the path \(\tilde{\delta}_3\) is the commutator

\[ \tilde{\alpha}_3 = [\gamma_1, \gamma_2] = \gamma_1^{-1} \cdot \gamma_2^{-1} \cdot \gamma_1 \cdot \gamma_2, \]

where \(\gamma_1\) is a loop along \(\tilde{\delta}_3\) starting and ending at some point in the middle of \(\tilde{\delta}_1\) which encircles \(t_1\) once anticlockwise, and \(\gamma_2\) is a similar loop with respect to \(t_2\). It is easy to see that the cycle \(\tilde{\alpha}_3\) lifts up to a closed path \(\alpha_3\) in \(E_t\) and if \(a, b \not\in \mathbb{Z}\) then

\[ \int_{\alpha_3} \frac{p(x)dx}{y} = (1 - e^{-2\pi ia})(1 - e^{-2\pi ib}) \int_{\tilde{\alpha}_3} \frac{p(x)dx}{f(x)} dx. \]

(see [7], Proposition 3.3.7). Note that in order to have

\[ \int_{\delta_i} \frac{p(x)}{f(x)} dx = 0, \quad \forall p \in \mathbb{C}[x], \quad i = 1, 2, 3 \]

we have to assume that \(\text{Re}(a), \text{Re}(b), \text{Re}(c) < 0\). But this is not necessary if we work with Pochhammer cycles.

### 4 The period map and the monodromy group

For a fixed \(a \in T\), the period map is given by:

\[ \text{pm} : (T, a) \to \text{GL}(2, \mathbb{C}), \quad t \mapsto \left( \begin{array}{cc} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} x dx y^{-1} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} x dx y^{-1} \end{array} \right), \]

where \((T, a)\) means a small neighborhood of \(a\) in \(T\). The map \(\text{pm}\) can be extended along any path in \(T\) with the starting point \(a\). We denote by \(P\) the union of images of the extensions of \(\text{pm}\) and call it the period domain. In order to study the analytic extensions of \(\text{pm}\) we have to calculate the monodromy group. In what follows we use the following convention: Two paths in \(E_t\) are equal if the integration of any differential form \(\frac{p(x)dx}{y}\), \(p \in \mathbb{C}[x]\) over them is equal. For instance, using this convention we have

\[ \delta_1 + \delta_2 + \delta_3 = 0. \]
Let
\[ A = e^{2\pi i a}, \quad B = e^{2\pi i b}, \quad C = e^{2\pi i c}. \]
We fix \( t_2 \) and \( t_3 \) and let \( t_1 \) turn around \( t_2 \) anti clockwise. We obtain three new paths \( h_3(\delta_1), h_3(\delta_2) \) and \( h_3(\delta_3) \) in \( E_t \) such that \( h_3(\delta_1) + h_3(\delta_2) + h_3(\delta_3) = 0 \) (this follows from (13)). Note that in the \( x \)-plane (resp. in \( E_t \)) the triangle formed by \( h_3(\delta_i) \)'s (resp. \( h_3(\delta_i) \)'s) does not intersect itself. We have
\[ h_3(\delta_2) = \delta_2 + (A - AB)\delta_3, \quad h_3(\delta_3) = -\delta_2 - A\delta_3 = \delta_1 + (1 - A)\delta_3, \quad h_3(\delta_3) = AB\delta_3 \]
(see Figure 2, A). We call \( h_3 \) the monodromy around the hyperplane \( t_1 = t_2 \). These formulas are compatible with the Picard-Lefschetz formula in the case \( a = b = c = \frac{1}{2} \). In a similar way
\[ h_1(\delta_3) = \delta_3 + B\delta_1 - BC\delta_1, \quad h_1(\delta_2) = -\delta_3 - B\delta_1, \quad h_1(\delta_1) = BC\delta_1 \]
and
\[ h_2(\delta_1) = \delta_1 + C\delta_2 - CA\delta_2, \quad h_2(\delta_3) = -\delta_1 - C\delta_2, \quad h_2(\delta_2) = CA\delta_2. \]
Therefore, the monodromies with respect to the basis \((\delta_1, \delta_2)\) are represented as
\[ M_3 = \begin{pmatrix} A & A - 1 \\ A(B - 1) & A(B - 1) + 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & C - CA \\ 0 & CA \end{pmatrix}. \]
Note that
\[ M_1 M_2 M_3 = \begin{pmatrix} ABC & 0 \\ 0 & ABC \end{pmatrix}, \]
and that for \( n \in \mathbb{N} \)
\[ h_3^n(\delta_2) = \delta_2 + (A - AB) \frac{(AB)^n - 1}{AB - 1} \delta_3, \quad h_3^n(\delta_1) = \delta_1 + (1 - A) \frac{(AB)^n - 1}{AB - 1} \delta_3, \quad h_3^n(\delta_3) = (AB)^n \delta_3. \]
The monodromy group \( \Gamma \) is defined to be the subgroup of \( \text{GL}(2, \mathbb{C}) \) generated by \( M_i, \ i = 1, 2, 3 \). For \( a = b = c = \frac{1}{2} \) we have
\[ M_3 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \]
and it is easy to see that \( \Gamma = \Gamma(2) := \{ A \in \text{SL}(2, \mathbb{Z}) \mid A \equiv 2 \text{ Id} \} \).

We discuss the monodromies for the family (10). In this family the monodromies change the place of the roots of \( f \). Therefore, the monodromy \( h_3 \) corresponding to the replacement of \( t_1 \) and \( t_2 \) is given by
\[ h_3(\delta_1) = -\delta_2 = \delta_1 + \delta_2, \quad h_3(\delta_2) = \delta_2 + A\delta_3, \quad h_3(\delta_3) = -A\delta_3, \]
(see Figure 2 B). The other monodromies are
\[ h_1(\delta_2) = -\delta_3, \quad h_1(\delta_3) = \delta_3 + A\delta_1, \quad h(\delta_1) = -A\delta_1, \quad h_2(\delta_3) = -\delta_1, \quad h_2(\delta_1) = \delta_1 + A\delta_2, \quad h(\delta_2) = -A\delta_2. \]
Therefore, in the basis \((\delta_1, \delta_2)\), the monodromies has the form
\[ M_3 = \begin{pmatrix} 0 & -1 \\ -A & 1 - A \end{pmatrix}, \quad M_1 = \begin{pmatrix} -A & 0 \\ 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & A \\ 0 & -A \end{pmatrix}. \]
Note that
\[ M_3^{-1}M_1M_3 = M_2 \]
For \( a = \frac{1}{2} \) we have
\[
M_3 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
and so \( \Gamma = \text{SL}(2, \mathbb{Z}) \).

Remark 1. In general it is hard to decide for which parameters \( a, b, c \) the group \( \Gamma \) is Kleinian, i.e. it acts discontinuously in some open subset of \( \mathbb{C} \cup \{\infty\} \). There is a necessary condition for such groups called Jorgensen's inequality (see [2]) but it is not sufficient\(^1\). For \( \nu_0 := 1 - a - c = \frac{1}{p} \), \( \nu_1 := 1 - b - c = \frac{1}{q} \), \( \nu_{\infty} := 1 - a - b = \frac{1}{r} \), where \( p, q, r \) are positive integers, the group \( \Gamma \) is the triangular group of type \( \langle p, q, r \rangle \) and it is Kleinian (see [2, 9, 14]). Despite the fact that \( \Gamma \backslash \mathcal{P} \) may not have any reasonable structure, the global period map \( \text{pm} : T \to \Gamma \backslash \mathcal{P} \) is well-defined.

5 A kind of Gauss-Manin connection

The Gauss-Manin connection is the art of derivation of differential forms on families of algebraic varieties and then simplifying the result. Despite the fact that the varieties considered in this article are not algebraic, the process of derivation and simplification is similar to the algebraic case (see for instance [10]). In what follows, derivation with respect to \( x \) is denoted by \( ' \).

First of all we have to simplify the integral (11). More precisely we want to reduce it to the integrals with \( p = 1, x \). Let \( \mathcal{R} = \mathbb{C}(t) \) and
\[
g = (x - t_1)(x - t_2)(x - t_3).
\]

Proposition 1. For all \( p \in \mathcal{R}[x] \), there is \( \tilde{p} \in \mathcal{R}[x] \), \( \deg(\tilde{p}) \leq 1 \) such that
\[
\int_\delta \frac{pdx}{y} = \int_\delta \frac{\tilde{p}dx}{y},
\]
where \( \delta \) is a path which connects two points of \( \{t_1, t_2, t_3\} \) and does not cross it elsewhere.

\(^1\)I would like to thank Katsuhiko Matsuzaki who informed me about the mentioned fact.
Proof. For $n > 1$ modulo exact forms we have
\[
0 = d\left(\frac{x^{n-2}g}{f}\right) = (-x^{n-2}g'f' + (x^{n-2}g')f') \frac{dx}{f}.
\]

Note that $g'f'$ is a polynomial in $x$. We set $p_n = a_n x^n + r_n(x)$, $a_n \in \mathbb{C}$, $\deg(r_n) \leq n - 1$ the polynomial in the parenthesis. We have $a_n \neq 0$ and so modulo exact forms we have:
\[
x^n f \frac{dx}{f} = -\frac{1}{a_n} r_n f^{-1} \frac{dx}{f}.
\]

By various applications of the above equality in $\int_{\delta}^{\infty} \frac{pdf}{y}$ we finally get the desired equality. 

Let us now differentiate the integrals:

**Proposition 2.** Let $t$ be one of the parameters $t_i$, $i = 0, 1, 2, 3$. We have
\[
\frac{\partial}{\partial t} \int_{\delta} f \frac{pdf}{f} = \int_{\delta} \nabla \frac{pdf}{f}, \quad p \in \mathbb{C}[x],
\]
where
\[
\nabla \frac{pdf}{f} := \frac{1}{\Delta} \left(\left(a_1 - \frac{\partial f}{\partial t} g \right)f + a_2 - \frac{\partial f}{\partial t} g f'p + \Delta \frac{\partial p}{\partial t} \right) \frac{dx}{f}.
\]

Proof. We can find two polynomials $a_1, a_2 \in \mathbb{R}[x]$ such that
\[
g'f'a_1 + ga_2 = \Delta.
\]

We have
\[
\frac{\partial}{\partial t} \int_{\delta} f \frac{pdf}{f} = \int_{\delta} \left(\frac{-\partial f}{\partial t} \frac{pdf}{f} + \frac{\partial p}{\partial t} \right) \frac{dx}{f}
\]
\[
= \int_{\delta} \left(\frac{-\partial f}{\partial t} p + \frac{\partial p}{\partial t} \right) \frac{dx}{f}
\]
\[
= \int_{\delta} \left(\frac{\tilde{p}}{g} + \frac{\partial p}{\partial t} \right) \frac{dx}{f}, \quad \tilde{p} = \frac{-\partial f}{\partial t} g p
\]
\[
= \frac{1}{\Delta} \int_{\delta} \left(\frac{g f' a_1 + ga_2}{f} \tilde{p} + \Delta \frac{\partial p}{\partial t} \right) \frac{dx}{f}
\]
\[
= \frac{1}{\Delta} \int_{\delta} \left(\frac{df}{f^2} a_1 \tilde{p} + a_2 \tilde{p} \frac{dx}{f} + \Delta \frac{\partial p}{\partial t} \right) \frac{dx}{f}
\]
\[
= \frac{1}{\Delta} \int_{\delta} \left(\frac{1}{f} d(a_1 \tilde{p}) + a_2 \tilde{p} \frac{dx}{f} + \Delta \frac{\partial p}{\partial t} \right) \frac{dx}{f}
\]
\[
= \frac{1}{\Delta} \int_{\delta} ((a_1 \tilde{p})' + a_2 \tilde{p} + \Delta \frac{\partial p}{\partial t} ) \frac{dx}{f}.
\]

\[\square\]
For the implementation of the algorithms of this section in SINGULAR [3] see the author’s web page. For the family (9) we have used these algorithms and we have obtained:

\[
\nabla \phi \omega = \frac{1}{(t_1-t_2)(t_1-t_3)} \begin{pmatrix}
-at_1 + (a+c-1)t_2 + (a+b-1)t_3 & -a-b+c+2 \\
-1/2 \\
0 \\
-1/2
\end{pmatrix}
\omega
\]

where

\[
\omega = \begin{pmatrix}
\frac{dx}{dy} \\
\frac{dx}{y} \\
\end{pmatrix}
\]

The derivations with respect to \( t_2 \) (resp \( t_3 \)) is obtained by permutation of \( t_1 \) with \( t_2 \) and \( a \) with \( b \) (resp. \( t_1 \) with \( t_3 \) and \( a \) with \( c \)). It is also easy to check by hand that

\[
\nabla \frac{\partial}{\partial \tau_i} \omega = \frac{1}{t_0} \begin{pmatrix}
-1/2 \\
0 \\
-1/2
\end{pmatrix} \omega
\]

We will simply denote by \( \nabla \frac{\partial}{\partial \tau_i} \omega = A \omega \).

For the family (10) we use \( g = (x-t_1)^3 - t_2(x-t_1) - t_3 \) and \( \Delta = t_0(4t_2^3 - 27t_3^2) \) and we have

\[
\nabla \frac{\partial}{\partial \tau_0} \omega = \frac{1}{t_0} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1/2 & 0 \\
-1/2 & 0 & 1/2
\end{pmatrix}, \quad \nabla \frac{\partial}{\partial \tau_1} \omega = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix},
\]

\[
\nabla \frac{\partial}{\partial \tau_2} \omega = \frac{1}{(4t_2^3 - 27t_3^2)} \begin{pmatrix}
-27at_1t_3 - 6at_2^2 + 18t_1t_3 + 2t_3^2 & 27at_3 - 18t_3 \\
-27at_1t_3^2 + 9at_2t_3 + 18t_1t_3^2 - 2t_3^2 + 3t_3^2 & 27at_3 - 6at_2^2 - 18t_3 + 4t_3^2 \\
18at_1t_2 + 27at_3 - 12t_1t_2 - 9t_3 & -18at_2 + 12t_2
\end{pmatrix},
\]

\[
\nabla \frac{\partial}{\partial \tau_3} \omega = \frac{1}{(4t_2^3 - 27t_3^2)} \begin{pmatrix}
18at_1t_2^2 - 6at_2^2 + 12t_1t_2^2 + 9t_1t_3 + 2t_3^2 & 18at_1t_2 + 27at_3 + 12t_1t_2 - 18t_3
\end{pmatrix}.
\]

6 Determinant of the period matrix

From Proposition 2 it follows that the period map satisfies the differential equation

\[
d(pm) = pmA^r, \quad A = \sum_{i=0}^3 (\nabla \frac{\partial}{\partial \tau_i}) dt_i.
\]

This and (14) imply that \( \det := \det(pm) \) satisfies

\[
\frac{\partial \det}{\partial t_1} = \frac{1}{(t_1-t_2)(t_1-t_3)} ((a+c-1)t_2 + (a+b-1)t_3 + (-2a-b-c+2)t_1) \det.
\]

Solving this differential equation one concludes that \( \det \) is of the form \( C(t_1-t_3)^{1-a-c}(t_1-t_2)^{1-a-b} \), where \( C \) does not depend on \( t_1 \). Repeating the same argument for \( t_0, t_2, t_3 \) we conclude that

\[
\det(pm) = \gamma \cdot t_0^{-1}(t_1-t_3)^{1-a-c}(t_1-t_2)^{1-a-b}(t_2-t_3)^{1-b-c},
\]

where \( \gamma \) is a constant depending only on \( a, b \) and \( c \). For the family (10) in a similar way we get

\[
\det(pm) = \gamma \cdot t_0^{-1} (27t_3^2 - 4t_2^3)^{1-a}.
\]
7 Redefining the period map and the monodromy group

We have calculated the determinant of the period map in (15). It depends on $t_1, t_2, t_3$ except for the case $a = b = c = \frac{1}{2}$. In order that the determinant of the period map to be equal to $t_0^{-1}$ and the monodromy group to be a subgroup of $\text{SL}(2, \mathbb{C})$, we have to multiply (12) by

$$p := \gamma^{-\frac{1}{2}}(t_1 - t_3)^{-\frac{1}{2}(1-a-c)}(t_1 - t_2)^{-\frac{1}{2}(1-a-b)}(t_2 - t_3)^{-\frac{1}{2}(1-b-c)}.$$

In other words, we have to redefine

$$f(x) := \gamma^{\frac{1}{2}} t_0^2 (t_1 - t_3)^{\frac{1}{2}(1-a-c)}(t_1 - t_2)^{\frac{1}{2}(1-a-b)}(t_2 - t_3)^{\frac{1}{2}(1-b-c)}(x - t_1)^a(x - t_2)^b(x - t_3)^c$$

for the family (9). We have to recalculate the Gauss-Manin connection. By Leibniz rule we have

$$\nabla(p \omega) = (dp) \cdot \omega + p \cdot A \otimes \omega = \left( \frac{dp}{p} I_{2 \times 2} + A \right) \otimes (p \omega)$$

and

$$\frac{dp}{p} = \frac{1}{2} (a + b - 1) \frac{dt_1 - dt_2}{t_1 - t_2} + \cdots = \left( \frac{1}{2} (a + b - 1) - \frac{1}{t_1 - t_2} + \frac{1}{2} (a + c - 1) \right) \frac{1}{t_1 - t_3} dt_1 + \cdots$$

After redefining the period map the monodromy matrices are changed as follows:

$$M_3 = -\frac{1}{\sqrt{AB}} \begin{pmatrix} A & A - 1 \\ A(B - 1) & A(B - 1) + 1 \end{pmatrix}, \quad M_1 = -\frac{1}{\sqrt{BC}} \begin{pmatrix} BC & 0 \\ 1 - B & 1 \end{pmatrix}, \quad M_2 = -\frac{1}{\sqrt{CA}} \begin{pmatrix} 1 & C - CA \\ 0 & C \end{pmatrix}$$

Note that $\sqrt{A} = e^{\pi i a}, \cdots$ are well-defined and $\Gamma := \langle M_1, M_2, M_3 \rangle = \langle M_1, M_2 \rangle \subset \text{SL}(2, \mathbb{C})$.

For the family (10) we redefine

$$f(x) = \gamma^{\frac{1}{2}} t_0^2 (27t_3^2 - 4t_2^2)^{\frac{1}{2}(\frac{1}{2} - 3)}((x - t_1)^3 - t_2(x - t_1) - t_3)^a,$$

which is the one in (1) with $t_0 = 1$. For $p = (27t_3^2 - 4t_2^2)^{-\frac{1}{2}(\frac{1}{2} - 3)}$ we have

$$\frac{dp}{p} = \frac{1}{2} (a - \frac{1}{2}) \frac{54t_3 dt_3 - 12t_2^2 dt_2}{27t_3^2 - 4t_2^2}$$

The new monodromy group is (4). For both families we conclude that $\text{det}(pm) = t_0^{-1}$.

**Remark 2.** A subgroup $A$ of $\text{SL}(2, \mathbb{R})$ is called arithmetic if $A$ is commensurable with $\text{SL}(2, \mathbb{Z})$, i.e. $A \cap \text{SL}(2, \mathbb{Z})$ has a finite index in both $\text{SL}(2, \mathbb{Z})$ and $A$. It is a natural question to classify all the cases such that the monodromy group is conjugated with an arithmetic group. For the case $a = b = 1 - c = \frac{1}{6}$ the monodromy group is conjugated with $\text{SL}(2, \mathbb{Z})$ (see [12]).

8 The inverse of the period map

First we note that the period map is a local biholomorphism. We consider $pm$ as a map sending the point $(t_0, t_1, t_2, t_3)$ to $(x_1, x_2, x_3, x_4)$. Its derivative at $t$ is a $4 \times 4$ matrix whose $i$-th column constitutes the first and second row of $x(\nabla_\omega)$. For $s := a + b + c - 2 \neq 0$, this is an invertible matrix. More precisely, we have

$$(dF)_x = (dpm)^{-1} = \frac{1}{\text{det}(x)}.$$
The algebraic group

\[ G_0 = \left\{ \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \mid k_3 \in \mathbb{C}, k_1, k_2 \in \mathbb{C}^* \right\} \]

acts on the period domain \( \mathcal{P} \subset \text{GL}(2, \mathbb{C}) \) from the right by the usual multiplication of matrices. It acts also in \( \mathbb{C}^4 \) as follows:

\[ t \cdot g := (t_0(k_1k_2)^{-1}, t_1k_1^{-1}k_2 + k_3k_1^{-1}, t_2k_1^{-1}k_2 + k_3k_1^{-1}, t_3k_1^{-1}k_2 + k_3k_1^{-1}) \]

(18)

\[ t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4, g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0. \]

The relation between these two actions of \( G_0 \) is given by:
Since \(\omega\) is holomorphic and compactly supported, for any \(u \in \mathcal{U}\) we have \(\langle \omega, u \rangle = 0\) \(\equiv (\omega, \cdot) = 0\). Now \(\cdot \cdot \cdot = \cdot \cdot \cdot\), \(1 \cdot \cdot \cdot = \cdot \cdot \cdot\) = \(1 \cdot \cdot \cdot\) where \(X, x\) means a small neighborhood of \(x\) in \(X\) and \(I\) is the identity \(2 \times 2\) matrix.

If \(t_a, s \in [0, 1]\) is a path in \(T\) and \(g_s, s \in [0, 1]\) is a path in \(G_0\) which connects \(I\) to \(g \in G_0\), by analytic continuation of the equality (19), it makes sense to say that (19) is true for an arbitrary \(g \in G_0\).

**Proof.** Let

\[
\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad (x, y) \mapsto (k_2^{-1}k_1x - k_3k_2^{-1}, k_2^{-1}k_1^2y).
\]

Then

\[
k_2k_1^{-2}\alpha^{-1}(y - f(x)) = y - (8\pi i\alpha)^2k_2k_1^{-2}(t_2 - t_3)\frac{1}{2}(t_2 - t_3) < (k_2^{-1}k_1x - k_3k_2^{-1} - t_1)^a \cdots = y - (8\pi i\alpha)^2k_2k_1^{-2}e^{a-b-c}k_2k_1^{-2} < (k_2^{-1}k_1x - k_3k_2^{-1} - t_1)^a \cdots
\]

This implies that \(\alpha\) induces an isomorphism

\[
\alpha : E_{t \cdot g} \rightarrow E_t.
\]

Now

\[
\alpha^{-1}\omega = \begin{pmatrix} k_1^{-1} & 0 \\ -k_3k_2^{-1}k_1 & k_2^{-1} \end{pmatrix} \omega = \begin{pmatrix} k_1 & 0 \\ k_3 & k_2 \end{pmatrix}^{-1} \omega,
\]

where \(\omega = \frac{dx}{y}, \frac{dx}{y}\) \(\alpha\) and so

\[
\text{pm}(t) = \text{pm}(t \cdot g), g^{-1}
\]

which proves (19). \(\square\)

In a similar way for the family (10) we have the action

\[
t \cdot g := (t_0k_1^{-1}k_2^{-1}, t_1k_1^{-1}k_2 + k_3k_1^{-1}, t_2k_1^{-3}k_2, t_3k_1^{-4}k_2)
\]

(20)
\[
t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4, g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0
\]

with the property (19).

**Remark 3.** The rational map

\[
\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad (x, y) \mapsto \left(x, \frac{y}{(x - t_1)(x - t_2)(x - t_3)^e}\right)
\]

sends \(E_{t, a, b, c}\) biholomorphically to \(E_{t, (a), (b), (c)}\). We use Proposition 1 and write

\[
\alpha^*\omega^\alpha = \omega^\alpha C, \quad C \in \text{Mat}(2, \mathbb{Q}[t]).
\]

The period map associated to \(E_{t, (a), (b), (c)}\) is the multiplication of the period map associated to \(E_{t, a, b, c}\) with \(C\). For this reason it is sometimes practical to assume that \(0 \leq \text{Re}(a), \text{Re}(b), \text{Re}(c) < 1\).
10 Automorphic properties of $g_i$'s

We continue the notation introduced in §8. We denote by

$$F = (F_0, F_1, F_2, F_3) : (\mathcal{P}, x_0) \to (T, a)$$

the local inverse of the period map. Taking $F$ of (19) we conclude that

$$F(xg) = F(x) \cdot g, \ g \in (G_0, I).$$

We get

$$F_0(xg) = F_0(x) k_1^{−1} k_2^{−1},$$

$$F_i(xg) = F_1(x) k_1^{−1} k_2 + k_3 k_1^{−1}, \ i = 1, 2, 3$$

The first equality also follows from $F_0(x) = \det(x)^{−1}$.

For any $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ there is a path $\gamma \in \pi_1(T, a)$ such that if $\tilde{p}m : (T, a) \to \mathcal{P}$ is the analytic continuation of $pm$ along $\gamma$ then

$$\tilde{p}m(t) = Apm(t), \ \forall t \in (T, a).$$

This implies that the analytic continuation of $F$ along the path $\delta = pm(\gamma)$, which connects $pm(a)$ to $Apm(a)$ satisfies

$$F(x) = F(Ax), \ x \in (\mathcal{P}, x_0).$$

Using the Schwarz function

$$D(t) = \frac{\int_{\delta_1} \frac{dz}{f}}{\int_{\delta_2} \frac{dz}{f}}$$

we define the path $\sigma = D(\gamma)$. If $cz_0 + d \neq 0$ then $Az_0$ is well-defined and the path $\sigma$ connects $z_0$ to $Az_0$ in $\mathbb{C}$. We claim that there is an analytic continuations of $g_i$'s along $\sigma$ such that

$$(cz + d)^{−2} g_i(Az) = g_i(z) + c(cz + d)^{−1}, \ i = 1, 2, 3, \ A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, z \in (\mathbb{C}, z_0).$$

We have

$$\begin{align*}
(1, g_1, g_2, g_3) &= F \begin{pmatrix} z & 1 \\ -1 & 0 \end{pmatrix} \\
&= (23) F \begin{pmatrix} (a & b \\ c & d) \begin{pmatrix} z & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \\
&= F \begin{pmatrix} Az & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} cz + d & -c \\ 0 & (cz + d)^{−1} \det(A) \end{pmatrix} \\
&= (21) F \begin{pmatrix} Az & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} cz + d & -c \\ 0 & (cz + d)^{−1} \end{pmatrix} \\
&= (1, (cz + d)^{−2} g_1(Az) - c(cz + d)^{−1}, \cdots) \end{align*}$$
The fourth equality makes sense in the following way: Let

\[
x_s := \begin{pmatrix} D(\gamma_s) & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{P}, \quad \tau_s := x_s^{-1} \text{pm}(\gamma_s) \in G_0, \quad s \in [0, 1].
\]

\(\tau\) is a path in \(G_0\) which connects \(I\) to \(\begin{pmatrix} cz + d & -c \\ 0 & (cz + d)^{-1} \end{pmatrix}\). For \(s\) near enough to \(0\) we have \(F(x_s\tau_s) = F(x_s) \cdot \tau_s\) and so by analytic continuation we have it for \(s = 1\).

In a similar way we prove the third part of Theorem 1. Note that for the family (10), \(F_2\) and \(F_3\) satisfy:

\[
F_2(xg) = F_2(x)k_1^{-3}k_2, \quad F_3(xg) = F_3(x)k_1^{-4}k_2^2, \quad \forall x \in \mathcal{L}, \ g \in G_0.
\]

References


