REDUCTION IN K-THEORY OF SOME INFINITE EXTENSIONS OF NUMBER FIELDS

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ABSTRACT. For the cyclotomic extension \( F(\mu_{\infty}) = \bigcup_{m \geq 1} F(\mu_m) \) of a number field \( F \), we prove that the reduction map \( K_{2n+1}(F(\mu_\infty)) \rightarrow K_{2n+1}(\kappa_{\bar{v}}) \), when restricted to nontorsion elements, is surjective. Here \( \kappa_{\bar{v}} \) denotes the residue field at a prime \( \bar{v} \) of \( F(\mu_\infty) \).

1. Statement of the result

Let \( F \) be a number field and let \( \mathcal{O}_F \) denote its ring of algebraic integers. Fix an algebraic closure \( \bar{F} \) of \( F \). We will denote by \( F(\mu_\infty) \) the subfield of \( \bar{F} \) obtained by adding to \( F \) all roots of unity. Thus \( F(\mu_\infty) = \bigcup_{m \geq 1} F(\mu_m) \), where \( F(\mu_m) \) is the smallest extension of \( F \) containing all \( m \)th roots of unity. We put \( \mathcal{O}_{F(\mu_\infty)} = \bigcup_{m \geq 1} \mathcal{O}_{F(\mu_m)} \). We fix a prime ideal \( \bar{v} \) of \( \mathcal{O}_{F(\mu_\infty)} \) and denote by \( v \) the prime ideal of \( \mathcal{O}_F \) lying below \( \bar{v} \). We denote by \( \kappa_{\bar{v}} = \mathcal{O}_{F(\mu_\infty)}/\bar{v} \) and by \( \kappa_v = \mathcal{O}_F/v \) the respective residue fields. It is clear that \( \kappa_{\bar{v}} \) is the algebraic closure of the finite field \( \kappa_v \). In the paper \( n \) denotes a fixed natural number.

We investigate the map

\[ r_{\bar{v}} : K_{2n+1}(F(\mu_\infty)) \rightarrow K_{2n+1}(\kappa_{\bar{v}}) \]

induced on the Quillen’s K-theory by the arithmetic reduction at the prime \( \bar{v} \).

Theorem.

For any element \( t \) of the group \( K_{2n+1}(\kappa_{\bar{v}}) \) there exists a nontorsion element \( x \) in the group \( K_{2n+1}(F(\mu_\infty)) \), such that \( r_{\bar{v}}(x) = t \).

We also prove (see Remark below) that the reduction map \( r_{\bar{v}} \) is surjective when restricted to the torsion part of \( K_{2n+1}(F(\mu_\infty)) \). Together with the Theorem it shows that there are two disjoint subsets in the group \( K_{2n+1}(F(\mu_\infty)) \), which are mapped onto \( K_{2n+1}(\kappa_{\bar{v}}) \) by the reduction at \( \bar{v} \).

In the paper [3] the author investigated the reduction \( K_{2n+1}(\mathbb{Z}) \rightarrow K_{2n+1}(\mathbb{F}_p) \), on the odd dimensional K-theory of rational integers. One of the results of [3] gives the density of primes \( p \) for which the reduction is nontrivial. The density was computed by using special elements in K-groups of \( \mathbb{Z} \) and the arithmetic of

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certain Kummer extensions of cyclotomic fields. The method applied in the present
note is different. At a critical point in the proof of the Theorem we use Borel’s
computations of ranks of the K-groups of number rings and a result of Harris and
Segal (cf. [4], Theorem 1.1).

2. Proof of the Theorem

Observe that

\[ K_{2n+1}(\kappa_\tilde{v}) = \lim_{\kappa} K_{2n+1}(\kappa) \]

where the direct limit is taken over all finite fields \( \kappa \) contained in \( \kappa_\tilde{v} \). For each \( \kappa \) as
above we choose a prime \( w \) of the number field \( \mathcal{O}_L \) contained in \( F(\mu_n) \) such that \( \kappa_w \) is \( \kappa \). Note that maps of the direct system are inclusions of finite
cyclic groups cf. [5]. In order to prove the Theorem it is enough to show that for
any \( w \) and for every \( t \in K_{2n+1}(\kappa_w) \) there exists a nontorsion \( x \in K_{2n+1}(F(\mu_n)) \)
such that \( r_\tilde{v}(x) \) equals the value at \( t \) of the inclusion \( K_{2n+1}(\kappa_w) \to K_{2n+1}(\kappa_\tilde{v}) \).
Let us put \( q_w := \# \kappa_w \). Consider the cyclic group \( K_{2n+1}(\kappa_w) = \mathbb{Z}/m_w \), where
\( m_w = q_w^{n+1} - 1 \) by the theorem of Quillen [5]. Let

\[ \mathbb{Z}/m_w = \bigoplus_{i=1}^s \mathbb{Z}/l_i^{k_i} \]

be the decomposition of the group into its Sylow primary subgroups. For every
\( 1 \leq i \leq s \) we pick a generator \( t_i \) of the \( l_i \)-primary part \( \mathbb{Z}/l_i^{k_i} \) of \( K_{2n+1}(\kappa_w) \). Next
we choose the finite field extension \( L' / F \) with the following two properties:

1. \( F(\mu_{m_w}) \subset L' \subset F(\mu_\infty) \)
2. the rank of the group \( K_{2n+1}(L') \) is bigger than \( s \).

By the well-known results of Borel [2] and Quillen [6], for any number field \( L \),
the rank of \( K_{2n+1}(L) \) is not smaller than the number of complex places of \( L \). The
number of complex places of \( F(\mu_m) \) is an increasing and unbounded function of
\( m \). This shows that the field \( L' \) with properties (1) and (2) exists, since clearly the
natural map:

\[ K_{2n+1}(F(\mu_{m_w})) \otimes \mathbb{Q} \to K_{2n+1}(L') \otimes \mathbb{Q} \]

is injective (cf. [1], Theorem 2, p.68). Let \( w' \) be the prime of \( \mathcal{O}_{L'} \) which lies below \( \tilde{v} \). Consider the commutative diagram:

\[
\begin{array}{c}
K_{2n+1}(F(\mu_\infty)) & \xrightarrow{r_\tilde{v}} & K_{2n+1}(\kappa_\tilde{v}) \\
\alpha \downarrow & & \downarrow \\
K_{2n+1}(L') & \xrightarrow{r_{\kappa_\tilde{v}'}} & K_{2n+1}(\kappa_\tilde{v}') \\
\beta \downarrow & & \downarrow \\
K_{2n+1}(L) & \xrightarrow{r_w} & K_{2n+1}(\kappa_w) \\
\end{array}
\]

with the reduction maps as the horizontal arrows. The vertical arrows on the right
hand side of the diagram (2.1) are embeddings of the direct system of K-groups.
of finite fields. Since $F(\mu_{m_w}) \subset L'$, it follows by [4], Theorem 1.1, p.21, that for
every $1 \leq i \leq s$ there exists an element $t_i \in K_{2n+1}(L')$ of order $l_i^w$ such that
$r_w'(t_i) = \beta(t_i)$. Let us choose elements $y_1, y_2, \ldots, y_s$ of $K_{2n+1}(L')$ in such a way that
their images in the rational $K$-group $K_{2n+1}(L')$ are linearly independent. It is possible because of the property (2) of the field $L'$. Clearly, $y_1, y_2, \ldots, y_s$ are nontorsion. Put $y_i' := t_i + y_i$, for every $1 \leq i \leq s$. Let $\pi_i$ denote the projection of
the cyclic group $K_{2n+1}(\kappa_w')$ onto its $l_i$-primary summand. It is clear that either
$\pi_i(r_w'(y_i))$ or $\pi_i(r_w'(y_i'))$ is of order $l_i^w$. Without loss of generality, we can assume that $\pi_i(r_w'(y_i'))$ is of order $l_i^w$, for every $1 \leq i \leq s$. We choose integers $a_1, a_2, \ldots, a_s$ such that, for every $1 \leq i \leq s$, the element
$$y_i'' = a_i \frac{m_w}{l_i^w} y_i'$$
meets the condition:
$$r_w'(y_i'') = \beta(t_i).$$
It follows that the element $r_w'(\sum_{i=1}^s y_i'')$ generates the cyclic group $\beta(K_{2n+1}(\kappa_w))$.
By the choice of the elements $y_1, y_2, \ldots, y_s$, we know that $\sum_{i=1}^s y_i''$ is nontorsion.
Since the map $\alpha \otimes \mathbb{Q} : K_{2n+1}(L') \otimes \mathbb{Q} \to K_{2n+1}(F(\mu_\infty)) \otimes \mathbb{Q}$ is injective (cf. [1],
Corollary 1, p.70), it follows that the element $\alpha(\sum_{i=1}^s y_i'')$ is nontorsion in the group $K_{2n+1}(F(\mu_\infty))$. To finish the proof it is enough to put
$$x := \alpha(\sum_{i=1}^s y_i'')$$
and apply the commutativity of the diagram (2.1).

Corollary. Let $L$ be an algebraic extension of $F$ which contains $F(\mu_\infty)$. For any
prime $\bar{v}$ of $L$ the reduction map $r_{\bar{v}} : K_{2n+1}(L) \to K_{2n+1}(\kappa_{\bar{v}})$ is surjective, when
restricted to the subset of nontorsion elements.

Proof. The claim of the corollary follows by the Theorem, because $K_{2n+1}(\kappa_{\bar{v}}) = K_{2n+1}(\kappa_\infty)$ and the map $K_{2n+1}(F(\mu_\infty)) \to K_{2n+1}(L)$ is injective on nontorsion elements by [1] loc. cit.

Remark. Applying [4], Theorem 1.1, to the torsion part of $K_{2n+1}(F(\mu_\infty))$ in the
same way as it was done in the proof of the Theorem, one can see that the restric-
tion of the reduction $K_{2n+1}(F(\mu_\infty)) \to K_{2n+1}(\kappa_{\bar{v}})$ to the torsion, is surjective.

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