ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 8.
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To 100th birthday of Professor A.O. Gelfond.

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§8.0. Foreword.

Let $|z| \geq 1$, $-3\pi/2 < \arg(z) \leq \pi/2$, $\log(z) = \ln(|z|) + i \arg(z)$.

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$.

Let

\begin{equation}
 f_{l,1}(z, \nu) = f_{l,1}(z, \nu) = \sum_{k=0}^{\nu} (-1)^{\nu+k} \binom{\nu}{k} (z)^{2l} \binom{\nu+k}{\nu}^{2l},
\end{equation}

where $l = 0, 1, 2$, $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

\begin{equation}
 R(t, \nu) = \frac{\prod_{j=1}^{\nu} (t - j)}{\prod_{j=0}^{\nu} (t + j)},
\end{equation}

where $\nu \in [0, +\infty) \cap \mathbb{Z}$,
\[ f_{l,2}(z, \nu) = f_{l,3}(z, \nu) = \sum_{t=1}^{+\infty} z^{-t} (R(t, \nu))^{2+l}, \]

where \( l = 0, 1, 2 \) and \( \nu \in [0, +\infty) \cap \mathbb{Z} \), and since \((R(t, \nu))^{2+l}\) for \( \nu \in \mathbb{N} \) has in the points \( t = 1, \ldots, \nu \), the zeros of the order \( 2+l \), it follows that

\[ f_{l,2}(z, \nu) = \sum_{t=1}^{+\infty} z^{-t} (R(t, \nu))^{2+l}, \]

for \( l = 0, 1, 2 \) and \( \nu \in [0, +\infty) \cap \mathbb{Z} \). Let

\[ f_{i,3}(z, \nu) = f_{i,3}(z, \nu) = (\log(z)) f_{l,2}(z, \nu) + f_{l,4}(z, \nu), \]

where

\[ f_{l,4}(z, \nu) = -\sum_{t=1}^{+\infty} z^{-t} \left( \frac{\partial}{\partial t} (R^{2+l}) \right)(t, \nu), \]

\( l = 0, 1, 2 \) and \( \nu \in [0, +\infty) \cap \mathbb{Z} \), and since \((R(t, \nu))^{2+l}\) for \( \nu \in \mathbb{N} \) has in the points \( t = 1, \ldots, \nu \), the zeros of the order \( 2+l \), it follows that

\[ f_{i,4}(z, \nu) = -\sum_{t=1}^{+\infty} z^{-t} \left( \frac{\partial}{\partial t} (R^{2+l}) \right)(t, \nu) \]

for \( l = 0, 1, 2 \) and \( \nu \in [0, +\infty) \cap \mathbb{Z} \). Let

\[ f_{i,5}(z, \nu) = -i\pi f_{i,3}(z, \nu) + f_{i,5}(z, \nu), \]

with \( l = 1, 2, \nu \in [0, +\infty) \cap \mathbb{Z} \) and

\[ f_{i,5}(z, \nu) = 2^{-1} (\log(z))^2 f_{l,2}(z, \nu) + (\log(z)) f_{l,4}(z, \nu) + f_{l,6}(z, \nu) = \]

\[ = -2^{-1} (\log(z))^2 f_{l,2}(z, \nu) + (\log(z)) f_{l,3}(z, \nu) + f_{l,6}(z, \nu), \]

where

\[ f_{l,6}(z, \nu) = 2^{-1} \sum_{t=1}^{+\infty} z^{-t} \left( \frac{\partial}{\partial t} \right)^2 (R^{2+l})(t, \nu), \]

and since \((R(t, \nu))^{2+l}\) for \( \nu \in \mathbb{N} \) has in the points \( t = 1, \ldots, \nu \), the zeros of the order \( 2+l \), and \( l = 1, 2 \) now, it follows that

\[ f_{l,6}(z, \nu) = 2^{-1} \sum_{t=1}^{+\infty} z^{-t} \left( \frac{\partial}{\partial t} \right)^2 (R^{2+l})(t, \nu) \]

for \( l = 1, 2 \) and \( \nu \in [0, +\infty) \cap \mathbb{Z} \). Let
\[(12) \quad f_{l,7}^\nu(z, \nu) = f_{l,7}(z, \nu) + (2\pi^2/3)f_{l,3}(z, \nu).\]

with \(l = 2, \nu \in [0, +\infty) \cap \mathbb{Z}\) and

\[(13) \quad f_{l,7}(z, \nu) =
-3^{-1}(\log(z))^3 f_{l,2}(z, \nu) + 2^{-1}(\log(z))^2 f_{l,3}(z, \nu) + f_{l,8}(z, \nu) +
(\log(z))(f_{l,5}(z, \nu) + 2^{-1}(\log(z))^2 f_{l,2}(z, \nu) - (\log(z))f_{l,3}(z, \nu)) =
6^{-1}(\log(z))^3 f_{l,2}(z, \nu) - 2^{-1}(\log(z))^2 f_{l,3}(z, \nu) + (\log(z))f_{l,5}(z, \nu) + f_{l,8}(z, \nu) =
(1/6)(\log(z))^3 f_{l,2}(z, \nu) + (1/2)(\log(z))^2 f_{l,4}(z, \nu) +
(\log(z))f_{l,0}(z, \nu) + f_{l,8}(z, \nu),
\]

where

\[(14) \quad f_{l,8}(z, \nu) = -6^{-1} \sum_{t=\nu+1}^{\infty} z^{-t} \left( \left( \frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right) (t, \nu),\]

and, since \((R(t, \nu))^{2+l}\) for \(\nu \in \mathbb{N}\) have in the points \(t = 1, \ldots, \nu\), the zeros of the order \(2 + l\), and \(l = 2\) now, it follows that

\[(15) \quad f_{l,8}(z, \nu) = -6^{-1} \sum_{t=1}^{\infty} z^{-t} \left( \left( \frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right) (t, \nu).\]

Let \(\mathfrak{R}_0 = \{1, 2, 3\}, \mathfrak{R}_1 = \{1, 2, 3, 5\}, \mathfrak{R}_2 = \{1, 2, 3, 5, 7\}\).

Let \(\lambda\) be a variable. We denote by \(T_{n,\lambda}\) the diagonal \(n \times n\)-matrix, \(i\)-th diagonal element of which is equal to \(\lambda^{i-1}\) for \(i = 1, \ldots, n\). We denote by \(\delta\) the operator \(z \frac{d}{dz}\). Let further \(l = 0, 1, 2, k \in \mathfrak{R}_l, |z| > 1, \nu \in \mathbb{N}\), and let \(Y_{l,k}(z; \nu)\) be the columnn with \(4 + 2l\) elements, \(i\)-th of which is equal to \((\nu^{-1}\delta)^{-1}f_{l,k}^\nu(z, \nu)\) for \(i = 1, \ldots, 4 + 2l\).

**Theorem 1.** The following equalities hold

\[(16) \quad A_i^\sim(z; \nu)Y_{i,k}(z; \nu) = T_{4+2l,1-\nu-1}Y_{i,k}(z; \nu-1),\]

\[(17) \quad Y_{l,k}(z; \nu) = T_{4+2l,-1}A_i^\sim(z; -\nu)T_{4+2l,-1+\nu-1}Y_{l,k}(z; \nu-1),\]

where \(l = 0, 1, 2, k \in \mathfrak{R}_l, |z| > 1, \nu \in \mathbb{N}, \nu \geq 2,\)

\[(18) \quad A_i^\sim(z; \nu) = S^\sim + z \sum_{i=0}^{1+l} \nu^{-i}V_i^\sim(i)\]

with

\[(19) \quad S^\sim = \begin{pmatrix} 1 & -4 & 8 & -12 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}\]
$$S_1^\sim = \begin{pmatrix}
-1 & 6 & -18 & 38 & -66 & 102 \\
0 & -1 & 6 & -18 & 38 & -66 \\
0 & 0 & -1 & 6 & -18 & 38 \\
0 & 0 & 0 & -1 & 6 & -18 \\
0 & 0 & 0 & 0 & -1 & 6 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix},$$

$$S_2^\sim = \begin{pmatrix}
1 & -8 & 32 & -88 & 192 & -360 & 608 & -952 \\
0 & 1 & -8 & 32 & -88 & 192 & -360 & 608 \\
0 & 0 & 1 & -8 & 32 & -88 & 192 & -360 \\
0 & 0 & 0 & 1 & -8 & 32 & -88 & 192 \\
0 & 0 & 0 & 0 & 1 & -8 & 32 & -88 \\
0 & 0 & 0 & 0 & 0 & 1 & -8 & 32 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$V_0^{\sim*}(0) = 4 \begin{pmatrix}
4 & -5 & -2 & 3 \\
-3 & 4 & 1 & -2 \\
2 & -3 & 0 & 1 \\
-1 & 2 & -1 & 0
\end{pmatrix},$$

$$V_0^{\sim*}(1) = 4 \begin{pmatrix}
3 & -6 & 3 & 0 \\
-2 & 4 & -2 & 0 \\
1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

$$V_1^{\sim*}(0) = \begin{pmatrix}
146 & -198 & -180 & 268 & 66 & -102 \\
-102 & 146 & 108 & -180 & -38 & 66 \\
66 & -102 & -52 & 108 & 18 & -38 \\
-38 & 66 & 12 & -52 & -6 & 18 \\
18 & -38 & 12 & 12 & 2 & -6 \\
-6 & 18 & -20 & 12 & -6 & 2
\end{pmatrix},$$

$$V_1^{\sim*}(1) = \begin{pmatrix}
240 & -516 & 108 & 372 & -204 & 0 \\
-160 & 348 & -84 & -236 & 132 & 0 \\
96 & -212 & 60 & 132 & -76 & 0 \\
-48 & 108 & -36 & -60 & 36 & 0 \\
16 & -36 & 12 & 20 & -12 & 0 \\
0 & -4 & 12 & -12 & 4 & 0
\end{pmatrix},$$

$$V_1^{\sim*}(2) = \begin{pmatrix}
102 & -306 & 306 & -102 & 0 & 0 \\
-66 & 198 & -198 & 66 & 0 & 0 \\
38 & -114 & 114 & -38 & 0 & 0 \\
-18 & 54 & -54 & 18 & 0 & 0 \\
6 & -18 & 18 & -6 & 0 & 0 \\
-2 & 6 & -6 & 2 & 0 & 0
\end{pmatrix}. $$
The above matrices $A_i^\sim(z; \nu), S_i^\sim$ and $V_i^{\sim*}(i)$ have the following properties:

\begin{align*}
V_i^{\sim*}(0) &= 8 
\begin{pmatrix}
176 & -249 & -364 & 545 & 280 & -431 & -76 & 119 \\
-119 & 176 & 227 & -364 & -169 & 280 & 45 & -76 \\
76 & -119 & -128 & 227 & 92 & -169 & -24 & 45 \\
-45 & 76 & 61 & -128 & -43 & 92 & 11 & -24 \\
24 & -45 & -20 & 61 & 16 & -43 & -4 & 11 \\
-11 & 24 & -1 & -20 & -5 & 16 & 1 & -4 \\
4 & -11 & 8 & -1 & 4 & -5 & 0 & 1 \\
-1 & 4 & -7 & 8 & -7 & 4 & -1 & 0
\end{pmatrix}, \\
V_i^{\sim*}(1) &= 8 
\begin{pmatrix}
455 & -1020 & -113 & 1552 & -603 & -628 & 357 & 0 \\
-300 & 682 & 44 & -996 & 404 & 394 & -228 & 0 \\
185 & -428 & -3 & 592 & -253 & -228 & 135 & 0 \\
-104 & 246 & -16 & -316 & 144 & 118 & -72 & 0 \\
51 & -124 & 19 & 144 & -71 & -52 & 33 & 0 \\
-20 & 50 & -12 & -52 & 28 & 18 & -12 & 0 \\
5 & -12 & 1 & 16 & -9 & -4 & 3 & 0 \\
0 & -2 & 8 & -12 & 8 & -2 & 0 & 0
\end{pmatrix}, \\
V_i^{\sim*}(2) &= 8 
\begin{pmatrix}
400 & -1243 & 972 & 542 & -1028 & 357 & 0 & 0 \\
-259 & 808 & -642 & -332 & 653 & -228 & 0 & 0 \\
156 & -489 & 396 & 186 & -384 & 135 & 0 & 0 \\
-85 & 268 & -222 & -92 & 203 & -72 & 0 & 0 \\
40 & -127 & 108 & 38 & -92 & 33 & 0 & 0 \\
-15 & 48 & -42 & -12 & 33 & -12 & 0 & 0 \\
4 & -13 & 12 & 2 & -8 & 3 & 0 & 0 \\
-1 & 4 & -6 & 4 & -1 & 0 & 0 & 0
\end{pmatrix}, \\
V_i^{\sim*}(3) &= 8 
\begin{pmatrix}
119 & -476 & 714 & -476 & 119 & 0 & 0 & 0 \\
-76 & 304 & -456 & 304 & -76 & 0 & 0 & 0 \\
45 & -180 & 270 & -180 & 45 & 0 & 0 & 0 \\
-24 & 96 & -144 & 96 & -24 & 0 & 0 & 0 \\
11 & -44 & 66 & -44 & 11 & 0 & 0 & 0 \\
-4 & 16 & -24 & 16 & -4 & 0 & 0 & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}

Proof. Full proof can be found in [62]. In [60] I had promised to give arithmetical applications of the Theorem 1. In [62] I had given short
deduction of the Apery’s equation from the Theorem 1. In Part 7 of this work I begin the proof of the Theorem 2, which joins the Apery’s Theorem and my result in [23], [43] in one Theorem. Now I complete this proof. Let me to formulate the Theorem 2 again. Let

\[ z \in \mathbb{Q}, \ |z| \geq 1, \ x = 1/z, \ b \in \mathbb{N}, \ a = bz \in \mathbb{Z}, \]

\[ \tilde{\eta}_i(z) = \left( \sum_{k=0}^{1} \sqrt{|z| + k(-1)^i} \right)^2 = 2\sqrt{|z| + (-1)^i} + 2\sqrt{|z| + (-1)^i \sqrt{|z|}} \]

for \( i = 0, 1, \)

\[ \tilde{\eta}_2(z) = \sqrt{|z| + \sqrt{|z| + 1 + \sum_{k=0}^{1} \sqrt{|z|^2 + |z| + (-1)^k \sqrt{|z|}} = \sqrt{|z| + \sqrt{|z| + 1 + 2(\sqrt{|z|^2 + |z| + |z|}) = r + \sqrt{r^2 + 1 + 2(\sqrt{r^2 + r^2 + r^2})}, \]

where \( r = \sqrt{|z|}. \)

\[ \beta_k(z) = \frac{\ln((\tilde{\eta}_{2k/2}(z))^2 e^{3b})}{\ln((\tilde{\eta}_k(z))^2 e^{3b})}, \]

where \( k = 0, 1, 2 \)

\[ \alpha_k(z) = \beta_k + \frac{(1 - (-1)^k) (\ln(\tilde{\eta}_0(z))/\tilde{\eta}_1(z))}{\ln((\tilde{\eta}_1(z))^2 e^{3b})}, \]

\[ D_k(b) = \{ y \in \mathbb{R}: (-1)^{k/2} y > (\sqrt{e^{3b} + 1})^4/(e^{3b} + 1)^{k/2}/16 e^{3b} \}, \]

where \( k = 0, 1, 2, \)

\[ L_{i,s}(x) = (i/x + (-1)^i) \sum_{n=1}^{+\infty} x^n/n^s, \]

where \( i = 0, 1, \ s \in \mathbb{N}, \ |x| \leq 1, \ |x - 1| + s > 1, \)

\[ L_{1,1}(1) = 0, \]

\[ x_1 \in \mathbb{R}, \ x_2 \in \mathbb{Q}, \ |x_1| + |x_2| > 0, \]

\[ \varphi_i = \phi_i(x_1, x_2, x) = \tilde{\varphi}_i(z, x_1, x_2) = \]
where $i = 1, 2$. Let further,

\[
\varphi_3(x_1, x_2, x) = \varphi_3(z, x_1, x_2, x) = x_1, \quad \tilde{\alpha}_0(x) = \alpha_0(z),
\]

\[
\tilde{\alpha}_i(x) = \alpha_1(z) \text{ for } i = 1, 2
\]

$\tilde{\alpha}_0(x) = \alpha_2(z)$, $\varepsilon > 0$, and $\|\psi\|$ denotes the distance from $\psi$ to $\mathbb{Z}$.

**Theorem 2.** There exist effective positive

\[
\hat{\gamma}_i(x_1, x_2, x) = \gamma^*_i(z, x_1, x_2 \varepsilon),
\]

where $i = 1, 2$;

\[
\hat{\gamma}_0(x, \varepsilon) = \gamma_0(z, \varepsilon),
\]

\[
\hat{\gamma}_1(x, \varepsilon) = \gamma_1(z, \varepsilon), \quad \hat{\gamma}_0(x, \varepsilon) = \gamma_2(z, \varepsilon),
\]

such that,

if

\[
z \in D_0(b), \quad x_1 = \ln(z), \quad x_2 = 1,
\]

then

\[
\max_{i=1,2,3} \|q\hat{\phi}_i\|q^{\alpha_0(z) + \varepsilon} \geq \gamma_0(z, \varepsilon)
\]

for any $q \in \mathbb{N}$;

if $k = 1, 2$,

\[
z \in D_k(b), \quad x_1 \in \mathbb{Z}, \quad x_2 \in \mathbb{Z}, \quad x_2 \neq 0,
\]

then

\[
\max_{i=1,2,3} \|q\hat{\phi}_i\|q^{\beta \kappa(z) + \varepsilon} \geq \gamma^*_k(z, x_1, x_2, \varepsilon)
\]

for any $q \in \mathbb{N}$,

\[
\max_{i=1,2} \|\hat{\phi}_i(z, x_1, x_2)\|(|x_1| + |x_2|)^{\alpha_k(z) + \varepsilon} \geq \gamma_0(z, \varepsilon)
\]

for any $x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}$, for which

\[
|x_1| + |x_2| > 0.
\]

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§8.1. Properties of the roots of characteristic polynomial in the case $l = 0$. 

\[x_1 L_{2-i,i}(x) + ix_2 L_{2-i,i+1}(x) =
\]

\[x_1 L_{2-i,i}(1/z) + ix_2 L_{2-i,i+1}(1/z),\]
The considered in the §7.3 of [63] difference equation

\[ \sum_{\kappa=-2}^{2} -1/(16(4z-1)(\nu + 1/2)^{11})a_{0,\kappa}(z; \nu)y(z; \nu + \kappa) = 0, \]

where \(|z| \geq 1\), \(\nu \in \mathbb{N}_0\), is a difference equation of Poincaré type, and characteristic polynomial of this equation is equal to

\[ T_0(z; \lambda) = 1 - 4(8z + 1)\lambda + (256z^2 - 192z + 6)\lambda^2 - 4(8z + 1)\lambda^3 + \lambda^4. \]

When \(\eta\) runs through the roots of the polynomial

\[ D^\wedge(z; \eta) = (\eta + 1)^4 - 2^4z\eta^2 = \eta^4 + 4\eta^3 + (6 - 16z)\eta^2 + 4\eta + 1, \]

then \(\lambda = \eta^2\) runs through the roots of the polynomial \(T_0(z; \lambda)\) If

\[ r \geq 1, \ \varphi \in [0, \pi], \ z = r^2 \exp(i2\varphi), \]

then we can represent polynomial \(D^\wedge(z; \eta)\) in the form

\[ D^\wedge(z; \eta) = \prod_{\kappa=0,1} (\eta + 1)^2 + 4\sqrt{|z|} \exp(i(\varphi - \kappa\pi))\eta. \]

So, we must study the roots of the polynomial

\[ D_1^\wedge(r, \psi; \eta) = (\eta + 1)^2 + 4r \exp i\psi\eta = \eta^2 + 2(1 + 2r \exp i\psi)\eta + 1 = (\eta + 1 + 2r \exp i\psi)^2 - 4r \exp(i\psi)(1 + r \exp(i\psi), \]

where \(r = \sqrt{|z|} \geq 1\) and \(\psi \in [-\pi, \pi]\). If \(\varphi = 0\) in (41) then we must consider for \(\psi\) two values \(\psi = 0\) and \(\psi = \pi\). If \(\varphi = \pi/2\) in (41) then we must consider for \(\psi\) two values \(\psi = -\frac{\pi}{2}\) and \(\psi = \frac{\pi}{2}\).

In the case \(r = 1\) the roots \(\eta\) of the polynomial \(D_1^\wedge(r, \psi; \eta)\) are studied in §2 of [43]. Let

\[ r > 1, \ \phi_1(r, \psi) = \arccos \left( \frac{1 + r \cos(\psi)}{\sqrt{1 + 2r \cos(\psi) + r^2}} \right) \]

where \(\psi \in [0, \pi]\). If \(\psi \in [-\pi, 0]\), then we let \(\varphi_1(r, \psi) = -\varphi_1(r, -\psi)\). Clearly,

\[ \cos(\varphi_1(r, \psi)) = \frac{1 + r \cos(\psi)}{\sqrt{1 + 2r \cos(\psi) + r^2}}, \]

\[ \sin(\varphi_1(r, \psi)) = \frac{r \sin(\psi)}{\sqrt{1 + 2r \cos(\psi) + r^2}}, \]

If we consider the circumference with radius equal to \(r\) and center in the point \((1, 0)\), and consider the triangle with the apices \(1 + r \exp(i\psi), 0\) and 1
then we find easily that the value $0 < \varphi_1(r, \psi) < \psi$, if $\psi \in (0, \pi)$, and the value $\varphi_1(r, \psi)$ increases in $(0, \pi)$ with increasing of $\psi$ in $(0, \pi)$. Let

$$\varphi_2(r, \psi) = \frac{\psi + \varphi_1(r, \psi)}{2}, \quad \varphi_3(r, \psi) = \frac{\psi - \varphi_1(r, \psi)}{2} < \frac{\pi}{4}.$$  

Then, clearly,

$$0 \leq \varphi_2(r, \psi) \leq \psi, \quad 0 \leq \varphi_3(r, \psi) < \pi/2,$$

and the value $\varphi_2(r, \psi)$ increases in $(0, \pi)$ with increasing of $\psi$ in $(0, \pi)$,

$$\varphi_2(r, 0) = \varphi_3(r, 0) = \varphi_3(r, \pi) = 0, \quad \varphi_2(r, \pi) = \pi.$$  

Clearly,

$$\cos(2\varphi_3(r, \psi)) = \cos(\psi - \varphi_1(r, \psi))) = \frac{r + \cos(\psi)}{\sqrt{1 + 2r \cos(\psi) + r^2}} > 0.$$  

Therefore

$$2\varphi_3(r, \psi) < \frac{\pi}{2} \quad \text{and} \quad \varphi_3(r, \psi) < \frac{\pi}{4}.$$  

Clearly, the roots $\eta$ of the polynomial $D_1(r; \psi; \eta)$ are

$$\eta = \eta_k(r, \psi) = -1 - 2r \exp i\psi - (-1)^{k/2} \sqrt{r} \exp(i(\varphi_2(\psi))(1 + r^2 + 2r \cos(\psi))^{1/4}},$$

where $\psi \in [-\pi, \pi]$, $k = 0, 1$. Therefore

$$\eta_k(r, -\psi) = \overline{\eta_k(r, \psi)} \quad \text{for} \quad \psi \in [-\pi, \pi].$$  

In view of (48),

$$|\eta_k(r, \psi)|^2 = 1 + 4r^2 + 4r(1 + r^2 + 2r \cos(\psi))^{1/2} +$$

$$4r \cos(\psi) + (-1)^k 4\sqrt{r}(1 + r^2 + 2r \cos(\psi))^{1/4} \cos(\varphi_2(\psi)) +$$

$$(-1)^k 8r^{3/2}(1 + r^2 + 2r \cos(\psi))^{1/4} \cos(\varphi_3(\psi)) =$$

$$(2r - 1)^2 + 8r \cos^2(\psi/2) + 4r((r - 1)^2 + 4r \cos^2(\psi/2))^{1/2} +$$

$$(-1)^k 4\sqrt{r}((r - 1)^2 + 4r \cos^2(\psi/2))^{1/4} \times$$

$$2r \cos(\varphi_3(\psi)) + \cos(\varphi_2(\psi))$$

where $\psi \in [-\pi, \pi]$, $k = 0, 1$. In view of (47),

$$\Delta(r, \psi) :=$$

$$4\sqrt{r}(1 + r^2 + 2r \cos(\psi))^{1/4} \cos(\varphi_2(\psi)) +$$

$$8r^{3/2}(1 + r^2 + 2r \cos(\psi))^{1/4} \cos(\varphi_3(\psi)) =$$

$$4r^{1/2}(1 + r^2 + 2r \cos(\psi))^{1/4}(2r \cos(\varphi_3(\psi) + \cos(\varphi_2(\psi))) \geq$$

$$4r^{1/2}(1 + r^2 + 2r \cos(\psi))^{1/4}(r\sqrt{2} + \cos(\varphi_2(\psi))) \geq 0,$$
if $r \geq 1$, $\psi \in [-\pi/2, \pi/2]$. It follows from (50) that

\begin{equation}
1 = |\eta_1(r, \psi)|^2 |\eta_0(r, \psi)|^2 = (1 + 4r^2 + 4r(1 + r^2 + 2r \cos(\psi))^{1/2} + 4r \cos(\psi))^2 - (\Delta(r, \psi))^2.
\end{equation}

Since

\[(2r - 1)^2 + 8r \cos^2(\psi/2) + 4r((r - 1)^2 + 4r \cos^2(\psi/2))^{1/2}\]

decreases together with increasing $\psi \in [0, \pi]$, when $r$ is fixed in $[1, +\infty)$, it follows from (52), (51), that $\Delta(r, \psi)$ decreases with increasing $\psi \in [0, \pi]$. Therefore, in view of (50), if $r$ is fixed in $[1, +\infty)$, then the value $|\eta_0(r, \psi)|^2$ decreases with increasing of $\psi \in (0, \pi)$, and $|\eta_1(r, \psi)|^2 = 1/|\eta_0(r, \psi)|^2$ increases with increasing of $\psi \in [0, +\pi]$. Since

\[(2r - 1)^2 + 8r \cos^2(\psi/2) + 4r((r - 1)^2 + 4r \cos^2(\psi/2))^{1/2}\]

increases with increasing $r \in [0, +\infty)$, when $\psi$ is fixed in $[0, +\pi]$, it follows from (52), (51), that $\Delta(r, \psi)$ decreases with increasing $r \in [1, +\infty)$. Therefore, in view of (50), if $\psi$ is fixed in $[0, \pi]$, then $|\eta_0(r, \psi)|^2$ decreases with increasing of $r \in [1, +\infty)$, and $|\eta_1(r, \psi)|^2 = 1/|\eta_0(r, \psi)|^2$ increases with increasing of $r \in (1, +\infty)$. Therefore

\begin{equation}
1 \leq |\eta_0(r, \pi)|^2 = (2r - 1)^2 + 4r(1 - 1) + 4\sqrt{r(1 - 1)}(2r - 1) < \end{equation}

\[|\eta_0(r, \psi)|^2 < |\eta_0(r, 0)|^2 = (2r + 1)^2 + 4r(1 + 1) + 4(2r + 1)\sqrt{r(1 + 1)}
\]

and

\begin{equation}
\eta_1(r, 0) = \frac{1}{\eta_0(r, 0)} < |\eta_1(r, \psi)| = \frac{1}{|\eta_0(r, \psi)|} < \end{equation}

\[(2r - 1)^2 + 4r(1 - 1) - 4\sqrt{r(1 - 1)}(2r - 1) = \eta_1(r, \pi) = \frac{1}{\eta_0(r, \pi) \leq 1}, \]

if $\psi \in [-\pi, 0] \cup (0, \pi]$, $r \geq 1$.

§8.2. Comparison of the functions $f_{0,2}(z; \nu)$ and $f_{0,3}(z; \nu)$.

Lemma 8.1.1. Let $z \geq 1$, $\lambda_1 \in (0, 1/2)$ Then

\begin{equation}
f_{0,3}(z; \nu) = f_{0,2}(z; \nu)^{-\lambda_1} O(1),
\end{equation}

where $\nu \in \mathbb{N}$, and $O(1)$ depends only from $z$.

Proof. Since $|z| = z$ now it follows that $r = \sqrt{z} \geq 1$. Let

\[f_0 = f_0(r, \eta) = D_1^\nu(r, \pi, \eta) = (\eta + 1)^2 - 4r \eta,
\]

\[\eta_k = \eta_k(r) = \eta_k(r, \pi) = 2r - 1 + (-1)^k 2\sqrt{r^2 - r} \text{ where } k = 0, 1.
\]

Then

\begin{equation}
f_0(r, \eta_1(r)) = 0, \frac{\partial f_0}{\partial \eta}(r, \eta_1(r)) = -4\sqrt{r^2 - r},
\end{equation}
\( \frac{\partial^2 f_0}{\partial \eta^2}(r, \eta) = 2, \)

\( 0 < \eta_1(r) = 1/\eta_2(r) < 1, \frac{\partial f_0}{\partial \eta}(r, \eta_1(r)) < 0, \)

if \( r > 1, \)

\[ \eta_1(1) = \eta_2(1) = 1, \frac{\partial f_0}{\partial \eta}(1, 1) = 0. \]

Let

\( \tau = (1 + \eta)/(1 - \eta) \) where \( 0 \leq \eta < 1, \)

In view of (58), let

\( \tau_1 = \tau_1(r) = (1 + \eta_1(r))/(1 - \eta_1(r)) \) where \( r > 1, \)

Clearly, \( \tau_1(r) \in (1, +\infty) \), if \( r > 1 \). In view of (232) in [63], let

\[ g(z, \tau) = \tau^4/((\tau^2 - 1)^2 z) - 1, \text{ where } z \geq 1, 1 < \tau \]

\( h(z, \eta) = (D^\eta(z; \eta))/(16z^2\eta^2) = \)

\[ f_0(\sqrt{z}, \eta)((\eta + 1)^2 + 4\sqrt{z}\eta)/(16z\eta^2), \]

where \( z \geq 1, 0 < \eta \leq 1. \) Clearly,

\( h(z, \eta) = g(z, (1 + \eta)/(1 - \eta)) \) where \( z \geq 1, 0 < \eta < 1. \)

Let further

\( u(z, \tau) = 2 \ln((\tau - 1)/\tau + 1)) + \tau \ln(1 + g(z, \tau)), \)

where \( z \geq 1, \tau > 1, \)

\( w(z, \eta) = u(z, (1 + \eta)/(1 - \eta)) = 2 \ln(\eta) + \)

\( (\ln(1 + h(z, \eta))(1 + \eta)/(1 - \eta)), \text{ where } z \geq 1, 0 < \eta < 1, \)

\( w_1(z) = w(z, \eta_1(\sqrt{z})) = 2 \ln(\eta_1(\sqrt{z}) + \)

\( (\ln(1 + h(z, \eta_1(\sqrt{z}))(1 + \eta_1(z))/(1 - \eta_1(z))), \)

where \( z > 1. \) Since \( \eta_0(\sqrt{z}) \geq 1, \) it follows that

\( h(z, \eta_1(\sqrt{z})) = 0, \quad h(z, \eta)(\eta - \eta_1(\sqrt{z})) < 0, \)

if \( z \geq 1, 0 < \eta < 1, \eta \neq \eta_1(\sqrt{z}). \) Therefore in view of (59), (62),

\( g(z, \tau_1(\sqrt{z})) = 0, g(z, \tau)(\tau - \tau_1(\sqrt{z})) < 0, \)}
if \( z \geq 1, 1 < \tau, \tau \neq \tau_1(\sqrt{z}) \). In view of (56) and (61),

\[
\frac{\partial h}{\partial \eta}(z, \eta_1(\sqrt{z})) = -\frac{(\eta_1(\sqrt{z}) + 1)^2 + 4\sqrt{z}\eta_1(\sqrt{z})}{4z\eta_1(\sqrt{z})}\sqrt{z - \sqrt{z}},
\]

where \( z \geq 1 \),

\[
\varepsilon_0(z) := -\frac{1}{2}\frac{\partial h}{\partial \eta}(z, \eta_1(\sqrt{z})) > 0, \text{ if } z > 1.
\]

In view of (68), \( \frac{\partial h}{\partial \eta}(1, 1) = 0 \); therefore, in view of (56), (57) and (61),

\[
\frac{\partial^2 h}{\partial \eta^2}(1, 1) = 1.
\]

In view of (63), (64),

\[
\frac{\partial u}{\partial \tau}(z, \tau) = \frac{4}{(\tau^2 - 1)} + 4 - \frac{4\tau^2}{(\tau^2 - 1)} + \ln(1 + g(z, \tau)) = \ln(1 + g(z, \tau)),
\]

\[
\frac{\partial w}{\partial \eta}(z, \eta) = (2/(1 - \eta)^2)\frac{\partial u}{\partial \tau}(z, (1 + \eta)/(1 - \eta)) = \frac{2}{(1 - \eta)^2}(1 + h(z, \eta)),
\]

where \( z \geq 1, 0 < \eta < 1 \),

\[
\frac{\partial^2 w}{\partial \eta^2}(z, \eta) = (2/(1 - \eta)^2)\frac{\partial h}{\partial \eta}(z, \eta)/(1 + h(z, \eta)) + (4/(1 - \eta)^3)\ln(1 + h(z, \eta)),
\]

where \( z \geq 1, 0 < \eta < 1 \). Therefore, if \( z > 1 \), then, in view of (64) – (72),

\[
\frac{\partial w}{\partial \eta}(z, \eta_1(\sqrt{z}))) = 0, -\varepsilon_1(z) = -\frac{1}{2}f\frac{\partial^2 w}{\partial \eta^2}(z, \eta_1(\sqrt{z})) = \varepsilon_0(z)(2/(1 - \eta_1(\sqrt{z})^2) > 0.
\]

If \( z = 1 \), then, in view of (65), (61), (66), (64) – (72),

\[
w_1(1) := \lim_{z \to 1^+} w_1(z) = \lim_{z \to 1^+} (\ln(1 + h(z, \eta_1(\sqrt{z}))/h(z, \eta_1(\sqrt{z}))) \times
\]
\[
\lim_{z \to 1^+} \left( h(z, \eta_1(\sqrt{z})(1 + \eta_1(z))/(1 - \eta_1(z)) \right) = 2 \times \\
\lim_{z \to 1^+} \left( h(z, \eta_1(\sqrt{z}))(1 - \eta_1(z)) \right) = 0,
\]

(75) \[ w(1, 1) := \lim_{\eta \to 1^-} (w(1, \eta)) = \lim_{\eta \to 1^-} (2\ln(\eta)) + \]

\[
\lim_{\eta \to 1^-} \left( \ln(1 + h(1, \eta))/h(1, \eta) \times \right)
\lim_{\eta \to 1^-} \left( h(1, \eta)(1 + \eta)/(1 - \eta) = 2 \times \right)
\lim_{\eta \to 1^-} \left( h(1, \eta)/(1 - \eta) = 2 \times \right)
\left( \lim_{\eta \to 1^-} \left( ((\eta + 1)^2 + 4\sqrt{z}\eta)/(16z\eta^2) \right) \times \right)
\lim_{\eta \to 1^-} \left( ((\eta - 1)^2/(1 - \eta)) = 0 \right)
\]

So, \( w_1(1) = w(1, \eta_1(1)) = w(1, 1) = 0 \). Further we have

(76) \[ \frac{\partial w}{\partial \eta}(1, 1) = \lim_{\eta \to 1^-} (w(1, \eta)/(\eta - 1)) = \]

\[
\lim_{\eta \to 1^-} \left( (2\ln(\eta))/(\eta - 1) - 2 \times \right)
\lim_{\eta \to 1^-} \left( \ln(1 + h(1, \eta))/h(1, \eta) \times \right)
\lim_{\eta \to 1^-} \left( h(1, \eta)/(\eta - 1)^2 \right) = \]

\[ 2 - 2 \left( \lim_{\eta \to 1^-} \left( ((\eta + 1)^2 + 4\sqrt{z}\eta)/(16z\eta^2) \right) = 2 - 1 = 1. \]

In view of (61), (71), (66),

(77) \[ \lim_{\eta \to 1^-} \left( \frac{\partial w}{\partial \eta}(1, \eta) \right) = \]

\[
\left( \lim_{\eta \to 1^-} \left( \frac{\ln(1 + h(1, \eta))}{h(1, \eta)} \right) \right) \times \]

\[
\left( \lim_{\eta \to 1^-} \left( \frac{2h(1, \eta)}{(\eta - 1)^2} \right) \right) = 1. \]

I use below the results of [39] with

\[ d^\nu = d^\nu = 1, \ m = n = 1. \]

We represent \((R(t, \nu))^2\), where \( R(t, \nu) \) is defined in (2) in the form

(78) \[ (R(t, \nu))^2 = R_1(t, \nu)R_2(t, \nu), \]

where

(79) \[ R_1(t, \nu) = \prod_{j=1}^{\nu} (t - j) \]

\[ \frac{\prod_{j=0}^{\nu-1} (t + j)}{\prod_{j=1}^{\nu} (t + j)} \]
\[ \prod_{j=0}^{\nu-1} \frac{t-1-j}{t+j}, \quad R_2(t, \nu) = (t + \nu)^{-2}. \]

It follows from (79) that

\[ R_1(t, \nu)z^{-t} = \frac{(\Gamma(t))^4}{(\Gamma(t-\nu)^2(\Gamma(t+\nu))^2}z^{-t}. \]  

In view of (3), (6), we can take \( t \geq \nu + 1 \) in further calculations and use Stirlings formula in the form

\[ \ln \Gamma(x) = (x - \frac{1}{2})\log x - x + O(1). \]

with \( x \geq 1 \) and \( O(1) = \theta(x)C, \) where \(|\theta(x)| \leq 1\) and \( C \) is appropriate absolute constant. Below \( O(1) \) will be depend only from \( z \geq 1. \) We put \( t = \nu \tau \) now. Then

\[ (R_1(t+1, \nu))z^{-(t-1)} = (R_1(t, \nu))z^{-t} = t^4/z(t^2 - \nu^2)^2 = \]

\[ (R_1(t, \nu))(1 + g(z, \tau)) = (R_1(t, \nu))(1 + h(z, \eta)), \]

where \( t \in \mathbb{N}, \nu \in \mathbb{N}, \tau = t/nu, eta = (\tau - 1)/((\tau - 1). \)

In view of (80), (81),

\[ \ln(R_1(t, \nu)z^{-t}) = 4(t - 1/2)\ln(t) - 4t - \]

\[ 2(t - \nu - 1/2)\ln(t - \nu) + 2t - 2\nu - \]

\[ 2(t + \nu - 1/2)\ln(t + \nu) + 2t + 2\nu - t\ln(z) + O(1) = \]

\[ \nu\tau\ln(1 + g(1 + g(z, \tau)) + 2\nu\ln((\tau - 1)/((\tau + 1)) - \]

\[ \frac{1}{2}\ln(\tau^4/(\tau^2 - 1)^2) + O(1) = \]

\[ u(z, \tau)\nu - \frac{1}{2}\ln(\tau^4/(\tau^2 - 1)^2) + O(1) = \]

\[ w(z, \eta)\nu - \frac{1}{2}\ln(\tau^4/(\tau^2 - 1)^2) + O(1), \]

where \( \nu \in \mathbb{N}, \ t \in [\nu + 1, +\infty] \cap \mathbb{N}. \) Therefore

\[ \ln(R_1(\nu \tau, \nu)z^{-\nu \tau}) - \ln(R_1(\nu \tau_1(\sqrt{z}), \nu)z^{-\nu \tau_1(\sqrt{z})}) = \]

\[ \nu(u(z, \tau) - u(z, \tau_1(\sqrt{z}) - \frac{1}{2}\ln(\tau^4/(\tau^2 - 1)^2) + \]

\[ \frac{1}{2}\ln((\tau_1(\sqrt{z}))^4/((\tau_1(\sqrt{z}))^2 - 1)^2) - \]

\[ \nu(\tau - \tau_1(\sqrt{z})\ln(z) + O(1). \]

In view of (67), (83), (64),

\[ \max_{t \in [\nu + 1, +\infty] \cap \mathbb{N}} (R_1(t, \nu)z^{-t}) = \]
\[ R_1(\nu(\tau_1(\sqrt{z}) + \theta(z; \nu)/\nu))z^{-(\nu(\tau_1(\sqrt{z}) + \theta(z; \nu)))} = R_1(\nu\tau, \nu)z^{-\nu\tau} \]

with \( \tau = \tau_1(\sqrt{z}) + \theta(z; \nu)/\nu \), where \( 0 \leq \theta(z; \nu) < 1 \). In view (84) – (85), (64), (66), (83),

\( \max_{t \in [\nu + 1, +\infty) \cap \mathbb{N}} (R_1(t, \nu)z^{-t}) = (\eta_1(\sqrt{z}))^{2\nu}e^{O(1)} \),

where \( z > 1 \). If \( z > 1 \), then all summands in (3) are positive, and its sum, which is equal to \( f_2(z; \nu) \), is bigger, than a single summands with \( t = \nu(\tau_1(\sqrt{z}) + \theta(z; \nu)/nu) \); therefore, in view of (78), (79), (86),

\( \nu^{-2}(\eta_1(\sqrt{z})^{2\nu}e^{O(1)}) \leq f_{0,2}(z; \nu) \).

On the other hand, if \( z > 1 \), then, in view of (78), (79), (86),

\[ f_{0,2}(z; \nu) \leq \left( \max_{t \in [\nu + 1, +\infty) \cap \mathbb{N}} (R_1(t, \nu)z^{-t}) \right)^{+\infty} \sum_{t=1+\nu}^{+\infty} R_2(t, \nu) = \nu^{-1}(\eta_1(\sqrt{z})^{2\nu}e^{O(1)}) \]

Since \( (t - 1 - k)/(t + k) \) increases, when \( k \) is fixed in \([0, \nu - 1] \) and \( t \) increases in \((\nu + 1, \infty) \), it follows from (79) and (74) that

\[ \sup_{t \in [\nu + 1, +\infty) \cap \mathbb{N}} (R_1(t, \nu)z^{-t}) = 1 = e^{w_1(1)} \]

In view of (78), (79), (83), (89),

\[ \nu^{-4}e^{O(1)} \leq R^2(2\nu^2 - \nu, \nu) \leq f^2(1; \nu) \leq \left( \sup_{t \in [\nu + 1, +\infty) \cap \mathbb{N}} (R_1(t, \nu)) \right)^{+\infty} \sum_{t=1+\nu}^{+\infty} R_2(t; \nu) = \nu^{-1}e^{O(1)} \]

In view of (71), (72), (76), (77), there exists \( \delta_1(z) \in (0, \eta_1(\sqrt{z})) \) such that

\[ \left| \frac{\partial w}{\partial \eta}(z, \eta) - \frac{\partial w}{\partial \eta}(z, \eta_1(\sqrt{z})) \right| \leq 1/2 \]

for \( z \geq 1, |\eta - \eta_1(\sqrt{z})| < \delta_1(z), 0 < \eta < 1 \),

\[ \left| \frac{\partial h}{\partial \eta}(z, \eta) - \frac{\partial h}{\partial \eta}(z, \eta_1(\sqrt{z})) \right| \leq \varepsilon_0(z) \]

for \( z > 1, |\eta - \eta_1(\sqrt{z})| < \delta_1(z), 0 < \eta < 1 \),

\[ \left| \frac{\partial^2 w}{\partial \eta^2}(z, \eta) - \frac{\partial w}{\partial \eta}(z, \eta_1(\sqrt{z})) \right| \leq \varepsilon_1(z) \]

for \( z > 1, |\eta - \eta_1(\sqrt{z})| < \delta_1(z), 0 < \eta < 1 \). In view of (73) and (93),

\[ \frac{\partial^2 w}{\partial \eta^2}(z, \eta) \leq -\varepsilon_0(z) \]
for \( z > 1, |\eta - \eta_1(\sqrt{z})| < \delta_1(z), \eta < 1 \). In view of (69) and (92),
\[
(95) \quad \frac{\partial h}{\partial \eta}(z, \eta) \leq -\varepsilon_0(z)
\]
for \( z > 1, |\eta - \eta_1(\sqrt{z})| < \delta_1(z), \eta < 1 \). In view of (76) and (91),
\[
(96) \quad \frac{\partial w}{\partial \eta}(1, \eta) \geq \frac{1}{2} \text{ for } 1 - \delta_1(1) < \eta < 1.
\]
If \( \eta \geq \eta_1(\sqrt{z} - \delta_1(z) \) then
\[
\tau \geq \frac{1 + \eta_1(\sqrt{z} - \delta_1(z)}{1 - \eta_1(\sqrt{z} + \delta_1(z)}
\]
and \( \frac{1}{2} \ln(\tau^4/(\tau^2 - 1)^2) = O(1) \); therefore, in view of (83), (94), (73), (96), if
\[
z > 1, 0 < \eta = (t - \nu)/(t + \nu) < 1, |\eta - \eta_1(\sqrt{z})| < \delta_1(z),
\]
then
\[
(97) \quad \ln(R_1(t, \nu)z^{-t}) = w(z, \eta)\nu + O(1) =
\]
\[
w(z, \eta_1(\sqrt{z})\nu + (w(z, \eta) - w(z, \eta_1(\sqrt{z}))\nu + O(1) \leq
\]
\[
w_1(z)\nu - \frac{1}{2}\varepsilon_1(\eta - \eta_1(\sqrt{z})^2\nu + O(1),
\]
and, if \( 0 < \eta = (t - \nu)/(t + \nu) < 1, 1 - \delta_1(1) < \eta < 1 \), then
\[
(98) \quad \ln(R_1(t, \nu)) = w(1, \eta)\nu + O(1) =
\]
\[
w_1(1, \nu) + (w(1, \eta) - w_1(1, \nu) + O(1) \leq -(1 - \eta)\nu/2 + O(1).
\]
We fix \( \lambda_1 \in (0, 1/2) \), and let \( \lambda_2 = 2\lambda_1 \). Clearly, if \( z > 1 \), then
\[
(99) \quad \tau_1(\sqrt{z}) = \frac{1 + \eta_1(\sqrt{z} - \delta_1(z)}{1 - \eta_1(\sqrt{z} + \delta_1(z)) =
\]
\[
\frac{2\delta_1(z)}{(1 - \eta_1(\sqrt{z} + \delta_1(z)))(1 - \eta_1(\sqrt{z}) > 2\delta_1(z).
\]
Let \( \nu_1(z) \) is fixed in \( ((3/\delta_1(z))^{1/\lambda_1}, +\infty) \cap \mathbb{Z} \) for \( z \geq 1 \). If \( z > 1 \), then each of the sets
\[
\mathfrak{M}_1(z; \nu) = (\nu \tau_1(\sqrt{z}) - \nu 2\delta_1(z), \nu \tau_1(\sqrt{z}) - \nu \delta_1(z)) \cap \mathbb{Z},
\]
\[
\mathfrak{M}_2(z; \nu) = (\nu \tau_1(\sqrt{z}) + \nu \delta_1(z), \nu \tau_1(\sqrt{z}) + 2\nu \delta_1(z)) \cap \mathbb{Z}
\]
is not empty for \( \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z} \). Clearly, the set
\[
\mathfrak{M}_3(\nu) = (\nu(\nu^{2\lambda_1} - 1), \nu(2\nu^{2\lambda_1} - 1), \cap \mathbb{Z},
\]
is not empty for \( \nu \in [\nu_1(1), +\infty) \cap \mathbb{Z} \). Let \( t_k(z; \nu) \) is fixed in \( \mathfrak{M}_k(z, \nu) \) for \( z > 1, k = 1, 2, \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z} \), and let \( t_3(\nu) \) is fixed in \( \mathfrak{M}_3(\nu) \) for \( \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z} \). Then
\[
(100) \quad \nu < \nu \frac{1 + \eta_1(\sqrt{z} - \delta_1(z)}{1 - \eta_1(\sqrt{z} + \delta_1(z)} <
\]
\[ \nu \tau_1(\sqrt{z}) - \nu 2 \delta_1(z) < t_1(z; \nu) < \nu \tau_1(\sqrt{z}) - 2 \nu^{1-\lambda_1} < \]
\[ \nu \tau_1(\sqrt{z}) + \nu^{1-\lambda_1} < t_2(z; \nu) < \nu \tau_1(\sqrt{z}) + 2 \nu^{1-\lambda_1} < \nu (\tau_1(\sqrt{z}) + \delta_1(z)), \]

where \( z > 1, \nu \in \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}, \)

(101) \[ \nu + 1 < 4 \nu < \nu (\nu^{2\lambda_1} - 1) < t_3(\nu) < (2 \nu^{2\lambda_1} - 1), \]

where \( \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}. \) In view of (100),

(102) \[ \nu^{-\lambda_1} (\tau_1(\sqrt{z}) + \delta_1(z) + 1)^{-2} < \]
\[ \frac{|t_k(z; \nu)/\nu - \tau_1(\sqrt{z})|}{(t_2(z; \nu)/\nu + 1)^2} < \]
\[ < \frac{|t_k(z; \nu)/\nu - 1 - \eta_1(\sqrt{z})|}{t_k(z; \nu)/\nu + 1} = \]
\[ \frac{|t_k(z; \nu)/\nu - 1 - \tau_1(\sqrt{z}) - 1|}{\tau_1(\sqrt{z}) + 1} < \]
\[ |t_k(z; \nu)/\nu - \tau_1(\sqrt{z})| < 2 \nu^{-\lambda_1} < \delta_1(z) \]

where \( z > 1, \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}. \) In view of (101),

(103) \[ 1 - 2 \nu^{-2\lambda_1} < \frac{t_3(\nu) - \nu}{t_3(\nu) + \nu} < 1 - \nu^{-2\lambda_1} \]

where \( \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}. \) In view of (97), (102), if

\[ z > 1, \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z} \text{ and } t \in [\nu + 1, t_1(z; \nu)] \cup [t_2(z; \nu), +\infty), \]

then

(104) \[ 0 < R_1(t, \nu)z^{-t} \leq \max_{k=1,2} R_1(t_k(z, \nu), \nu)z^{-t_k(z, \nu)} = \]
\[ \exp(w_1(z)\nu - \varepsilon_2(z)\nu^{1-2\lambda_1})O(1), \]

where

\[ \varepsilon_2(z) = \frac{1}{2} \varepsilon_1(z)(\tau_1(\sqrt{z}) + \delta_1(z) + 1)^{-4}. \]

In view of (103),

\[ \nu^{-2\lambda_1} < 1 - \frac{t_3(\nu)/\nu - 1}{t_3(\nu)/\nu + 1} < 2 \nu^{-2\lambda_1}. \]

Therefore, if \( t \in [\nu + 1, t_3(\nu)], \) then

(105) \[ 0 < R_1(t, \nu) \leq R_1(t_3(\nu), \nu) \leq \exp(-\nu^{1-2\lambda_1}/2 + O(1)) = \]
\[ \exp(w_1(1)\nu - \nu^{1-2\lambda_1}/2)O(1). \]
\[
\text{In view of (2),}
\]
\[
(R(t, \nu))^{-2} \frac{\partial (R(t, \nu))^2}{\partial t} = \]
\[
2 \ln(t^2/(t^2 - \nu^2)) + O(1/(t - \nu) = O(\ln(\nu)),
\]
where \( \nu \in [2, +\infty) \cap \mathbb{Z}, t \in [\nu + 1, +\infty) \cap \mathbb{Z}, \)
\[
((R(t, \nu))^2 z^{-t})^{-1} \frac{\partial (R(t, \nu))^2 z^{-t}}{\partial t} = 
\]
\[
- \ln(z) + (R(t, \nu))^{-2} \frac{\partial (R(t, \nu))^2}{\partial t} = 
\]
\[
- \ln(z) + 2 \ln(t^2/(t^2 - \nu^2)) + O(1/(t - \nu) = \ln(1 + g(z, \tau)) + O(1/(t - \nu) = \ln(1 + h(z, \eta)) + O(1/(t - \nu) = O(\ln(\nu)),
\]
where
\[
\nu \in [2, +\infty) \cap \mathbb{Z}, \quad t \in [\nu + 1, +\infty) \cap \mathbb{Z}, \quad \tau = t/\nu, \quad \eta = \frac{\tau - 1}{\tau + 1}, \quad z \geq 1.
\]

In view of (107), (104), (65), (66), (87),
\[
\sum_{t=\nu+1}^{t_1(z, \nu)} |R(t, \nu))|^2 z^{-t} \ln(z) - \frac{\partial ((R(t, \nu))^2 z^{-t})}{\partial t} = 
\]
\[
\sum_{t=t_2(z, \nu)}^{+\infty} |R(t, \nu))|^2 z^{-t} \ln(z) - \frac{\partial ((R(t, \nu))^2 z^{-t})}{\partial t} = 
\]
\[
\left( \sum_{t=\nu+1}^{t_1(z, \nu)} - \frac{\partial ((R(t, \nu))^2 z^{-t})}{\partial t} \right) + 
\]
\[
\left( \sum_{t=t_2(z, \nu)}^{+\infty} - \frac{\partial ((R(t, \nu))^2 z^{-t})}{\partial t} \right) = 
\]
\[
(O(\ln(\nu))) \left( \sum_{t=\nu+1}^{t_1(z, \nu)} (R(t, \nu)^2 z^{-t}) \right) + 
\]
\[
(O(\ln(\nu))) \sum_{t=t_2(z, \nu)}^{+\infty} (R(t, \nu)^2 z^{-t}) = 
\]
\[
\exp(w_1(z)\nu - \varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu))) \times 
\]
\[
\left( \sum_{t=\nu+1}^{t_1(z, \nu)} R_2(t, \nu) \right) + 
\]
\[
\exp(w_1(z)\nu - \varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu))) \times 
\]
\[
\sum_{t=t_2(z, \nu)}^{+\infty} R_2(t, \nu) = \\
\exp(w_1(z)\nu - \varepsilon_2(z)\nu^{1-2\lambda_1}) \left( O\left( \frac{\ln(\nu)}{\nu} \right) \right) = \\
(\eta_1(\sqrt{z})^{2\nu} \exp(-\varepsilon_2(z)\nu^{1-2\lambda_1}) \left( O\left( \frac{\ln(\nu)}{\nu} \right) \right) = \\
f_2(z; \nu) \exp(-\varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu))\nu)
\]
for \( z > 1, \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z} \). In view of (107), (105), (66), (90),

\[
(109) \quad \left( \sum_{t=\nu+1}^{t_3(z, \nu)} -\partial (R(t, \nu)^2) \right) = \\
(O(\ln(\nu))) \sum_{t=\nu+1}^{t_3(z, \nu)} (R(t, \nu)^2 = \\
(O(\ln(\nu)) \exp(-\nu^{1-\lambda_2}/2 + O(1)) \sum_{t=\nu+1}^{t_3(z, \nu)} R_2(t, \nu) = \\
\exp(-\nu^{1-\lambda_2}/2) \left( O\left( \frac{\ln(\nu)}{\nu} \right) \right) = \\
f_2(1; \nu) \exp(-\varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu))\nu^3).
\]
If \( z > 1, z > 1, \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z} \), and \( t_1(z; \nu) \leq t \leq t_2(z; \nu) \), then, in view of (102),
\[
-\delta_1(z) < \frac{t_1(z; \nu) - \nu}{t_1(z; \nu) + \nu} - \eta_1(\sqrt{z}) \leq \frac{t - \nu}{t + \nu} - \eta_1(\sqrt{z} \leq \\
\frac{t_2(z; \nu) - \nu}{t_2(z; \nu) + \nu} - \eta_1(\sqrt{z}) < \delta_1(z),
\]
i.e., for \( \eta = (\tau - 1)/(\tau + 1) = (t - \nu)/(t + \nu) \) the inequality \( |\eta - \eta_1(\sqrt{z})| < \delta_1(z) \) holds; therefore, in view of (95)

\[
(110) \quad \frac{\partial \ln(1 + h(z, \eta)}{\partial \eta} \leq -\frac{1}{2} \varepsilon_0(z)
\]
and, in view of (107),

\[
(111) \quad ((R(t, \nu)^2 z^{-t}-1) \frac{\partial (R(t, \nu)^2 z^{-t}}{\partial t}(t, \nu) = O(\nu^{-\lambda_1}).
\]
If \( t_3(z; \nu) \leq t \) then, in view of (103),
\[
-\delta_1(1) < 2\nu^{-2\lambda_1} < \frac{t_3(\nu) - \nu}{t_3(\nu) + \nu} - \eta_1(1) = \\
\frac{t_3(\nu) - \nu}{t_3(\nu) + \nu} - 1 \leq \frac{t - \nu}{t + \nu} - 1 < \frac{t_1(z; \nu) - \nu}{t_1(z; \nu) + \nu} - 1 < -\nu^{-2\lambda_1} < 0,
\]
in view of (95),
\[
\frac{\partial \ln(1 + h(1, \eta))}{\partial \eta} \leq -\frac{1}{2} \varepsilon_0(z),
\]
and, in view of (107),
\[
((R(t, \nu))^2)^{-1} \frac{\partial (R(t, \nu))^2}{\partial t} = O(\nu^{-\lambda_1}).
\]
In view of (111),
\[
\sum_{t \in (t_1(z, \nu), t_2(z, \nu) \cap \mathbb{Z}} -\frac{\partial (R(t, \nu))^2}{\partial t} z^{-t} + \ln(z) (R(t, \nu))^2 z^{-t} = O(\nu^{-\lambda_1}) \sum_{t \in (t_1(z, \nu), t_2(z, \nu) \cap \mathbb{Z}} (R(t, \nu))^2 z^{-t}.
\]
In view of (113),
\[
\sum_{t \in (t_3(\nu), +\infty) \cap \mathbb{Z}} -\frac{\partial (R(t, \nu))^2}{\partial t} = O(\nu^{-\lambda_1}) \sum_{t \in (t_3(\nu), +\infty) \cap \mathbb{Z}} (R(t, \nu))^2 z^{-t}.
\]
The equality (55) follows from (108), (114), (109), (115), (4), (5), (6).

\section*{8.3 To what absolute values of roots of characteristic polynomial (38) correspond the obtained solutions of the equation (37).}

According to well-known classical Perron’s theorem, if \(y(\nu)\) is non-zero solution of difference equation of Poincaré type, then the following equality holds
\[
\lim_{\nu \to +\infty} \sup |y(\nu)|^{1/\nu} = \rho,
\]
where \(\rho = |\eta|\), and \(\eta\) is a root of characteristic polynomial of this equation. If (116) holds, then we will say that the solution \(y(\nu)\) corresponds to \(\rho\). It follows from (87), (88) and (90) that solution \(y(\nu) = f_{0,2}(z; \nu)\) corresponds to \((\eta_1(\sqrt{z}))^2 = (\eta_1(\sqrt{z}, \pi))^2\), if \(z \geq 1\).

\textbf{Lemma 8.3.1.} Let \(s \in \mathbb{N}_0, n \in \mathbb{N},\)
\[
a_i^\ast \in \mathbb{C}, \quad a_i(\nu) \in \mathbb{C},
\]
\[
a_n(\nu) = 1, \quad a_i(\nu) - a_i^\ast = O(1/(\nu + 1))
\]
for \(\nu \in \mathbb{N}_0\) and \(i = 0, \ldots, n\). Let us consider the following difference equation
\[
\sum_{k=0}^{n} a_k(\nu) y(\nu + k) = 0,
\]
where \( \nu \in \mathbb{N}_0 \). For any \( m^* \in \mathbb{N}_0 \) let \( V_{m^*} \) denotes the linear over \( \mathbb{C} \) space of solutions \( y = y(\nu) \) of the equation

\[
\sum_{k=0}^{n} a_k(\nu)y(\nu + k) = 0,
\]

where \( \nu \in [m^*, +\infty) \cap \mathbb{Z} \). Let the absolute values of all the roots of the characteristic polynomial

\[
T(z) = \sum_{k=0}^{n} a_k^* z^k
\]

are among the numbers \( \{\rho_i: 1 \leq i \leq 1 + s\} \) such that \( \rho_{s+1} = 0 \) and \( \rho_j < \rho_i \) for \( 1 \leq i < j \leq s + 1 \). Let \( e_i \) and \( k_i \) denote respectively the sum and the maximum of the multiplicities of those roots, whose absolute value is equal to the number \( \rho_i \), where \( i = 1, \ldots, s + 1 \), and let \( k^* = k_{s+1} \). We suppose that, if \( s > 0 \), then

\[
e_i > 0
\]

for \( i = 1 \ldots, s \). For given \( y = y(\nu) \) in \( \mathbb{C}^{[m^*, +\infty) \cap \mathbb{Z}} \), let

\[
\omega_{n,y}(\nu) = \max(|y(\nu)|, \ldots, |y(\nu + n - 1)|).
\]

a) Then there exist \( A > 0, m^* \in \mathbb{N}, \alpha(\nu) > 0 \) with \( \nu \in [m^*, +\infty) \cap \mathbb{Z} \) and the subspaces \( V_{m^*,1}^{\wedge} \oplus \ldots \oplus V_{m^*,s+1}^{\wedge} \) such that

\[
V_{m^*} = V_{m^*,1}^{\wedge} \oplus \ldots \oplus V_{m^*,s+1}^{\wedge}, \quad \dim_{\mathbb{C}}(V_{m^*,i}^{\wedge}) = e_i, 1 \leq i \leq s + 1,
\]

and, if \( y \in V_{m^*,\theta}^{\wedge} \) for some \( \theta \in \{1, \ldots, s\} \), then

\[
\exp(-A(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu\omega_n(y)(m^*) \leq \omega_{n,y}(\nu)
\]

for \( \nu \in [m^*, +\infty) \cap \mathbb{Z} \); moreover, the spaces

\[
V_{m^*,j}^{\wedge} = V_{m^*,j}^{\wedge} \oplus \ldots \oplus V_{m^*,s+1}^{\wedge},
\]

where \( j = 1 \ldots, s + 1 \), have the following properties:

if \( y \in V_{m^*,\theta}^{\wedge} \) for some \( \theta \in \{1, \ldots, s\} \), then

\[
\omega_{n,y}(\nu) \leq \exp(A(\ln(\nu) + \nu^{1-1/k_\theta}))((\rho_\theta)^\nu\omega_n(y)(m^*))
\]

if

\[
k^* > 0,
\]

and \( y \in V_{m^*,s+1}^{\wedge} (= V_{m^*,s+1}^{\wedge}) \), then

\[
|y(\nu)| \leq (A/\nu)^{\nu/k^*}\omega_{n,y}(m^*),
\]

where \( \nu \in m + \mathbb{N} - 1 \).
\[ V \cap V_{m^*,\theta+1} = \{0\}, \]

where \( \theta \in \{1, \ldots, s\} \), then for this \( V \) there exists a constant \( A^* = A^*(V) > 0 \) such that

\[ \exp(-A^*(\ln(\nu) + \nu^{1-1/k}))\rho(y)^\nu \omega_n(y)(m^*) \leq \omega_{n,y}(\nu) \]

where \( y \in V \), \( k = \max(k_1, \ldots, k_s) \) and \( \nu \in [m^*, +\infty) \cap \mathbb{N} \).

**Proof.** The proof can be found in [49] – [53].

**Remark 1.** It follows from the Lemma 8.3.1 that the linear space \( V_{m^*,\theta}^\wedge \), where \( \theta = 1, \ldots, s+1 \), does not depend from the construction and is defined uniquely by means of the equality

\[ V_{m^*,\theta}^\wedge = \{ y \in V_m : \limsup V_k = 0 \} \]

**Lemma 8.3.2.** Let \( V \) be a \( r \)-dimensional linear subspace of the linear space \( V_{m^*}^\wedge \), let \( r \geq 1 \) and let \( V \cap V_{m^*,s+1}^\wedge = \{0\} \). Let further \( \{y_1(\nu), \ldots, y_r(\nu)\} \) is a basis of the space \( V \). Let

\[ k_3(V) = \max\{k \in \mathbb{N} : 1 \leq k \leq s, \ V \subset V_{m^*,k}^\wedge \}, \]

and

\[ k_4(V) = \min\{k \in \mathbb{N} : 1 \leq k \leq s, \ V \cap V_{m^*,k+1}^\wedge = \{0\} \}. \]

For \( X \in \mathbb{C}^r \),

\[ X = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \]

let

\[ q_\infty(X) = \max\{|x_1|, \ldots, |x_r|\}, \]

\[ y = y^{\nu}(X, \nu) = x_1 y_1^{\nu}(\nu) + \ldots + x_r y_r^{\nu}(\nu). \]

Then for every \( \varepsilon \in (0, 1) \) there exist \( C_3(\varepsilon) > 0 \) and \( C_4(\varepsilon) > 0 \) such that

\[ C_3(\varepsilon)\rho_{k_4}(1-\varepsilon)^\nu q_\infty(X) \leq \omega_{n,y}(\nu) \leq C_4(\varepsilon)\rho_{k_3}(\varepsilon)^\nu q_\infty(X). \]

**Proof.** Any \( y \in V \) has an unique representation in the form (128) with column \( X = X(y) \). Let \( q_\infty(y) := q_\infty(X(y)) \). Then \( q_\infty(y) \) and \( \omega_{m^*}(y) \) are two norms on finite-dimensional linear over \( \mathbb{C} \) space \( V \). Therefore there exist constants \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[ C_2 q_\infty(y) \leq \omega_{m^*}(y) \leq C_1 q_\infty(y). \]

Hence, according to the Lemma 8.3.1, for every \( \varepsilon \in (0, 1) \) there exist constants \( C_5(\varepsilon) > 0 \) and \( C_6(\varepsilon) > 0 \) such that

\[ C_6(\varepsilon)\rho_{k_4}(1-\varepsilon)^\nu \omega_{m^*,y} \leq C_5(\varepsilon)(\rho_{k_3} + \varepsilon)^\nu \omega_{m^*,y}. \]
Then (129) holds with $C_3(\varepsilon) = C_1 C_5(\varepsilon)$ and $C_4(\varepsilon) = C_2 C_5(\varepsilon)$. □

We apply the Lemma 8.3.1 to our case $l = 0$. We have $n = 4$ for the equation (37). Let $z > 1$. Then it follows from (53) – (54) that

$$1 < \rho_2 = |\eta_0(r, \pi)|^2 = (2r - 1)^2 + 4r(r - 1) + 4\overline{r(r - 1)(2r - 1)} <$$

$$\rho_1 = |\eta_0(r, 0)|^2 = (2r + 1)^2 + 4r((r + 1) + 4\overline{r(r + 1)(2r + 1)},$$

$$\rho_4 = 1/\rho_1 < \rho_3 = 1/\rho_2 < 1,$$

$s = 4, e_1 = e_2 = e_3 = e_4 = k_1 = k_2 = k_3 = k_4 = 1$.

We note that, in view of (26),

$$(\tilde{\eta}_i(z))^2 =$$

$$(2r + (-1)^i + 2\sqrt{r^2 + (-1)^i})^2 =$$

$$(2r + (-1)^i)^2 + 4r((r + (-1)^i) + 4(2r + (-1)^i)\sqrt{r + (-1)^i}) = \rho_{i+1}$$

for $i = 0, 1$. Let is fixed the number $m^*$, which is specified in the Lemma 8.3.1. Then $V_{m^*, 5} = \{0\}$, and $V_{m^*, 4} = V_{m^*, 4}$. Since the solution $y(\nu) = f_{0, 2}(z; \nu)$ corresponds to $\rho_3$, it belongs to $V_{m^*, 3} \setminus V_{m^*, 4}$. Let $y_4(z; \nu)$ be non-zero solution in $V_{m^*, 4}$. Then $y(\nu) = f_{0, 2}(z; \nu)$ and $y_4(z; \nu)$ compose the basis of $V_{m^*, 3}$. In view of (55), the solution $y(\nu) = f_{0, 3}(z; \nu)$ belongs to $V_{m^*, 3}$. Hence,

$$f_{0, 3}(z; \nu) = \alpha f_{0, 2}(z; \nu) + \beta y_4(z; \nu),$$

where $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$. In view of (87) and (123),

$$f_{0, 3}(z; \nu) = f_{0, 2}(z; \nu)(\alpha + \beta y_4(z; \nu)/f_{0, 2}(z; \nu)) =$$

$$f_{0, 2}(z; \nu)(\alpha + O(1)\nu^{O(1)}(\rho_4/\rho_3)^\nu).$$

Hence, in view of (55), $\alpha = 0$ and $f_{0, 3}(z; \nu)$ belongs to $V_{m^*, 4}$ and, if it is non-zero solution of the equation (37), then it corresponds to $\rho_4$ in this case.

Let $z = 1$. Then it follows from (53) – (54) that

$s = 3, e_1 = e_3 = k_1 = k_3 = 1, e_2 = k_2 = 2$

$$\rho_1 = |\eta_0(r, 0)|^2 = 17 + 12\overline{2} > \rho_2 = |\eta_0(r, \pi)|^2 =$$

$$|\eta_1(r, \pi)|^2 = 1 > \rho_3 = |\eta_1(r, 0)|^2 = 1/\rho_1.$$

Let is fixed $m^*$, which is specified in the Lemma 8.3.1. Then $V_{m^*, 5} = \{0\}$, and $V_{m^*, 4} = V_{m^*, 4}$. The solution $y(\nu) = f_{0, 2}(1; \nu)$ corresponds to $\rho_2 = 1$ in this case. It is proved in the §7.4 of [63], that our difference equation has also solution $y(\nu) = 1$, which, clearly, corresponds to $\rho_2 = 1$; it is proved there also that the solutions $y(\nu) = f_{0, 2}(1; \nu)$ and $y(\nu) = 1$, compose a linearly independent system over $\mathbb{C}$; since each of these solutions correspond to $\rho_2$, it follows that they are contained in $V_{m^*, 2} \setminus V_{m^*, 3}$. Let $y_3(\nu)$ be non-zero solution in $V_{m^*, 3}$. Let

$$0 = \alpha f_{0, 2}(1; \nu) + \beta y_3(\nu) + \gamma,$$

where $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ and $\nu \in [m, +\infty) \cap \mathbb{N}$ Then, in view of 123 for $y = y_3(\nu)$ and (90), $\gamma = 0$. Then, in view of 123 for $y = y_3(\nu)$ and (90),

$$0 = f_{0, 2}(1; \nu)(\alpha + \beta y_3(1; \nu)/f_{0, 2}(1; \nu)) =$$
\[ \alpha f_{0,2}(1; \nu)(\alpha + O(1)\nu^{O(1)}(\rho_3/\rho_2)^\nu), \]
and therefore \( \alpha = \beta = 0. \) Then \( y(\nu) = f_{0,2}(1; \nu), \) \( y(\nu) = 1 \) and \( y_3(\nu) \) compose the linearly independent system over \( \mathbb{C}; \) according to the assertions of the Lemma 8.3.1, \( \dim_{\mathbb{C}}(V_{m^*,2}^\wedge) = 3; \) hence, \( y(\nu) = f_{0,2}(1; \nu), \) \( y(\nu) = 1 \) and \( y_3(\nu) \) compose the basis of \( V_{m^*,2}^\wedge. \) In view of (55), the solution \( y(\nu) = f_{0,3}(1; \nu) \) belongs to \( V_{m^*,2}^\wedge. \) Therefore

\[ f_{0,3}(1; \nu) = \alpha f_{0,2}(1; \nu) + \beta y(\nu) + \gamma, \]
where \( \alpha \in \mathbb{C}, \ \beta \in \mathbb{C} \) and \( \gamma \in \mathbb{C}. \) It follows from (87), (88), (90) and (55), that \( \gamma = 0; \) therefore

\[ f_{0,3}(1; \nu) = f_{0,2}(1; \nu)(\alpha + \beta y(1; \nu)/f_{0,2}(1; \nu)) = \]

\[ f_{0,2}(1; \nu)(\alpha + O(1)\nu^{O(1)}(\rho_3/\rho_2)^\nu). \]

Hence, in view of (55), \( \alpha = 0 \) and \( f_{0,3}(1; \nu) \) belongs to \( V_{m^*,3}^\wedge \) and, if it is non-zero solution of the equation (37, then it corresponds to \( \rho_3 \) in this case.

Let, finally, \( z \leq -1. \) Then \( \phi = \pi/2 \) in (40), and, as it has been mentioned already in §8.1, we must consider for \( \psi \) in (42) two values \( \psi = -\frac{\pi}{2} \) and \( \psi = \frac{\pi}{2}. \) In view of (49), (53), (54),

\[ s = 2, \ e_1 = e_2 = k_1 = k_2 = 2, \]

(130) \[ V_{m^*,2}^\wedge = V_{m^*,2}^\vee, \ V_{m^*,3}^\wedge = V_{m^*,3}^\vee = \{0\}, \dim_{\mathbb{C}} V_{m^*,2}^\vee = 2, \]

(131) \[ \rho_1 = |\eta_0(r, \pi/2)|^2 = |\eta_0(r, -\pi/2)|^2 > \]

\[ |\eta_0(r, \pi)|^2 \geq |\eta_1(r, \pi)|^2 > \]

\[ \rho_2 = 1/\rho_1 = |\eta_1(r, \pi/2)|^2, \]
where \( r = \sqrt{-z} \geq 1. \) Clearly,

\[ \cos(\varphi_1(\pi/2)) = \sin(2\varphi_3(\pi/2)) = 1/\sqrt{r^2 + 1}, \]

\[ \cos(2\varphi_3(\pi/2)) = \sin(\varphi_1(\pi/2)) = r/\sqrt{r^2 + 1}, \]

\[ \cos(2\varphi_2(\pi/2)) = -\sin(\varphi_1(\pi/2)) = -r/\sqrt{r^2 + 1}, \]

\[ \cos(\varphi_2(\pi/2)) = \sqrt{(1 - r/\sqrt{r^2 + 1})/2}, \]

\[ \cos(\varphi_3(\pi/2)) = \sqrt{(1 + r/\sqrt{r^2 + 1})/2}. \]

Therefore, in view of (50),

\[ \rho_k = |\eta_{k-1}(r, \pi/2)|^2 = 1 + 4r^2 + 4r\sqrt{1 + r^2} + \]

\[ (-1)^k2\sqrt{2r}\left(2r\sqrt{\sqrt{r^2 + 1} + r + \sqrt{r^2 + 1} - r}\right) = \]

\[ 1 + 4r^2 + 4r\sqrt{1 + r^2} + \]

\[ (-1)^{1-k}2\sqrt{2r}\sqrt{(4r^2 + 1)\sqrt{r^2 + 1} + r(4r^2 + 3),} \]
where \( k = 1, 2, \) \( r = \sqrt{-z} \geq 1. \) We note that, in view of (27),

\[
(\tilde{\eta}_2(z))^2 = 4r^2 + 1 + 4r^2 \sqrt{r^2 + 1} + 2\sqrt{2r} \sqrt{r + \sqrt{r^2 + 1}}^3 = 4r^2 + 1 + 4r^2 \sqrt{r^2 + 1} + \]

\[2\sqrt{2r} \sqrt{r(4r^2 + 3) + (4r^2 + 1)\sqrt{r^2 + 1}} = \rho_1,
\]

in this case. Since \( |f_{0,2}(z; \nu)| \leq f_{0,2}(|z|; \nu), \) and the solution \( y(\nu) = f_{0,2}(|z|; \nu) \)
correspond to \( |\eta_1(r, \pi)|^2, \) it follows from (38) that \( f_{0,2}(z; \nu) \) cannot correspond
to \( \rho_1 \) and, hence, if it is non-zero solution of the equation (37), then it
corresponds to \( \rho_2. \) If \( t \geq \nu + 1 \) then, in view of (2),

\[
((R^{2+l})(t, \nu))^{-1} \left( \frac{\partial}{\partial t} (R^{2+l}) \right) (t, \nu),
\]

where \( l = 0, 1, 2, |z| \geq 1. \) is sum of \( (2\nu + 1) \) summands, which are \( O(1), \)
where \( O(1) \) depends only from \( z. \) Therefore in view of (6),

\[
f_{l,4}(z, \nu) = f_{l,2}(|z|, \nu)(2\nu + 1)O(1),
\]

where \( l = 0, 1, 2, |z| \geq 1 \) and \( O(1), \) depends only from \( z. \) Consequently,
when \( y(\nu) = f_{0,4}(z; \nu) \) is non-zero solution of the equation (37), then it
corresponds to \( \rho_2. \) In view of (5), if \( y(\nu) = f_{0,3}(z; \nu) \) is non-zero solution of
the equation (37), then it corresponds to \( \rho_2. \) Moreover, if

\[
x_k \in \mathbb{R}, \text{ for } k = 1, 2,
\]

\[
y(x_1, x_2, z, \nu) = \sum_{k=1}^{2} x_k f_{2k}(z; \nu)
\]

and \( y(x_1, x_2, z, \nu) \) is non-zero solution of the equation (37), then it corre-
sponds to \( \rho_2. \)

\section*{8.4. Properties of some sequences.}

Here we prove, as a generalization of the Lemma 3.2.1 in [43], the following
\textbf{Lemma 8.4.1.} \textit{Let}

\[
z \in \mathbb{Q}, \ |z| \geq 1, \ z \neq 1, b \in \mathbb{N}, a = bz \in \mathbb{Z},
\]

\textit{Then the four sequences}

\[
(133) \quad \{\alpha_{0,i}^*(z; \kappa + k)\}_{k=1}^{+\infty}, \ \{\beta_{0,j}^*(z; \kappa + k)\}_{k=1}^{+\infty},
\]

where \( i = 1, 2, j = 0, 1, \) compose a linearly independent system over \( \mathbb{C} \) for
any \( \kappa \in \mathbb{N}. \)

\textbf{Proof.} The proof for \( |z| > 1 \) can be found in [39] (Lemma 14) in more
general situation. The proof for \( z = -1 \) can be found in [43] (Lemma 3.2.1).
Making use of the simplicity of the situation, which we consider now, I give
here more short proof. Let $\mathcal{O}_p = \{ u \in \mathbb{Q} : ord_p(u) \geq 0 \}$. According to (101) and (102) in [62] (see §8.6 below), the polynomials (133) have a form

\begin{equation}
\alpha_{0,i}^*(z; \nu) = \sum_{k=0}^{\nu} \alpha_{0,i,k,\nu} z^k,
\end{equation}

with $\alpha_{0,i,k,\nu} \in \mathbb{Q}$ for $i = 1, 2, k \in [0, \nu] \cap \mathbb{Z}, \nu \in [0, +\infty) \cap \mathbb{Z}$,

\begin{equation}
\beta_{0,j}^*(z; \nu) = \sum_{i=1}^{2} \left( \sum_{k=0}^{\nu} \alpha_{0,i,k,\nu} \left( \sum_{t=1}^{k} z^{k-t} \binom{i+j-1}{j} (t)^{-i-j} \right) \right) = \sum_{i=1}^{2} \left( \sum_{t=1}^{\nu} \left( \sum_{k=t}^{\nu} \alpha_{0,i,k,\nu} z^{k-t} \right) \right)
\end{equation}

for $j = 0, 1, \nu \in [0, +\infty) \cap \mathbb{Z}$. Let $p$ be an arbitrary prime greater than 3. In view of (70) – (80) in (23),

\begin{equation}
\alpha_{0,i,k,2p} \in p^i \mathcal{O}_p
\end{equation}

for $i = 1, 2, k \in [1, 2p-1] \cap \mathbb{Z} \setminus \{p\}$,

\begin{equation}
\alpha_{0,2,0,2p} = 1, \ \alpha_{0,1,0,2p} \in -6/p + \mathcal{O}_p
\end{equation}

\begin{equation}
\alpha_{0,2,p,2p} \in 36 + p\mathcal{O}_p, \ \alpha_{0,2,2p,2p} \in 36 + p\mathcal{O}_p,
\end{equation}

\begin{equation}
\alpha_{0,1,p,2p} \in -60/p + \mathcal{O}_p, \ \alpha_{0,1,2p,2p} \in 66/p + \mathcal{O}_p.
\end{equation}

Therefore if the prime $p$ isn’t a divisor of $2ab$ then

\begin{equation}
\alpha_{0,2}^*(z; 2p) \in 36(z + z^2) + 1 + p^2 \mathcal{O}_p,
\end{equation}

\begin{equation}
ord_p(\alpha_{0,1}^*(z; 2p) \in 66z^2/p - 60z/p - 6/p + \mathcal{O}_p.
\end{equation}

In view of (135) – (141),

\begin{equation}
\beta_{0,j}^*(z; 2p) \in \sum_{i=1}^{2} \left( \sum_{t=1}^{2} \left( \sum_{k=0}^{2p} \alpha_{0,i,k,2p} z^{k-p} \right) \right) +
\end{equation}

\begin{equation}
p^{-1} \mathcal{O}_p \subseteq \sum_{i=1}^{2} \left( \sum_{t=1}^{2} \left( \sum_{k=\tau}^{2} \alpha_{0,i,k,2p} z^{p(\kappa-\tau)} \right) \right) + p^i \mathcal{O}_p + p^{-1} \mathcal{O}_p =
\end{equation}

\begin{equation}
\sum_{i=1}^{2} \left( \sum_{t=1}^{2} \left( \sum_{k=\tau}^{2} \alpha_{0,i,k,2p} z^{p(\kappa-\tau)} \right) \right) + p^{-1} \mathcal{O}_p \subseteq
\end{equation}
\[
\left(1 + j - 1 \right) \left( p \right) - 1 - j ( -60/p + \mathcal{O}_p ) + \left( 2 + j - 1 \right) \left( p \right) - 2 - j (36 + p \mathcal{O}_p ) + \\
\left( 1 + j - 1 \right) \left( p \right) - 1 - j (66/p + \mathcal{O}_p ) (z + p \mathcal{O}_p ) + \\
\left( 2 + j - 1 \right) \left( p \right) - 2 - j (36 + p \mathcal{O}_p ) (z + p \mathcal{O}_p ) + \\
\left( 1 + j - 1 \right) \left( p \right) - 1 - j (66/p + \mathcal{O}_p ) + \\
\left( 2 + j - 1 \right) \left( p \right) - 2 - j (36 + p \mathcal{O}_p ) + p^{-1} \mathcal{O}_p = \\
6(p2)^{-2-j} \times \\
(11 - 2^{1+j} 10 + (1 + j)9(1 + 2^{2+j}) + 2^{1+j} 11 + (1 + j)9(2^{2+j})z) + \\
p^{-1-j} \mathcal{O}_p
\]

where \( j = 0, 1 \). Let

\[ F = 6ab(36a^2 + 36ab + b^2)(66a^2 - 60ab - 6b^2)(58a + 36b)(188a + 133b). \]

Clearly, if \( a \in \mathbb{Z}, b \in \mathbb{N}, |a| \geq b, a \neq b, \) then \( F \neq 0 \). Therefore, if \( p > |F| \), then

\[(143) \quad \text{ord}_p(\alpha_{0,i}^*(z; 2p)) = -2 + i, \]

where \( i = 1, 2, \) and

\[(144) \quad \text{ord}_p(\beta_{0,j}^*(z; 2p)) = -2 - j, \]

where \( j = 0, 1 \). As it was mentioned in §7.4 of [63], \( z - 1 \) is a divisor of the polynomial \( \alpha_{0,1}^*(z; \nu) \); let

\[(145) \quad P_0^*(z; \nu) := \frac{\alpha_{0,1}^*(z; \nu)}{z - 1} \in \mathbb{Q}[z, \nu]; \]

then

\[(146) \quad P_0^*(1; \nu) = \frac{d\alpha_{0,1}^*}{dz}(1; \nu). \]

Let

\[(147) \quad P_1^*(z; \nu) := \alpha_{0,2}^*(z; \nu), P_2^*(z; \nu) := \alpha_{0,1}^*(z; \nu), \]

\[(148) \quad P_{3+j}^*(z; \nu) := \beta_{0,j}^*(z; \nu) \text{ for } j = 0, 1. \]

Then, in view of (143) – (144)

\[(149) \quad \text{ord}_p(P_{0,i}^*(z; 2p)) = 1 - i \]

where \( i = 1, 2, 3, 4 \). We must prove that for any \( \kappa \in \mathbb{N} \) four sequences

\[(150) \quad \left\{ P_i^*(z; \kappa + k) \right\}_{k=1}^{+\infty}, \]
where \( i = 1, 2, 3, 4 \), compose a linearly independent system over \( \mathbb{C} \).

First we prove that for each \( \kappa \in \mathbb{N} \) four sequences (150) compose a linearly independent system over \( \mathbb{Q} \). Suppose the contrary. Then there exist \( \kappa \in \mathbb{N} \) and \( a_i \in \mathbb{Z} \), where \( i = 1, \ldots, 4 \), such that

\[
\sum_{i=1}^{4} |a_i| > 0
\]

and

\[
(151) \quad \sigma := \sum_{i=1}^{4} a_i P_i^\kappa(z; \nu) = 0,
\]

where \( \nu \in \mathbb{N}, \nu > \kappa \). Let \( k = \max \{i \in \{1, 2, 3, 4\}: a_i \neq 0\} \) Let \( p \) be any prime such that \( p > F + \sum_{i=1}^{4} |a_i| \). Then \( \text{ord}_p(\sigma) = 1 - k \), an we obtain a contradiction. So four sequences (150) compose a linearly independent system over \( \mathbb{Q} \). Hence, the composed by these sequences infinite \( 4 \times \mathbb{N} \)-matrix contain an invertible \( 4 \times 4 \)-submatrix. ■

**Lemma 8.4.2.** Let

\[
z \in \mathbb{Q}, \ |z| \geq 1, \ b \in \mathbb{N}, a = bz \in \mathbb{Z},
\]

Then the four sequences

\[
(152) \quad \{P_i(z; \kappa + k)^\kappa\}_{k=1}^{+\infty},
\]

where \( i = 0, 1, 3, 4 \) compose a linearly independent system over \( \mathbb{C} \) for any number \( \kappa \in \mathbb{N} \).

**Proof.** If \( z \neq 1 \) the assertion of the Lemma is direct Corollary of the Lemma 8.4.1. If \( z = 1 \) the assertion of the Lemma is Corollary of the Lemma 7.4.1 in [63]. ■

If \( \nu \in [2, +\infty) \cap \mathbb{Z} \), we let \( D_\nu \) denote the smallest number in \( \mathbb{N} \) with property that the following inequality holds for every \( k = 1, \ldots, 2\nu \) and for every prime \( p, 1 \leq p \leq \nu \):

\[
\text{ord}_p(k^{-1}D_\nu) \geq 0.
\]

It is clear that for any \( \varepsilon > 0 \)

\[
(153) \quad D_\nu = \prod_{p \leq \nu} p^{(\ln(2\nu))/\ln(p)} = \exp((\ln(2\nu))(\nu/\ln(\nu) + O(\nu/(\ln(\nu))^2)) = \exp(\nu(1 + O(1)/\ln(\nu)))).
\]

**Lemma 8.4.3.** Let

\[
(154) \quad P_i^\wedge(z; \nu) = P_i^\nu(z; \nu)(D_\nu)^3
\]

where \( i = 1, 2, 3, 4 \). Then

\[
(155) \quad P_i^\wedge(z; \nu) \in \mathbb{Z}[z]
\]
for \( i = 0, 1, 2, 3, 4 \) and \( \nu \in [2, +\infty) \cap \mathbb{Z} \).

**Proof.** For \( i = 1, 2, 3, 4 \) the proof can be found in [23], page 48. Since assertion of the Lemma holds for \( i = 1 \), it follows from (145) and Horner rule that it holds for \( i = 0 \) also. ■

**Lemma 8.4.4.** For any \( k \in \mathbb{N} \) four sequences

\[
\{P_i^\wedge(z; \kappa + k)\}_{k=0}^{+\infty},
\]

where \( i = 0, 1, 3, 4 \) compose a linearly independent system over \( \mathbb{C} \).

**Proof.** The infinite \( 4 \times \mathbb{N} \)-matrix produced by the sequences (152), contains 4 columns, which composed an invertible \( 4 \times 4 \)-matrix \( M^* \); we suppose that \( k_1, k_2, k_3, k_4 \) are the numbers of these columns. Let further \( M^\wedge \) be the corresponding matrix, composed by the columns with numbers \( k_1, k_2, k_3, k_4 \) in the \( 4 \times \mathbb{N} \)-matrix, produced by the sequences (156). Then

\[
\det(M^\wedge) = \det(M^*) \prod_{i=1}^{4} (D_{\kappa+k_i})^3 \neq 0.
\]

■

**§8.5. Proof of the Theorem 2.**

Let \( \{m, n\} \subset \mathbb{N} \),

\[
a_{i,k} \in \mathbb{R}
\]

for \( i = 1, \ldots, m, \ k = 1, \ldots, n \),

\[
\alpha_j^\wedge(\nu) \in \mathbb{Z}
\]

where \( j = 1, \ldots, m+n \) and \( \nu \in \mathbb{N} \). Let there are \( \gamma_0^\wedge, r_1^\wedge \geq 1, \ldots, r_m^\wedge \geq 1 \) such that

\[
|\alpha_i(\nu)| < \gamma_0^\wedge (r_i^\wedge)^\nu
\]

where \( i = 1, \ldots, m \) and \( \nu \in \mathbb{N} \). Let \( y_k(\nu) = -\alpha_{m+k}^\wedge(\nu) + \sum_{i=1}^{m} a_{i,k} \alpha_i^\wedge(\nu) \), where \( k = 1, \ldots, n \) and \( \nu \in \mathbb{N} \). If

\[
X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in \mathbb{R}^n,
\]

then let

\[
q_\infty(X) = \max(|Z_1|, \ldots, |Z_n|),
\]

\[
y^\wedge(X) = y^\wedge(X, \nu) = \sum_{k=1}^{n} y_k^\wedge(\nu) Z_k
\]

for \( \nu \in \mathbb{N} \), let

\[
\phi_i^\wedge(X) = \sum_{k=1}^{n} a_{i,k} Z_k
\]
for \( i = 1, \ldots, m \), and let

\[
\alpha_0^\wedge(X, \nu) = \sum_{k=1}^{n} \alpha_{m+k}^\wedge(\nu)Z_k
\]

for \( \nu \in \mathbb{N} \). Clearly,

\[
y^\wedge(X, \nu) = -\alpha_0^\wedge(X, \nu) + \sum_{i=1}^{m} \alpha_i^\wedge(\nu)\phi_i(X)
\]

for \( X \in \mathcal{R}^n \) and \( \nu \in \mathbb{N} \),

\[
\alpha_0^\wedge(X, \nu) \in \mathbb{Z}
\]

for \( X \in \mathbb{Z}^n \) and \( \nu \in \mathbb{N} \).

Lemma 8.5.1. Let \( \{l^\nu, n\} \subseteq \mathbb{N}, \gamma_1^\wedge > 0, \gamma_2^\wedge > 1/2, R_1 \geq R_2 > 1 \),

(160) \[ \alpha_i = (\log(r_i^\wedge R_1/R_2))/\log(R_2), \]

where \( i = 1, \ldots, m \), let \( X \in \mathbb{Z}^n \setminus \{0\} \),

\[
\gamma_3^\wedge = \gamma_1^\wedge(R_1)^{-\log(2\gamma_2 R_2)/\log(R_2)}, \gamma_4^\wedge = \gamma_3^\wedge \left( \sum_{i=1}^{m} \gamma_0^\wedge(r_i^\wedge)^{(\log(2\gamma_2^\wedge)/\log(R_2)+l^\nu)} \right)^{-1}
\]

and let for each \( \nu \in \mathbb{N} - 1 \) hold the inequalities

(161) \[ \gamma_1^\wedge(R_1)^{-\nu}\varphi_{\infty}(X) \leq \sup\{|y^\wedge(X, \kappa)|: \kappa = \nu, \ldots, \nu + l^\nu - 1\}, \]

(162) \[ |y^\wedge(X, \nu)| \leq \gamma_2^\wedge(R_2)^{-\nu}\varphi_{\infty}(X) \]

Then

(163) \[ \sup\{\|\phi_i^\wedge(X)\|(\varphi_{\infty}(X))^{\alpha_i}: i = 1, \ldots, m\} \geq \gamma_4^\wedge. \]

Proof. Proof may be found in [42], Theorem 2.3.1. □

Let now \( z = a/b \geq 1 \), where \( a \in \mathbb{N}, b \in \mathbb{N} \). In fiew of (5), (145) – (148), (32), (33) above and (99) in [62] (see §8.6 below),

(164) \[ f_{i,3}^\nu(z, \nu) = f_{i,3}(z, \nu) = (\ln(z))f_{i,2}(z, \nu) + f_{i,4}(z, \nu) = \]

\[
P_0^*(z; \nu)(\ln(z))L_{1,1}(1/z) + 1L_{1,2}(1/z) + \]

\[
P_1^*(z; \nu)(\ln(z))L_{0,2}(1/z) + 2L_{0,3}(1/z) - \]

\[
P_3^*(z; \nu)(\ln(z)) - P_4^*(z; \nu) = \]

\[
P_0^*(z; \nu)\tilde{\varphi}_1(z, \ln(z), 1) + \]

\[
P_1^*(z; \nu)\tilde{\varphi}_2(z, \ln(z), 1) - \]

\[
P_3^*(z; \nu)\tilde{\varphi}_3(z, \ln(z), 1) - P_4^*(z; \nu) \]

According to the Lemma 8.4.2, \( y(\nu) = f_3(z; \nu) \) iz non-zero solution of the equation (37), and, according to results of the §8.4, if \( r = \sqrt{z} > 1 \), then it corresponds to

\[
\rho_4 = |\eta_0(r, 0)|^{-2} =
\]
For any \( x \) corresponds to \( \eta_0(x, 0) \), then it corresponds to
\[
\rho_3 = |\eta_0(1, 0)|^{-2} = \frac{1}{17 + 12\sqrt{2}}.
\]
So, if \( z \geq 1 \), then \( y(\nu) = f_3(z; \nu) \) corresponds to \( |\eta_0(\sqrt{z}, 0)|^{-2} \).

We take in the Lemma 8.5.1 \( n = 1, m = 3 \),
\[
a_{1,1} = (\ln(z) - L_{1.1}(1/z) + L_{1.2}(1/z),
\]
\[
a_{2,1} = (\ln(z)) - L_{0.2}(1/z) + 2L_{0.3}(1/z), a_{3,1} = \ln(z),
\]
\[
\alpha_1(\nu) = b^* (D_{\nu})^3 P_0(z; \nu), \alpha_2(\nu) = b^* (D_{\nu})^3 P_3(z; \nu),
\]
\[
\alpha_3(\nu) = -b^* (D_{\nu})^3 P_0(z; \nu), \alpha_4(\nu) = b^* (D_{\nu})^3 P_4(z; \nu).
\]
For any \( k = 0, 1, 3, 4 \) the solution \( y(\nu) = P_k(z; \nu) \) of the equation (37) corresponds to
\[
\rho_k^\nu \leq \rho_1 = |\eta_0(r, 0)|^2 =
\]
\[
(-1 - 2r - 2\sqrt{r(r + 1)})^2,
\]
where \( r = \sqrt{z} \). Therfore, in view of (153), for any \( \varepsilon_1 \in (0, 1) \) there exists a constant \( \gamma_0^\nu = \gamma_0^{\nu}(\varepsilon_1) \) such that with
\[
r_i = (\rho_1 b^3)^{1+\varepsilon_1} = (|\eta_0(r, 0)| b^3)^{2(1+\varepsilon_1)},
\]
where \( i = 1, 2, 3, 4 \), the following inequality holds:
\[
|\alpha_i(\nu)| < \gamma_0^\nu r_i = \gamma_0^{\nu}(\rho_1 b^3)^{(1+\varepsilon_1)\nu},
\]
where \( i = 1, 2, 3 \) and \( \nu \in \mathbb{N} \). Since \( n = 1 \) now, it follows that
\[
y_1(\nu) = f_3(z; \nu), X = (q) \in \mathbb{R}^1, q_{\infty}(X) = |q|,
\]
\[
y^\wedge(X) = y^\wedge(X, \nu) = q f_3(z; \nu),
\]
\[
\varphi_1(X) = q \tilde{\varphi}_1(z, \ln(z), 1)
\]
\[
\varphi_2(X) = q \tilde{\varphi}_2(z, \ln(z), 1)
\]
\[
\varphi_3(X) = q \tilde{\varphi}_3(z, \ln(z), 1).
\]
Since the solution \( y(\nu) = f_3(z; \nu) \) corresponds to \( |\eta_0(\sqrt{z}, 0)|^{-2} \), it follows from the Lemma 8.3.1 that there exist constants
\[
\gamma_1 = \gamma_1(z, \varepsilon_1) > 0, \gamma_2 = \gamma_2(z, \varepsilon_1) > 1/2
\]
such that
\[
(168) \quad \gamma_1(R_1)^{-\nu} |q| \leq \sup\{|q f_3(z; \nu + \kappa)| : \kappa = 0, ..., 3\},
\]
\[
(169) \quad \{|q f_3(z; \nu + \kappa)| \leq |q \gamma_2(R_2)^{-\nu}
\]
if
\[
(170) \quad R_1 = (\rho_1 b^3)^{1+\varepsilon_1} = (|\eta_0(r, 0)| b^3)^{2(1+\varepsilon_1)} \geq \]
The condition $R_2 > 1$ will be fulfilled, if
\begin{align}
\rho_1/(be^3) &= (|\eta_0(r,0)|^2)/(be^3) = \\
&= (-1 - 2r - 2\sqrt{r(r + 1)})^2/(be^3) > 1.
\end{align}

The condition (171) is equivalent to the condition
\begin{align}
\sqrt{be^3} < 1 + 2r + 2\sqrt{r(r + 1)}.
\end{align}

Since ($\sqrt{be^3} > 1 > 1 + 2r - 2\sqrt{r(r + 1)} = 1/(1 + 2r + 2\sqrt{r(r + 1)})$, it follows that the condition (171) is equivalent to the condition
\begin{align}
(\sqrt{be^3} - 1)^2 - 4\sqrt{z}\sqrt{be^3} < 0.
\end{align}

The last inequality is equivalent to the condition
\begin{align}
z = (-1)^{[k/2]}z > (\sqrt{be^3} - 1)^4/(16be^3) = \\
(\sqrt{be^3} - (-1)^k)^4 \times \\
(1 + (-1)^k)^4/(e^3b + 1)^{[k/2]}(16e^3b)
\end{align}

with $k = 0$, i.e. to the condition $z \in D_0(b)$. So, if $z \in D_0(b)$, then in view of (163),
\begin{align}
q^{-\alpha_4} \gamma_4^\wedge \leq \\
\max(||q\tilde{\varphi}_1(z, \ln(z), 1)||, ||q\tilde{\varphi}_2(z, \ln(z), 1)||, ||q\tilde{\varphi}_3(z, \ln(z), 1)||),
\end{align}

where $\gamma_4^\wedge$ is a positive constant, which depends from $z$ and $\varepsilon_1$, and where
\begin{align}
\alpha = \alpha(\varepsilon_1) = \\
\frac{(1 + \varepsilon_1) \ln(\rho_1(be^3)) + 2\varepsilon_1) \ln(\rho_1/(be^3))}{(1 - \varepsilon_1) \ln(\rho_1/(be^3))} = \\
\frac{(1 + \varepsilon_1) \ln((\tilde{\eta}_0(z))^2(be^3)) + 2\varepsilon_1) \ln((\tilde{\eta}_0(z))^2/(be^3))}{(1 - \varepsilon_1) \ln((\tilde{\eta}_0(z))^2/(be^3))}.
\end{align}

Since $\alpha(0) = \beta_0(z)$, where the value $\beta_0(z)$ is specified in (28), it follows that for any $\varepsilon > 0$ the inequality $\alpha(\varepsilon_1) < \beta_0(z) + \varepsilon$ holds for sufficiently small $\varepsilon_1$ and, when $z \in D_0(b)$, then, according to (29), (26), the inequality (34) holds with $\gamma_0(z, \varepsilon)$ equal to $\gamma_4^\wedge$ in (173). Let
\begin{align}
x_k \in \mathbb{R}, \text{ for } k = 1, 2, \sum_{k=1}^{2} |x_k| > 0,
\end{align}

\begin{align}
y(x_1, x_2, z, \nu) = \sum_{k=1}^{2} x_k f_{2k}(z; \nu) = \\
P_{\mu}^\gamma(z; \nu)(x_1 L_{1,1}(1/z) + x_2 L_{1,2}(1/z)) +
\end{align}
\[ P_1^*(z; \nu)(x_1L_{0,2}(1/z) + 2x_2L_{0,3}(1/z)) - P_3^*(z; \nu)x_1 - P_4^*(z; \nu) = \]
\[ P_0^*(z; \nu)\tilde{\varphi}_1(z, x_1, x_2) + P_1^*(z; \nu)\tilde{\varphi}_2(z, x_1, x_2) - P_3^*(z; \nu)x_1 - P_4^*(z; \nu)x_2. \]

If \( a \in \mathbb{N}, b \in \mathbb{N}, a \geq b, z = -a/b \), then, as it follows from the assertion of the Lemma 8.4.2, \( y(x_1, x_2, z, \nu) \) is non-zero solution of the equation (37); in view of (132) and according to results of the §8.3, it corresponds to \((\tilde{\eta}_2(z))^{-2}\).

If \( a \in \mathbb{N}, b \in \mathbb{N}, a > b, z = a/b > 1 \), then, in view of (5),
\[ y(x_1, x_2, z, \nu) = x_1f_2(z; \nu) + x_2(f_3(z; \nu) - (\ln(z))f_2(z; \nu) = \]
\[ (x_1 - x_2 \ln(z))f_2(z; \nu) + x_2f_3(z; \nu); \]

according to the results of the §8.3, if \( x_1 \neq x_2 \ln(z) \), then \( y(x_1, x_2, z, \nu) \) corresponds to \(|\eta_0(r, \pi)|^{-2}\) and, if \( x_1 = x_2 \ln(z) \), then \( y(\nu) = y(x_1, x_2, z, \nu) \) corresponds to \(|\eta_0(0, 0)|^{-2}\), where \( r = \sqrt{|z|} \). We want to consider first the case, when \( x_1 \in \mathbb{R}, x_2 \in \mathbb{Z} \setminus \{0\} \) and \( x_1 \neq x_2 \ln(z) \) now.

We apply Lemma 8.5.1 with \( n = 1, m = 3 \),
\[ a_{1,1} = x_1L_{1,1}(1/z) + x_2L_{1,2}(1/z) = \tilde{\varphi}_1(z, x_1, x_2), \]
\[ a_{2,1} = x_1L_{0,2}(1/z) + 2x_2L_{0,3}(1/z) = \tilde{\varphi}_1(z, x_1, x_2), \]
\[ a_{3,1} = x_1 = \varphi_3(z, x_1, x_2), \]
\[ \alpha_1^\wedge(\nu) = b'(D_\nu)3P_0^*(z; \nu), \alpha_2^\wedge(\nu) = b'(D_\nu)3P_3^*(z; \nu), \]
\[ \alpha_3^\wedge(\nu) = -b'(D_\nu)3P_3^*(z; \nu), \alpha_4^\wedge(\nu) = x_2b'(D_\nu)3P_4^*(z; \nu). \]

As above, for any \( \varepsilon_1 \in (0, 1) \) there exists a constant \( \gamma_0^\wedge = \gamma_0^\wedge(\varepsilon_1) \) such that with \( r_1 \) from (165) the inequality (166) holds. Since \( n = 1 \) in Lemma 8.5.1 now, it follows that
\[ y_1(\nu) = y(x_1, x_2, z, \nu), X = (q) \in \mathbb{R}^1, q_\infty(X) = \]
\[ |q|, y^\wedge(X) = y^\wedge(X, \nu) = qy(x_1, x_2, z; \nu), \]
\[ \varphi_1^*(X) = q\tilde{\varphi}_1(z, x_1, x_2) \]
\[ \varphi_2(X) = q\tilde{\varphi}_2(z, x_1, x_2) \]
\[ \varphi_3(X) = q\tilde{\varphi}_3(z, x_1, x_2) = x_1. \]

Since the solution \( y_1(\nu) = y(x_1, x_2, z, \nu) \) corresponds to \(|\eta_0(\sqrt{z}, \pi)|^{-2}\), it follows from Lemma 8.3.1 that there exist constants
\[ \gamma_1 = \gamma_1(z, \varepsilon_1) > 0, \gamma_2 = \gamma_2(z, \varepsilon_1) > 1/2 \]

such that
\[ \gamma_1(R_1)^{-\nu}|q| \leq \sup\{|qy(x_1, x_2, z, \nu + \kappa)| : \kappa = 0, ..., 3\}. \]
The condition \((179)\) is equivalent to the condition
\[
R_1 = (|\eta_0(\sqrt{z}, \pi)|^2 / be^3)^{1+\varepsilon_1} \geq R_2 = (|\eta_0(\sqrt{z}, \pi)|^{-2} / be^3)^{1-\varepsilon_1} > 1, \ \nu \in \mathbb{N}.
\]
The condition \(R_2 > 1\) will be fulfilled, if
\[
(179) \quad (|\eta_0(r, 0)|^2 / be^3) = \sqrt{be^3} < 2r - 1 + 2\sqrt{r(r-1)}.
\]
Since \((\sqrt{be^3} > 1 \geq 2r - 1 - 2\sqrt{r(r-1)} = 1/(2r - 1 + 2\sqrt{r(r-1)})\), it follows that the condition \((179)\) is equivalent to the condition
\[
(\sqrt{be^3} + 1)^2 - 4\sqrt{z}\sqrt{be^3} < 0.
\]

The last inequality is equivalent to the condition
\[
z = (-1)^{\lfloor k/2 \rfloor} z > (\sqrt{be^3} + 1)^4 / (16be^3) =
(\sqrt{be^3} - (-1)^k)4 \times (e^{3/2}b^{1/2} + (-1)^k)^4 / (e^3b + 1))^{\lfloor k/2 \rfloor} / (16e^3b)
\]
with \(k = 1\), i.e. to the condition \(z \in D_1(b)\).
So, if \(z \in D_1(b)\), then in view of \((163)\),
\[
q^{-\alpha} \gamma_4^\wedge \leq \max(||q\tilde{\varphi}_1(z, x_1, x_2)||, ||q\tilde{\varphi}_2(z, x_1, x_2)||, ||q\tilde{\varphi}_3(z, x_1, x_2)||),
\]
where \(\gamma_4^\wedge\) is a positive constant, which depends from \(z\) and \(\varepsilon_1\), and where
\[
\alpha = \alpha(\varepsilon_1) =
(1 + \varepsilon_1) \ln((2r - 1 + 2\sqrt{r(r-1)})^2(\sqrt{be^3}) + 2\varepsilon_1) \ln \left(\frac{(2r-1+2\sqrt{r(r-1)})^2}{be^3}\right)
(1 - \varepsilon_1) \ln((2r - 1 + 2\sqrt{r(r-1)})^2 / (be^3))
(1 + \varepsilon_1) \ln((\tilde{\eta}_1(z))^2(\sqrt{be^3}) + 2\varepsilon_1) \ln((\tilde{\eta}_1(z))^2 / (be^3))
(1 - \varepsilon_1) \ln((\tilde{\eta}_1(z))^2 / (be^3)).
\]
Since \(\alpha_0 = \beta_1(z)\), where the value \(\beta_1(z)\) is specified in \((28)\), it follows that for any \(\varepsilon > 0\) the inequality \(\alpha(\varepsilon_1) < \beta_1(z) + \varepsilon\) holds for sufficiently small \(\varepsilon_1\) and, when \(z \in D_1(b)\), then, according to \((29)\), \((26)\) the inequality \((35)\) holds with \(k = 1\), \(\gamma_4^\wedge(z, x_1, x_2, \varepsilon)\) equal to \(\gamma_4^\wedge\) in \((181)\). Since
\[
(\tilde{\eta}_1(z))^2 = (|\eta_0(\sqrt{z}, \pi)|^2) < (|\eta_0(\sqrt{z}, 0)|^2) = (\tilde{\eta}_0(z))^2,
\]
where \( z \geq 1 \), it follows from (28), (29), that \( \alpha_0(z) = \beta_0(z) < \beta_1(z) \); on the other hand, in view of (30), \( D_1(b) \subset D_0(b) \). Consequently, (35) is a corollary of (34), if \( x_1 = x_2 \ln(z), x_2 \in \mathbb{Z} \setminus \{0\} \).

Let \( a \in \mathbb{N}, b \in \mathbb{N}, z = -a/b = -r^2, r \geq 1 \). In view of (43) – (45)

\[
\cos(\varphi_1(r, \pi/2)) = \sin(2\varphi_2(r, \pi/2)) = 0, 
\sin(2\varphi_3(r, \pi/2)) = \frac{1}{\sqrt{1 + r^2}},
\]

\[
\sin(\varphi_1(r, \pi/2)) = -\cos(2\varphi_2(r, \pi/2)) = \cos(2\varphi_3(r, \pi/2)) = \frac{r}{\sqrt{1 + r^2}}.
\]

Therefore, in view (50),

\[
|\eta_k(r, \pi/2)|^2 = 4r^2 + 1 + 4r(r^2 + 1)^{1/2} + (-1)^k 4\sqrt{r} \left(2r \sqrt{(r^2 + 1)}/2 + \sqrt{(r^2 + 1 - r)/2}\right),
\]

where \( r \geq 1, k = 0, 1 \). First I want to check this equality directly. In view of (42),

\[
D^\gamma(r, \pi/2; \eta) = (\eta + 1)^2 + 4r \exp i\pi/\eta = 
\eta^2 + 2(1 + 2ri)\eta + 1 = (\eta + 1 + 2ri)^2 + 4r(r - i).
\]

Therefore the roots \( \eta \) of this polynomial are

\[
-1 - 2ri - 2\sqrt{r}e^{i\pi/2} \left(\sqrt{r^2 + 1} - r/2 + i \sqrt{r^2 + 1 + r/2}\right),
\]

where \( \varepsilon^2 = 1 \), and the squares of their absolute values are

\[
1 + 4r^2 + 4r\sqrt{r^2 + 1} + 
\varepsilon^4 \sqrt{r}((\sqrt{r^2 + 1} - r)/2 + 2r \sqrt{r^2 + 1 + r/2}).
\]

Since \( |\eta_1(r, \pi/2)|^2 < |\eta_0(r, \pi/2)|^2 \) it follows that \( \varepsilon \) for \( \eta_k(r, \pi/2)|^2 \) is equal to \((-1)^k\). So, (184) is checked. In view of (27),

\[
(\tilde{\eta}_2(z))^2 = 2r^2 + 1 + 2r\sqrt{r^2 + 1} + 2r^2 + 2r\sqrt{r^2 + 1} + 
2(r + \sqrt{r^2 + 1})\sqrt{2(r\sqrt{r^2 + 1} + r^2)} = 
4r^2 + 1 + 4r\sqrt{r^2 + 1} + 2(r + \sqrt{r^2 + 1})\sqrt{2(r\sqrt{r^2 + 1} + r^2)},
\]

\[
\left(2(r + \sqrt{r^2 + 1})\sqrt{2(r\sqrt{r^2 + 1} + r^2)}\right)^2 = 
8r(r + \sqrt{r^2 + 1})^3 = 8r(4r^3 + 3r^2 + (4r^2 + 1)\sqrt{r^2 + 1}),
\]
(188) \[
\left(4\sqrt{r} \left(2r\sqrt{\frac{1+r^2+r}{2}} + \sqrt{\frac{1+r^2-r}{2}}\right)\right)^2 =
\]
\[16r(2r^2\sqrt{1+r^2} + 2r^3 + (\sqrt{1+r^2})/2 - r/2 + 2r =
8r(4r^3 + 3r + (4r^2 + 1)\sqrt{1+r^2}.
\]

In view of (187), (188), (186) and (184),
\[(\tilde{\eta}_2(z))^2 = |\eta_0(r, \pi/2)|^2,
\]
where \(z = -r^2, r \geq 1\). The function \(\rho_0(r) = |\eta_0(r, \pi/2)|^2\) is a continuous increasing function which maps \([0, +\infty)\) onto \([1, +\infty)\). We want to find the inverse map \(r = r_0(\rho)\) of \([1, +\infty)\) onto \([0, +\infty)\).

In view of (184),
\[|\eta_0(r, \pi/2)|^2 \geq 1 \geq \frac{1}{|\eta_0(r, \pi/2)|^2} = |\eta_1(r, \pi/2)|^2,
\]
and
\[|\eta_k(r, \pi/2)|^2, k = 0, 1\]
are roots \(\rho\) of the trinomial
\[\rho^2 - 2(4r^2 + 1 + 4r(r^2 + 1)^{1/2})\rho + 1,
\]
moreover \(\rho = \rho_0(r) \geq 1\). Hence, for \(r = r_0(\rho)\) we have
\[r^2 + \sqrt{r^4 + r^2} = \frac{1}{2}(\rho + 1/rho) - 1 = \frac{(\rho - 1)^2}{8\rho},
\]
\[r^4 + r^2 = r^4 + \frac{(\rho - 1)^4}{64\rho^2} - r^2(\rho - 1)^2/4\rho,
\]
\[r^2 = \frac{(\rho - 1)^4/(64\rho^2)}{1 + (\rho - 1)^2/(4\rho)} =
(\rho - 1)^4/(16\rho(\rho + 1)^2),
\]
and, finally,
(190) \[r_0(\rho) = (\rho - 1)^2/(4(\rho + 1)\sqrt{\rho}).\]

We apply the Lemma 8.5.1 to the function \(y_1(\nu) = y(x_1, x_2, z, \nu)\) again, but now for \(z = -r^2\) with \(r \geq 1\). The inequality (166) holds with
\[r_i = |\eta_0(r, \pi/2)|^{2(1+\varepsilon_1)},
\]
where \(i = 1, 2, 3, 4\). Since \(y(x_1, x_2, z, \nu)\) corresponds to \(|\eta_0(\sqrt{-z}, \nu)/2)|^{-2}\), it follows from Lemma 8.3.1 that there exist constants
\[\gamma_1 = \gamma_1(z, \varepsilon_1) > 0, \gamma_2 = \gamma_2(z, \varepsilon_1) > 1/2\]
such that (176) - (177) hold with
(192) \[R_1 = (|\eta_0(\sqrt{-z}, \pi/2)|^{2}/be^3)^{1+\varepsilon_1} \geq
\]
\[ R_2 = (|\eta_0(\sqrt{-z}, \pi/2)|^2/\beta e^3)^{1-\varepsilon_1} > 1, \nu \in \mathbb{N}. \]

In view of (190) with \( \rho = e^3b \), the condition \( R_2 > 1 \) will be fulfilled, if

\[ (193) \]

\[ -z = (-1)^{[k/2]} z = r^2 > \]

\[ (r_0(e^3b))^2 = (e^3b - 1)^4/(16e^3b(e^3b + 1))^2 = \]

\[ (e^3b)^{1/2} - (-1)^k \times \]

\[ (e^3b)^{1/2} + (-1)^k/((e^3b + 1))|k/2|/(16e^3b), \]

where \( k = 2 \), i.e. if \( z \in D_2(b) \). So, if \( z \in D_2(b) \), then in view of (163),

\[ (194) \]

\[ q^{-\alpha} \gamma_4^\wedge \leq \]

\[ \max(\|q\tilde{\varphi}_1(z, x_1, x_2)\|, \|q\tilde{\varphi}_2(z, x_1, x_2)\|, \|q\tilde{\varphi}_3(z, x_1, x_2)\|), \]

where \( \gamma_4^\wedge \) is a positive constant, which depends from \( z \) and \( \varepsilon_1 \), and where, in view of (189),

\[ (195) \]

\[ \alpha = \alpha(\varepsilon_1) = \]

\[ \frac{(1 + \varepsilon_1) \ln((\tilde{\eta}_2(z))^2(\beta e^3) + +2\varepsilon_1) \ln ((\tilde{\eta}_2(z))^2/(\beta e^3))}{(1 - \varepsilon_1) \ln((\tilde{\eta}_2(z))^2/(\beta e^3))}. \]

Since \( \alpha(0) = \beta_2(z) \), where the value \( \beta_2(z) \) is specified in (28), it follows that for any \( \varepsilon > 0 \) the inequality \( \alpha(\varepsilon_1) < \beta_2(z) + \varepsilon \) holds for sufficiently small \( \varepsilon_1 \) and, when \( z \in D_2(b) \), then, according to (29), (26) the inequality (35) holds with \( k = 2, \gamma_4^\wedge(z, x_1, x_2, \varepsilon) \) equal to \( \gamma_4^\wedge \) in (181).

In previous results \( x_1 \) and \( x_2 \) were fixed. Let we consider the case when \( x_1 \) and \( x_2 \) change. Let \( a \in \mathbb{N}, b \in \mathbb{N}, z = -a/b = -r^2, r \geq 1, \) and let \( z \in D_2(b) \). We apply Lemma 8.5.1 with \( n = 2, m = 2, \)

\[ a_{1,1} = L_{1,1}(1/z), a_{1,2} = L_{1,2}(1/z), \]

\[ a_{2,1} = L_{0,2}(1/z), a_{2,2} = 2L_{0,3}(1/z), \]

\[ \alpha_1^\wedge(\nu) = b^\nu(D_\nu)^3P_0^*(z; \nu), \]

\[ \alpha_2^\wedge(\nu) = b^\nu(D_\nu)^3P_3^*(z; \nu), \]

\[ \alpha_3^\wedge(\nu) = b^\nu(D_\nu)^3P_4^*(z; \nu), \]

\[ y_1(\nu) = b^\nu(D_\nu)^3f_{0,2}^\nu(z, \nu) = b^\nu(D_\nu)^3P_0^*(z; \nu)L_{1,1}(1/z) + b^\nu(D_\nu)^3P_1^*(z; \nu)L_{0,2}(1/z) - b^\nu(D_\nu)^3P_3^*(z; \nu) = \]

\[ \alpha_1^\wedge(\nu)a_{1,1} + \alpha_2^\wedge(\nu)a_{2,1} - \alpha_3^\wedge(\nu), \]

\[ y_2(\nu) = b^\nu(D_\nu)^3f_{0,4}^\nu(z, \nu) = b^\nu(D_\nu)^3P_0^*(z; \nu)L_{1,2}(1/z) + b^\nu(D_\nu)^3P_1^*(z; \nu)2L_{0,3}(1/z) - b^\nu(D_\nu)^3P_4^*(z; \nu) = \]

\[ \alpha_1^\wedge(\nu)a_{1,2} + \alpha_2^\wedge(\nu)a_{2,2} - \alpha_3^\wedge(\nu), \]

\[ (196) \]

\[ X = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2, \]

\[ (197) \]

\[ \frac{1}{2}(|x_1| + |x_2|) \leq q_\infty(X) := \max(|x_1|, |x_2|) \leq |x_1| + |x_2| \]
where \( i = 1, 2 \),

\[
\alpha_0(X, \nu) = \alpha_3^*(\nu)x_1 + \alpha_4^*(\nu)x_2.
\]

According to the Lemma 8.4.2, \( y(\nu) = \{P^*_i(z; \nu)\} \), where \( i = 0, 1, 3, 4 \) is non-zero solution of the equation (37); hence, in view of (184) it correspond to some \( \rho_i^* \leq (\tilde{\eta}_2(z))^2 = |\eta_0(r, \pi/2)|^2 \). Therefore the inequality (166) holds with \( r_i \) specified in (191).

In view of (31), if \( z \leq -1 \), \( k = 0, 1 \), \( s > 0 \) then

\[
(-1)^k L_{k,s}(1/z) > 0 = (-kz + 1) \sum_{n=1}^{+\infty} (1/z)^n/n^s < 0.
\]

Therefore, according to the Lemma 8.4.2,

\[
f_{0,2}^\nu(z, \nu) = P_0^*(z; \nu)L_{1,1}(1/z) + P_1^*(z; \nu)L_{0,2}(1/z) - P_3^*(z; \nu),
\]

\[
f_{0,4}^\nu(z, \nu) = P_0^*(z; \nu)L_{1,2}(1/z) + P_1^*(z; \nu)2L_{0,3}(1/z) - P_4^*(z; \nu)
\]

compose the basis of the space \( V = V_{m^*;2}^\nu = V_{m^*;2}^\wedge \) from the Lemma 8.3.1. Let

\[
y^*(X) = y^*(X, z, \nu) = y(x_1, x_2, z, \nu) = x_1f_2(z, \nu) + x_2f_4(z, \nu)
\]

with \( z \in D_2(b), \nu \in \mathbb{N} \) and \( X \) in (196). We apply Lemma 8.3.2 now. Then we have \( r = 2, k_3(V) = k_4(V) = 2 \). Therefore, according to the Lemma 8.3.2, for any \( \varepsilon_1 \in (0, 1) \) there exist \( C_7 = C_7(z, \varepsilon_1) > 0 \) and \( C_8 = C_8(z, \varepsilon_1) > 0 \) such that

\[
C_8(R_1be^3)^{-\nu}q_\infty(X) \leq \sup\{|y^*(X, z, \nu + \kappa)| : \kappa = 0, ..., 3\},
\]

\[
\{|y^*(X, z, \nu)| \leq |q_\infty(X)C_7(R_2be^3)^{-\nu}
\]

with \( R_1 \) and \( R_2 \) in (192). In view of (198), and (204),

\[
y^\wedge(X, z, \nu) = (D_\nu)^3y^*(X, z, \nu).
\]

Therefore, in view of (153), there exist constants

\[
\gamma_1 = \gamma_1(z, \varepsilon_1) > 0, \ \gamma_2 = \gamma_2(z, \varepsilon_1) > 1/2
\]
such that

\[(207)\]
\[\gamma_1(R_1 b e^3)^{-\nu} q_\infty(X) \leq \sup\{|y^\wedge(X, z, \nu + \kappa)| : \kappa = 0, \ldots, 3\},\]

\[(208)\]
\[\{\text{verty}^*(X, z, \nu)| \leq |q_\infty(X)(R_2 b e^3)^{-\nu}\]

with \(R_1\) and \(R_2\) in (192). So, if \(z \in D_2(b)\), then in view of (163),

\[(209)\]
\[
\max(\|\tilde{\varphi}_1(z, x_1, x_2)\|, \|\tilde{\varphi}_2(z, x_1, x_2)\|)(|x_1| + |x_1|)^\alpha \geq \\
\max(\|\tilde{\varphi}_1(z, x_1, x_2)\|, \|\tilde{\varphi}_2(z, x_1, x_2)\|)q_\infty(X) \geq \gamma_4^\wedge
\]

where \(\gamma_4^\wedge\) is a positive constant, which depends from \(z\) and \(\varepsilon_1\), and \(\alpha = \alpha(z, \varepsilon_1)\) is specified in (195). In view of (28) and (29), \(\alpha(0) = \beta_2(z) = \alpha_2(z)\); where \(\beta z_2(\varepsilon)\) is specified in (28); therefore it follows that for any \(\varepsilon > 0\) the inequality \(\alpha(\varepsilon_1) < \alpha_2(z) + \varepsilon\) holds for sufficiently small \(\varepsilon_1\) and, if \(z \in D_2(b)\), then (36) holds with \(k = 2, \gamma_0^\wedge(z, \varepsilon)\) equal to \(\gamma_4^\wedge\) in (207).

Let, finally, Let \(a \in \mathbb{N}, b \in \mathbb{N}, z = a/b = r^2, r \geq 1,\) and let \(z \in D_2(b)\). We apply Lemma 8.5.1 with \(n = 2, m = 2\) again. Then the inequality (166) holds with \(\gamma_1\) in (165). If \(z > 1\), then \(f_{0,2}^\wedge(z, \nu)\) and \(f_{0,4}^\wedge(z, \nu)\) compose the basis of the space \(V = V_{m,3}^\wedge = V_{m,3}^\wedge \oplus V_{m,4}^\wedge\) from the Lemma 8.3.1; \(\dim\mathbb{C}(V_{m,3}^\wedge) = 1\) for \(k = 1, 2, k_3(V) = 3, k_4(V) = 4\) If \(z = 1\), then \(f_{0,2}^\wedge(z, \nu)\) and \(f_{0,4}^\wedge(z, \nu)\) compose the basis of the subspace \(V = V_{m,2}^\wedge \oplus V_{m,4}^\wedge\) from the Lemma 8.3.1; \(\dim\mathbb{C}(V_{m,2}^\wedge) = 2, \dim\mathbb{C}(V_{m,3}^\wedge) = 1, k_3(V) = 2, k_4(V) = 3\). In both cases

\[\rho_{k_3(V)} = (\eta_0(\sqrt{z}, \pi) - 2, \rho_{k_4(V)} = (\eta_0(\sqrt{z}, 0) - 2,\]

the inequalities (205) and (207) hold with \(R_1\) in (170) and \(R_2\) in (178). Hence, if \(z \in D_1(b)\), then (209) holds with a positive constant \(\gamma_4^\wedge\), which depends from \(z\) and \(\varepsilon_1\), and with

\[(210)\]
\[\alpha = \alpha(\varepsilon_1) =
\]
\[
\frac{(1 + \varepsilon_1) \ln((\eta_0(z))^2/(b e^3)) - (1 - \varepsilon_1) \ln((\eta_1(z))^2/(b e^3)) + (1 + \varepsilon_1) \ln((\eta_0(z))^2/(b e^3)) - (1 - \varepsilon_1) \ln((\eta_1(z))^2/(b e^3))}{(1 - \varepsilon_1) \ln((\eta_1(z))^2/(b e^3))}
\]

In view of (29), we have the equality \(\alpha(0) = \alpha_1(z)\); therefore for any \(\varepsilon > 0\) the inequality \(\alpha(\varepsilon_1) < \alpha_1(z) + \varepsilon\) holds for sufficiently small \(\varepsilon_1\) and, if \(z \in D_1(b)\), then (36) holds with \(k = 1, \gamma_0^\wedge(z, \varepsilon)\) equal to \(\gamma_4^\wedge\) in (207).

The Theorem 2 is proved.

§8.6. Corrections in the previous my papers.

The last equation in §6.5 of Part 6 must have the form

\[(\nu+1)^3y(1; \nu+1) + \nu^3y(\nu-1) = (17\nu^3 + 51\nu^2 + 27\nu + 5)y(\nu),\]

instead of

\[(\nu+1)^3y(1; \nu+1) + \nu^3y(\nu-1) = (17\nu^3 + 51\nu^2 + 27\nu + 5)y(\nu)(34\nu^3 + 85\nu^2).\]
On the page 6 in [63] must stand

\[ \tilde{\eta}_2(z) = \sqrt{|z|} + \sqrt{|z| + 1} + \sum_{k=0}^{1} \sqrt{|z|^2 + |z| + (-1)^k \sqrt{|z|}} = \]

\[ \sqrt{|z|} + \sqrt{|z| + 1} + \sqrt{2(\sqrt{|z|^2 + |z| + |z|})}, \]

instead of what is written there. On the page 8 in [63] must stand

\[ \beta_2 = \alpha_2 = 1 + \frac{6}{2 \ln \left(1 + \sqrt{2} + \sqrt{\sqrt{2} + 1 + \sqrt{\sqrt{2} - 1}}\right) - 3} = 106, 00187... \]

instead of what is written there. On the page 6 in [63] must stand

\[ \tilde{\eta}_i(z) = \left( \sum_{k=0}^{1} \sqrt{|z| + k(-1)^i} \right)^2 = \]

\[ 2 \sqrt{|z|} + (-1)^i + 2 \sqrt{|z| + (-1)^i \sqrt{|z|}} \]

for \( i = 0, 1 \), instead of what is written there.

In the formulation of the Theorem 2 in the [63] must stand \( a = b z \in \mathbb{Z} \) instead of \( b z \in \mathbb{Z} \).

The equality (99) in [62] must have the form

\[ f_{l,2+2j}(z, \nu) = \]

\[ \sum_{i=1}^{2+l} \left( \sum_{t=1}^{\nu} \sum_{k=0}^{2+l} \alpha_{l,i,k,\nu} z^k z^{-t-k} \binom{i + j - 1}{j} (t + k)^{-i-j} \right) = \]

\[ \sum_{i=1}^{2+l} \left( \sum_{k=0}^{2+l} \alpha_{l,i,k,\nu} z^k \left( \sum_{t=1}^{\infty} z^{-t-k} \binom{i + j - 1}{j} (t + k)^{-i-j} \right) \right) = \]

\[ \sum_{i=1}^{2+l} \left( \sum_{k=0}^{2+l} \alpha_{l,i,k,\nu} z^k \left( \binom{i + j - 1}{j} L_{i+j}(1/z) - \sum_{t=1}^{k} z^{-t} \binom{i + j - 1}{j} (t)^{-i-j} \right) \right) = \]

\[ \left( \sum_{i=1}^{2+l} \alpha_{i,j}(z; \nu) \binom{i + j - 1}{j} L_{i+j}(1/z) \right) - \beta_{i,j}(z; \nu) = \]

\[ \left( \sum_{i=1}^{\infty} \alpha_{i,j}(z; \nu) \binom{i + j - 1}{j} L_{i+j}(1/z) \right) - \beta_{i,j}(z; \nu), \]

instead of what is written there. The equality (102) in [62] must have the form

\[ \beta_{i,j}(z; \nu) = \]

\[ \sum_{i=1}^{2+l} \left( \sum_{k=0}^{\nu} \alpha_{l,i,k,\nu} \left( \sum_{t=1}^{\nu} \binom{i + j - 1}{j} z^{-k-t}(t)^{-i-j} \right) \right), \]
The expression for $\alpha_k$ on the page 6 in [63] must have a form

$$\alpha_k = \beta_k + \frac{(1 - (-1)^k)(\ln(\tilde{\eta}_0(z)/\tilde{\eta}_1(z)))}{\ln((\tilde{\eta}_1(z))^2/e^{3b})},$$

instead of what is written there.

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