TWO RESULTS ON CENTRALISERS OF NILPOTENT ELEMENTS

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INTRODUCTION

Let $X$ and $Y$ be commuting nilpotent endomorphisms of a finite-dimensional vector space $V$ over a field $\mathbb{k}$. In [4, Sect. 3], McNinch shows that, for all but finitely many points $(a : b) \in \mathbb{P}_{\mathbb{k}}^1$, both $X$ and $Y$ belong to the nilpotent radical of the centraliser of $aX + bY$ in $GL(V)$. (There is an additional restriction on $aX + bY$ if $\text{char } \mathbb{k} =: p > 0$; namely, $(aX + bY)^{p-1}$ has to be zero.) From this, he deduces a similar result for commuting nilpotent elements of arbitrary semisimple Lie algebras if $\text{char } \mathbb{k}$ is sufficiently large, see [4, Theorem 26 and Prop. 28]. However, the proof for $GL(V)$ is rather tedious. It requires lengthy manipulations with Jordan normal forms of $X$ and $Y$ and consideration of nilpotent elements over the field $\mathbb{k}(t)$.

The goal of this note is two-fold. First, we provide a very short alternative proof of McNinch’s results if $\mathbb{k}$ is algebraically closed and $p = 0$ or sufficiently large. We use only standard properties of $\mathfrak{sl}_2$-triples and centralisers of nilpotent elements, and work with an arbitrary simple Lie algebra. Second, we characterise the nilpotent elements $e$ such that $G \cdot e$ is the largest nilpotent orbit meeting the centraliser of $e$. Such nilpotent elements (orbits) are said to be self-large. In the last section, we discuss some problems related to self-large orbits.

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1. A SHORT PROOF OF McNINCH’S RESULT

Throughout, $G$ is a connected simple algebraic group over $\mathbb{k}$, where $\mathbb{k}$ is algebraically closed and $\text{char } \mathbb{k} = 0$, and $\mathfrak{g} = \text{Lie } G$. Write $\mathfrak{g}_e$ for the centraliser of $x \in \mathfrak{g}$ and $\mathcal{N}$ for the nilpotent cone in $\mathfrak{g}$. The nilpotent radical of a Lie algebra $\mathfrak{q}$ is denoted by $\mathfrak{q}^u$.

Let us start with a reformulation of the McNinch’s result. Given commuting (non-proportional) elements $x, y \in \mathcal{N}$, we consider the “commutative nilpotent” plane $\mathcal{P} = \mathbb{k}x + \mathbb{k}y \subset \mathcal{N} \subset \mathfrak{g}$. It is then claimed that, for almost all $e = ax + by \in \mathcal{P}$, $x$ and $y$ belong to $(\mathfrak{g}_e)^u$. Let us give a more precise meaning to the words “almost all”. Since the closure of $G \cdot \mathcal{P}$ is irreducible, there is a unique nilpotent $G$-orbit, $\mathcal{O}$, such that $\mathcal{O} \cap \mathcal{P}$ is dense in $\mathcal{P}$. So we will actually require that $e \in \mathcal{O}$.

Theorem 1.1. Suppose \( e, x \in \mathcal{N}, \left[e, x\right] = 0 \), and the intersection of the orbit \( G.e \) with \( \mathcal{P} = \mathbb{k}e + \mathbb{k}x \) is dense in \( \mathcal{P} \). Then \( x \in \left( g_e \right)^u \).

Before giving a proof, we fix some notation and state an auxiliary result. Let \( \{e, h, f\} \) be an \( \mathfrak{sl}_2 \)-triple containing \( e \) and \( g = \bigoplus_{i \in \mathbb{Z}} g(i) \) the corresponding \( \mathbb{Z} \)-grading of \( g \). Here \( g(i) \) is the \( i \)-eigenspace of \( \text{ad} \ h \). In particular, \( g(0) = g_h \). Then \( p = \bigoplus_{i \geq 0} g(i) =: g_{\geq 0} \) is a parabolic subalgebra and \( p^u = g_{\geq 1} \). Set \( g_e(i) = g(i) \cap g_e \). As is well known, \( g_e = \bigoplus_{i \geq 0} g_e(i) \) and \( g_e(0) \) is a Levi subalgebra of \( g_e \). Furthermore, \( g_e(0) = g_e \cap g_f \) [1, Ch. 3]. Let \( \alpha_h : \mathbb{k}^X \to G \) be the one-parameter subgroup such that \( \alpha_h(t) \cdot y = t^iy \) for any \( y \in g(i) \).

The following observation is extracted from the proof of Proposition 1.2 in [6].

Lemma 1.2 (Premet). If \( x_0 \in g_e(0) \) is nonzero and nilpotent, then \( e + x_0 \) and \( e \) are not conjugate. Moreover, \( e \) lies in the closure of \( G.e + (e + x_0) \).

Proof. For the sake of convenience, we recall Premet’s argument. Since \( x_0 \in g_e(0) \) is nilpotent, there is an \( \mathfrak{sl}_2 \)-triple \( \{x_0, h', y\} \) contained in \( g_e(0) \). It follows that \( \{e + x_0, h + h', f + y\} \) is also an \( \mathfrak{sl}_2 \)-triple. Being a member of an \( \mathfrak{sl}_2 \)-triple, \( h' \) lies in \( [g(0), g(0)] \). Therefore \( h \) and \( h' \) are orthogonal with respect to the Killing form, \( \kappa \), on \( g \) and hence \( \kappa(h + h', h + h') > \kappa(h, h) \). It follows that \( h \not
\cdot G \not\quad h + h' \) and hence \( e \not
\cdot G \not e + x_0 \) [1]. Finally, we have

\[
\alpha_{h + h'}(t) \alpha_h(-t)(e + x_0) = e + t^2x_0,
\]

which implies that \( e \in \overline{G.e + (e + x_0)} \).

Proof of Theorem 1.1. Using the above notation, write \( x = x_0 + x_1 + \ldots \), where \( x_i \in g(i) \). Our goal is to prove that \( x_0 = 0 \). Since \( e \in g(2) \), we have \( [e, x_i] = 0 \) for all \( i \).

Consider the commutative nilpotent planes \( \mathcal{P}_t = \alpha_h(t) : \mathcal{P} \) for \( t \in \mathbb{k}X \). Clearly, \( \mathcal{P}_t \) is spanned by \( e \) and \( \alpha_h(t) \cdot x = x_0 + tx_1 + t^2x_2 + \ldots \). The limit \( \lim_{t \to 0} \mathcal{P}_t \) exists in the Grassmannian of 2-planes in \( g \) and for \( x_0 \neq 0 \) it is equal to \( \mathcal{P}_0 := \mathbb{k}e + \mathbb{k}x_0 \). We thus obtain another commutative plane, \( \mathcal{P}_0 \). Furthermore, \( \mathcal{P}_0 \subset \mathcal{N} \) (as the limit of \( \{\mathcal{P}_t\} \)), hence \( x_0 \) is nilpotent.

By Lemma 1.2, \( e + ax_0 \) is not conjugate to \( e \) for every \( a \neq 0 \). Hence \( G.e \cap \mathcal{P}_0 \) is not dense in \( \mathcal{P}_0 \). Since \( \lim_{t \to 0} \mathcal{P}_t = \mathcal{P}_0 \), we conclude that \( G.e \cap \mathcal{P}_t \) is not dense in \( \mathcal{P}_t \) for almost all \( t \in \mathbb{k}X \), and because all \( \mathcal{P}_t \) are \( G \)-conjugate, this is also true for \( \mathcal{P} = \mathcal{P}_1 \). This contradiction shows that \( x_0 = 0 \), i.e., \( x \in \left( g_e \right)^u \).

Remark 1.3. a) Under the assumptions of the theorem, we proved that \( x_0 = 0 \). One may ask whether it is true that \( x_1 = 0 \) as well. In general, the answer is negative. This follows from Proposition 2.4 below.

b) The previous proof certainly works, if \( \text{char} \ k \) is sufficiently large. E.g. if \( \text{char} \ k > 4h - 1 \), where \( h \) is the Coxeter number of \( g \).
Recall that $e \in \mathcal{N}$ or $G \cdot e$ is said to be even if the eigenvalues of $\text{ad } h$ are even; it is called distinguished if $g_e(0) = \{0\}$. It is known that “distinguished” implies “even” [1, Thm. 8.2.3].

Following Premet [6], we say that $e$ is almost distinguished if $g_e(0)$ is toral (= Lie algebra of a torus). Let $\mathcal{N}(g_e)$ denote the set of nilpotent elements of $g_e$. It is easily seen that $\mathcal{N}(g_e) = \mathcal{N}(g_e(0)) \times (g_e)_{\geq 1} = \mathcal{N}(g_e(0)) \times (g_e)^u$. Therefore $(g_e)^u = \mathcal{N}(g_e)$ if and only if $e$ is almost distinguished.

Definition 1. A nilpotent element $e$ (orbit $G \cdot e$) is said to be self-large if $G \cdot e \cap g_e$ is dense in $\mathcal{N}(g_e)$. In other words, this means that $G \cdot e$ is the largest nilpotent orbit meeting $g_e$.

Our consideration of self-large orbits was motivated by attempts to better understand Premet’s results on “nilpotent commuting variety” [6, Sect. 1] and generalise it to some other situations.

In this section, we give a characterisation of self-large elements. The answer is being given in terms of the $\mathbb{Z}$-grading associated with an $\mathfrak{sl}_2$-triple $\{e, h, f\}$.

Theorem 2.1. Suppose $e \in \mathcal{N}$, and let $g_e = \bigoplus_{i \geq 0} g_e(i)$ be the $\mathbb{N}$-grading determined by $h$. Then $e$ is self-large if and only if $g_e(0)$ is toral and $g_e(1) = 0$.

For future use, we record the following simple assertion:

\begin{equation}
(2.1) \quad \text{ad } f : g_e(1) \to g_f(-1) \text{ is a bijection, and the inverse map is just ad } e.
\end{equation}

From this one readily deduce the following

Lemma 2.2. For any nonzero $\xi \in g_f(-1)$ there is $\eta \in g_f(-1)$ such that $\kappa(e, [\xi, \eta]) \neq 0$. In particular, $(\xi, \eta) \mapsto \kappa(e, [\xi, \eta])$ is a non-degenerate skew-symmetric $g_e(0)$-invariant bilinear form on $g_f(-1)$.

Lemma 2.3. Assume that there is $z \in g_f(-1)$ such that $[z, [z, e]] \neq 0$. Then $[z, e] \in g_e(1)$ and the orbit $G \cdot (e + [z, e])$ is larger than $G \cdot e$.

Proof. Set $v_z = [z, e]$. By Eq. (2.1), $v_z \in g_e(1)$ and also $z = [v_z, f]$. Then

\[
\exp(-z)(e + v_z) = e + v_z - [z, e + v_z] + \frac{1}{2}[z, [z, e + v_z]] + \ldots
\]

\[
= e - [z, v_z] + \frac{1}{2}[z, v_z] + \ldots = e - \frac{1}{2}[z, v_z] + \text{(terms in } g_{\leq -1}).
\]

Here the element $[z, v_z]$ lies in $g(0)$ and an easy computation shows that it commutes with $e$. Hence it also commutes with $f$. Thus, we have shown that $\exp(-z)(e + v_z) \in e + \mathfrak{p}^-$, where $\mathfrak{p}^- = g_{\leq 0}$, and the component of degree zero lies in $g_e(0) = g_f(0)$.

Set $N = \exp(g_{\leq -2})$. It is a unipotent group and $e + \mathfrak{p}^-$ is an $N$-stable subvariety of $g$. There is an isomorphism of $N$-varieties

\[
e + \mathfrak{p}^- \simeq N \times (e + g_f),
\]
where the $N$-action on $e + g_f$ is trivial, and $N$ acts on itself by left translations. In other words, for every $y \in p^-$, the $N$-orbit of $e + y$ is isomorphic to $N$ and contains a unique element from $e + g_f$. For regular nilpotent elements, this is implicit in [3, Sect. 4]. A general proof is given by Katsylo [2, § 5]. Let $\psi(e+y)$ denote the unique point in $N \cdot (e + y) \cap (e + g_f)$. It is important that the $N$-action does not affect the zero component of $y$, $y_0$, whenever $y_0 \in g_e(0)$. It follows that

\[(2.2) \quad \psi(\exp(-z)(e + v_z)) = e - \frac{1}{2}[z, v_z] + \text{terms in } (g_f)_{\leq -1}\]

The affine subspace $e + g_f$ is the transverse (or Slodowy) slice to $G \cdot e$ at $e$. It follows from [7, 7.4] that $G \cdot e \cap (e + g_f) = \{ e \}$. If $[z, v_z] \neq 0$, then Eq. (2.2) shows that $G \cdot (e + v_z) \cap (e + g_f)$ contains a point different from $e$, which implies that $e + v_z \notin G \cdot e$. Since $G \cdot (e + v_z) \supset e + k^x v_z$ (cf. Proof of Lemma 1.2), we actually have $e \in G \cdot (e + v_z)$. \hfill $\square$

**Proof of Theorem 2.1.** (a) The sufficiency is easy. If $g_e(0)$ is toral and $g_e(1) = 0$, then $N \cdot (e) = (g_e)^u \subset g_{\geq 2}$. Since $P \cdot e$ is dense in $g_{\geq 2}$, the assertion follows.

(b) Let us prove the necessity. If $g_e(0)$ is not toral, then there is a nilpotent element $x_0 \in g_e(0)$. Then $\tilde{e} = e + x_0 \in N \cdot (e)$ and $\tilde{e} \notin G \cdot e$, see Lemma 1.2.

In the rest of the proof we assume that $g_e(0)$ is toral. If $g_e(1) \neq 0$, then our goal is to find an element $v \in g_e(1)$ such that $e + v$ lies in a larger orbit. By Lemma 2.3, it suffices to find $z \in g_f(-1)$ such that $[z, [z, e]] \neq 0$.

**Claim 1.** The space of $h$-fixed vectors in $g_f(-1)$ is trivial.

For, consider the semisimple Lie algebra $s = [l, l]$, where $l = g^b$. Then $e, h, f \in s$ and $e$ is distinguished as element of $s$. In particular, $e$ is even in $s$. Since $l = h \oplus s$ and $h \subset g(0)$, we have $0 = s(-1) = l(-1) = g(-1)^b$.

It follows from Claim 1 and Lemma 2.2 that the weight decomposition of $g_f(-1)$ with respect to $h = g_e(0)$ can be written as

$$g_f(-1) = \bigoplus_{\gamma \in A} (V_{\gamma} \oplus V_{-\gamma}),$$

where $A$ is a subset of $\mathfrak{x}(\mathfrak{h})$ such that $A \cap (-A) = \emptyset$.

**Claim 2.** There are $\mu \in A$ and weight vectors $\xi \in V_{\mu}$, $\eta \in V_{-\mu}$ such that $\kappa(e, [\xi, \eta]) \neq 0$.

By Lemma 2.2, there are some $\tilde{\xi}, \tilde{\eta} \in g_f(-1)$ such that

$$\kappa(e, [\tilde{\xi}, \tilde{\eta}]) \neq 0. \quad (2.3)$$

Let $\tilde{\xi} = \sum_{\gamma \in A} a_{\gamma} \xi_{\gamma}, a_{\gamma} \in k$, be the weight decomposition, and likewise for $\tilde{\eta}$. Substituting this to Eq. (2.3), one readily finds that for some $\gamma$, the components $\xi_{\gamma}$ and $\eta_{-\gamma}$ satisfies the required property.

Having found such weight vectors, we take $t \in h$ such that $[t, \xi] = \xi$ and $[t, \nu] = -\nu$. Then $\kappa([[e, \xi + \eta], \xi + \eta], t) = 2\kappa(e, [\xi, \eta]) \neq 0$, which shows that $[[e, \xi + \eta], \xi + \eta] \neq 0$. Hence $z = \xi + \eta$ is a required element. \hfill $\square$
Notice that in order to construct a suitable element \( v \in \mathfrak{g}_e(1) \), we take the sum of two different weight vectors: \( v = [e, \xi] + [e, \eta] \). The reason is that a single weight vector is not suitable, as shows the following

**Proposition 2.4.** Suppose \( \mathfrak{h} = \mathfrak{g}_e(0) \) is toral and \( v \in \mathfrak{g}_e(1) \) is an \( \mathfrak{h} \)-weight vector. Then \( e + v \in \mathcal{G}.e \).

**Proof.** Let \( z \in \mathfrak{g}_f(-1) \) be the unique element such that \( v = [z, e] \). Then \( [z, v] \in \mathfrak{g}(0) \) and \([z, v, e] = [[z, e], v] = 0\). Thus, \([z, v] \in \mathfrak{h} \) is semisimple. Let \( \gamma \in \mathfrak{X}(\mathfrak{h}) \) be the \( \mathfrak{h} \)-weight of \( v \). Then \( \gamma \neq 0 \) (Claim 1), \( z \) has the same weight, and the weight of \([z, v]\) equals \( 2\gamma \). If follows that \([z, v]\) is nilpotent as well. Hence \([z, v] = 0\). Therefore \( \exp(z) \cdot e = e + [z, e] = e + v \). \( \Box \)

**Example 2.5.** We describe the almost distinguished orbits in all simple Lie algebras and point out the self-large ones among them.

1. For \( \mathfrak{g} = \mathfrak{g}(V) \) classical, the nilpotent orbits are parametrized via partitions of \( n = \dim V \). If \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s) \) is a partition of \( n \), then \( \mathcal{O}_\lambda \) stands for the corresponding orbit. For \( e \in \mathcal{O}_\lambda \), a description of \( \mathfrak{g}_e(0) \) via \( \lambda \) is due to Springer and Steinberg, see e.g. [1, Thm. 6.1.3]. This allows us to quickly find all almost distinguished orbits.

\( (a) \) \( \mathfrak{g} = \mathfrak{sl}(V) \). Here \( \lambda \) is an arbitrary partition and \( \mathcal{O}_\lambda \) is almost distinguished if and only if all parts of \( \lambda \) are distinct. Furthermore, \( \mathfrak{g}_e(1) \neq 0 \) if and only if \( \lambda_i = \lambda_{i+1} + 1 \) for some \( i < s \) [5, Prop. 3.4]. Thus, the self-large orbits are those satisfying the property \( \lambda_i - \lambda_{i+1} \geq 2 \) for each \( i < s \).

\( (b) \) \( \mathfrak{g} = \mathfrak{so}(V) \). Here each even part of \( \lambda \) must occur an even number of times. The orbit \( \mathcal{O}_\lambda \) is almost distinguished if and only if \( \lambda \) has no even parts and each odd part occurs at most twice. Such orbits are even, hence self-large.

\( (c) \) \( \mathfrak{g} = \mathfrak{sp}(V) \). Here each odd part of \( \lambda \) must occur an even number of times. The orbit \( \mathcal{O}_\lambda \) is almost distinguished if and only if \( \lambda \) has no odd parts and each even part occurs at most twice. Such orbits are even, hence self-large.

2. For \( \mathfrak{g} \) exceptional, we only indicate the almost distinguished orbits with non-trivial toral part \( \mathfrak{g}_e(0) \). Such orbits exist only in type \( \mathbf{E} \), see Table 1.

<table>
<thead>
<tr>
<th>( \mathfrak{g} )</th>
<th>label</th>
<th>diagram</th>
<th>( \mathfrak{g}_e(0) )</th>
<th>( \dim \mathfrak{g}_e(1) )</th>
<th>( \dim \mathfrak{g}_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{E}_8 )</td>
<td>( D_7(a_1) )</td>
<td>2–0–0–2–0–0–2</td>
<td>( t_1 )</td>
<td>0</td>
<td>26</td>
</tr>
<tr>
<td>( \mathbf{E}_6(a_1)+\mathbf{A}_1 )</td>
<td>2–0–1–0–1–0–1</td>
<td>( t_1 )</td>
<td>2</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>( \mathbf{D}_7(a_2) )</td>
<td>1–0–1–0–1–0–1</td>
<td>( t_1 )</td>
<td>2</td>
<td>32</td>
<td></td>
</tr>
</tbody>
</table>
Almost distinguished orbits in $E_n$, cont.

<table>
<thead>
<tr>
<th>Group</th>
<th>Equation</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_7$</td>
<td>$E_6(a_1)$</td>
<td>$t_1$</td>
<td>0, 15</td>
</tr>
<tr>
<td>$A_4 + A_1$</td>
<td>$0-1-0-1-0$</td>
<td>$t_1$</td>
<td>4, 29</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$D_5$</td>
<td>$t_1$</td>
<td>0, 10</td>
</tr>
<tr>
<td>$D_5(a_1)$</td>
<td>$1-1-0-1-1$</td>
<td>$t_1$</td>
<td>2, 14</td>
</tr>
<tr>
<td>$A_4 + A_1$</td>
<td>$1-1-0-1-1$</td>
<td>$t_1$</td>
<td>2, 16</td>
</tr>
<tr>
<td>$D_4(a_1)$</td>
<td>$0-0-2-0$</td>
<td>$t_2$</td>
<td>0, 20</td>
</tr>
</tbody>
</table>

Remark. It turns out, a posteriori, that for $g \neq sl_n$, every self-large orbit is even.

3. PROBLEMS AND EXAMPLES

Results of Section 2 show that there is a hierarchy of nilpotent $G$-orbits:

\[
\text{distinguished orbits} \subset \text{self-large orbits} \subset \text{almost distinguished orbits},
\]

where all inclusions are proper.

Lemma 3.1. Suppose $e, e' \in \mathcal{N}$ are self-large and $[e, e'] = 0$. Then $e \sim_G e'$.

Proof. Consider an $sl_2$-triple containing $e$ and the related $\mathbb{Z}$-grading, as above. Since $e' \in \mathcal{N}(g_e) = (g_e)u$ and $g_e(1) = 0$, we have $e' \in g_{\geq 2} = \overline{D}e$. The assertion follows by the symmetry of $e$ and $e'$.

Below we discuss several related problems.

Since $\mathcal{N}(g_e)$ is irreducible, there is always a unique maximal nilpotent orbit meeting $g_e$. That is, we obtain the mapping $\mathcal{D} : \mathcal{N}/G \to \mathcal{N}/G$ which assigns the dense $G$-orbit in $G \mathcal{N}(g_e)$ to $G \cdot e$.

Problem 1. Determine explicitly $\mathcal{D}$, i.e., for every $G \cdot e \in \mathcal{N}/G$ describe the orbit $\mathcal{D}(G \cdot e)$.

For classical Lie algebras, one should expect a recipe in terms of partitions. However, this seems to be a non-trivial task. Note that if $\mathcal{O}_{\text{min}} \subset \mathcal{N}$ is the minimal nonzero orbit and $v \in \mathcal{O}_{\text{min}}$, then $g_v$ contains the nilpotent radical of a Borel subalgebra. Hence, for any $e \in \mathcal{N}$, the unique minimal nonzero nilpotent orbit meeting $g_e$ is always $\mathcal{O}_{\text{min}}$.

Problem 2. Describe the image of $\mathcal{D}$.
By definition, the self-large orbits are those having the property that $\mathcal{D}(O) = O$. In particular, they belong to $\text{Im} \mathcal{D}$. Are there some other orbits? Equivalently, is it true that $\mathcal{D}^2 = \mathcal{D}$? At least, my direct computations of $\mathcal{D}$ for small ranks provide only self-large orbits in $\text{Im} \mathcal{D}$.

**Problem 3.** Describe all nilpotent $G$-orbits meeting $\mathfrak{g}_e$.

The answer should be helpful for better understanding the structure of the nilpotent commuting variety. By Lemma 3.1, if $e$ is self-large, then no other self-large orbits meet $\mathfrak{g}_e$.

**Example 3.2.** Suppose $\mathfrak{g} = \mathfrak{sl}_n$, $\lambda = (\lambda_1, \ldots, \lambda_s)$, and $e \in O\lambda$. If $e$ is not self-large, then it is easy to indicate larger nilpotent orbits meeting $\mathfrak{g}_e$. Namely, if $\lambda_i - \lambda_{i+1} \leq 1$ for some $i$, then one can replace two parts $\lambda_i, \lambda_{i+1}$ with one part $\lambda_i + \lambda_{i+1}$ (with eventual rearranging the resulting parts). More generally,

\[
\text{(*)} \quad \begin{cases} \text{a substring } \ldots, a^k, (a - 1)^k, \ldots \text{ of } \lambda \text{ can be replaced} \\ \text{with the single part } ka + l(a - 1). \end{cases}
\]

One can do the same thing with other parts of the initial partition, if possible, but it is not allowed to apply this to newly obtained parts. However, concatenation of such steps is not sufficient for constructing $\mathcal{D}(O\lambda)$. For instance, take $\lambda = (3, 1, 1)$ for $\mathfrak{sl}_5$. Then

\[
(3, 1, 1) \leftrightarrow (3, 2) \not\leftrightarrow (5).
\]

That is, $O_{(3,2)}$ meets the centraliser of $e \in O_{(3,1,1)}$. However, a direct verification shows that $\mathcal{D}(O_{(3,1,1)}) = O_{(4,1)}$. Note that $O_{(4,1)}$ is self-large, while $O_{(3,2)}$ is not. Similarly, for $\mathfrak{g} = \mathfrak{sl}_7$, we have $\mathcal{D}(O_{(4,2,1)}) = O_{(5,2)}$.

Let us justify rule ($\ast$). Taking the respective Jordan subspaces, it suffices to assume that $\lambda = (a^k, (a - 1)^l)$. Let $e$ be a regular nilpotent element of $\mathfrak{sl}_n$ with $n = ka + l(a - 1)$. Then $O\lambda$ is the orbit of $e^{k+l}$, hence the assertion.

**Example 3.3.** For some classes of orbits, the description of all orbits meeting $N(\mathfrak{g}_e)$ is available. If $e \in \mathfrak{g} = \mathfrak{sl}_n$ is regular nilpotent, then $e, e^2, \ldots, e^{n-1}$ form a basis for $\mathfrak{g}_e$. It is easily seen that if $O$ meets $\mathfrak{g}_e$, then $O = SL_n-e^k$ for some $k$. The partition of $e^k$ has $k$ nonzero parts; $n-k \left\lfloor \frac{n}{k} \right\rfloor$ parts are of size $\left\lfloor \frac{n}{k} \right\rfloor + 1$ and the remaining parts are of size $\left\lfloor \frac{n}{k} \right\rfloor$.

Similar situation occurs for $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$, where one has to take odd powers of $e$.

**Example 3.4.** For $\mathfrak{g} = \mathfrak{sl}_7$, we have $\text{Im} \mathcal{D} = \{O_{(7)}, O_{(6,1)}, O_{(5,3)}\}$, i.e., precisely the set of self-large orbits. The full description of $\mathcal{D}$ is given by the following data:

\[
\mathcal{D}^{-1}(O_{(7)}) = \{O_{(7)}, O_{(4,3)}, O_{(3,2,2)}, O_{(2^3,1)}, O_{(2^2,1^3)}, O_{(2,1^5)}\};
\]

\[
\mathcal{D}^{-1}(O_{(6,1)}) = \{O_{(6,1)}, O_{(3,3,1)}, O_{(3,2,1,1)}, O_{(3,1^4)}\};
\]

\[
\mathcal{D}^{-1}(O_{(5,2)}) = \{O_{(5,2)}, O_{(5,1,1)}, O_{(4,2,1)}, O_{(4,1^3)}\}.
\]

**Example 3.5.** For $\mathfrak{g} = \mathfrak{so}_7$, we again have 3 self-large orbits and

\[
\mathcal{D}^{-1}(O_{(7)}) = \{O_{(7)}, O_{(3,2,2)}, O_{(2^2,1^3)}\},
\]

\[
\mathcal{D}^{-1}(O_{(5,1,1)}) = \{O_{(5,1,1)}, O_{(3,1^4)}\}, \quad \mathcal{D}^{-1}(O_{(3,3,1)}) = \{O_{(3,3,1)}\}.\]
REFERENCES


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