THE FANO SURFACE OF THE KLEIN CUBIC THREEFOLD

XAVIER ROULLEAU

Abstract. We prove that the Klein cubic threefold $F$ is the only one cubic threefold which has an order 11 automorphism. We calculate the period lattice of the intermediate Jacobian of $F$ and study its Fano surface $S$. We compute the set of fibrations of $S$ on a curve of positive genus and the intersection between the fibres of these fibrations. These fibres generate an index 2 sub-group of the Néron-Severi group and we obtain the generators of this group. The Néron-Severi group of $S$ has rank 25 = $h^{1,1}$ and discriminant $11^{10}$.

MSC: 14J29, 14J50 (primary); 14J70, 32G20 (secondary).

Key-words: Fano surface of a cubic threefold, Automorphisms, Surfaces with maximal Picard number.

0.1. Introduction. Let $F \hookrightarrow \mathbb{P}^4$ be a smooth cubic threefold. Its intermediate Jacobian

$$J(F) := H^{2,1}(F, \mathbb{C})^*/H_3(F, \mathbb{Z})$$

is a principally polarised Abelian variety $(J(F), \Theta)$ of dimension 5 that has the role in the analysis of curves on $F$ similar to the role of the Jacobian variety in the study of divisors on a curve.

The Hilbert scheme of lines on $F$ is a smooth surface $S$ called the Fano surface of $F$; the Abel-Jacobi map $\vartheta : S \rightarrow J(F)$ is an embedding that induce an isomorphism $\text{Alb}(S) \rightarrow J(F)$ where

$$\text{Alb}(S) := H^0(\Omega_S)^*/H_1(S, \mathbb{Z})$$

is the Albanese variety of $S$ and $H^0(\Omega_S) := H^0(S, \Omega_S)$ ([5] 0.6, 0.8).

The cotangent bundle theorem ([5] Lemma 12.5) ables us to recover the cubic $F$ if we know only its Fano surface and moreover it gives a natural isomorphism between the spaces $H^0(\Omega_S)$ and $H^0(F, \mathcal{O}_F(1)) = H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$. As we mainly work with the Fano surface, we will identify the homogenous coordinates $x_1, \ldots, x_5$ of $\mathbb{P}^4$ with elements of $H^0(\Omega_S)$. We will also identify the Abelian variety $J(F)$ with $\text{Alb}(S)$.

In [10], we give the classification of elliptic curve configurations on a Fano surface. It is proved that this classification is equivalent to the classification of the automorphism sub-groups of $S$ that are generated by certain involutions. Moreover, it is also proved that the automorphism groups $\text{Aut}(F)$ and $\text{Aut}(S)$ of a cubic and its Fano surface are isomorphic. In the present paper, we pursue the study of these groups. By [10] Corollary 1.19, the order of $\text{Aut}(S)$ divides
This legitimates the study of the Fano surfaces which have an order 11 automorphism. A. Adler [1] has proved that automorphism group of the Klein cubic:

\[ F : x_1x_5^2 + x_5x_3^2 + x_3x_4^2 + x_4x_2^2 + x_2x_1^2 = 0 \]

is isomorphic to \( PSL_2(\mathbb{F}_{11}) \). We prove in the present paper that:

**Proposition 0.1.** The Klein cubic is the only one smooth cubic threefold which has an order 11 automorphism.

If a curve admits a sufficiently large group of automorphisms, Bolza has found a method to compute a period matrix of its Jacobian (see [4], 11.7). As for the case of curves, we will use the fact that the Klein cubic admits a large group of automorphisms to compute the period lattice of its intermediate Jacobian \( J(F) \) or, what is the same thing, the period lattice \( H^1(S, \mathbb{Z}) \subset H^0(\Omega_S) \) of the two dimensional variety \( S \).

\[
\nu = -\frac{1+i\sqrt{-11}}{2}
\]

and let \( E \) be the elliptic curve \( E = \mathbb{C}/\mathbb{Z}[\nu] \). Let us denote by \( e_1,..,e_5 \in H^0(\Omega_S)^* \) the dual basis of \( x_1,..,x_5 \). Let \( \xi \) be a primitive 11-th root of unity and let \( v_i \in H^0(\Omega_S)^* \) be:

\[
v_i = \xi^i e_1 + \xi^{2i} e_2 + \xi^{3i} e_3 + \xi^{4i} e_4 + \xi^{5i} e_5.
\]

For \( s \) a point of the Hilbert sheme \( S \), we denote by \( C_s \) the divisor on \( S \) that parametrizes the lines on \( F \) that cut the line corresponding to the point \( s \).

**Theorem 0.2.** The period lattice \( H_1(S, \mathbb{Z}) \subset H^0(\Omega_S)^* \) is equal to:

\[
\frac{\mathbb{Z}[\nu]}{1 + 2^\nu}(v_0 - 3v_1 + 3v_2 - v_3) + \frac{\mathbb{Z}[\nu]}{1 + 2^\nu}(v_1 - 3v_2 + 3v_3 - v_4) + \bigoplus_{k=0}^{2} \mathbb{Z}[\nu]v_k.
\]

The Néron-Severi group \( \text{NS}(S) \) of \( S \) has rank 25 = \( h^{1,1}(S) \) and discriminant \( 11^{10} \). The image of the morphism

\[
\vartheta^* : \text{NS}(\text{Alb}(S)) \to \text{NS}(S)
\]

is sub-lattice of index 2 and \( \text{NS}(S) \) is generated by this lattice and the class of the incidence divisor \( C_s \).

The set of numerical classes of fibres of fibrations of \( S \) in a curve of positive genus is in natural bijection with \( \mathbb{P}^4_2(\mathbb{Z}[\nu]) \) and generate \( \vartheta^* \text{NS}(\text{Alb}(S)) \).

We remark that \( J(F) \cong \text{Alb}(S) \) is isomorphic to \( \mathbb{E}^5 \) but by [5] (0.12), this isomorphism is not an isomorphism of principally polarised Abelian varieties. The fact that \( J(F) \) is isomorphic to \( \mathbb{E}^5 \) is mentioned in [2].

The main properties used to prove Theorem 0.2 are the fact that the class of \( S \hookrightarrow \text{Alb}(S) \) is equal to \( \frac{1}{3} \Theta^3 \) and the fact that the action of the group \( \text{Aut}(S) \) on \( \text{Alb}(S) \) preserves the polarisation \( \Theta \).
To close this introduction, let us mention two known facts: (1) as the plane Klein quartic, the Klein cubic threefold has a modular interpretation see [7], (2) the cotangent sheaf of its Fano surface is ample [10].

0.2. Properties of the Fano surfaces. Let us recall some facts proved in [10] and fix the notations:

An automorphism \( f \) of \( F \) preserves the lines on \( F \) and induces an automorphism \( \rho(f) \) of the Fano surface \( S \).

An automorphism \( \sigma \) of \( S \) induces an automorphism \( \sigma' \) of the Albanese variety of \( S \) such that the following diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{\vartheta} & \text{Alb}(S) \\
\downarrow \sigma & & \downarrow \sigma' \\
S & \xrightarrow{\vartheta} & \text{Alb}(S)
\end{array}
\]

is commutative (where \( \vartheta : S \rightarrow \text{Alb}(S) \) is a fixed Albanese morphism). We denote by \( M_\sigma \in GL(H^0(\Omega S)^*) \) the linear part of the differential of \( \sigma' \) and we denote by

\[
q : GL(H^0(\Omega S)^*) \rightarrow PGL(H^0(\Omega S)^*)
\]

the natural quotient map. We have ([10], Theorem 1.15):

**Theorem 0.3.** A) The morphism \( q(M_\sigma) \) is an automorphism of \( F \hookrightarrow \mathbb{P}(H^0(\Omega S)^*) \).

B) The morphisms \( \rho : \text{Aut}(F) \rightarrow \text{Aut}(S) \) and \( \sigma \rightarrow q(M_\sigma) \) are reciprocal isomorphisms.

C) For all \( \sigma \in \text{Aut}(S) \), the automorphism \( \sigma' \) is an automorphism of the principally polarised Abelian variety \( (\text{Alb}(S), \Theta) \).

0.3. The unique cubic with an order 11 automorphism. Let us prove that the Klein cubic is the only one that possesses an order 11 automorphism.

Suppose that a cubic threefold \( F \) has an order 11 automorphism \( f \). Let \( \tau = \rho(f) \) be the induced automorphism of the Fano surface \( S \). The Proposition 13.2.5 and the Theorem 13.2.8 of [4] imply that the eigenvalues of \( M_\tau \) are 5 pairwise non-complex conjugate 11-th primitive root of unity.

We denote by \( \mathcal{O} \) the set of sets of 5 pairwise non-complex conjugate 11-th primitive root of unity: \( \mathcal{O} \) contains \( 2^5 \) elements. The group \( \text{Aut}(\mathbb{C}) \) of automorphisms of \( \mathbb{C} \) acts on \( \mathcal{O} \).

Let \( \xi \) be a 11-th primitive root of unity. For \( i \in \{0, 1, \ldots, 10\} \), we denote by \( \chi_i \) the 1 dimensional representation:

\[
x \mapsto \xi^ix \in \mathbb{C}.
\]

Let us suppose that \( \{\xi, \xi^9, \xi^3, \xi^4, \xi^5\} \in \mathcal{O} \) is the set of eigenvalues of \( M_\tau \).

The third symmetric power of the dimension 5 representation:

\[
(x_1, x_2, x_3, x_4, x_5) \rightarrow (\xi x_1, \xi^9 x_2, \xi^3 x_3, \xi^4 x_4, \xi^5 x_5)
\]
is decomposed in the following direct sum:

\[ H^n(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 5\chi_0 + 2\chi_1 + 3\chi_2 + 3\chi_3 + 4\chi_4 + 3\chi_5 + 3\chi_6 + 3\chi_7 + 3\chi_8 + 3\chi_9 + 3\chi_{10}. \]

The space 5\chi_0 is generated by:

\[ x_1x_2^2, \ x_5x_3^2, \ x_3x_4^2, \ x_4x_2^2, \ x_2x_1^2 \]

By an appropriate variable change, we see that any smooth cubic of this space is isomorphic to the Klein cubic:

\[ x_1x_2^2 + x_5x_3^2 + x_3x_4^2 + x_4x_2^2 + x_2x_1^2 = 0. \]

The other stable spaces define singular cubic threefolds.

Let us study the representation \( \chi_1 + \chi_2 + \chi_3 + \chi_4 + \chi_5 \). Its third symmetric power is:

\[ H^3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 4\chi_0 + 3\chi_1 + 2\chi_2 + 3\chi_3 + 2\chi_4 + \chi_5 + 3\chi_6 + 4\chi_7 + 4\chi_8 + 5\chi_9 + 4\chi_{10}. \]

A basis of 5\chi_9 is:

\[ x_2x_3x_4, \ x_1x_3x_5, \ x_5x_2^2, \ x_1x_4^2, \ x_3^3 \]

and any cubic of this space is singular at the point: \( (1:0:0:0:0) \). As we can verify, the other stable spaces give also singular threefolds.

Hence there is no Fano surface with an order 11 automorphism \( \tau \) such that the eigenvalues of \( M_\tau \) is the set \( \{\xi, \xi^2, \xi^3, \xi^4, \xi^5\} \).

The orbit \( O_0 \subset \mathcal{O} \) of the element \( \{\xi, \xi^2, \xi^3, \xi^4, \xi^5\} \) \( \in \mathcal{O} \) by \( \text{Aut}(\mathbb{C}) \) has order 10. Hence we have studied 10 representations and no one gives a smooth cubic threefold.

The set \( \{\xi, \xi^2, \xi^3, \xi^4, \xi^6\} \) \( \in \mathcal{O} \) is not an element of the orbits \( O_0 \) and \( O_1 \).

The third symmetric power of the representation:

\[(x_1, x_2, x_3, x_4, x_5) \rightarrow (\xi x_1, \xi^2 x_2, \xi^3 x_3, \xi^4 x_4, \xi^6 x_5)\]

is decomposed in:

\[ H^6(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 3\chi_0 + 3\chi_1 + 2\chi_2 + 3\chi_3 + 2\chi_4 + 3\chi_5 + 3\chi_6 + 4\chi_7 + 4\chi_8 + 4\chi_9 + 4\chi_{10}. \]

As we can verify, no element of these 11 spaces gives a smooth cubic threefold. The orbit \( O_2 \) of the set \( \{\xi, \xi^2, \xi^3, \xi^4, \xi^5\} \) by the action of \( \text{Aut}(\mathbb{C}) \) has order 10.

The set \( \{\xi, \xi^2, \xi^3, \xi^4, \xi^7\} \) \( \in \mathcal{O} \) is not an element of the orbits \( O_0, O_1 \) and \( O_2 \). The third symmetric power of the representation:

\[(x_1, x_2, x_3, x_4, x_5) \rightarrow (\xi x_1, \xi^2 x_2, \xi^3 x_3, \xi^5 x_4, \xi^7 x_5)\]

is decomposed in:

\[ H^8(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 4\chi_0 + 2\chi_1 + 3\chi_2 + 2\chi_3 + 4\chi_4 + 3\chi_5 + 4\chi_6 + 3\chi_7 + 3\chi_8 + 4\chi_9 + 3\chi_{10}. \]

No one elements of these 11 spaces gives a smooth cubic threefold. The orbit \( O_3 \) of the set \( \{\xi, \xi^2, \xi^3, \xi^4, \xi^7\} \) by the action of \( \text{Aut}(\mathbb{C}) \) has order 10.
The union of the orbits $O_0, O_1, O_2, O_3$ is equal to $O$. This shows that the Klein cubic
\[ x_1x_5^2 + x_5x_3^2 + x_3x_4^2 + x_4x_2^2 + x_2x_1^2 = 0 \]
is (up to isomorphism) the only one smooth cubic on which $\mathbb{Z}/11\mathbb{Z}$ acts. $\square$

Remark 0.4. By the same method, we can prove that there is no smooth cubic threefold with an order 7 automorphism.

0.4. The period lattice of the intermediate Jacobian of the Klein cubic. Let $F$ be the Klein cubic and let $S$ be its Fano surface. Let $\vartheta : S \to \text{Alb}(S)$ be a fixed Albanese morphism; it is an embedding. We compute here the period lattice $H_1(S, \mathbb{Z}) \subset H^0(\Omega_S)^*$ of the Albanese variety of $S$.

The order 5 automorphism:
\[ g : (x_1 : x_2 : x_3 : x_4 : x_5) \to (x_5 : x_1 : x_4 : x_2 : x_3) \]
acts on $F$. Let be $\sigma = \rho(g)$. By Theorem 0.3, there exists a 5-th root of unity $\theta$ such that $M_\sigma \in GL(H^0(\Omega_S)^*)$ is equal to:
\[ M_\sigma : x \to \theta(x_5, x_1, x_4, x_2, x_3). \]
Since the Klein cubic and $g$ are defined over $\mathbb{Q}$, we have $\theta = 1$.
Moreover, we know that $M_\tau$ verifies:
\[ M_\tau : x \to (\xi x_1, \xi^9 x_2, \xi^3 x_3, \xi^4 x_4, \xi^5 x_5) \]
where $\tau = \rho(f)$ is defined in paragraph 0.2.

Let $q_1$ be the endomorphism of $\text{Alb}(S)$ defined by the linear part of:
\[ \sum_{k=0}^{k=4} (\sigma')^k \]
(\text{where } \sigma' \circ \vartheta = \vartheta \circ \sigma). Its differential is:
\[ dq_1 : x \to (x_1 + x_2 + x_3 + x_4 + x_5)(e_1 + e_2 + e_3 + e_4 + e_5) \]
and its image is an elliptic curve which we denote by $E$. Let us take $\xi = e^{2\pi i}$ where $i^2 = -1$. The restriction of the linear part of $q_1 \circ \tau' : \text{Alb}(S) \to E$ to $E$ is the multiplication by:
\[ \nu = \xi + \xi^9 + \xi^3 + \xi^4 + \xi^5 = \frac{-1 + i\sqrt{11}}{2}. \]
The curve $E$ has complex multiplication by the principal ideal domain $\mathbb{Z}[\nu]$.
Up to a normalization of the basis $e_1, \ldots, e_5$ by a multiplication by a scalar, we can suppose that:
\[ H_1(S, \mathbb{Z}) \cap \mathbb{C}v_0 = \mathbb{Z}[\nu]v_0 \]
(such normalization preserves the equation of $F$).
For $i \in \mathbb{Z}/11\mathbb{Z}$, let $v_i$ be the vector:
\[ v_i = (M_\tau)^i v_0 \in H^0(\Omega_S)^* \]
\[ = \xi^i e_1 + \xi^{9i} e_2 + \xi^{3i} e_3 + \xi^{4i} e_4 + \xi^{5i} e_5. \]
Let $\Lambda_0 \subset H^0(\Omega_S)^* \subset H^0(\Omega_S)^*$ be the $\mathbb{Z}$-module generated by the $v_i$, $i \in \mathbb{Z}/11\mathbb{Z}$. The $\mathbb{Z}$-module $\Lambda_0$ is leaved stable by $M_\tau$ and $\Lambda_0 \subset H_1(S, \mathbb{Z})$.

**Lemma 0.5.** The $\mathbb{Z}$-module $\Lambda_0$ is equal to the lattice:

$$R_0 = \mathbb{Z}[\nu]v_0 + \mathbb{Z}[\nu]v_1 + \mathbb{Z}[\nu]v_2 + \mathbb{Z}[\nu]v_3 + \mathbb{Z}[\nu]v_4.$$ 

**Proof.** We have:

$$\nu v_0 = v_1 + v_3 + v_4 + v_5 + v_9,$$

hence $\nu v_0$ is an element of $\Lambda_0$. This implies that the vectors $\nu v_i = (M_\tau)^i v_0$ are elements of $\Lambda_0$ for all $i$, hence: $R_0 \subset \Lambda_0$.

Reciprocally, we have:

$$v_5 = v_0 + (1 + \nu)v_1 - v_2 + v_3 + \nu v_4.$$ 

This proves that $R_0$ is leaved stable by $M_\tau$ and that the lattice $R_0$ contains the vectors $v_i = (M_\tau)^i v_0$. Hence we have: $R_0 = \Lambda_0$. \square

We need to know the first Chern class $c_1(\Theta)$ of the $\Theta$ divisor of $\text{Alb}(S)$.

**Lemma 0.6.** Let $H$ be the matrix of the Hermitian form of the polarisation $\Theta$ in the basis $e_1, \ldots, e_5$. There exists a positive integer such that:

$$H = a \frac{2}{\sqrt{\Pi}} I_5$$

where $I_5$ is the size 5 identity matrix.

**Proof.** The automorphism $\tau'$ preserves the polarisation $\Theta$ (see [10], Lemma 1.18). By [4] Lemma 2.17, this implies that:

$$^t M_\tau H \bar{M}_\tau = H$$

where $\bar{M}_\tau$ is the matrix in the basis $e_1, \ldots, e_5$ whose coefficients are conjugated of those of $M_\tau$ and where $^t$ is the transposition. The only Hermitian matrices that verify this equality are the diagonal matrices. By the same reasoning with $\sigma$ instead of $\tau$, we obtain that these diagonal coefficients are equal and:

$$H = a \frac{2}{\sqrt{\Pi}} I_5$$

where $a$ is a positive real ($H$ is a positive definite Hermitian form). As $H$ is a polarisation, the alternating form $c_1(\Theta) = \Im(H)$ take integer values on $H_1(S, \mathbb{Z})$. But $v_1$ and $v_2$ are elements of $H_1(S, \mathbb{Z})$ and:

$$\Im(H(v_1, v_2)) = -a$$

hence $a$ is an integer. \square

Let be $k \in \mathbb{Z}/11\mathbb{Z}$. The differential of the linear part of the morphism $q_1 \circ (\tau')^k$ is:

$$x \mapsto \ell_k(x)(e_1 + \ldots + e_5)$$

where $\ell_k$ is the linear form:

$$\ell_k = \xi^k x_1 + \xi^9 k x_2 + \xi^{3k} x_3 + \xi^{4k} x_4 + \xi^{5k} x_5 \in H^0(\Omega_S).$$
Let be $\lambda \in H_1(S, \mathbb{Z})$. As

$$H_1(S, \mathbb{Z}) \cap \mathbb{C}v_0 = \mathbb{Z}[\nu]v_0,$$

the scalar $\ell_k(\lambda)$ is an element of $\mathbb{Z}[\nu]$. Put:

$$\Lambda_4 = \{ u \in \mathbb{C}^5/\forall k, 0 \leq k \leq 4, \ell_k(u) \in \mathbb{Z}[\nu] \}. $$

The set $\Lambda_4$ contains $H_1(S, \mathbb{Z})$.

**Lemma 0.7.** The $\mathbb{Z}$-module $\Lambda_4$ is equal to the lattice:

$$R_1 = \sum_{i=3}^{12} \frac{\mathbb{Z}[\nu]}{1 + 2\nu} (v_i - v_{i+1}) + \mathbb{Z}[\nu]v_0.$$

Moreover $M_\tau$ leaves stable $\Lambda_4$.

**Proof.** The element $u = \sum u_i e_i \in H^0(\Omega_S)^*$ is in $\Lambda_4$ if and only if

$$
\begin{pmatrix}
\xi & \xi^9 & \xi^3 & \xi^4 & \xi^5 \\
\xi^2 & \xi^7 & \xi^6 & \xi^8 & \xi^{10} \\
\xi^3 & \xi^5 & \xi^9 & \xi^4 & 1 \\
\xi^4 & \xi^3 & \xi^5 & \xi^9 & 1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5
\end{pmatrix}
\in
\begin{pmatrix}
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu]
\end{pmatrix}.
$$

The group $\Lambda_4$ is writted in the basis $b = v_0, ..., v_4$:

$$
\begin{pmatrix}
-1 & -\nu & 0 & -1 & 1 - \nu \\
-\nu & 2 & 0 & -\nu & 3 + \nu \\
0 & 0 & 0 & 1 & -1 \\
-1 & -\nu & 1 & -2 & 1 - \nu \\
1 - \nu & 3 + \nu & -1 & 1 - \nu & 2 + 2\nu
\end{pmatrix}
\begin{pmatrix}
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu] \\
\mathbb{Z}[\nu]
\end{pmatrix}
$$

and by elementary operations with coefficients in $\mathbb{Z}[\nu]$ on the rows, we obtain that $\Lambda_4$ is equal to $R_1$.

Now, we use the fact that:

$$v_5 = v_0 + (1 + \nu)v_1 - v_2 + v_3 + \nu v_4$$

to prove that $\frac{1}{1 + 2\nu}(v_4 - v_5)$ is an element of $\Lambda_4$. As $M_\tau(v_j) = v_{j+1}$, this ables us to conclude that $\Lambda_4$ is stable by $M_\tau$. \qed

We denote by $\phi : \Lambda_4 \rightarrow \Lambda_4/\Lambda_0$ the quotient map. The ring $\mathbb{Z}[\nu]/(1 + 2\nu)$ is the finite field with 11 elements.

**Lemma 0.8.** The quotient $\Lambda_4/\Lambda_0$ is a $\mathbb{Z}[\nu]/(1 + 2\nu)$-vector space with basis:

$$
t_1 = \frac{1}{1 + 2\nu}(v_4 - v_1) + \Lambda_0, \quad t_2 = \frac{1}{1 + 2\nu}(v_1 - v_2) + \Lambda_0 \\
t_3 = \frac{1}{1 + 2\nu}(v_2 - v_3) + \Lambda_0, \quad t_4 = \frac{1}{1 + 2\nu}(v_3 - v_4) + \Lambda_0.
$$

**Proof.** The quotient $\Lambda_4/\Lambda_0$ is an hyperplane of the 5 dimensional $\mathbb{Z}[\nu]/(1 + 2\nu)$-vector space $\frac{1}{1 + 2\nu}\Lambda_0/\Lambda_0$. \qed
Let $R$ be a lattice such that $\Lambda_0 \subset R \subset \Lambda_4$. The group $\phi(R)$ is a sub-vector space of $\Lambda_4/\Lambda_0$ and:

$$\phi^{-1}(R) = R + \Lambda_0 = R.$$ 

Because $M_\tau$ preserves $\Lambda_0$, the morphism $M_\tau$ induces a morphism $\tilde{M}_\tau$ on the quotient $\Lambda_4/\Lambda_0$ such that $\phi \circ M_\tau = \tilde{M}_\tau \circ \phi$. As $M_\tau$ leaves stable $H_1(S,\mathbb{Z})$, the vector space $\phi(H_1(S,\mathbb{Z}))$ is stable by $\tilde{M}_\tau$. We denote:

$$w_1 = -t_1 + 3t_2 - 3t_3 + t_4$$
$$w_2 = t_1 - 2t_2 + t_3$$
$$w_3 = -t_1 + t_2$$
$$w_4 = t_1.$$

The matrix of $\tilde{M}_\tau$ in the basis $w_1,\ldots, w_4$ of $\Lambda_4/\Lambda$ is:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

The sub-spaces left stable by $\tilde{M}_\tau$ are the space $W_0 = \{0\}$ and the spaces $W_i$, $1 \leq i \leq 4$ generated by $w_1,\ldots, w_i$. Let $\Lambda_i$ be the lattice $\phi^{-1}W_i$, then:

**Theorem 0.9.** The lattice $H_1(S,\mathbb{Z})$ is equal to $\Lambda_2$, and $\Lambda_2$ is equal to

$$R_2 = \frac{\mathbb{Z}[\nu]}{1 + 2\nu}(v_0 - 3v_1 + 3v_2 - v_3) + \frac{\mathbb{Z}[\nu]}{1 + 2\nu}(v_1 - 3v_2 + 3v_3 - v_4) + \bigoplus_{k=0}^2 \mathbb{Z}[\nu]v_k.$$

Moreover, the Hermitian matrix associated to $\Theta$ is equal to $\frac{2}{\sqrt{11}}I_5$ in the basis $e_1,\ldots, e_5$.

**Proof.** Let $c_1(\Theta) = \Im m(H) = i \frac{2}{\sqrt{11}}\sum dx_k \wedge dx_k$ be the alternating form of the principal polarisation $\Theta$. Let $\lambda_1,\ldots, \lambda_{10}$ be a basis of a lattice $\Lambda$. The Pfaffian $P_{f\Theta}(\Lambda)$ of $\Lambda$ is the determinant of the matrix

$$(c_1(\Theta)(\lambda_j, \lambda_k))_{1 \leq j,k \leq 10}.$$

Since $\Theta$ is a principal polarisation, we have $P_{f\Theta}(H_1(S,\mathbb{Z})) = 1$.

It is easy to find a basis of $\Lambda_j$ ($j \in \{0,\ldots, 4\}$). For example, the space $W_2$ is generated by $w_2 = t_1 - 2t_2 + t_3$ and $w_1 + w_2 = t_2 - 2t_3 + t_4$ and as

$$\phi\left(\frac{1}{1 + 2\nu}(v_0 - 3v_1 + 3v_2 - v_3)\right) = w_2$$
$$\phi\left(\frac{1}{1 + 2\nu}(v_1 - 3v_2 + 3v_3 - v_4)\right) = w_1 + w_2,$$

the lattice $R_2$ (that contains $\Lambda_0$) is equal to $\Lambda_2$.

Then, with the help of a computer, we can calculate the Pfaffian $P_j$ of the lattice $\Lambda_j$ and verify that it is equal to:

$$a^{10}11^{4-2j}.$$
where $a$ is the integer of Lemma 0.6. As $a$ is a positive, the only possibility that $P_j$ equals 1 is $j = 2$ and $a = 1$. □

0.5. The Néron-Severi Group of the Fano surface of the Klein cubic.

Let us define:

\[ u_1 = \frac{1}{1 + 2^v} (v_0 - 3v_1 + 3v_2 - v_3), \quad u_2 = \frac{1}{1 + 2^v} (v_1 - 3v_2 + 3v_3 - v_4) \]
\[ u_3 = v_0, \quad u_4 = v_1, \quad u_5 = v_2. \]

Let $y_1, \ldots, y_5 \in H^0(\Omega_S)$ be the linear forms such that:

\[ \sum_{k=1}^{k=5} x_k e_k = \sum_{k=1}^{k=5} y_k u_k. \]

Let be $k, 1 \leq k \leq 5$. The image of $H_1(S, \mathbb{Z})$ by $y_k \in H^0(\Omega_S)$ is $\mathbb{Z}[\nu]$, and this form is the differential of an Abelian varieties morphism

\[ r_k : \text{Alb}(S) \rightarrow \mathbb{E} = \mathbb{C}/\mathbb{Z}[[\nu]] \]

The morphisms $r_1, \ldots, r_5$ form a basis of the $\mathbb{Z}[\nu]$-module $\text{Hom}_{\text{Ab var}}(\text{Alb}(S), \mathbb{E})$.

We denote by $\Lambda_A^*$ the free $\mathbb{Z}[\nu]$-module of rank 5 generated by $y_1, \ldots, y_5$ and for $\ell \in \Lambda_A^*$, we denote by $\Gamma_{\ell} : \text{Alb}(S) \rightarrow \mathbb{E}$ the morphism whose differential is $\ell : H^0(\Omega_S)^* \rightarrow \mathbb{C}$.

Let $\vartheta : S \rightarrow \text{Alb}(S)$ be a fixed Albanese morphism. We denote by $\gamma_{\ell} : S \rightarrow \mathbb{E}$ the morphism $\gamma_{\ell} = \Gamma_{\ell} \circ \vartheta$ and we denote by $F_\ell$ the numerical equivalence class of a fibre of $\gamma_{\ell}$ (this class is independant of the choice of $\vartheta$).

We define the scalar product of two forms $\ell, \ell' \in \Lambda_A^*$ by:

\[ \langle \ell, \ell' \rangle = \sum_{k=1}^{k=5} \ell(e_k) \overline{\ell'(e_k)} \]

and the norm of $\ell$ by:

\[ \|\ell\| = \sqrt{\langle \ell, \ell \rangle}. \]

We denote by $\text{NS}(X)$ the Néron-Severi group of a variety $X$. The aim of this paragraph is to prove the following result:

**Theorem 0.10.** Let $\ell, \ell'$ be elements of $\Lambda_A^*$. The fibre $F_\ell$ has genus:

\[ g(F_\ell) = 1 + 3 \|\ell\|^2, \]

verifies $F_\ell C_s = 2 \|\ell\|^2$ and :

\[ F_\ell F_{\ell'} = \|\ell\|^2 \|\ell'\|^2 - \langle \ell, \ell' \rangle \langle \ell', \ell \rangle. \]

The image of the morphism $\vartheta^* : \text{NS}(\text{Alb}(S)) \rightarrow \text{NS}(S)$ is a rank 25 sub-lattice of discriminant $2^2 11^{10}$.

The following 25 fibres

\[ F_{y_k}, \quad k \in \{1, \ldots, 5\} \quad F_{y_k + y_l}, \quad 1 \leq k < l \leq 5 \quad F_{y_k + vy_l}, \quad 1 \leq k < l \leq 5 \]

are a $\mathbb{Z}$-basis of $\vartheta^* \text{NS}(\text{Alb}(S))$ and together with the class of the incident divisor $C_s$ ($s \in S$) they generate the Néron-Severi group of $S$. 

We begin by the following lemma:

**Lemma 0.11.** The Néron-Severi group of $\text{Alb}(S)$ is generated by the 25 forms:

\[
\frac{i}{\sqrt{11}} dy_k \wedge d\bar{y}_k, \ k \in \{1, \ldots, 5\} \quad \frac{i}{\sqrt{11}} (dy_k \wedge d\bar{y}_l + dy_l \wedge d\bar{y}_k), \ 1 \leq k < l \leq 5
\]

\[
\frac{i}{\sqrt{11}} (udy_k \wedge d\bar{y}_l + \bar{u}dy_l \wedge d\bar{y}_k), \ 1 \leq k < l \leq 5
\]

**Proof.** The Hermitian form $H' = \frac{2}{\sqrt{11}} I_5$ in the basis $u_1, \ldots, u_5$ defines a principal polarisation of $\text{Alb}(S)$. Let $\text{End}^s(\text{Alb}(S))$ be the group of symmetric morphisms for the Rosati involution associated to $H'$. There exists an isomorphism

\[
\phi_{H'} : \text{NS}(\text{Alb}(S)) \rightarrow \text{End}^s(\text{Alb}(S)).
\]

The group $\text{End}^s(\text{Alb}(S))$ is easily calculated and we obtain the lemma when we take the inverse morphism of $\phi_{H'}$ (see [4] Proposition 5.2.1). \hfill \Box

The Néron-Severi group of the curve $E = \mathbb{C}/\mathbb{Z}[\nu]$ is the $\mathbb{Z}$-module generated by the form:

\[
\eta = \frac{i}{\sqrt{11}} dz \wedge d\bar{z}.
\]

Let $\ell \in \Lambda^*_A$, we have:

\[
\Gamma^*_\ell \eta = \frac{i}{\sqrt{11}} d\ell \wedge d\bar{\ell}
\]

and this form is the Chern class of the divisor $\Gamma^*_\ell 0$.

**Lemma 0.12.** The 25 forms:

\[
\eta_k = \Gamma^*_y \eta, \ k \in \{1, \ldots, 5\} \quad \eta^1_{k,l} = \Gamma^*_{y_k+y_l} \eta, \ 1 \leq k < l \leq 5
\]

\[
\eta^\nu_{k,l} = \Gamma^*_{y_k+\nu y_l} \eta, \ 1 \leq k < l \leq 5
\]

are a basis of the Néron-Severi group of $\text{Alb}(S)$.

**Proof.** Let $1 \leq k \leq 5$ be an integer. The element $\Gamma^*_y \eta = \frac{i}{\sqrt{11}} dy_k \wedge d\bar{y}_k$ is in the basis of Lemma 0.11. Let $1 \leq l < k \leq 5$ be integers, let be $a \in \{1, \nu\}$, and $\ell = y_k + ay_l$. We have:

\[
\Gamma^*_\ell \eta = \frac{i}{\sqrt{11}} (dy_k \wedge d\bar{y}_k + \bar{a}dy_k \wedge d\bar{y}_l + ady_l \wedge d\bar{y}_k + a\bar{a}dy_l \wedge d\bar{y}_l),
\]

this proves, when we take $a = 1$ and next $a = \nu$, that the forms of the basis of Lemma 0.11 are $\mathbb{Z}$-linear combinations of the forms $\eta_k, \eta^1_{k,l}, \eta^\nu_{k,l}, \ 1 \leq k, l \leq 5$. \hfill \Box

Let us prove the Theorem 0.10.

**Proof.** As the homology class of $S$ in $\text{Alb}(S)$ is equal to $\frac{\Theta^3}{3!}$, the intersection of the fibres $F_\ell$ and $F_{\ell'}$ is equal to:

\[
\int_A \frac{1}{3!} \wedge^3 c_1(\Theta) \wedge \Gamma^*_\ell \eta \wedge \Gamma^*_{\ell'} \eta.
\]
Write $\ell$ in the basis $x_1, \ldots, x_5 : \ell = a_1x_1 + \ldots + a_5x_5$ and $\ell' = b_1x_1 + \ldots + b_5x_5$, then:

$$\frac{1}{3!} \left( \frac{i}{\sqrt{11}} \right)^2 d\ell \wedge d\ell' \wedge d\ell'' \wedge (\wedge^3 c_1(\Theta))$$

is equal to:

$$\left( \frac{i}{\sqrt{11}} \right)^3 (\sum a_jx_j) \wedge (\sum \bar{a}_jx_j) \wedge (\sum b_jx_j) \wedge (\sum \bar{b}_jx_j)$$

$$\wedge \sum_{h<j<k} dx_h \wedge dx_j \wedge dx_k$$

that is equal to:

$$\left( \sum a_k\bar{a}_k\bar{b}_j - a_k\bar{a}_jb_jb_k \right) \frac{1}{5!} \wedge^5 c_1(\Theta).$$

But: $\int_A \frac{1}{3!} \wedge c_1(\Theta) = 1$ because $\Theta$ is a principal polarisation of $\text{Alb}(S)$, hence:

$$F_{\ell}F_{\ell'} = \int_A \frac{1}{3!} \wedge^3 c_1(\Theta) \wedge \Gamma_{\ell}^* \eta \wedge \Gamma_{\ell'}^* \eta = \sum_{k\neq j} a_k\bar{a}_kb_j\bar{b}_j - a_k\bar{a}_jb_jb_k$$

$$= ||\ell||^2 ||\ell'||^2 - \langle \ell, \ell' \rangle \langle \ell', \ell \rangle.$$

By [5] (10.9) and Lemma 11.27, $\frac{3}{2} \partial^* c_1(\Theta)$ is the Poincaré dual of a canonical divisor $K$ of $S$, hence:

$$KF_{\ell} = \frac{3}{2} \partial^* c_1(\Theta) \partial^* \Gamma_{\ell}^* \eta = \frac{3}{2} \int_A \frac{1}{3!} \wedge^4 c_1(\Theta) \wedge \Gamma_{\ell}^* \eta$$

and:

$$KF_{\ell} = \int_A 6 \left( \frac{i}{\sqrt{11}} \right)^5 (\sum a_jdx_j) \wedge (\sum \bar{a}_jdx_j) \wedge \sum_{1\leq k\leq 5} (\wedge_{j\neq k}(dx_j \wedge d\bar{x}_j))$$

so $KF = 6 \sum_{k=1}^5 a_k\bar{a}_k = 6 ||\ell||^2$. Hence we have $g(F_{\ell}) = (KF_{\ell} + 0)/2 + 1 = 3 ||\ell||^2 + 1$.

Lemma 0.12 gives us a basis $\eta_1, \ldots, \eta_{25}$ of $\text{NS}(\text{Alb}(S))$ and we know the intersections $\partial^* \eta_k \partial^* \eta_l$ in the Fano surface. With the help of a computer, we can verify that the determinant of the intersection matrix:

$$(\partial^* \eta_k \partial^* \eta_l)_{1 \leq k, l \leq 25}$$

is equal to $2^2 11^{10}$. By general results of [10], Proposition 1.17, the index of $\partial^* \text{NS}(\text{Alb}(S))$ is 2 and $\text{NS}(S)$ is generated by $\partial^* \text{NS}(\text{Alb}(S))$ and the class of an incidence divisor $C_s$.

We obtain also the following corollary:

**Corollary 0.13.** Let $C$ be a smooth curve of genus $> 0$ and let $\gamma : S \to C$ be a fibration with connected fibres. Then there exists an isomorphism $j : \mathbb{E} \to C$ and an $\ell \in \Lambda^*_3$ such that $\gamma = j \circ \gamma_{\ell}$.

The connected fibrations (in a curve of genus $> 0$) up to isomorphism are in bijection with $\mathbb{P}_2^4(\mathbb{Z}[\nu])$. \hfill $\square$
Proof. The natural morphism \( \wedge^2 H^0(\Omega_S) \to H^0(S, \wedge^2 \Omega_S) \) is an isomorphism, hence if \( \gamma : S \to C \) is fibration on a curve of genus > 0, the curve \( C \) has genus 1. This implies that there is a morphism \( \Gamma : \text{Alb}(S) \to C \) such that \( \gamma = \Gamma \circ \vartheta \). Moreover \( \Gamma \) has connected fibres hence \( C \) is isomorphic to \( \mathbb{E} \) (here we use the fact that \( \mathbb{Z}[\nu] \) is principal).

Let \( \ell \in \Lambda^*_A, \ell = t_1y_1 + \ldots + t_5y_5 \), the fibration \( \Gamma_\ell \) has connected fibres if and only if \( t_1, \ldots, t_5 \) generates \( \mathbb{Z}[\nu] \). \( \square \)

References


Xavier Rouleau
Max Planck Institute für Mathematik,
Vivatgasse 7,
53111 Bonn,
Germany

rouleau@mpim-bonn.mpg.de