The Auslander-Reiten sequences ending at Gabriel-Roiter factor modules over tame hereditary algebras

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Abstract. Let $\Lambda = kQ$ be a finite dimensional hereditary algebra over an algebraically closed field k with Q a quiver of Euclidean type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, or $\widetilde{\mathbb{E}}_{6,7,8}$. We study the Auslander-Reiten sequences terminating at Gabriel-Roiter factor modules and show that for almost all but finitely many Gabriel-Roiter factor modules, the Auslander-Reiten sequences have indecomposable middle terms.

Keywords. Tame hereditary algebra, Gabriel-Roiter measure, irreducible maps, Auslander-Reiten sequences.

Mathematics Subject Classification 2000. 16G70

1 Introduction

Throughout the paper, we assume that k is an algebraically closed field and Λ is a finite dimensional basic connected k-algebra. We denote by $\operatorname{mod}\Lambda$ the category of finite dimensional left Λ-modules and by $\operatorname{ind}\Lambda$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of indecomposable Λ-modules. We denote by $|M|$ the length of a Λ-module M. We use the symbol $\subset$ to denote proper inclusion. For any two Λ-modules X and Y, $\operatorname{Ext}^i(X,Y)$ always stands for $\operatorname{Ext}^i_\Lambda(X,Y)$ for any $i \geq 0$.

The Gabriel-Roiter measure has been introduced by Gabriel (under the name 'Roiter measure', [6]) in 1973, in order to clarify the induction scheme used by Roiter in his proof of the first Brauer-Thrall conjecture. Ringel used the Gabriel-Roiter measure as a foundation tool for representation theory of artin algebras ([8],[9]). So-called Gabriel-Roiter submodules of an indecomposable module are indecomposable submodules with a certain maximality condition. Gabriel-Roiter submodules of an indecomposable module Y always exist in case Y is not simple, and one of the most interesting properties is that if Y is an indecomposable non-simple module and X is a Gabriel-Roiter submodule of Y, then Y/X, the Gabriel-Roiter factor module, is indecomposable ([8],[9]). Therefore, any indecomposable non-simple module Y is an extension of indecomposable modules. A known interesting application of Gabriel-Roiter submodules is that they can be used to construct orthogonal exceptional pairs to indecomposable modules over representation-directed algebras ([3],[4],[10]).

For each indecomposable module M, there exists a minimal right almost split map $\oplus_{i=1}^n X_i \to M$ with X_i indecomposable. We denote by $\alpha(M) = n$ the number of the indecomposable summands and say M has n middle terms. We say M has an indecomposable middle term in case $\alpha(M) = 1$. In [4], we have shown that any non-injective Gabriel-Roiter factor module over a representation-finite hereditary algebra has an indecomposable middle term. Our purpose of this paper is the study of the Auslander-Reiten sequences terminating at Gabriel-Roiter factor modules over tame hereditary algebras. It turns out that all but finitely many Gabriel-Roiter factor modules have indecomposable middle terms. More precisely, if Λ is of type $\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n$, then the Gabriel-Roiter factor modules possessing decomposable middle terms have dimension vectors smaller than δ, the minimal radical vector (Theorem 4.1,4.3); if Λ is of type $\widetilde{\mathbb{E}}_{6,7,8}$, then any non-injective Gabriel-Roiter factor module has indecomposable middle term (Theorem 4.7).

2 Preliminaries

In this section, we present some preliminaries which will be used later on. For details, we refer to [1], [5], [7]. Let $\Lambda = kQ$ be a path algebra with the underlying graph of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, or $\widetilde{\mathbb{E}}_{6,7,8}$. The dimension vector for a Λ-module M is denoted by $\underline{\dim}\, M$. We call a module M **sincere** if $(\underline{\dim}\, M)_i \geq 1$ for each i, and **thin** if $(\underline{\dim}\, M)_i \leq 1$ for each i.

We have a bilinear form $\langle a, b\rangle = aC^{-t}b$ for all $a, b \in \mathbb{Z}^n$ where C is the Cartan matrix and t denotes the transpose of a matrix. Moreover, given two modules $X, Y \in \operatorname{mod}\Lambda$, we have

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle = \dim \operatorname{Hom}(X, Y) - \dim \operatorname{Ext}^1(X, Y)$$

We denote by q the quadratic form on $\mathbb{Z}$ defined by $q(a) = \langle a, a\rangle$. Then q is positive semi-definite with one-dimensional **radical** $\mathbb{Z}\delta$, that is, $q(\delta) = 0$ and $h = r\delta$ for some $r \in \mathbb{Z}$ whenever $q(h) = 0$. We list the underlying graphs of the quivers of tame hereditary algebras and indicate δ for each case.

$$\widetilde{\mathbb{A}}_n: \begin{array}{cccccc} & & 1-1 & \cdots-1 & & \\ & / & & & \backslash & \\ 1 & & & & & 1 \\ & \backslash & & & / & \\ & & 1-1 & \cdots-1 & & \end{array} \qquad \widetilde{\mathbb{D}}_n: \begin{array}{ccccc} 1 & & & & 1 \\ & \backslash & & / & \\ & & 2-2-\cdots-2 & & \\ & / & & \backslash & \\ 1 & & & & 1 \end{array} \qquad \widetilde{\mathbb{E}}_6: \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 1-2-3-2-1 \end{array}$$

$$\widetilde{\mathbb{E}}_7: \begin{array}{ccccccc} & & & 2 & & & \\ & & & | & & & \\ 1 & - 2 & - 3 & - 4 & - 3 & - 2 & - 1 \end{array} \qquad \widetilde{\mathbb{E}}_8: \begin{array}{cccccccc} & & 3 & & & & & \\ & & | & & & & & \\ 2 & - 4 & - 6 & - 5 & - 4 & - 3 & - 2 & - 1 \end{array}$$

We have a decomposition of the Auslander-Reiten quiver Γ_Λ for Λ into the preprojective part $\mathcal{P}$, the preinjective part $\mathcal{I}$ and the regular one $\mathcal{R}$, where $\mathcal{R}$ is a sum of stable tubes $\mathcal{T}_\lambda$ of ranks $r_\lambda \geq 1$, for $\lambda \in \mathbb{P}^1(k) = k \cup \{\infty\}$. A tube of rank 1 is called **homogeneous** and that of rank > 1 is called **exceptional**. Note that $\mathcal{T}_\lambda$ is exceptional for at most three $\lambda \in \mathbb{P}^1(k)$. For indecomposable Λ-modules X, Y, if $\operatorname{Hom}(X, Y) \neq 0$ and X and Y do not belong to the same connected component of Γ_Λ, then X is preprojective or Y is preinjective. For each tube $\mathcal{T}_\lambda$, we call the modules lying on the mouth regular simple modules. For each regular simple module $E \in \mathcal{T}_\lambda$, we denote by $E = E[1] \to E[2] \to E[3] \to \cdots$ the unique infinite path in $\mathcal{T}_\lambda$ of irreducible monomorphisms.

The **defect** of a Λ-module X is defined to be $\langle \delta, \underline{\dim}\, X\rangle = -\langle \underline{\dim}\, X, \delta\rangle$. We thus get a **defect function** which is also denoted by $\delta : \delta(X) = \langle \delta, \underline{\dim}\, X\rangle$. It is well-known that an indecomposable Λ-module X is preprojective, (resp. regular, preinjective) if and only if $\delta(X)$ is negative (resp. zero, positive).

Lemma 2.1 ([2]). *Assume that X and Y are indecomposable preprojective modules such that the defect of X is $\delta(X) = -1$. If $0 \neq f \in \operatorname{Hom}(X, Y)$, then f is injective.*

Proof. Since $\operatorname{Im} f$ is a submodule of Y, it is a preprojective module. We thus have $-1 = \delta(X) = \delta(\operatorname{Im} f)+\delta(\operatorname{Ker} f)$. It follows that either $\delta(\operatorname{Im} f) = 0$ or $\delta(\operatorname{Ker} f) = 0$. But $f \neq 0$ implies $\delta(\operatorname{Ker} f) = 0$. Therefore, $\operatorname{Ker} f = 0$ and f is injective. $\square$

Corollary 2.2. *Assume that Λ is of type $\widetilde{\mathbb{A}}_n$.*
(1) *All non-zero maps between indecomposable preprojective modules are injective and the corresponding factors are regular modules. In particular, all irreducible maps between indecomposable preprojective modules are monomorphisms.*
(2) *All non-zero maps between indecomposable preinjective modules are surjective and the corresponding kernels are regular modules. In particular, all irreducible maps between indecomposable preinjective modules are epimorphisms.*

Proof. Note that in this case, all indecomposable preprojective modules are of defect -1. (2) is dual to (1). $\square$

3 The Gabriel-Roiter measure

In this section, we assume that Λ is a finite dimensional k-algebra. Let $\mathbb{N}_1=\{1, 2, \cdots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N}_1)$ the set of all subsets of $\mathbb{N}_1$. We consider the set $\mathcal{P}(\mathbb{N}_1)$ as a totally ordered set as follows: If I,J are two different subsets of $\mathbb{N}_1$, write $I < J$ provided the smallest element in $(I\backslash J) \cup (J\backslash I)$ belongs to J. Also we write $I \ll J$ provided $I \subset J$ and for all elements $a \in I$, $b \in J\backslash I$, we have $a < b$. We say that J **starts with** I provided $I = J$ or $I \ll J$. It is easy to check that:
(1) If $I \subseteq J \subseteq \mathbb{N}_1$, then $I \leq J$.
(2) If $I_1 \leq I_2 \leq I_3$, and I_3 starts with I_1, then I_2 starts with I_1.

For each Λ-module M, we denote by $|M|$ the length of M. Let $\mu(M)$ be the maximum of the sets $\{|M_1|, |M_2|, \cdots , |M_t|\}$ where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of indecomposable submodules of M. We call $\mu(M)$ the **Gabriel-Roiter measure** (briefly **GR measure**) of M. If M is an indecomposable Λ-module, then a chain of indecomposable submodules $M_1 \subset M_2 \subset \cdots \subset M_t = M$ with $\mu(M) = \{|M_1|, |M_2|, \cdots |M_t|\}$ is called a **Gabriel-Roiter filtration** (briefly **GR filtration**) of M. We call an inclusion $T \subset M$ of indecomposable Λ-modules a **Gabriel-Roiter inclusion** (briefly **GR inclusion**) provided $\mu(M) = \mu(T)\cup\{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(T)$. In this case, we call T a **Gabriel-Roiter submodule** (briefly, **GR submodule**) of M. Note that a chain $M_1 \subset M_2 \subset \cdots \subset M_t = M$ is a GR filtration if and only if all the inclusions $M_i \subset M_{i+1}$ are GR inclusions. The factor module of a GR inclusion is called **Gabriel-Roiter factor** (briefly **GR factor**). A short exact sequence $0\longrightarrow T \xrightarrow{f} M \xrightarrow{g} X\longrightarrow 0$ is called a **GR sequence** provided f is a GR inclusion.

We now begin to present some basic properties of the Gabriel-Roiter measure. We fix a finite dimensional k-algebra Λ.

Main Property (Gabriel). *Let X, Y_1, $\cdots$,Y_t be indecomposable modules and assume that there is a monomorphism $f : X \longrightarrow \oplus_{i=1}^{t} Y_i$. Then*
(1) $\mu(X) \leq \max\{\mu(Y_i)\}$.

(2) *If* $\mu(X) = \max\{\mu(Y_i)\}$*, then* f *splits.*
(3) *If* $\max\{\mu(Y_i)\}$ *starts with* $\mu(X)$*, then there is some* j *such that* $\pi_j f$ *is injective, where* $\pi_j : \oplus_{i=1}^t Y_i \longrightarrow Y_j$ *is the canonical projection.*

For the proof, we refer to [8].

From the Main Property, we obtain many interesting consequences of the Gabriel-Roiter inclusions. We now collect some properties which will be useful later on.

Proposition 3.1. *Let* $\delta : 0 \longrightarrow T \stackrel{l}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} M/T \longrightarrow 0$ *be a GR sequence. Then the following statements hold:*
(1) T *is a direct summand of all proper submodules of* M *containing* T*.*
(2) M/T *is indecomposable.*
(3) *Any map to* M/T *which is not an epimorphism factors through* π*.*
(4) *All irreducible maps to* M/T *are epimorphisms.*
(5) *If all irreducible maps to* M *are monomorphisms, then* l *is an irreducible map.*
(6) M/T *is a factor module of* $\tau^{-1}T$ *and* $M/T \cong \tau^{-1}T$ *if and only if* δ *is an Auslander-Reiten sequence.*

The proof can be found in [8] and [4].

Proposition 3.2. *Assume that* T *is a GR submodule of* M*. Then there is an irreducible monomorphism* $T \rightarrow X$ *with* X *indecomposable and an epimorphism* $X \rightarrow M$*.*

Proof. Assume that $l : T \rightarrow M$ is the inclusion map and $T \stackrel{f=(f_i)}{\longrightarrow} \oplus_{i=1}^r X_i$ is the minimal left almost split map. Then we obtain the following commutative diagram:

$$\begin{tikzcd} T \arrow[r, "f=(f_i)"] \arrow[d, "l"] & \oplus X_i \arrow[dl, "g=(g_i)"] \\ M & \end{tikzcd}$$

Assume that g_i is not an epimorphism for each i. The induced monomorphism $(g_i f_i) : T \rightarrow \oplus X_i \rightarrow \oplus_i \operatorname{Im} g_i$ implies $\mu(T) \leq \max\{\mu(\operatorname{Im} g_i)\} \leq \mu(T)$ since T is a GR submodule of Y and Im g_i is a proper submodule of M. By the Main Property, we obtain that the map $(g_i f_i)$ splits, thus (f_i) splits. But $f = (f_i)$ is an almost split map. The contradiction implies there is an index j such that g_j is an epimorphism. Since $|X| \geq |M| > |T|$, we obtain that f_j is a monomorphism. Now we take $X = X_j$. $\square$

Remark. We may also require that the composition of the maps $T \rightarrow X \rightarrow M$ obtained in the proposition is a monomorphism. This follows from the fact that the subset consisting of all non-monomorphisms in $\operatorname{Hom}(T, M)$ is a subgroup, see [11] for details.

4 The Auslander-Reiten sequences terminating at Gabriel-Roiter factor modules

In this section, we study the Auslander-Reiten sequences terminating at Gabriel-Roiter factor modules. Throughout the section, Λ will be a tame hereditary algebra.

Theorem 4.1. *Assume that Λ is of type $\widetilde{\mathbb{A}}_n$. If $0 \to T \to M \to M/T \to 0$ is a GR sequence, then $\underline{\dim}\, M/T \leq \delta$. In particular, all but finitely many GR factor modules have indecomposable middle term.*

Proof. We keep in mind that any irreducible map to a GR factor is an epimorphism (Proposition 3.1). Corollary 2.2 implies there is no preprojective GR factor module. If X is an indecomposable regular module but not regular simple, then there is always an irreducible monomorphism to X. Therefore, a GR factor module is either a regular simple module or a preinjective module. If M/T is regular simple, then $\underline{\dim}\, M/T \leq \delta$.

Since an indecomposable preinjective module always has decomposable middle term with exactly two summands, we need to show that there are only finitely many preinjective GR factor modules. If M/T is preinjective, we have the following two cases to consider: (1) T is preprojective, M is regular and (2) T is regular, M is preinjective. In both cases, we get an irreducible monomorphism $T \to X$ and an epimorphism $X \to M$ by Proposition 3.2. Note that X/T is a regular module. Since X/T is a factor of irreducible monomorphism $T \to X$ and any irreducible map to X/T is surjective, then X/T is a regular simple module. In particular $\underline{\dim}\, X/T \leq \delta$. The epimorphism $X \to M$ implies $(\underline{\dim}\, X)_i \geq (\underline{\dim}\, M)_i$, and hence $(\underline{\dim}\, X/T)_i \geq (\underline{\dim}\, M/T)_i$. Therefore, we have $\underline{\dim}\, M/T \leq \delta$. Since an indecomposable preinjective module is uniquely, up to isomorphism, determined by its dimension vector, there are only finitely many indecomposable preinjective modules with dimension vector $< \delta$. Thus, all but finitely many GR factor modules have indecomposable middle term. $\square$

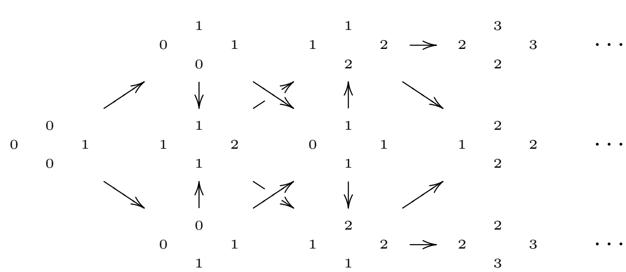
The next example shows that there exists non-injective preinjective Gabriel-Roiter factor module which has decomposable middle term.

Example. Let $\Lambda = k\widetilde{\mathbb{A}}_{2,2}$ with $\widetilde{\mathbb{A}}_{2,2} =$

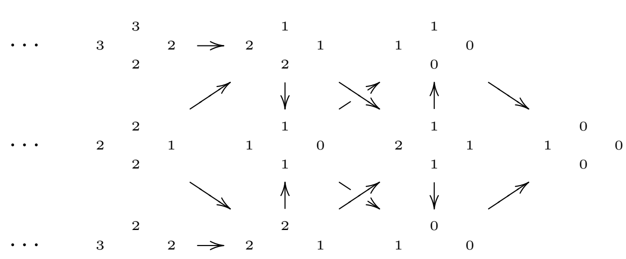
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 4 \\ & 3 \arrow[ur] & \end{tikzcd}$$

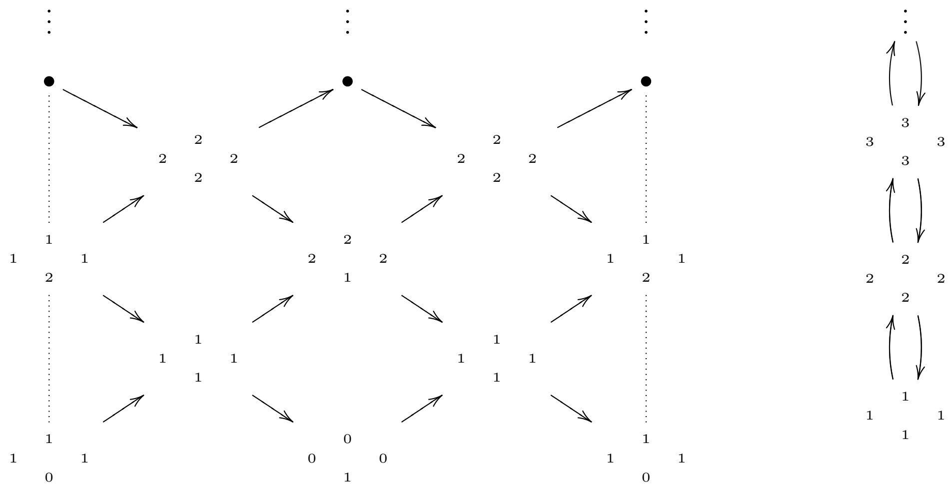
. We list the components of the AR-quiver in the following (only one exceptional regular component is presented here since the other one is "symmetric"):

preprojective component:



preinjective component:





We now show the non-injective preinjective module $\tau I_1 = \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}$ is a GR factor module.

Since all irreducible maps in the preprojective component are monomorphisms, all GR inclusions of preprojective modules are irreducible maps (3.1). We can easily calculate the GR measures for preprojective modules. Set $E_1[1] = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 0 & \end{smallmatrix}$, $E_2[1] = \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}$. These are two regular simple modules. Clearly, $P_2 = \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}$ is, up to isomorphism, the unique GR submodule of both $E_1[1]$ and $E_2[2]$, whereas $E_1[2]$ contains $E_1[1]$ as a GR submodule. Thus we have $\mu(E_2[2]) = \{1, 2, 4\}$ and $\mu(E_1[2]) = \{1, 2, 3, 4\}$. It is easy to see that $\tau^{-1}P_4 = \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}$ is a GR submodule of $E_2[3]$ and any homogeneous regular simple module. Thus $\mu(E_2[3]) = \{1, 2, 3, 5\}$ and $\mu(H_1) = \{1, 2, 3, 4\}$. Here we denote by H_1 any homogeneous regular simple module. These are the GR measures for all indecomposable modules whose dimension vectors are smaller than $X = \begin{smallmatrix} & 2 & \\ 2 & & 1 \\ & 2 & \end{smallmatrix}$.

Corollary 2.2 implies that the GR submodules of X are regular modules. An easy calculation shows all homogeneous regular simple modules are submodules of X, thus they are GR submodules by comparing the GR measures. Thus, $\mu(X) = \{1, 2, 3, 4, 7\}$ with GR factor module τI_1.

We now introduce some notations. Assume that Λ is of type $\widetilde{\mathbb{D}}_n$ or type $\widetilde{\mathbb{E}}_{6,7,8}$. Fix a vertex $j \in Q_0$, a **sectional sequence** to j is a sequence of vertices $i_1, i_2, \cdots, i_s = j$ such that i_k and i_{k+1} are connected by an edge. The sectional sequence to j is **complete** if i_1 is an ending vertex of Q. If $M = \tau^{-r}P_j$ is an indecomposable preprojective module, then we say a sectional path $X_{i_1} \to X_{i_2} \to \cdots \to X_{i_s} = M$ is complete if $\tau^{r_k}X_{i_k} = P_{i_k}$ and $i_i, i_2, \cdots, i_s$ is a complete sectional sequence to j. A sectional path $X_1 \to X_2 \to \cdots X_s = M$ to M is said to be **maximal** if any path $Y \to X_1 \to \cdots X_s = M$ of irreducible maps is not a sectional path for any Y. Then, a maximal sectional path $X_1 \to X_2 \to \cdots \to X_s = M$ being not complete implies that X_1 is projective.

For each indecomposable preprojective module M, we denote by $(\to M)$ the subquiver of the AR quiver consisting of all maximal sectional paths to M. We say $(\to M)$ is complete if each maximal sectional path to M is complete. We may also define $(M \to)$ for a preprojective module M. Note that $(M \to)$ is complete if M is not projective. Similarly, we have $(N \to)$ and $(\to N)$ for an indecomposable preinjective module and $(\to N)$ is complete if N is not injective.

The following lemma will be used in the proof of the theorems that follow:

Lemma 4.2. *Let*

$$0 \to A_1 \xrightarrow{\binom{f_1}{g_1}} B_1 \oplus A_2 \xrightarrow{(h_1, f_2)} B_2 \to 0$$

and

$$0 \to A_2 \xrightarrow{\binom{f_2}{g_2}} B_2 \oplus A_3 \xrightarrow{(h_2, f_3)} B_3 \to 0$$

be two exact sequences. Then the sequence

$$0 \to A_1 \xrightarrow{\binom{f_1}{g_2 g_1}} B_1 \oplus A_3 \xrightarrow{(h_2 h_1, -f_3)} B_3 \to 0$$

is exact.

Proof. Straightforward. □

Theorem 4.3. *Assume that Λ is of type $\widetilde{\mathbb{D}}_n$ and N is a GR factor module.*
(1) *If N is preprojective or regular, then $\alpha(N) = 1$.*
(2) *If N is preinjective with $\alpha(N) \geq 2$, then $\underline{\dim}\, N \leq \delta$.*
(3) *All but finitely many GR factor Λ-modules have indecomposable middle terms.*

Proof. We only prove (a) and (b), because the statement (c) easily follows from (a) and (b),

Let $0 \to T \to M \to N = M/T \to 0$ be a GR sequence. If M/T is regular, then it is a regular simple module and thus $\alpha(N) = 1$. Since there are only $n + 1$ indecomposable injective module and any indecomposable injective module is a thin module, we may assume that N is not injective. Clearly, for any indecomposable non-injective module X with $\alpha(X) = 3$ or 4, there exists an irreducible monomorphism ending at X. As an upshot, we assume that $\alpha(N) \leq 2$ and N is neither regular nor injective.

We first consider the case that $N = M/T$ is a preprojective module. If $\alpha(N) = 2$, then we get the following full subquiver of the AR quiver:

$$\begin{tikzcd}
X_{t+1} \arrow[dr] & & Y_1 \arrow[dr] & & & & Y_t \arrow[dr, "s"] & \\
Y'_0 \arrow[r] & X_t \arrow[ur] \arrow[r] \arrow[dr] & Y'_1 \arrow[r] & \bullet & \cdots & \bullet \arrow[ur] \arrow[r] \arrow[dr] & Y'_t \arrow[r, "h"] & Z_t \\
& & X_{t-1} \arrow[ur] \arrow[dr, dash, dashed] & & & & Z_{t-1} \arrow[ur] & \\
& & & X_1 \arrow[dr, "g"] & & Z_1 \arrow[ur, dash, dashed] & & \\
& & & & N \arrow[ur] \arrow[dr] & & & \\
& & & V_1 \arrow[ur, "f"] & & W_1 \arrow[dr, dash, dashed] & & \\
& & V_{r-1} \arrow[ur, dash, dashed] \arrow[dr] & & & & W_{r-1} \arrow[dr] & \\
U'_0 \arrow[r] & V_r \arrow[ur] \arrow[r] \arrow[dr] & U'_1 \arrow[r] & \bullet & \cdots & \bullet \arrow[ur] \arrow[r] \arrow[dr] & U'_r \arrow[r] & W_r \\
V_{r+1} \arrow[ur] & & U_1 \arrow[ur] & & & & U_r \arrow[ur] &
\end{tikzcd}$$

First assume that the maximal sectional path $X_{t+1}\to X_r\cdots\to X_1\to N$ is complete. Then X_{t+1} is obviously a preprojective module with defect -1. It follows from Lemma 2.1 that the composition of the irreducible maps $X_{t+1}\to\cdots\to X_1\to N$ is a monomorphism. Then it factors through M which implies $M\cong X_i$ for some $1\leq i\leq t$. If the sectional path $X_{t+1}\to X_r\cdots\to X_1\to N$ is not complete (X_{t+1} is zero), then some X_j is projective and by the property of projective modules, the composition $X_j\to\cdots\to X_1\to N$ factors through M. We obtain again that $M\cong X_i$ for some $1\leq i\leq j$. By the same reason, we have $M\cong V_s$ for some $1\leq s\leq r$. This contradiction tells N is not a GR factor. Therefore a preprojective GR factor has an indecomposable middle term and (a) follows.

Now we show (b) and assume that $N=M/T$ is a preinjective module with $\alpha(N)=2$. We use the above quiver and consider it as a full subquiver of the preinjective component.

Since N is not injective, we have that $(\to N)$ is complete, thus $X_{t+1}\neq 0$. If the composition of the irreducible maps $X_{t+1}\to\cdots\to X_1\to N$ is a monomorphism, then it factors through M. Therefore $M\cong X_i$ for some $1\leq i\leq t$. In particular, M and N both have defect 2. Then T has defect 0 and thus it is a regular module. Thus the unique irreducible monomorphism $T\to R$ satisfies that there is an epimorphism $R\to M$. It follows that $(\underline{\dim}\, R)_i\geq(\underline{\dim}\, M)_i$ and hence, $(\underline{\dim}\, R/T)_i\geq(\underline{\dim}\, M/T)_i=(\underline{\dim}\, N)_i$ for each i. Note that R/T is a regular simple module with dimension vector $\underline{\dim}\, R/T\leq\delta$. Thus, $\underline{\dim}\, N<\delta$.

Now we assume that the compositions $X_{t+1}(Y_0')\to X_t\to\cdots\to X_1\to N$ and $V_{r+1}(U_0')\to V_r\to\cdots V_1\to N$ are all epimorphisms. We first assume that Z_t, W_r are not zero. Starting with two short exact sequences:

$$0\to X_{t+1}\to Y_1'\oplus N\to Z_1\to 0$$

$$0\to Y_1'\to Y_2\oplus Z_1\to Z_2\to 0$$

we obtain the following short exact sequence by using Lemma 4.2:

$$\begin{cases} 0\to X_{t+1}\to N\oplus Y_t\to Z_t\to 0 & \text{if } t \text{ is even}\\ 0\to X_{t+1}\to N\oplus Y_t'\to Z_t\to 0 & \text{if } t \text{ is odd}\end{cases}$$

So $|Z_t|-|Y_t|=|N|-|X_{t+1}|<0$ or $|Z_t|-|Y_t'|=|N|-|X_{t+1}|<0$. It follows that $Y_t(Y_t'), Z_t$ are injective. By the same reason, $Y_t, Y_t', Z_t, U_r, U_r', W_r$ are all injective. An easy calculation shows that $(\underline{\dim}\, N)_j=\dim\mathrm{Hom}(N,I_j)\leq 1$ for all $j\in Q_0$.

Now assume that Z_t or W_r is zero, then some Z_i or W_j is injective. An easy calculation shows that $\dim\mathrm{Hom}(N,I_j)\leq 2$ for all non-extending vertices $j\in Q_0$ and $\dim\mathrm{Hom}(N,I_i)\leq 1$ for all extending vertices $i\in Q_0$. Hence $\underline{\dim}\, N\leq\delta$.

Note that there are only finitely many indecomposable preinjective modules whose dimension vectors are smaller than δ. The proof is complete. □

Lemma 4.4. *Let Λ be a hereditary algebra and $0\to T\to M\to M/T\to 0$ be a GR sequence such that M/T is not injective. Let $0\to\tau(M/T)\to X\to M/T\to 0$ be an Auslander-Reiten sequence. Then $|\tau^{-1}M|\geq|\tau^{-1}X|$ and equality holds if and only if $X\cong M$.*

Proof. Since M/T is not injective, by using $\tau^{-1}\cong\mathrm{Ext}^1(\mathrm{D}-,\Lambda)$, we get the following two short exact sequences:

$$0\to M/T\to\tau^{-1}X\to\tau^{-1}(M/T)\to 0$$

$$0 \to \tau^{-1}T \to \tau^{-1}M \to \tau^{-1}(M/T) \to 0$$

Therefore, $|\tau^{-1}M| = |\tau^{-1}X| - |M/T| + |\tau^{-1}T| \geq |\tau^{-1}X|$, and equality holds if and only if $|\tau^{-1}T| = |M/T|$. Recall that if T is a GR submodule of M, then M/T is a factor module of $\tau^{-1}T$ and $\tau^{-1}T \cong M/T$ if and only if $0 \to T \to M \to M/T \to 0$ is an Auslander-Reiten sequence (3.1). Thus, $|\tau^{-1}T| = |M/T|$ if and only if $\tau^{-1}T \cong M/T$, if and only if $M \cong X$. $\square$

Lemma 4.5. *Assume that Λ is of type $\widetilde{\mathbb{E}}_{6,7,8}$ and Y is an indecomposable preprojective Λ-module with $\alpha(Y) \geq 2$. Then there is an irreducible monomorphism ending at Y.*

Proof. We may assume that Y is not projective. Then $(Y \to)$ is complete and we can find in the AR quiver a complete sectional path $Y \to Y_1 \to \cdots \to Y_{r+1}$ with $\alpha(Y_i) = 2$ for all $1 \leq i < r+1$ and $\alpha(Y_{r+1}) = 1$

$$\begin{tikzcd}
& Y \arrow[dr] & & & \\
X \arrow[ur, "f"] \arrow[dr] & & Y_1 \arrow[dr, dash, dashed] & & \\
& X_1 \arrow[ur] \arrow[dr, dash, dashed] & & Y_r \arrow[dr] & \\
& & X_r \arrow[ur] & & Y_{r+1}
\end{tikzcd}$$

The irreducible map $f : X \to Y$ is an epimorphism implies the irreducible map $X_r \to Y_r$ is an epimorphism. This contradiction shows that f is injective. $\square$

Lemma 4.6. *Assume that $0 \to T \to M \to M/T \to 0$ is a GR sequence over a representation-finite hereditary algebra of type A_n. Then M/T is a uniserial module.*

See [4] for a proof.

Theorem 4.7. *Assume that Λ is of type $\widetilde{\mathbb{E}}_{6,7,8}$ and $0 \to T \to M \to M/T \to 0$ is a GR sequence of Λ-modules. Then $\alpha(M/T) = 1$ if M/T is not injective.*

Proof. We give a detailed proof for $\widetilde{\mathbb{E}}_8$. The proofs for the cases $\widetilde{\mathbb{E}}_6$ and $\widetilde{\mathbb{E}}_7$ are similar.

If M/T is a preprojective module with $\alpha(M/T) \geq 2$, we can always obtain an irreducible monomorphism to M/T by Lemma 4.5. Hence $\alpha(M/T) = 1$.

Now we assume that M/T is a preinjective module. We assume for a contradiction $\alpha(M/T) \geq 2$. If $\alpha(M/T) = 3$, there is an irreducible monomorphism to M/T since M/T is not injective (Note that for $\widetilde{\mathbb{E}}_6$, we may not get an irreducible monomorphism to M/T. But we can get irreducible monomorphisms $Z_1 \to Z_2 \to M/T$ such that the composition is monomorphism). We thus consider the case $\alpha(M/T) = 2$ and look at the following subquiver of the AR quiver:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & X_3 \arrow[dr] & & \bullet \arrow[dr] & & & & \\
 & & X_2 \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & & & \\
 & & & X_1 \arrow[ur] \arrow[r] \arrow[dr] & U \arrow[r] & V \arrow[dr] & & \\
 & & & & X \arrow[ur] \arrow[dr, "g_x"] & & \tau^{-1}X \arrow[dr] & \\
 & & & Z_1 \arrow[ur] \arrow[dr] & & N \arrow[ur] \arrow[dr] & & \tau^{-1}N \\
 & & Z_2 \arrow[ur] \arrow[dr] & & Y \arrow[ur, "g_y"] \arrow[dr] & & \tau^{-1}Y \arrow[ur] \arrow[dr, dotted] & \\
 & Z_3 \arrow[ur] \arrow[dr] & & Y_1 \arrow[ur, "g"] \arrow[dr] & & \tau^{-1}Y_1 \arrow[ur, "\tau^{-1}g"] \arrow[dr, dotted] & & \tau^{-2}Y_1 \\
Z_4 \arrow[ur] & & Y_2 \arrow[ur, "h"] & & \tau^{-1}Y_2 \arrow[ur] & & \tau^{-2}Y_2 \arrow[ur, dotted] &
\end{tikzcd}$$

(1). We use above quiver and assume that $M/T = X$. Since X is not injective, we easily obtain a short exact sequence $0 \to Z_4 \to X \to N \to 0$ by using Lemma 4.2. In particular, the composition of irreducible maps $Z_4 \to \cdots \to Z_1 \to X$ is a monomorphism. It follows that $M \cong Z_i$ for some $1 \le i \le 3$. On the other hand, again by Lemma 4.2, we have a short exact sequence $0 \to X_3 \to X \oplus U \to V \to 0$. Thus $|X| - |X_3| = |V| - |U| > 0$ if U is not injective. In this case, the composition of irreducible map $X_3 \to X_2 \to X_1 \to X$ is a monomorphism. Again, $M \cong X_j$ for some $1 \le j \le 2$. This contradiction implies that $M/T = X$ is not a GR factor module. If U is injective, then V and $\tau^{-1}X$ are injective. Since the irreducible map $Z_1 \to X$ is surjective (3.1), we have that N (or $\tau^{-1}Y$, or $\tau^{-2}Y_1$) is injective. (Namely, the irreducible map $Z_1 \to X$ is surjective implies g_y is surjective. Since $\tau^{-1}Y = 0$ implies N is injective, we may assume that $\tau^{-1}Y \neq 0$. It follows that $\tau^{-1}g$ is surjective. If $\tau^{-1}Y$ is not injective, we have $\tau^{-2}Y_1 \neq 0$ and the irreducible map $\tau^{-2}Y_2 \to \tau^{-2}Y_1$ is an epimorphism. It follows that $\tau^{-2}Y_1$ and $\tau^{-2}Y_2$ are both injective modules.) Thus the original quiver of $\widetilde{\mathbb{E}}_8$ is of the following form with vertex 5 (or 6,7) a source.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & 1 & & & & & \\
2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[u] & 5 \arrow[l] \arrow[r, dash] & 6 \arrow[r, dash] & 7 \arrow[r, dash] & 8 \arrow[r, dash] & 9
\end{tikzcd}$$

We can easily get $(\underline{\dim}\, M)_i = \dim \operatorname{Hom}(M, I_i) \le 1$ for all i. Therefore we may view M and $M/T = X$ as kA_8-modules, where A_8 is obtained from the graph $\widetilde{\mathbb{E}}_8$ by deleting the vertex 1.

By the above observation, $X = M/T$ is not a uniserial module. This contradicts Lemma 4.6 .

(2). Now we consider the case that $M/T = N$ as in the above full subquiver of the AR quiver.

As before, the composition $g_y g h$ is injective and hence, M lies on the sectional path $Y_2 \to Y_1 \to Y \to N$. Thus $M \cong Y$ or $M \cong Y_1$. If $M \cong Y$, then $|\tau^{-1}M| = |\tau^{-1}Y_1| < |\tau^{-1}Y| + |\tau^{-1}X|$ which contradicts Lemma 4.4. So we assume that $M \cong Y_1$.

Case 1. $\tau^{-1}Y_1$ and $\tau^{-1}Y_2$ are both injective.

In this case, we have $\tau^{-1}Y$ and $\tau^{-1}N$ are injective modules and the irreducible map $\tau^{-1}Y_1 \overset{\tau^{-1}g}{\to} \tau^{-1}Y$ is an epimorphism. Then $|\tau^{-1}Y_1| - |\tau^{-1}Y| = 1$ since $\tau^{-1}Y_1$ is injective. X is not injective implies $\tau^{-1}X \neq 0$. Also the irreducible map $\tau^{-1}X \to \tau^{-1}N$ is an epimorphism since $Y_1 \to \tau^{-1}Y_2$ is surjective. Thus $|\tau^{-1}X| > |\tau^{-1}N| \neq 0$ and $|\tau^{-1}M| = |\tau^{-1}Y_1| = |\tau^{-1}Y| + 1 < |\tau^{-1}Y| + |\tau^{-1}X|$.

Case 2. $\tau^{-1}Y_1$ is injective but $\tau^{-1}Y_2$ is not.
In this case, there is an irreducible map from $\tau^{-1}Y_1$ to the simple injective module $\tau^{-2}Y_2$ which means $\tau^{-1}Y_1/\mathrm{soc}\tau^{-1}Y_1$ has two direct summands. So $|\tau^{-1}Y_1| - |\tau^{-1}Y| = |\tau^{-2}Y_2| + 1 = 2$. we have $|\tau^{-1}M| = |\tau^{-1}Y_1| = |\tau^{-1}Y| + 2 \leq |\tau^{-1}Y| + |\tau^{-1}X|$ since $\tau^{-1}X$ is not simple.

Case 3. $\tau^{-1}Y_1$ is not injective.
In this case, $\tau^{-2}Y_2$ and $\tau^{-2}Y_1$ are not zero. g_y is an epimorphism implies the irreducible map $\tau^{-2}Y_2 \to \tau^{-2}Y_1$ is an epimorphism and hence $\tau^{-2}Y_2$, $\tau^{-2}Y_1$ are injective modules. $|\tau^{-1}Y_1| - |\tau^{-1}Y| = |\tau^{-2}Y_2| - |\tau^{-2}Y_1| = 1$. Therefore $|\tau^{-1}M| = |\tau^{-1}Y_1| = |\tau^{-1}Y| + 1 < |\tau^{-1}Y| + |\tau^{-1}X|$.

Note that the above three cases are all the possibilities since g_y is an epimorphisms. In all the cases, we get $|\tau^{-1}M| \leq |\tau^{-1}Y| + |\tau^{-1}X|$ which contradicts Lemma 4.4.

(3). Now we consider the case that $M/T = Y$ in the above picture. Then Y is not injective implies $0 \to Y_2 \to Y \to \tau^{-}Y_1 \to 0$ is a short exact sequence. In particular the composition of irreducible maps $gh : Y_2 \to Y$ is a monomorphism, thus factors through M. It follows that $M \cong Y_1$. On the other hand, $|\tau^{-}M| = |\tau^{-}Y_1| < |\tau^{-}Y_1| + |N|$ since $n \neq 0$, and we get a contradiction.

(4). In case $M/T = Y_1$, M/T is not injective implies that the irreducible map $Y_2 \to Y_1$ is a monomorphism, a contradiction. Thus M/T is not a GR factor. For the same reason, M/T cannot be at the position of X_2.

This finishes the proof. □

Acknowledgments. I am grateful to the referees for valuable comments and helpful suggestions which make the article more readable. I thank Steffen König for his hospitality during my stay at Universität zu Köln. I also wish to thank for the support from the Leverhulme Trust through the Academic Interchange Network "Algebras, Representation and Applications".

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