Homological algebra of noncommutative 'spaces' I.

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Introduction

A considerable part of this manuscript is based on the notes of a lecture course in noncommutative algebraic geometry given at Kansas State University during the Fall of 2005 and Spring of 2006 and on (the second half of) my lectures delivered at the School on Algebraic K-Theory and Applications which took place at the International Center for Theoretical Physics (ICTP) in Trieste during the last two weeks of May of 2007 (see [R2], or [R3]). The starting point of the actual lectures in Kansas was the homological algebra of exact categories as it is viewed by Keller and Vossieck [KeV]. Besides an optimization of the Quillen's definition of an exact category, they observed that the stable categories of exact categories with enough injectives have a *suspension* functor and *triangles* whose properties give a 'one-sided' version of Verdier's triangulated category, which they call a *suspended* category. A short account on this subject is given in Appendix K.

The main body of the text reflects attempts to find natural frameworks for fundamental homological theories which appear in noncommutative algebraic geometry. The first move in this direction is the replacing exact categories with a much wider class of *right exact* categories. These are categories endowed with a Grothendieck pretopology whose covers are strict epimorphisms. The dual structures, *left* exact categories, appear naturally and play a crucial role in a version of K-theory sketched in Section 7 of this work.

Sections 1 and 2 contain generalities on right exact categories. In Section 1, we introduce *right exact* (not necessarily additive) categories and sketch their basic properties. We define *Karoubian* right exact categories and prove the existence (under certain conditions) of the *Karoubian envelope* of a right exact category. We observe that any k-linear right exact category is canonically realized as a subcategory of an exact k-linear category – its *exact envelope*. In Section 2, we consider right exact categories with initial objects. The existence of initial (resp. final) objects allows to introduce the notion of the kernel (resp. cokernel) of a morphism. Most of the section is devoted to some elementary properties of the kernels of morphisms, which are well known in the abelian case.

Section 3 is dedicated to *satellites* on right exact categories. Its content might be regarded as a non-abelian and non-additive (that is not necessarily abelian or additive) version of the classical theory of derived functors. We introduce ∂^*-functors and prove the existence of the universal ∂^*-functors on a given right exact 'space' with values in categories with kernels of morphisms and limits of filtered diagrams. We establish the existence of a *universal 'exact' ∂^*-functor* on a given right exact 'space'.

The latter subject naturally leads to a general notion of the *costable* category of a right exact category, which appears in Section 4. We obtain (by turning properties of costable categories into axioms) the notion of a (not necessarily additive) *cosuspended* category. We introduce the notion of a *homological* functor on a cosuspended category and prove the existence of a universal homological functor.

In Section 5, we introduce projective objects of right exact categories (and injective objects of left exact categories). They play approximately the same role as in the classical case: every universal ∂^*-functor annihilates *pointable projectives* (we call this way projectives which have morphisms to initial objects); and if the right exact category has enough projectives, then every 'exact' ∂^*-functor which annihilates all projectives is universal.

Starting from Section 6, a (noncommutative) geometric flavor becomes a part of the picture: we interpret svelte right exact categories as dual objects to (noncommutative) *right exact* 'spaces' and 'exact' functors between them as inverse image functors of morphisms of 'spaces'. We introduce a natural left exact structure on the category of right exact 'spaces'. Inverse image functors of its *inflations* are certain localizations functors.

Section 7 is dedicated to the first applications: the *universal* K-theory of right exact 'spaces'. We define the functor K_0 and then introduce higher K-functors as satellites of K_0. More precisely, the K-functor appears as a universal contravariant ∂^*-functor on a left exact category *over* the left exact category of right exact 'spaces'. Here 'over' means an 'exact' functor to the category left exact 'spaces'. Our K-functor has exactness properties which are expessed by the long 'exact' sequences corresponding to those 'exact' localizations which are inflations. In the abelian case, every exact localization is an inflation. Quillen's localization theorem states that the restriction of his K-functor to abelian categories has a natural structure of a an 'exact' ∂-functor. It follows from the universality of the K-theory defined here, that there exists a unique morphism from the Quillen's K-functor $K^{\mathfrak{Q}}_\bullet$ to the universal K-functor $K^{\mathfrak{q}}_\bullet$ defined on the left exact category of 'spaces' represented by abelian categories. In Section 8, we introduce infinitesimal 'spaces'. We establish some general facts about satellites and then, as an application, obtain the devissage theorem in K-theory. It is worth to mention that infinitesimal 'spaces' is a serious issue in noncommutative (and commutative) geometry: they serve as a base of a noncommutative version of Grothendieck-Berthelot crystalline theory and are of big importance for the D-module theory on noncommutative 'spaces'. We make here only a very little use of them leaving a more ample development to consequent papers.

The remaining five sections appear under the general title "complementary facts".

In Section C1 (which complements Section 3), we look at some examples, which acquire importance somewhere in the text. In Section C2, we pay tribute to standard techniques of homological algebra by expanding the most popular facts on diagram chasing to right exact categories. They appear here mainly as a curiosity and are used only once in the main body of the manuscript. Section C3 is dedicated to localizations of exact and (co)suspended categories. In particular, t-structures of (co)suspended categories appear on the scene. Again, a work by Keller and Vossieck, [KV1], suggested the notions. Section C4 is dedicated to cohomological functors on suspended categories and can be regarded as a natural next step after the works [KeV] and [Ke1]. It is heavily relied on Appendix K, where the basic facts on exact and suspended categories are gathered, following the approach of B. Keller and D. Vossieck [KeV], [KV1], [Ke2], except for some complements and most of proofs, which are made more relevant to the rest of the work. We consider cohomological functors on suspended categories with values in exact categories and prove the existence of a universal cohomological functor. The construction of the universal functor gives, among other consequences, an equivalence between the bicategory of Karoubian suspended svelte

categories with triangle functors as 1-morphisms and the bicategory of exact svelte $\mathbb{Z}_+$-categories with enough injectives whose 1-morphisms are 'exact' functors. We show that if the suspended category is triangulated, then the universal cohomological functor takes values in an abelian category, and our construction recovers the abelianization of triangulated categories by Verdier [Ve2]. It is also observed that the *triangulation* of suspended categories induces an abelianization of the corresponding exact $\mathbb{Z}_+$-categories. We conclude with a discussion of homological dimension and resolutions of suspended categories and exact categories with enough injectives. These resolutions suggest that the 'right' objects to consider from the very beginning are exact (resp. abelian) and (co)suspended (resp. triangulated) $\mathbb{Z}_+^n$-categories. All the previously discussed facts (including the content of Appendix K) extend easily to this setting. In Section C5, we define the *weak costable* category of a right exact category as the localization of the right exact category at a certain class of arrows related with its projectives. If the right exact category in question is exact, then its costable category is isomorphic to the costable category in the conventional sense (reminded in Appendix K). If a right exact category has enough *pointable* projectives (in which case all its projectives are pointable), then its weak costable category is naturally equivalent to the *costable* category of this right exact category defined in Section 4. We study right exact categories of modules over monads and associated stable and costable categories. The general constructions acquire here a concrete shape. We introduce the notion of a *Frobenius* monad. The category of modules over a Frobenius monad is a Frobenius category, hence its stable category is triangulated. We consider the case of modules over an augmented monad which includes as special cases most of standard homological algebra based on complexes and their homotopy and derived categories.

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1. Right exact categories.

1.1. Right exact categories and (right) 'exact' functors. We define a *right exact* category as a pair $(C_X, \mathfrak{E}_X)$, where C_X is a category and $\mathfrak{E}_X$ is a pretopology on C_X whose covers are *strict epimorphisms*; that is for any element $M \longrightarrow L$ of $\mathfrak{E}$ (– a cover), the diagram $M \times_L M \rightrightarrows M \longrightarrow L$ is exact. This requirement means precisely that the pretopology $\mathfrak{E}_X$ is subcanonical; i.e. every representable presheaf is a sheaf. We call the elements of $\mathfrak{E}_X$ *deflations* and assume that all isomorphisms are deflations.

1.1.1. The coarsest and the finest right exact structures. The coarsest right exact structure on a category C_X is the discrete pretopology: the class of deflations coincides with the class $Iso(C_X)$ of all isomorphisms of the category C_X.

Let $\mathfrak{E}_X^{\mathfrak{s}}$ denote the class of all *universally strict* epimorphisms of C_X; i.e. elements of $\mathfrak{E}_X^{\mathfrak{s}}$ are strict epimorphisms $M \xrightarrow{\mathfrak{e}} N$ such that for any morphism $\widetilde{N} \xrightarrow{f} N$, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle\widetilde{\mathfrak{e}}}\downarrow & cart & \downarrow{\scriptstyle\mathfrak{e}} \\ \widetilde{N} & \xrightarrow{f} & N \end{array}$$

whose left vertical arrow is a strict epimorphism. It follows that $\mathfrak{E}_X^{\mathfrak{s}}$ is the finest right exact structure on the category C_X. We call this structure *canonical*.

If C_X is an abelian category or a topos, then $\mathfrak{E}_X^{\mathfrak{s}}$ consists of all epimorphisms.

If C_X is a quasi-abelian category, then $\mathfrak{E}_X^{\mathfrak{s}}$ consists of all strict epimorphisms.

1.1.2. Right 'exact' and 'exact' functors. Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories. A functor $C_X \xrightarrow{F} C_Y$ will be called *right 'exact'* (resp. *'exact'*) if it maps deflations to deflations and for any deflation $M \xrightarrow{\mathfrak{e}} N$ of $\mathfrak{E}_X$ and any morphism $\widetilde{N} \xrightarrow{f} N$, the canonical arrow

$$F(\widetilde{N} \times_N M) \longrightarrow F(\widetilde{N}) \times_{F(N)} F(M)$$

is a deflation (resp. an isomorphism). Thus, the functor F is 'exact' if it maps deflations to deflations and preserves pull-backs of deflations.

1.1.3. Weakly right 'exact' and weakly 'exact' functors. A functor $C_X \xrightarrow{F} C_Y$ is called *weakly right 'exact'* (resp. *weakly 'exact'*) if it maps deflations to deflations and for any arrow $M \longrightarrow N$ of $\mathfrak{E}_X$, the canonical morphism

$$F(M \times_N M) \longrightarrow F(M) \times_{F(N)} F(M)$$

is a deflation (resp. an isomorphism). In particular, weakly 'exact' functors are weakly right 'exact'.

1.1.4. Note. Of cause, 'exact' (resp. right 'exact') functors are weakly 'exact' (resp. weakly right 'exact'). In the additive (actually, a more general) case, weakly 'exact' functors are 'exact' (see 2.5 and 2.5.2).

1.2. Proposition. *(a) Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category. The Yoneda embedding induces an 'exact' fully faithful functor $(C_X, \mathfrak{E}_X) \xrightarrow{\mathfrak{j}_X^*} (C_{X_\mathfrak{E}}, \mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}})$, where $C_{X_\mathfrak{E}}$ is the category of sheaves of sets on the presite $(C_X, \mathfrak{E}_X)$ and $\mathfrak{E}^{\mathfrak{s}}_{X_\mathfrak{E}}$ the family of all universally strict epimorphisms of $C_{X_\mathfrak{E}}$ (– the canonical structure of a right exact category).*

(b) Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and $(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$ a right weakly 'exact' functor. There exists a functor $C_{X_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_\mathfrak{E}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ {\scriptstyle \mathfrak{j}_X^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}_Y^*} \\ C_{X_\mathfrak{E}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_\mathfrak{E}} \end{array}$$

quasi commutes, i.e. $\widetilde{\varphi}^* \mathfrak{j}_X^* \simeq \mathfrak{j}_Y^* \varphi^*$. The functor $\widetilde{\varphi}^*$ is defined uniquely up to isomorphism and has a right adjoint, $\widetilde{\varphi}_*$.

Proof. (a) The argument is the same as the part (i) of the proof of K5.2.
(b) The argument coincides with the proof of K5.4. ∎

1.3. Interpretation: 'spaces' represented by right exact categories. Right weakly 'exact' functors will be interpreted as inverse image functors of morphisms between 'spaces' represented by right exact categories. We consider the category $\mathfrak{Esp}^{\mathfrak{w}}_{\mathfrak{r}}$ whose objects are pairs $(X, \mathfrak{E}_X)$, where $(C_X, \mathfrak{E}_X)$ is a svelte right exact category. A morphism from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ is a morphism of 'spaces' $X \xrightarrow{\varphi} Y$ whose inverse image functor $C_Y \xrightarrow{\varphi^*} C_X$ is a right weakly 'exact' functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_X, \mathfrak{E}_X)$. The map which assigns to every 'space' X the pair $(X, Iso(C_X))$ is a full embedding of the category $|Cat|^o$ of 'spaces' into the category $\mathfrak{Esp}^{\mathfrak{w}}_{\mathfrak{r}}$. This full embedding is a right adjoint functor to the forgetful functor

$$\mathfrak{Esp}^{\mathfrak{w}}_{\mathfrak{r}} \longrightarrow |Cat|^o, \quad (X, \mathfrak{E}_X) \longmapsto X.$$

1.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be additive right exact categories and $C_X \xrightarrow{F} C_Y$ an additive functor. Then*

(a) The functor F is right weakly 'exact' iff it maps deflations to deflations and the sequence

$$F(Ker(\mathfrak{e})) \longrightarrow F(M) \xrightarrow{F(\mathfrak{e})} F(N) \longrightarrow 0$$

is exact for any deflation $M \xrightarrow{\mathfrak{e}} N$.

(b) The functor F is weakly 'exact' iff it maps deflations to deflations and the sequence

$$0 \longrightarrow F(Ker(\mathfrak{e})) \longrightarrow F(M) \xrightarrow{F(\mathfrak{e})} F(N) \longrightarrow 0$$

is 'exact' for any deflation $M \xrightarrow{\mathfrak{e}} N$.

Proof. (a) Notice that each arrow of $\mathfrak{E}_X$ has a kernel, because the square

$$\begin{array}{ccc} Ker(\mathfrak{e}) & \longrightarrow & M \\ \downarrow & cart & \downarrow \mathfrak{e} \\ 0 & \longrightarrow & N \end{array}$$

is cartesian, and it exists when $\mathfrak{e} \in \mathfrak{E}_X$. This observation allows to use the argument of K4.3 which proves the assertion.

(b) If the category C_X is additive and there exists the kernel of $M \xrightarrow{\mathfrak{e}} N$, then $M \times_N M$ is canonically isomorphic to the coproduct of M and $Ker(\mathfrak{e})$. In fact, we have a commutative diagram

$$\begin{array}{ccccccc} & & & & \overset{\sim}{\longrightarrow} & & \\ & & Ker(p_2) & & \underset{0}{\longrightarrow} & Ker(\mathfrak{e}) & \\ & & \mathfrak{k}_2 \downarrow & & & \downarrow \mathfrak{j} & \\ M & \xrightarrow{\Delta_M} & M \times_N M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \qquad (1)$$

(borrowed from the argument of K4.3). Its left vertical arrow and the diagonal morphism $M \xrightarrow{\Delta_M} M \times_N M$ determine an isomorphism $M \oplus Ker(p_2) \xrightarrow{\sim} M \times_N M$, which implies an isomorphism $M \oplus Ker(\mathfrak{e}) \xrightarrow{\sim} M \times_N M$. Since the functor F is additive, we have canonical isomorphisms $F(M \times_N M) \simeq F(M \oplus Ker(\mathfrak{e})) \simeq F(M) \oplus F(Ker(\mathfrak{e}))$. There is a commutative diagram

$$\begin{array}{ccc} F(M \times_N M) & \xrightarrow{\alpha} & F(M) \times_{F(N)} F(M) \\ \wr \downarrow & & \downarrow \wr \\ F(M) \oplus F(Ker(\mathfrak{e})) & \xrightarrow{id \oplus \beta} & F(M) \oplus Ker(F(\mathfrak{e})) \end{array} \qquad (2)$$

in which $F(M \times_N M) \xrightarrow{\alpha} F(M) \times_{F(N)} F(M)$ and $F(Ker(\mathfrak{e})) \xrightarrow{\beta} Ker(F(\mathfrak{e}))$ are natural morphisms. Since the vertical arrows of (2) are isomorphisms, this shows that α is an isomorphism iff β is an isomorphism. This and (a) imply that the functor F is weakly 'exact' iff the sequence

$$0 \longrightarrow F(Ker(\mathfrak{e})) \longrightarrow F(M) \xrightarrow{F(\mathfrak{e})} F(N) \longrightarrow 0$$

is exact for every deflation $M \xrightarrow{\mathfrak{e}} N$. ∎

1.5. Karoubian envelopes of categories and right exact categories.

1.5.1. Lemma. *Let M be an object of a category C_X and $M \xrightarrow{p} M$ an idempotent (i.e. $p^2 = p$). The following conditions are equivalent:*

(a) The idempotent p splits, i.e. p is the composition of morphisms $M \xrightarrow{\mathfrak{e}} N \xrightarrow{\mathfrak{j}} M$ such that $\mathfrak{e} \circ \mathfrak{j} = id_N$.

(b) There exists a cokernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$.

(c) There exists a kernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$.

If the equivalent conditions above hold, then $Ker(id_{_M}, p) \simeq Coker(id_{_M}, p)$.

Proof. $(b) \Leftarrow (a) \Rightarrow (c)$. If the idempotent $M \xrightarrow{p} M$ is the composition of $M \xrightarrow{\mathfrak{e}} N$ and $N \xrightarrow{\mathfrak{j}} M$ such that $\mathfrak{e} \circ \mathfrak{j} = id_N$, then $M \xrightarrow{\mathfrak{e}} N$ is a cokernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$, because $\mathfrak{e} \circ p = \mathfrak{e} \circ \mathfrak{j} \circ \mathfrak{e} = \mathfrak{e}$ and if $M \xrightarrow{\mathfrak{t}} L$ any morphism such that $\mathfrak{t} \circ p = \mathfrak{t}$, then $\mathfrak{t} = (\mathfrak{t} \circ \mathfrak{j}) \circ \mathfrak{e}$. Since $\mathfrak{e}$ is an epimorphism, there is only one morphism g such that $\mathfrak{t} = g \circ \mathfrak{e}$. This shows that $(a) \Rightarrow (b)$. The implication $(a) \Rightarrow (c)$ follows by duality.

$(b) \Rightarrow (a)$. Let $M \xrightarrow{\mathfrak{e}} N$ be a cokernel of the pair $M \overset{id_M}{\underset{p}{\rightrightarrows}} M$. Since $p \circ p = p$, there exists a unique morphism $N \xrightarrow{\mathfrak{j}} M$ such that $p = \mathfrak{j} \circ \mathfrak{e}$. Since $\mathfrak{e} \circ \mathfrak{j} \circ \mathfrak{e} = \mathfrak{e} \circ p = \mathfrak{e} = id_N \circ \mathfrak{e}$ and $\mathfrak{e}$ is an epimorphism, $\mathfrak{e} \circ \mathfrak{j} = id_N$.

The implication $(c) \Rightarrow (a)$ follows by duality. ∎

1.5.2. Definition. A category C_X is called *Karoubian* if each idempotent in C_X splits. It follows from 1.5.1 that C_X is a Karoubian category iff for every idempotent $M \xrightarrow{p} M$ in C_X, there exists a kernel (equivalently, a cokernel) of the pair $(id_{_M}, p)$.

1.5.3. Proposition. *For any category* C_X*, there exists a Karoubian category* C_{X_K} *and a fully faithful functor* $C_X \xrightarrow{\mathfrak{k}^*_X} C_{X_K}$ *such that any functor from* C_X *to a Karoubian category factors uniquely up to a natural isomorphism through* $\mathfrak{k}^*_X$*. Every object of* C_{X_K} *is a retract of an object* $\mathfrak{k}^*_X(M)$ *for some* $M \in ObC_X$.

Proof. Objects of the category C_{X_K} are pairs (M, p), where M is an object of the category C_X and $M \xrightarrow{p} M$ is an idempotent endomorphism, i.e. $p^2 = p$. Morphisms $(M, p) \longrightarrow (M', p')$ are morphisms $M \xrightarrow{f} M'$ such that $fp = f = p'f$. The composition of $(M, p) \xrightarrow{f} (M', p')$ and $(M', p') \xrightarrow{g} (M'', p'')$ is $(M, p) \xrightarrow{gf} (M'', p'')$. It follows from this definition that $(M, p) \xrightarrow{p} (M, p)$ is the identical morphism. If $(M, p) \xrightarrow{q} (M, p)$ is an idempotent, then it splits into the composition of $(M, p) \xrightarrow{q} (M, q)$ and $(M, q) \xrightarrow{q} (M, p)$. The composition of $(M, q) \xrightarrow{q} (M, p) \xrightarrow{q} (M, q)$ is $(M, q) \xrightarrow{q} (M, q)$, which is the identical morphism. The functor $C_X \xrightarrow{\mathfrak{k}^*_X} C_{X_K}$ assigns to each object M of C_X the pair $(M, id_{_M})$ and to each morphism $M \xrightarrow{g} N$ the morphism $(M, id_{_M}) \xrightarrow{g} (N, id_N)$.

For any functor $C_X \xrightarrow{F} C_Z$ to a Karoubian category C_Z, let $C_{X_K} \xrightarrow{F_K} C_Z$ denote a functor which assigns to every object (M, p) of the category C_{X_K} the kernel of the pair $(id_{F(M)}, F(p))$. It follows that $F_K \circ \mathfrak{K}^*_X \simeq F$. In particular, for any functor $C_X \xrightarrow{F} C_Y$,

there exists a natural functor $C_{X_K} \xrightarrow{F_K} C_{X_K}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{F} & C_Y \\ {\scriptstyle \mathfrak{K}_X^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{K}_Y^*} \\ C_{X_K} & \xrightarrow{F_K} & C_{Y_K} \end{array}$$

quasi-commutes. The map $F \longmapsto F_K$ defines a (pseudo) functor from Cat to the category $KCat$ of Karoubian categories which is a left adjoint to the inclusion functor. This implies, in particular, the universal property of the correspondence $C_X \longmapsto C_{X_K}$.

For every object (M,p) of the category C_{X_K}, the morphism $(M,p) \xrightarrow{p} (M, id_M)$ splits; i.e. (M,p) is a retract of $\mathfrak{k}_X^*(M) = (M, id_M)$. ∎

The category C_{X_K} in 1.5.3 is called the *Karoubian envelope* of the category C_X.

1.5.4. Karoubian envelopes of right exact categories.

1.5.4.1. Definition. We call a right exact category $(C_X, \mathfrak{E}_X)$ *Karoubian* if the category C_X is Karoubian and any split epimorphism of the category C_X is a deflation.

1.5.4.2. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category. Suppose that for every idempotent $M \xrightarrow{p} M$ in C_X and every morphism $N \xrightarrow{f} M$ such that $f = pf$, there exists a cartesian square*

$$\begin{array}{ccc} N' & \xrightarrow{f'} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array}$$

Then the Karoubian envelope C_{X_K} of C_X has a structure $\mathfrak{E}_{X_K}$ of a right exact Karoubian category such that the canonical functor $C_X \xrightarrow{\mathfrak{k}_X^*} C_{X_K}$ is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_{X_K}, \mathfrak{E}_{X_K})$. The right exact Karoubian category $(C_{X_K}, \mathfrak{E}_{X_K})$ is universal in the following sense: every (weakly) right exact functor from the right exact category $(C_X, \mathfrak{E}_X)$ to a right exact Karoubian category $(C_Y, \mathfrak{E}_Y)$ is uniquely represented as the composition of the canonical exact, hence 'exact', functor from $(C_X, \mathfrak{E}_X)$ to its Karoubian envelope $(C_{X_K}, \mathfrak{E}_{X_K})$ and a (weakly) right exact furnctor from $(C_{X_K}\mathfrak{E}_{X_K})$ to $(C_Y, \mathfrak{E}_Y)$.

Proof. (a) Let C_X be a category and $M \xrightarrow{\mathfrak{e}} L$ a split epimorphism; i.e. there exists a morphism $L \xrightarrow{\mathfrak{j}} M$ such that $\mathfrak{e} \circ \mathfrak{j} = id_L$. Let $N \xrightarrow{g} L$ be a morphism. Since $\mathfrak{j}$ is a monomorphism, a pullback of $N \xrightarrow{g} L \xleftarrow{\mathfrak{e}} M$ exists iff a pullback of $N \xrightarrow{\mathfrak{j}g} M \xleftarrow{\mathfrak{j}\mathfrak{e}} M$. exists and they are isomorphic to each other. Notice that $p = \mathfrak{j}\mathfrak{e}$ is an idempotent and a morphism $N \xrightarrow{f} M$ factors through $L \xrightarrow{\mathfrak{j}} M$ iff $f = pf$. Thus, we have cartesian squares

$$\begin{array}{ccc} N' & \xrightarrow{f'} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} N' & \xrightarrow{f'} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ N & \xrightarrow{g} & L \end{array}$$

It follows from the right cartesian square that the morphism $\mathfrak{e}'$ is a split epimorphism, because it is a pullback of a split epimorphism.

(b) Suppose that the condition of 1.5.4.2 holds, and consider a pair of morphisms $(N,u) \xrightarrow{f} (M,q) \xleftarrow{q} (M,p)$ of the Karoubian envelope C_{X_K}. By definition, $fu = qf = f$ and $qp = pq = q$. By the hypothesis, there exists a pullback $N \times_{f,q} M$. The equality $qf = f$ implies that the projection $N \times_{f,q} M \xrightarrow{q'} N$ splits, i.e. there exists a morphism $N \xrightarrow{\mathfrak{j}'} N \times_{f,q} M$ such that $q'\mathfrak{j}' = id_N$. Set $u' = \mathfrak{j}'q'$. Then

$$\begin{array}{ccc} (N \times_{f,q} M, u') & \xrightarrow{f'} & (M,p) \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ (N,u) & \xrightarrow{f} & (M,q) \end{array}$$

is a cartesian square in C_{X_K}. This shows that split epimorphisms of C_{X_K} are stable under base change. The class of deflations $\mathfrak{E}_{X_K}$ consists of all possible compositions of morphisms of $\mathfrak{k}_X^*(\mathfrak{E}_X)$ and split epimorphisms.

(c) By the universal property of Karoubian envelopes, any functor $C_X \xrightarrow{F} C_Y$ is represented as the composition of the canonical embedding $C_X \longrightarrow C_{X_K}$ and a uniquely determined functor $C_{X_K} \xrightarrow{\widetilde{F}} C_Y$. If F is a (weakly) right 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$, then $\widetilde{F}$ is a (resp. weakly) right 'exact' morphism from the Karoubian envelope $(C_{X_K}, \mathfrak{E}_{X_K})$ of $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. ∎

1.5.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories. Suppose that $\mathfrak{E}_X$ consists of split deflations. Then a functor $C_X \xrightarrow{F} C_Y$ is a weakly right 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ iff it maps deflations to deflations.*

Proof. Let $M \xrightarrow{\mathfrak{e}} N$ be a split epimorphism in C_X and $N \xrightarrow{\mathfrak{j}} M$ its section. Set $\mathfrak{p} = \mathfrak{j} \circ \mathfrak{e}$. Suppose that $M \times_N M$ exists (which is the case if $\mathfrak{e} \in \mathfrak{E}_X$). Then we have a commutative diagram

$$\begin{array}{ccccc} M & \overset{\mathfrak{p}}{\underset{id_M}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle \mathfrak{t}}\downarrow & & \downarrow{\scriptstyle id_M} & & \downarrow{\scriptstyle id_N} \\ M \times_L M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \qquad (1)$$

whose left vertical arrow, $\mathfrak{t}$, is uniquely determined. A functor $C_X \xrightarrow{F} C_Y$ maps (1) to the commutative diagram

$$\begin{array}{ccccc} F(M) & \overset{F(\mathfrak{p})}{\underset{id}{\rightrightarrows}} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(N) \\ {\scriptstyle F(\mathfrak{t})}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ F(M \times_L M) & \overset{F(p_1)}{\underset{F(p_2)}{\rightrightarrows}} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(N) \end{array} \qquad (2)$$

whose upper row is an exact diagram (by 1.5.1). Therefore, the lower row of (2) is an exact diagram. The assertion follows now from the definition of a weakly right 'exact' functor. ∎

1.5.6. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category whose deflations are split. Then every presheaf of sets on $(C_X, \mathfrak{E}_X)$ is a sheaf.*

2. Right exact categories with initial objects.

2.1. Kernels and cokernels of morphisms. Let C_X be a category with an initial object, x. For a morphism $M \xrightarrow{f} N$ we define the *kernel of f* as the upper horizontal arrow in a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \longrightarrow & N \end{array}$$

when the latter exists.

Cokernels of morphisms are defined dually, via a cocartesian square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{c}(f)} & Cok(f) \\ {\scriptstyle f}\uparrow & cocart & \uparrow{\scriptstyle f'} \\ M & \longrightarrow & y \end{array}$$

where y is a final object of C_X.

If C_X is a pointed category (i.e. its initial objects are final), then the notion of the kernel is equivalent to the usual one: the diagram $Ker(f) \xrightarrow{\mathfrak{k}(f)} M \underset{0}{\overset{f}{\rightrightarrows}} N$ is exact.

Dually, the cokernel of f makes the diagram $M \underset{0}{\overset{f}{\rightrightarrows}} N \xrightarrow{\mathfrak{c}(f)} Cok(f)$ exact.

2.1.1. Lemma. *Let C_X be a category with an initial object x.*

(a) Let a morphism $M \xrightarrow{f} N$ of C_X have a kernel. The canonical morphism $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism, if the unique arrow $x \xrightarrow{i_N} N$ is a monomorphism.

(b) If $M \xrightarrow{f} N$ is a monomorphism, then $x \xrightarrow{i_M} M$ is the kernel of f.

Proof. (a) By definition of the kernel of f, we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

Therefore, $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism if $x \xrightarrow{i_N} N$ is a monomorphism.

(b) Suppose that $M \xrightarrow{f} N$ is a monomorphism. If

$$\begin{array}{ccc} L & \xrightarrow{\phi} & x \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle i_N} \\ M & \xrightarrow{f} & N \end{array}$$

is a commutative square, then f equalizes the pair of arrows $(\psi, i_M \circ \phi)$. If f is a monomorphism, the latter implies that $\psi = i_M \circ \phi$. Therefore, in this case, the square

$$\begin{array}{ccc} x & \xrightarrow{id_x} & x \\ {\scriptstyle i_M}\downarrow & & \downarrow{\scriptstyle i_N} \\ M & \xrightarrow{f} & N \end{array}$$

is cartesian. ■

2.1.2. Corollary. *Let C_X be a category with an initial object x. The following conditions are equivalent:*

(a) If $M \xrightarrow{f} N$ has a kernel, then the canonical arrow $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ is a monomorphism.

(b) The unique arrow $x \xrightarrow{i_M} M$ is a monomorphism for any $M \in ObC_X$.

Proof. $(a) \Rightarrow (b)$, because, by 2.1.1(b), the unique morphism $x \xrightarrow{i_M} M$ is the kernel of the identical morphism $M \longrightarrow M$. The implication $(b) \Rightarrow (a)$ follows from 2.1.1(a). ■

2.1.3. Note. The converse assertion is not true in general: a morphism might have a trivial kernel without being a monomorphism. It is easy to produce an example in the category of pointed sets.

2.2. Examples.

2.2.1. Kernels of morphisms of unital k-algebras. Let C_X be the category Alg_k of associative unital k-algebras. The category C_X has an initial object – the k-algebra k. For any k-algebra morphism $A \xrightarrow{\varphi} B$, we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\mathfrak{k}(\varphi)}\uparrow & & \uparrow \\ k \oplus K(\varphi) & \xrightarrow{\epsilon(\varphi)} & k \end{array}$$

where $K(\varphi)$ denote the kernel of the morphism φ in the category of non-unital k-algebras and the morphism $\mathfrak{k}(\varphi)$ is determined by the inclusion $K(\varphi) \longrightarrow A$ and the k-algebra structure $k \longrightarrow A$. This square is cartesian. In fact, if

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\gamma}\uparrow & & \uparrow \\ C & \xrightarrow{\psi} & k \end{array}$$

is a commutative square of k-algebra morphisms, then C is an augmented algebra: $C = k \oplus K(\psi)$. Since the restriction of $\varphi \circ \gamma$ to $K(\psi)$ is zero, it factors uniquely through $K(\varphi)$. Therefore, there is a unique k-algebra morphism $C = k \oplus K(\psi) \xrightarrow{\beta} Ker(\varphi) = k \oplus K(\varphi)$ such that $\gamma = \mathfrak{k}(\varphi) \circ \beta$ and $\psi = \epsilon(\varphi) \circ \beta$.

This shows that each (unital) k-algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel $Ker(\varphi)$ equal to the augmented k-algebra corresponding to the ideal $K(\varphi)$.

It follows from the description of the kernel $Ker(\varphi) \xrightarrow{\mathfrak{k}(\varphi)} A$ that it is a monomorphism iff the k-algebra structure $k \longrightarrow A$ is a monomorphism.

Notice that cokernels of morphisms are not defined in Alg_k, because this category does not have final objects.

2.2.2. Kernels and cokernels of maps of sets. Since the only initial object of the category $Sets$ is the empty set $\emptyset$ and there are no morphisms from a non-empty set to $\emptyset$, the kernel of any map $X \longrightarrow Y$ is $\emptyset \longrightarrow X$. The cokernel of a map $X \xrightarrow{f} Y$ is the projection $Y \xrightarrow{\mathfrak{c}(f)} Y/f(X)$, where $Y/f(X)$ is the set obtained from Y by the contraction of $f(X)$ into a point. So that $\mathfrak{c}(f)$ is an isomorphism iff either $X = \emptyset$, or $f(X)$ is a one-point set.

2.2.3. Presheaves of sets. Let C_X be a svelte category and $C_X^\wedge$ the category of non-trivial presheaves of sets on C_X (that is we exclude the *trivial* presheaf which assigns to every object of C_X the empty set). The category $C_X^\wedge$ has a final object which is the constant presheaf with values in a one-element set. If C_X has a final object, y, then $\widehat{y} = C_X(-, y)$ is a final object of the category $C_X^\wedge$. Since $C_X^\wedge$ has small colimits, it has cokernels of arbitrary morphisms which are computed object-wise, that is using 2.2.2.

If the category C_X has an initial object, x, then the presheaf $\widehat{x} = C_X(-, x)$ is an initial object of the category $C_X^\wedge$. In this case, the category $C_X^\wedge$ has kernels of all its morphisms (because $C_X^\wedge$ has limits) and the Yoneda functor $C_X \xrightarrow{h} C_X^\wedge$ preserves kernels.

Notice that the initial object of $C_X^\wedge$ is not isomorphic to its final object unless the category C_X is pointed, i.e. initial objects of C_X are its final objects.

2.2.4. Sheaves of sets. Let τ be a pretopology on C_X and C_{X_τ} denote the category of sheaves of sets on (C_X, τ). Similarly to $C_X^\wedge$, the category C_{X_τ} has a final object. If C_X has an initial object x, then the sheaf associated with the presheaf $C_X(-, x)$ is an initial object of C_{X_τ}. In particular, $C_X(-, x)$ is an initial object of C_{X_τ} if it is a sheaf (say, the pretopology τ is subcanonical).

2.3. Some properties of kernels. Fix a category C_X with an initial object x.

2.3.1. Proposition. *Let $M \xrightarrow{f} N$ be a morphism of C_X which has a kernel pair, $M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M$. Then the morphism f has a kernel iff p_1 has a kernel.*

Proof. Suppose that f has a kernel, i.e. there is a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array} \qquad (1)$$

Then we have the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{\gamma} & M \times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array} \qquad (2)$$

which is due to the commutativity of (1) and the fact that the unique morphism $x \xrightarrow{i_N} N$ factors through the morphism $M \xrightarrow{f} N$. The morphism γ is uniquely determined by the equality $p_2 \circ \gamma = \mathfrak{k}(f)$. The fact that the square (1) is cartesian and the equalities $p_2 \circ \gamma = \mathfrak{k}(f)$ and $i_N = f \circ i_M$ imply that the left square of the diagram (2) is cartesian, i.e. $Ker(f) \xrightarrow{\gamma} M \times_N M$ is the kernel of the morphism p_1.

Conversely, if p_1 has a kernel, then we have a diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_2} & M \\ {\scriptstyle p_1'}\downarrow & cart & {\scriptstyle p_1}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_M} & M & \xrightarrow{f} & N \end{array}$$

which consists of two cartesian squares. Therefore the square

$$\begin{array}{ccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle p_1'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with $\mathfrak{k}(f) = p_2 \circ \mathfrak{k}(p_1)$ is cartesian. ∎

2.3.2. Remarks. (a) Needless to say that the picture obtained in (the argument of) 2.3.1 is symmetric, i.e. there is an isomorphism $Ker(p_1) \xrightarrow{\tau_f'} Ker(p_2)$ which is an arrow in the commutative diagram

$$\begin{array}{ccccc} Ker(p_1) & \xrightarrow{\mathfrak{k}(p_1)} & M \times_N M & \xrightarrow{p_1} & M \\ {\scriptstyle \tau_f'}\downarrow\wr & & {\scriptstyle \tau_f}\downarrow\wr & & \downarrow{\scriptstyle id_M} \\ Ker(p_2) & \xrightarrow{\mathfrak{k}(p_2)} & M \times_N M & \xrightarrow{p_2} & M \end{array}$$

(b) Let a morphism $M \xrightarrow{f} N$ have a kernel pair, $M \times_N M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M$, and a kernel. Then, by 2.3.1, there exists a kernel of p_1, so that we have a morphism $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M$ and the diagonal morphism $M \xrightarrow{\Delta_M} M \times_N M$. Since the left square of the commutative diagram

$$\begin{array}{ccccc} x & \longrightarrow & Ker(p_1) & \xrightarrow{p_1'} & x \\ \downarrow & cart & \mathfrak{c}(p_1)\downarrow & & \downarrow \\ M & \xrightarrow{\Delta_M} & M \times_N M & \xrightarrow{p_1} & M \end{array}$$

is cartesian and compositions of the horizontal arrows are identical morphisms, it follows that its left square is cartesian too. Loosely, one can say that the intersection of $Ker(p_1)$ with the diagonal of $M \times_N M$ is zero. If there exists a coproduct $Ker(p_1) \coprod M$, then the pair of morphisms $Ker(p_1) \xrightarrow{\mathfrak{k}(p_1)} M \times_N M \xleftarrow{\Delta_M} M$ determine a morphism

$$Ker(p_1) \coprod M \longrightarrow M \times_N M.$$

If the category C_X is additive, then this morphism is an isomorphism, or, what is the same, $Ker(f) \coprod M \simeq M \times_N M$. In general, it is rarely the case, as the reader can find out looking at the examples of 2.2.

2.3.3. Proposition. *Let*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ \widetilde{g}\downarrow & cart & \downarrow g \\ M & \xrightarrow{f} & N \end{array} \tag{3}$$

be a cartesian square. Then $Ker(f)$ exists iff $Ker(\widetilde{f})$ exists, and they are naturally isomorphic to each other.

Proof. Suppose that $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ exists, i.e. we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ f'\downarrow & cart & \downarrow f \\ x & \xrightarrow{i_N} & N \end{array} \tag{4}$$

Since $x \longrightarrow N$ factors through $N' \xrightarrow{g} N$ and the square (3) is cartesian, there is a unique morphism $Ker(f') \xrightarrow{\gamma} N'$ such that the diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{\gamma} & \widetilde{M} & \xrightarrow{\widetilde{g}} & M \\ f'\downarrow & & \downarrow \widetilde{f} & cart & \downarrow f \\ x & \longrightarrow & \widetilde{N} & \xrightarrow{g} & N \end{array} \tag{5}$$

commutes and $\mathfrak{k}(f) = \widetilde{g} \circ \gamma$. Therefore the left square of (5) is cartesian.

If $Ker(\widetilde{f})$ exists, then we have the diagram

$$\begin{array}{ccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & \widetilde{M} & \xrightarrow{\widetilde{g}} & M \\ {\scriptstyle \widetilde{f}'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} & cart & \downarrow{\scriptstyle f} \\ x & \longrightarrow & \widetilde{N} & \xrightarrow{g} & N \end{array}$$

whose both squares are cartesian. Therefore, their composition

$$\begin{array}{ccc} Ker(\widetilde{f}) & \xrightarrow{\widetilde{g}\circ\mathfrak{k}(\widetilde{f})} & M \\ {\scriptstyle \widetilde{f}'}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

is a cartesian square.

It follows that the unique morphism $Ker(\widetilde{f}) \xrightarrow{g'} Ker(f)$ making the diagram

$$\begin{array}{ccccc} Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle g'}\downarrow & & {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M & \xrightarrow{f} & N \end{array} \qquad (6)$$

commute is an isomorphism. ■

2.3.4. The kernel of a composition and related facts. Fix a category C_X with an initial object x.

2.3.4.1. The kernel of a composition. Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be morphisms such that there exist kernels of g and $g \circ f$. Then the argument similar to that of 2.3.3 shows that we have a commutative diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\ {\scriptstyle \mathfrak{k}(gf)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & cart & \downarrow{\scriptstyle i_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array} \qquad (1)$$

whose both squares are cartesian and all morphisms are uniquely determined by f, g and the (unique up to isomorphism) choice of the objects $Ker(g)$ and $Ker(gf)$.

Conversely, if there is a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{u} & Ker(g) & \xrightarrow{g'} & x \\ {\scriptstyle t}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & & \downarrow{\scriptstyle i_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

whose left square is cartesian, then its left vertical arrow, $K \xrightarrow{t} L$, is the kernel of the composition $L \xrightarrow{g\circ f} N$.

2.3.4.2. Remarks. (a) It follows from 2.3.3 that the kernel of $L \xrightarrow{f} M$ exists iff the kernel of $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ exists and they are isomorphic to each other. More precisely, we have a commutative diagram

$$\begin{array}{ccccccc}
Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) & \xrightarrow{g'} & x \\
\wr\downarrow & & \mathfrak{k}(gf)\downarrow & cart & \downarrow\mathfrak{k}(g) & cart & \downarrow i_N \\
Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M & \xrightarrow{g} & N
\end{array}$$

whose left vertical arrow is an isomorphism.

(b) Suppose that $(C_X, \mathfrak{E}_X)$ is a right exact category (with an initial object x). If the morphism f above is a deflation, then it follows from this observation that the canonical morphism $Ker(gf) \xrightarrow{\widetilde{f}} Ker(g)$ is a deflation too. In this case, $Ker(f)$ exists, and we have a commutative diagram

$$\begin{array}{ccccc}
Ker(\widetilde{f}) & \xrightarrow{\mathfrak{k}(\widetilde{f})} & Ker(gf) & \xrightarrow{\widetilde{f}} & Ker(g) \\
\wr\downarrow & & \mathfrak{k}(gf)\downarrow & cart & \downarrow\mathfrak{k}(g) \\
Ker(f) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M
\end{array}$$

whose rows are conflations.

The following observations is useful (and will be used) for analysing diagrams.

2.3.4.3. Proposition. *(a) Let $M \xrightarrow{g} N$ be a morphism with a trivial kernel. Then a morphism $L \xrightarrow{f} M$ has a kernel iff the composition $g \circ f$ has a kernel, and these two kernels are naturally isomorphic one to another.*

(b) Let

$$\begin{array}{ccc}
L & \xrightarrow{f} & M \\
\gamma\downarrow & & \downarrow g \\
\widetilde{M} & \xrightarrow{\phi} & N
\end{array}$$

be a commutative square such that the kernels of the arrows f and ϕ exist and the kernel of g is trivial. Then the kernel of the composition $\phi \circ \gamma$ is isomorphic to the kernel of the morphism f, and the left square of the commutative diagram

$$\begin{array}{ccccccc}
Ker(f) & \xrightarrow{\sim} & Ker(\phi\gamma) & \xrightarrow{\mathfrak{k}(f)} & L & \xrightarrow{f} & M \\
 & & \widetilde{\gamma}\downarrow & cart & \gamma\downarrow & & \downarrow g \\
 & & Ker(\phi) & \xrightarrow{\mathfrak{k}(\phi)} & \widetilde{M} & \xrightarrow{\phi} & N
\end{array}$$

is cartesian.

Proof. (a) Since the kernel of g is trivial, the diagram 2.3.4.1(1) specializes to the diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{\widetilde{f}} & x & \xrightarrow{id_x} & x \\ {\scriptstyle \mathfrak{k}(gf)}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & & \downarrow{\scriptstyle i_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

with cartesian squares. The left cartesian square of this diagram is the definition of $Ker(f)$. The assertion follows from 2.3.4.1.

(b) Since the kernel of g is trivial, it follows from (a) that $Ker(f)$ is naturally isomorphic to the kernel of $g \circ f = \phi \circ \gamma$. The assertion follows now from 2.3.4.1. ∎

2.3.4.4. Corollary. *Let C_X be a category with an initial object x. Let $L \xrightarrow{f} M$ be a strict epimorphism and $M \xrightarrow{g} N$ a morphism such that $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ exists and is a monomorphism. Then the composition $g \circ f$ is a trivial morphism iff g is trivial.*

Proof. The morphism $g \circ f$ being trivial means that there is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{i_N} & N \end{array}$$

By 2.3.4.3(a), $Ker(g \circ f) \simeq Ker(\gamma) = L$. Thus, we have a commutative diagram

$$\begin{array}{ccccc} Ker(gf) & \xrightarrow{f} & Ker(g) & \xrightarrow{g'} & x \\ {\scriptstyle \wr}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(g)} & cart & \downarrow{\scriptstyle i_N} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

(cf. 2.3.4.1). Since f is a strict epimorphism, it follows from the commutativity of the left square that $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ is a strict epimorphism. Since, by hypothesis, $\mathfrak{k}(g)$ is a monomorphism, it is an isomorphism, which implies the triviality of g. ∎

2.3.4.4.1. Note. The following example shows that the requirement "$Ker(g) \longrightarrow M$ is a monomorphism" in 2.3.4.4 cannot be omitted.

Let C_X be the category Alg_k of associative unital k-algebras, and let $\mathfrak{m}$ be an ideal of the ring k such that the epimorphism $k \longrightarrow k/\mathfrak{m}$ does not split. Then the identical morphism $k/\mathfrak{m} \longrightarrow k/\mathfrak{m}$ is non-trivial, while its composition with the projection $k \longrightarrow k/\mathfrak{m}$ is a trivial morphism.

2.3.5. The coimage of a morphism. Let $M \xrightarrow{f} N$ be an arrow which has a kernel, i.e. we have a cartesian square

$$\begin{array}{ccc} Ker(f) & \xrightarrow{\mathfrak{k}(f)} & M \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ x & \xrightarrow{i_N} & N \end{array}$$

with which one can associate a pair of arrows $Ker(f) \underset{0_f}{\overset{\mathfrak{k}(f)}{\rightrightarrows}} M$, where 0_f is the composition of the projection f' and the unique morphism $x \xrightarrow{i_M} M$. Since $i_N = f \circ i_M$, the morphism f equalizes the pair $Ker(f) \underset{0_f}{\overset{\mathfrak{k}(f)}{\rightrightarrows}} M$. If the cokernel of this pair of arrows exists, it will be called the *coimage of* f and denoted by $Coim(f)$, or. loosely, $M/Ker(f)$.

Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. Then f is the composition of the canonical strict epimorphism $M \xrightarrow{p_f} Coim(f)$ and a uniquely defined morphism $Coim(f) \xrightarrow{j_f} N$.

2.3.5.1. Lemma. *Let $M \xrightarrow{f} N$ be a morphism such that $Ker(f)$ and $Coim(f)$ exist. There is a natural isomorphism $Ker(f) \xrightarrow{\sim} Ker(p_f)$.*

Proof. The outer square of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (1)$$

is cartesian. Therefore, its left square is cartesian which implies, by 2.3.3, that $Ker(p_f)$ is isomorphic to $Ker(f')$. But, $Ker(f') \simeq Ker(f)$. ∎

2.3.5.2. Note. By 2.3.4.1, all squares of the commutative diagram

$$\begin{array}{ccccc} Ker(f) & \xrightarrow{f'} & x & & \\ {\scriptstyle id}\downarrow & cart & \downarrow & & \\ Ker(j_f p_f) & \xrightarrow{\widetilde{p_f}} & Ker(j_f) & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}(f)}\downarrow & cart & \downarrow & cart & \downarrow \\ M & \xrightarrow{p_f} & Coim(f) & \xrightarrow{j_f} & L \end{array} \qquad (2)$$

are cartesian.

If C_X is an additive category and $M \xrightarrow{f} L$ is an arrow of C_X having a kernel and a coimage, then the canonical morphism $Coim(f) \xrightarrow{j_f} L$ is a monomorphism. Quite a few non-additive categories have this property.

2.3.5.3. Example. Let C_X be the category Alg_k of associative unital k-algebras. Since cokernels of pairs of arrows exist in Alg_k, any algebra morphism has a coimage. It follows from 2.2.1 that the coimage of an algebra morphism $A \xrightarrow{\varphi} B$ is $A/K(\varphi)$, where $K(\varphi)$ is the kernel of ϕ in the usual sense (i.e. in the category of non-unital algebras). The canonical decomposition $\varphi = j_\varphi \circ p_\varphi$ coincides with the standard presentation of φ as the composition of the projection $A \longrightarrow A/K(\varphi)$ and the monomorphism $A/K(\varphi) \longrightarrow B$. In particular, φ is strict epimorphism of k-algebras iff it is isomorphic to the associated coimage map $A \xrightarrow{p_\varphi} Coim(\varphi) = A/K(\varphi)$.

2.4. Conflations and fully exact subcategories of a right exact category. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x. We denote by $\mathcal{E}_X$ the class of all sequences of the form $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$, where $\mathfrak{e} \in \mathfrak{E}_X$ and $K \xrightarrow{\mathfrak{k}} M$ is a kernel of $\mathfrak{e}$. Expanding the terminology of exact additive categories, we call such sequences *conflations*.

2.4.1. Fully exact subcategories of a right exact category. We call a full subcategory $\mathcal{B}$ of C_X a *fully exact* subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, if $\mathcal{B}$ contains the initial object x and is *closed under extensions*; i.e. if objects K and N in a conflation $K \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} N$ belong to $\mathcal{B}$, then M is an object of $\mathcal{B}$.

In particular, fully exact subcategories of $(C_X, \mathfrak{E}_X)$ are strictly full subcategories.

2.4.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its fully exact subcategory. Then the class $\mathfrak{E}_{X,\mathcal{B}}$ of all deflations $M \xrightarrow{\mathfrak{e}} N$ such that M, N, and $Ker(\mathfrak{e})$ are objects of $\mathcal{B}$ is a structure of a right exact category on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an 'exact' functor $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}}) \longrightarrow (C_X, \mathfrak{E}_X)$.*

Proof. (a) We start with the invariance of $\mathfrak{E}_{X,\mathcal{B}}$ under base change. Let

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{N} \\ {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\mathfrak{e}} & N \end{array}$$

be a cartesian square such that $\mathfrak{e}$ (hence $\widetilde{\mathfrak{e}}$) is a deflation and the objects M, N, $Ker(\mathfrak{e})$, and $\widetilde{N}$ belong to $\mathcal{B}$. The claim is that the remaining object, $\widetilde{M}$, belongs to $\mathcal{B}$.

In fact, consider the diagram

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{N} \\ {\scriptstyle g'}\downarrow & & {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \qquad (7)$$

Since its right square is cartesian, it follows from 2.3.3 that the canonical morphism $Ker(\widetilde{\mathfrak{e}}) \xrightarrow{g'} Ker(\mathfrak{e})$ is an isomorphism; i.e. the upper row of the diagram (7) is a conflation whose ends, $Ker(\widetilde{\mathfrak{e}})$ and $\widetilde{N}$, are objects of $\mathcal{B}$. Since $\mathcal{B}$ is fully exact, the middle object, $\widetilde{M}$, belongs to $\mathcal{B}$, which means that the deflation $\widetilde{M} \xrightarrow{\widetilde{\mathfrak{e}}} \widetilde{N}$ belongs to $\mathfrak{E}_{X,\mathcal{B}}$.

(b) The invariance of $\mathfrak{E}_{X,\mathcal{B}}$ under base change implies that it is closed under composition. In fact, let $L \xrightarrow{\mathfrak{s}} M \xrightarrow{\mathfrak{t}} N$ be morphisms of $\mathfrak{E}_{X,\mathcal{B}}$. By 2.3.4.1, we have a commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{ts}) & \xrightarrow{\widetilde{\mathfrak{s}}} & Ker(\mathfrak{s}) & \xrightarrow{\mathfrak{t}'} & x \\ {\scriptstyle \mathfrak{k}(\mathfrak{ts})}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{s})} & cart & \downarrow{\scriptstyle i_N} \\ L & \xrightarrow{\mathfrak{s}} & M & \xrightarrow{\mathfrak{t}} & N \end{array} \tag{8}$$

whose squares are cartesian. Since $\mathfrak{s}$ belongs to $\mathfrak{E}_{X,\mathcal{B}}$, its kernel $Ker(\mathfrak{s}) \xrightarrow{\mathfrak{k}(\mathfrak{s})} M$ is an arrow of $\mathcal{B}$. Applying (a) to the left cartesian square of (8), we obtain that $Ker(\mathfrak{ts}) \xrightarrow{\mathfrak{k}(\mathfrak{ts})} L$ is an arrow of $\mathcal{B}$, which means that $\mathfrak{ts} \in \mathfrak{E}_{X,\mathcal{B}}$.

(c) Each isomorphism of the category $\mathcal{B}$ belongs to the class $\mathfrak{E}_{X,\mathcal{B}}$, because each isomorphism is a deflation and its kernel is an initial object, and, by hypothesis, initial objects belong to $\mathcal{B}$. ∎

2.4.3. Remark. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and $\mathcal{B}$ its strictly full subcategory containing x. Let $\mathfrak{E}$ be a right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \xrightarrow{\mathfrak{J}} C_X$ maps deflations to deflations and preserves kernels of deflations. Then $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$. In particular, $\mathfrak{E}$ is contained in $\mathfrak{E}_{X,\mathcal{B}}$ if the inclusion functor is an 'exact' functor from $(\mathcal{B}, \mathfrak{E})$ to $(C_X, \mathfrak{E}_X)$. This shows that if $\mathcal{B}$ is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$, then $\mathfrak{E}_{X,\mathcal{B}}$ is the finest right exact structure on $\mathcal{B}$ such that the inclusion functor $\mathcal{B} \longrightarrow C_X$ is an exact functor from $(\mathcal{B}, \mathfrak{E}_{X,\mathcal{B}})$ to $(C_X, \mathfrak{E}_X)$.

2.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories and F a functor $C_X \longrightarrow C_Y$ which maps conflations to conflations. Suppose that the category C_Y is additive. Then the functor F is 'exact'.*

Proof. Let F be a functor $C_X \longrightarrow C_Y$ which preserves conflations. We need to show that the functor F preserves arbitrary pull-backs of deflations.

(a) Let $M \xrightarrow{\mathfrak{e}} N$ be a deflation and $\widetilde{N} \xrightarrow{f} N$ a morphism of C_X. Consider the associated with this data diagram

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{N} \\ {\scriptstyle f''}\downarrow & & {\scriptstyle \widetilde{f}}\downarrow & cart & \downarrow{\scriptstyle f} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \tag{3}$$

whose right square is cartesian. Therefore, by 2.3.3, the left vertical arrow of the diagram (3), $Ker(\widetilde{\mathfrak{e}}) \xrightarrow{f''} Ker(\mathfrak{e})$, is an isomorphism. Since the rows of the diagram (3) are conflations and, by hypothesis, F preserves conflations, the rows of the commutative diagram

$$\begin{array}{ccccc} F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(\widetilde{M}) & \xrightarrow{F(\widetilde{\mathfrak{e}})} & F(\widetilde{N}) \\ F(f'') \downarrow \wr & & F(\widetilde{f}) \downarrow & & \downarrow F(f) \\ F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(N) \end{array} \tag{4}$$

are conflations. The diagram (4) can be decomposed into a commutative diagram

$$\begin{array}{ccccc} F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(\widetilde{M}) & \xrightarrow{F(\widetilde{\mathfrak{e}})} & F(\widetilde{N}) \\ \beta \downarrow & & \gamma \downarrow & & \downarrow id \\ Ker(\mathfrak{e}'') & \xrightarrow{\mathfrak{k}(\mathfrak{e}'')} & \mathcal{M} & \xrightarrow{\mathfrak{e}''} & F(\widetilde{N}) \\ \psi \downarrow & & \phi \downarrow & cart & \downarrow F(f) \\ F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(N) \end{array} \tag{5}$$

where the right lower square is cartesian, γ is a morphism uniquely determined by the equalities $\mathfrak{e}'' \circ \gamma = F(\widetilde{\mathfrak{e}})$ and $\phi \circ \gamma = F(\widetilde{f})$; and $\psi \circ \beta = F(f'')$. Since the lower row of (5) is a conflation, it follows from 2.3.3 that the morphism $Ker(\mathfrak{e}'') \xrightarrow{\psi} F(Ker(\mathfrak{e}))$ is an isomorphism. Therefore, $\beta = \psi^{-1} \circ F(f'')$ is an isomorphism. Thus, we have a commutative diagram

$$\begin{array}{ccccc} F(Ker(\mathfrak{e})) & \xrightarrow{F(\mathfrak{k}(\widetilde{\mathfrak{e}}))} & F(\widetilde{M}) & \xrightarrow{F(\widetilde{\mathfrak{e}})} & F(\widetilde{N}) \\ \beta \downarrow \wr & & \gamma \downarrow & & \wr \downarrow id \\ Ker(\mathfrak{e}'') & \xrightarrow{\mathfrak{k}(\mathfrak{e}'')} & \mathcal{M} & \xrightarrow{\mathfrak{e}''} & F(\widetilde{N}) \end{array} \tag{6}$$

whose rows are conflations and two vertical arrows are isomorphisms.

(b) The claim is that then the third vertical arrow, $F(\widetilde{M}) \xrightarrow{\gamma} \mathcal{M}$, is an isomorphism.

In fact, applying the canonical 'exact' embedding of $(C_Y, \mathfrak{E}_Y)$ to the category $C_{Y_{\mathfrak{E}}}$ of sheaves of $\mathbb{Z}$-modules on the presite $(C_Y, \mathfrak{E}_Y)$, we reduce the assertion to the case when the category is abelian (with the canonical exact structure); and the fact is well known for the abelian categories. ∎

2.5.1. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be additive k-linear right exact categories and F an additive functor $C_X \longrightarrow C_Y$. Then the functor F is weakly 'exact' iff it is 'exact'.*

Proof. By 1.4, a k-linear functor $C_X \xrightarrow{F} C_Y$ is a weakly 'exact' iff it maps conflations to conflations. The assertion follows now from 2.5. ∎

2.5.2. The property (†)**.** In Proposition 2.5, the assumption that the category C_Y is additive is used only at the end of the proof (part (b)). Moreover, additivity appears there only because it garantees the following property:

(†) if the rows of a commutative diagram

$$\begin{array}{ccccc} \widetilde{L} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{N} \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & M & \longrightarrow & N \end{array}$$

are conflations and its right and left vertical arrows are isomorphisms, then the middle arrow is an isomorphism.

So that the additivity of C_Y in 2.5 can be replaced by the property (†) for $(C_Y, \mathfrak{E}_Y)$.

2.5.3. An observation. The following obvious observation helps to establish the property (†) for many non-additive right exact categories:

If $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ are right exact categories and $C_X \xrightarrow{F} C_Y$ is a conservative functor which maps conflations to conflations, then the property (†) holds in $(C_X, \mathfrak{E}_X)$ provided it holds in $(C_Y, \mathfrak{E}_Y)$.

2.5.3.1. Example. Let $(C_Y, \mathfrak{E}_Y)$ are right exact k-linear category, $(C_X, \mathfrak{E}_X)$ a right exact category, and $C_X \xrightarrow{F} C_Y$ is a conservative functor which maps conflations to conflations. Then the property (†) holds in $(C_X, \mathfrak{E}_X)$.

For instance, the property (†) holds for the right exact category $(Alg_k, \mathfrak{E}^{\mathfrak{s}})$ of associative unital k-algebras with strict epimorphisms as deflations, because the forgetful functor $Alg_k \xrightarrow{\mathfrak{f}_*} k-mod$ is conservative, maps deflations to deflations (that is to epimorphisms) and is left exact. Therefore, it maps conflations to conflations.

2.6. Digression: right exact additive categories and exact categories.

2.6.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be an additive k-linear right exact category. Then there exists an exact category $(C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ and a fully faithful k-linear 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{\gamma_X^*} (C_{X_\mathfrak{e}}, \mathfrak{E}_{X_\mathfrak{e}})$ which is universal; that is any 'exact' k-linear functor from $(C_X, \mathcal{E}_X)$ to an exact k-linear category factorizes uniquely through γ_X^*.*

Proof. We take as $C_{X_\mathfrak{e}}$ the smallest fully exact subcategory of the category $C_{X_\mathfrak{E}}$ of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$ containing all representable sheaves. By 8.3.1, the subcategory $C_{X_\mathfrak{e}}$ coincides with $\widetilde{C}_X^{(\infty)}$, where $\widetilde{C}_X$ denotes the image of C_X in $C_{X_\mathfrak{E}}$. Therefore, objects of the category $C_{X_\mathfrak{e}}$ are sheaves $\mathcal{F}$ such that there exists a finite filtration

$$0 = \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \ \ldots \ \longrightarrow \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_m/\mathcal{F}_{m-1}$ is representable for $1 \leq m \leq n$. By K5.1, the subcategory $C_{X_\mathfrak{e}}$, being a fully exact subcategory of an abelian category, is exact.

Let $(C_Y, \mathfrak{E}_Y)$ be an exact k-linear category and $(C_X, \mathfrak{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathfrak{E}_Y)$ an exact k-linear functor. The functor φ^* extends to a continuous (i.e. having a right adjoint) functor $C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ {\scriptstyle \mathfrak{j}_X^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}_Y^*} \\ C_{X_{\mathfrak{E}}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_{\mathfrak{E}}} \end{array}$$

is quasi-commutative (see 1.2). Since the functor φ^* is 'exact', it preserves pullbacks of deflations. In particular, it preserves kernels of deflations. Therefore, the restriction of φ^* to the Gabriel square, $C_{X^{(2)}}$, of C_X regarded as a subcategory of the exact category $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$, preserves conflations, hence it is 'exact'. This implies that the restriction of φ^* to the n-th Gabriel power $C_{X^{(n)}}$, of C_X (in $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$) is 'exact' for all n, whence the assertion. ∎

2.6.2. The bicategories of exact and right exact k-linear categories. Right exact svelte k-linear categories are objects of a bicategory $\mathfrak{Rex}_k$. Its 1-morphisms are right weakly 'exact' k-linear functors and 2-morphisms are morphisms between those functors.

We denote by $\mathfrak{Ex}_k$ the full subbicategory of $\mathfrak{Rex}_k$ whose objects are exact k-linear categories. It follows from 2.6.1 that the inclusion functor $\mathfrak{Ex}_k \longrightarrow \mathfrak{Rex}_k$ has a left adjoint (in the bicategorical sense).

3. Satellites in right exact categories.

3.1. Preliminaries: trivial morphisms, pointed objects, and complexes. Let C_X be a category with initial objects. We call a morphism of C_X *trivial* if it factors through an initial object. It follows that an object M is initial iff $id_{_M}$ is a trivial morphism. If C_X is a pointed category, then the trivial morphisms are usually called *zero morphisms.*

3.1.1. Trivial compositions and pointed objects. If the composition of arrows $L \xrightarrow{f} M \xrightarrow{g} N$ is trivial, i.e. there is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{i_N} & N \end{array}$$

where x is an initial object, and the morphism g has a kernel, then f is the composition of the canonical arrow $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a morphism $L \xrightarrow{f_g} Ker(g)$ uniquely determined by f and ξ. If the arrow $x \xrightarrow{i_N} N$ is a monomorphism, then the morphism ξ is uniquely determined by f and g ; therefore in this case, the arrow f_g does not depend on ξ.

3.1.1.1. Pointed objects. In particular, f_g does not depend on ξ, if N is a *pointed* object. The latter means that therre exists an arrow $N \longrightarrow x$.

3.1.2. Complexes. A sequence of arrows

$$\ldots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \ldots \tag{1}$$

is called a *complex* if each its arrow has a kernel and the next arrow factors *uniquely* through this kernel.

3.1.3. Lemma. *Let each arrow in the sequence*

$$\ldots \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \xrightarrow{f_1} M_1 \xrightarrow{f_0} M_0 \tag{2}$$

of arrows have a kernel and the composition of any two consecutive arrows is trivial. Then

$$\ldots \xrightarrow{f_4} M_4 \xrightarrow{f_3} M_3 \xrightarrow{f_2} M_2 \tag{3}$$

is a complex. If M_0 is a pointed object, then (2) is a complex.

Proof. The composition $M_2 \xrightarrow{f_0 \circ f_1} M_0$ factors through an initial object; in particular, there exist morphisms from M_i to an initial object x of C_X for all $i \geq 2$. Therefore, the unique morphism $x \longrightarrow M_i$ is a (split) monomorphism for all $i \geq 2$. By 2.1.1(a), this implies that $Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ is a monomorphism. Therefore, there exists a unique arrow $M_{i+2} \xrightarrow{f'_{i+1}} Ker(f_i)$ whose composition with $Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ equals to f_{i+1}.

By the similar reason, if there exists a morphism from M_0 (resp. M_1) to x, then $Ker(f_i) \xrightarrow{\mathfrak{k}(f_i)} M_{i+1}$ is a monomorphism for $i \geq 0$ (resp. for $i \geq 1$). ∎

3.1.4. Corollary. *A sequence of morphisms*

$$\ldots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \ldots$$

unbounded on the right is a complex iff the composition of any pair of its consecutive arrows is trivial and for every i, there exists a kernel of the morphism f_i.

3.1.4.1. Example. Let C_X be the category Alg_k of unital associative k-algebras. The algebra k is its initial object, and every morphism of k-algebras has a kernel. Pointed objects of C_X which have a morphism to initial object are precisely augmented k-algebras. If the composition of pairs of consecutive arrows in the sequence

$$\ldots \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0$$

is trivial, then it follows from the argument of 3.1.2 that A_i is an augmented k-algebra for all $i \geq 2$. And any unbounded on the right sequence of algebras with trivial compositions of pairs of consecutive arrows is formed by augmented algebras.

3.1.5. The categories of complexes. Let C_X be a category with initial objects. For any integer m, we denote by $\mathcal{K}_m(C_X)$ the category whose objects are complexes of the form

$$\ldots \xrightarrow{f_{m+2}} M_{m+2} \xrightarrow{f_{m+1}} M_{m+1} \xrightarrow{f_m} M_m$$

and morphisms are defined as usual. Every finite complex

$$M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_{m+2}} M_{m+2} \xrightarrow{f_{m+1}} M_{m+1} \xrightarrow{f_m} M_m \tag{3}$$

is identified with an object of $\mathcal{K}_m(C_X)$ by adjoining on the left the infinite sequence of trivial objects and (unique) morphisms from them.

We call an object (3) of the category $\mathcal{K}_m(C_X)$ a *bounded* complex if M_n is an initial object for all $n \gg m$. We denote by $\mathcal{K}^b_m(C_X)$ the full subcategory of $\mathcal{K}_m(C_X)$ generated by bounded complexes.

The categories $\mathcal{K}_m(C_X)$ (resp. $\mathcal{K}^b_m(C_X)$) are naturally isomorphic to each other via obvious *translation* functors.

We denote by $\mathcal{K}(C_X)$ the category whose objects are complexes

$$\ldots \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_{n-2}} \ldots \tag{4}$$

which are infinite in both directions. Unless C_X is a pointed category, there are no natural embeddings of the categories $\mathcal{K}_m(C_X)$ into $\mathcal{K}(C_X)$. There is a natural embedding into $\mathcal{K}(C_X)$ of the full subcategory $\mathcal{K}_{m,*}(C_X)$ of $\mathcal{K}_m(C_X)$ generated by all complexes (3) with M_m equal to an initial object.

We say that an object (4) of the category $\mathcal{K}(C_X)$ is a complex bounded on the left (resp. on the right) if M_n is an initial object for all $n \gg 0$ (resp. $n \ll 0$). We denote by $\mathcal{K}^+(C_X)$ (resp. by $\mathcal{K}^-(C_X)$) the full subcategory of $\mathcal{K}(C_X)$ whose objects are complexes bounded on the left (resp. on the right). Finally, we set $\mathcal{K}^b(C_X) = \mathcal{K}^-(C_X) \bigcap \mathcal{K}^+(C_X)$ and call objects of the subcategory $\mathcal{K}^b(C_X)$ *bounded* complexes.

3.1.6. 'Exact' complexes. Let $(C_X, \mathcal{E}_X)$ be a right exact category with an initial object. We call a sequence of two arrows $L \xrightarrow{f} M \xrightarrow{g} N$ in C_X 'exact' if the arrow g has a kernel, and f is the composition of $Ker(g) \xrightarrow{\mathfrak{k}(g)} M$ and a deflation $L \xrightarrow{f_g} Ker(g)$. A complex is called 'exact' if any pair of its consecutive arrows forms an 'exact' sequence.

3.2. ∂^*-functors. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object. A *∂^*-functor* from $(C_X, \mathcal{E}_X)$ to C_Y is a system of functors $C_X \xrightarrow{T_i} C_Y$, $i \geq 0$, together with a functorial assignment to every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ and every $i \geq 0$ a morphism $T_{i+1}(L) \xrightarrow{\mathfrak{d}_i(E)} T_i(N)$ which depends functorially on the conflation E and such that the sequence of arrows

$$\ldots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is a complex. Taking the trivial conflation $x \longrightarrow x \longrightarrow x$, we obtain that $T_i(x) \xrightarrow{id_{T_i(x)}} T_i(x)$ is a trivial morphism, or, equivalently, $T_i(x)$ is an initial object, for every $i \geq 1$.

Let $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ and $T' = (T'_i, \mathfrak{d}'_i |\ i \geq 0)$ be a pair of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y. A morphism from T to T' is a family $f = (T_i \xrightarrow{f_i} T'_i \mid i \geq 0)$ of functor morphisms such that for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of the exact category C_X and every $i \geq 0$, the diagram

$$\begin{array}{ccc} T_{i+1}(L) & \xrightarrow{\mathfrak{d}_i(E)} & T_i(N) \\ {\scriptstyle f_{i+1}(L)}\downarrow & & \downarrow{\scriptstyle f_i(N)} \\ T'_{i+1}(L) & \xrightarrow{\mathfrak{d}'_i(E)} & T'_i(N) \end{array}$$

commutes. The composition of morphisms is naturally defined. Thus, we have the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$ of ∂^*-functors from $(C_X, \mathcal{E}_X)$ to C_Y.

3.2.1. Trivial ∂^*-functors. We call a ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ *trivial* if all T_i are functors with values in initial objects. One can see that trivial ∂^*-functors are precisely initial objects of the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Once an initial object y of the category C_Y is fixed, we have a canonical trivial functor whose components equal to the constant functor with value in y – it maps all arrows of C_X to id_y.

3.2.2. Some natural functorialities. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and C_Y a category with initial object. If C_Z is another category with an initial object and $C_Y \xrightarrow{F} C_Z$ a functor which maps initial objects to initial objects, then for any ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$, the composition $F \circ T = (F \circ T_i, F\mathfrak{d}_i |\ i \geq 0)$ of T with F is a ∂^*-functor. The map $(F, T) \longmapsto F \circ T$ is functorial in both variables; i.e. it extends to a functor

$$Cat_*(C_Y, C_Z) \times \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \longrightarrow \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Z). \qquad (1)$$

Here Cat_* denotes the subcategory of Cat whose objects are categories with initial objects and morphisms are functors which map initial objects to initial objects.

On the other hand, let $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ be another right exact category with an initial object and Φ a functor $C_\mathfrak{X} \longrightarrow C_X$ which maps conflations to conflations. In particular, it maps initial objects to initial objects (because if x is an initial object of $C_\mathfrak{X}$, then $x \longrightarrow M \xrightarrow{id_M} M$ is a conflation; and $\Phi(x \longrightarrow M \xrightarrow{id_M} M)$ being a conflation implies that $\Phi(x)$ is an initial object). For any ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y, the composition $T \circ \Phi = (T_i \circ \Phi, \mathfrak{d}_i\Phi |\ i \geq 0)$ is a ∂^*-functor from $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ to C_Y. The map $(T, \Phi) \longmapsto T \circ \Phi$ extends to a functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \times \mathcal{E}x_*((C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}), (C_X, \mathfrak{E}_X)) \longrightarrow \mathcal{H}om^*((C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}), C_Y), \qquad (2)$$

where $\mathcal{E}x_*((C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X}), (C_X, \mathfrak{E}_X))$ denotes the full subcategory of $\mathcal{H}om(C_\mathfrak{X}, C_X)$ whose objects are preserving conflations functors $C_\mathfrak{X} \longrightarrow C_X$.

3.3. Universal ∂^*-functors. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object x and a category C_Y with an initial object y.

A ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y is called *universal* if for every ∂^*-functor $T' = (T'_i, \mathfrak{d}'_i |\ i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y and every functor morphism $T'_0 \xrightarrow{g} T_0$, there exists a unique morphism $f = (T'_i \xrightarrow{f_i} T_i \mid i \geq 0)$ from T' to T such that $f_0 = g$.

3.3.1. Interpretation. Consider the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \tag{3}$$

which assigns to every ∂^*-functor (resp. every morphism of ∂^*-functors) its zero component. For any functor $C_X \xrightarrow{F} C_Y$, we have a presheaf of sets $\mathcal{H}om(\Psi^*(-), F)$ on the category $\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$. Suppose that this presheaf is representable by an object (i.e. a ∂^*-functor) $\Psi_*(F)$. Then $\Psi_*(F)$ is a universal ∂^*-functor.

Conversely, if $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ is a universal ∂^*-functor, then $T \simeq \Psi_*(T_0)$.

3.3.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x; and let C_Y be a category with initial objects, kernels of morphisms, and limits of filtered systems. Then, for any functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ such that $T_0 = F$.*

In other words, the functor

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \tag{3}$$

which assigns to each morphism of ∂^*-functors its zero component has a right adjoint, Ψ_*.

Proof. (a) For an arbitrary functor $C_X \xrightarrow{F} C_Y$, we set $S_-(F)(L) = \lim Ker(F(\mathfrak{k}(\mathfrak{e})))$, where the limit is taken by the (filtered) system of all deflations $M \xrightarrow{\mathfrak{e}} L$. Since deflations form a pretopology, the map $L \longmapsto S_-(F)(L)$ extends naturally to a functor $C_X \xrightarrow{S_-(F)} C_Y$. By the definition of $S_-(F)$, for any conflation $E = (N \xrightarrow{\mathrm{j}} M \xrightarrow{\mathfrak{e}} L)$, there exists a unique morphism $S_-(F)(L) \xrightarrow{\widetilde{\partial}^0_{F}(E)} Ker(F(\mathrm{j}))$. We denote by $\partial^0_F(E)$ the composition of $\widetilde{\partial}^0_F(E)$ and the canonical morphism $Ker(F(j)) \longrightarrow F(N)$.

(b) Notice that the correspondence $F \longmapsto S_-(F)$ is functorial. Applying the iterations of the functor S_- to F, we obtain a ∂^*-functor $S^\bullet_-(F) = (S^i_-(F) | i \geq 0)$. The claim is that this ∂^*-functor is universal.

In fact, let $T = (T_i, \mathfrak{d} | i \geq 0)$ be a ∂^*-functor and $T_0 \xrightarrow{\lambda_0} F$ a functor morphism. For any conflation $E = (N \xrightarrow{\mathrm{j}} M \xrightarrow{\mathfrak{e}} L)$, we have a commutative diagram

$$\begin{array}{ccccccc}
T_1(L) & \xrightarrow{\mathfrak{d}_0(E)} & T_0(N) & \xrightarrow{T_0(\mathrm{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L) \\
 & & {\scriptstyle \lambda_0(N)}\downarrow & & \downarrow{\scriptstyle \lambda_0(M)} & & \downarrow{\scriptstyle \lambda_0(L)} \\
 & & F(N) & \xrightarrow{F(\mathrm{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array} \tag{4}$$

Since $T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$ is a complex, the morphism $\mathfrak{d}_0(E)$ is the composition of a uniquely defined morphism $T_1(L) \xrightarrow{\widetilde{\mathfrak{d}}_0(E)} Ker(T_0(\mathfrak{j}))$ and the canonical arrow $Ker(T_0(\mathfrak{j})) \longrightarrow T_0(N)$. We denote by $\widetilde{\lambda}_1(E)$ the composition of the morphism $\widetilde{\mathfrak{d}}_0(E)$ and the morphism $Ker(T_0(\mathfrak{j})) \xrightarrow{\lambda_1'} Ker(F(\mathfrak{j})$ uniquely determined by the commutativity of the diagram

$$\begin{array}{ccccc} Ker(T_0(\mathfrak{j})) & \xrightarrow{\mathfrak{k}(T_0(\mathfrak{j}))} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) \\ \lambda_1' \downarrow & & \lambda_0(N) \downarrow & & \downarrow \lambda_0(M) \\ Ker(F(\mathfrak{j})) & \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) \end{array}$$

Thus, we have a commutative diagram

$$\begin{array}{ccccccc} T_1(L) & \xrightarrow{\mathfrak{d}_0(E)} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L) \\ \widetilde{\lambda}_1(E) \downarrow & & \lambda_0(N) \downarrow & & \downarrow \lambda_0(M) & & \downarrow \lambda_0(L) \\ Ker(F(\mathfrak{j})) & \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L) \end{array}$$

with the morphism $\widetilde{\lambda}_1(E)$ uniquely determined by the arrows of the diagram (4). Since the *connecting* morphism $T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N)$ depends on the conflation E functorially, same is true for $\widetilde{\lambda}_1(E)$; that is the morphisms $T_1(L) \xrightarrow{\widetilde{\lambda}_1(E)} Ker(F(\mathfrak{j}))$, where E runs through conflations $N \longrightarrow M \longrightarrow L$ (with fixed L and morphisms of the form (h, g, id_L)), form a cone. This cone defines a unique morphism $T_1(L) \xrightarrow{\lambda_1(L)} S_-(F)(L)$. It follows from the universality of this construction that $\lambda = (\lambda_1(L)|\ L \in ObC_X)$ is a functor morphism $T_1 \xrightarrow{\lambda_1} S_-(F)$ such that the diagram

$$\begin{array}{ccccccc} T_1(L) & \xrightarrow{\mathfrak{d}_0(E)} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L) \\ \lambda_1(L) \downarrow & & \lambda_0(N) \downarrow & & \downarrow \lambda_0(M) & & \downarrow \lambda_0(L) \\ S_-(F)(L) & \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L) \end{array}$$

commutes. Iterating this construction, we obtain uniquely defined functor morphisms $T_i \xrightarrow{\lambda_i} S_-^i(F)$ for all $i \geq 1$. ∎

3.3.3. Remark. Let the assumptions of 3.3.2 hold. Then we have a pair of adjoint functors

$$\mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y) \xrightarrow{\Psi^*} \mathcal{H}om(C_X, C_Y) \xrightarrow{\Psi_*} \mathcal{H}om^*((C_X, \mathfrak{E}_X), C_Y)$$

By 3.3.2, the adjunction morphism $\Psi^*\Psi_* \longrightarrow Id$ is an isomorphism which means that Ψ_* is a fully faithful functor and Ψ^* is a localization functor at a left multiplicative system.

3.3.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a ∂^*-functor from $(C_X, \mathcal{E}_X)$ to C_Y. Let C_Z be another category with an initial object and F a functor from C_Y to C_Z which preserves initial objects, kernels of morphisms and limits of filtered systems. Then*

(a) If T is a universal ∂^*-functor, then $F \circ T = (F \circ T_i, F\mathfrak{d}_i \mid i \geq 0)$ is universal.

(b) If, in addition, the functor F is fully faithful, then the ∂^*-functor $F \circ T$ is universal iff T is universal.

Proof. (a) Suppose that the ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ is universal. Then it follows from the argument of 3.3.2 that $T_i \simeq S^i_-(T_0)$ for all $i \geq 0$, where $S_-(G)(L) = \lim Ker(G(\mathfrak{k}(\mathfrak{e})))$ and the limit is taken by the system of all deflations $M \xrightarrow{\mathfrak{e}} L$. Since the functor F preserves kernels of morphisms and filtered limits (that is all types of limits which appear in the construction of $S_-(G)(L)$), the natural morphism

$$F \circ S_-(G)(L) \longrightarrow S_-(F \circ G)(L)$$

is an isomorphism for any functor $C_X \xrightarrow{G} C_Y$ such that $S_-(G)(L) = \lim Ker(G(\mathfrak{k}(\mathfrak{e})))$ exists. Therefore, the natural morphism $F \circ S^i_-(T_0)(L) \longrightarrow S^i_-(F \circ T_0)(L)$ is an isomorphism for all $i \geq 0$ and all $L \in ObC_X$.

(b) Suppose that the functor F is fully faithful and the ∂^*-functor $F \circ T$ is universal. Then

$$F \circ T_{i+1}(L) \simeq S_-(F \circ T_i)(L) = \lim Ker(F \circ T_i(\mathfrak{k}(\mathfrak{e}))) \simeq$$
$$\lim F(Ker(T_i(\mathfrak{k}(\mathfrak{e})))) \simeq F(\lim Ker(T_i(\mathfrak{k}(\mathfrak{e})))) = F(S_-(T_i)(L)),$$

where the isomorphisms are due to compatibility of F with kernels of morphisms and filtered limits. Since all these isomorphisms are natural (i.e. functorial in L), we obtain a functor isomorphism $F \circ T_{i+1} \xrightarrow{\sim} F \circ S_-(T_i)$. Since the functor F is fully faithful, the latter implies an isomorphism $T_{i+1} \xrightarrow{\sim} S_-(T_i)$ for all $i \geq 0$. The assertion follows now from (the argument of) 3.3.2. ∎

3.3.5. An application. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category with an initial object. Consider the Yoneda embedding

$$C_Y \xrightarrow{h_Y} C_Y^\wedge, \quad M \longmapsto \widehat{M} = C_Y(-, M).$$

of the category C_Y into the category $C_Y^\wedge$ of presheaves of sets on C_Y. The functor h_Y is fully faithful and preserves all limits. In particular, it satisfies the conditions of 3.3.4(b). Therefore, a ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to C_Y is universal iff the ∂^*-functor $\widehat{T} \stackrel{\text{def}}{=} h_Y \circ T = (\widehat{T}_i, \widehat{\mathfrak{d}}_i \mid i \geq 0)$ from $(C_X, \mathcal{E}_X)$ to $C_Y^\wedge$ is universal.

By 3.3.2, for any functor $C_X \xrightarrow{G} C_Y^\wedge$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0) = \Psi_*(G)$ such that $T_0 = G$. In particular, for every functor $C_X \xrightarrow{F} C_Y$, there exists a unique up to isomorphism universal ∂^*-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ such that $T_0 = h_Y \circ F = \widetilde{F}$. It follows from 3.3.4(b) that there exists a universal ∂^*-functor whose zero component coincides with F iff for all $L \in ObC_X$ and all $i \geq 1$, the presheaves $T_i(L)$ are representable.

3.3.6. Remark. Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with an initial object x and C_Y a category with an initial object y and limits. Then, by the argument of 3.3.2, we have an endofunctor S_- of the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y, together with a cone $S_- \xrightarrow{\lambda} \mathfrak{y}$, where $\mathfrak{y}$ is the constant functor with the values in the initial object y of the category C_Y. For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of $(C_X, \mathfrak{E}_X)$ and any functor $C_X \xrightarrow{F} C_Y$, we have a commutative diagram

$$\begin{array}{ccccc} S_-F(L) & \xrightarrow{\lambda(L)} & y & & \\ {\scriptstyle \mathfrak{d}_0(E)}\downarrow & & \downarrow & & \\ F(N) & \xrightarrow{F\mathfrak{j}} & F(M) & \xrightarrow{F\mathfrak{e}} & F(L) \end{array}$$

3.3.7. Digression: deflations with trivial kernels. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial objects. We denote by $\mathfrak{E}_X^{\circledast}$ the class of all arrows of $\mathfrak{E}_X$ whose kernel is an initial object.

3.3.7.1. Proposition. *The class of arrows $\mathfrak{E}_X^{\circledast}$ is a right exact structure on the category C_X.*

Proof. The class $\mathfrak{E}_X^{\circledast}$ contains all isomorphisms of the category C_X. It is closed under compositions, because, by 2.3.4.3, if $Ker(\mathfrak{s})$ is trivial (i.e is an initial object of C_X), then $Ker(\mathfrak{s} \circ \mathfrak{t})$ is naturally isomorphic to $Ker(\mathfrak{t})$. In particular, $Ker(\mathfrak{s} \circ \mathfrak{t})$ is trivial, if both $Ker(\mathfrak{s})$ and $Ker(\mathfrak{t})$ are trivial. Finally, if

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{p_1} & M \\ {\scriptstyle \mathfrak{t}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} \\ N & \xrightarrow{f} & L \end{array}$$

is a cartesian square, then, by 2.3.3, $Ker(\mathfrak{s}) \simeq Ker(\mathfrak{t})$, which shows that $\mathfrak{E}_X^{\circledast}$ is stable under base change. ∎

3.3.7.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x; and let C_Y be a category with initial objects, kernels of morphisms, and limits of filtered systems. Let $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ be a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. If the functor T_0 maps all arrows of $\mathfrak{E}_X^{\circledast}$ to isomorphisms, then all functors T_i, $i \geq 0$, have this property.*

Proof. By the argument of 3.3.2, the assertion is equivalent to the following one:

If a functor $C_X \xrightarrow{F} C_Y$ maps arrows of $\mathfrak{E}_X^{\circledast}$ to isomorphisms, then its satellite, S_-F, has the same property.

In fact, let $\mathcal{L} \xrightarrow{\mathfrak{s}} L$ be an arrow of $\mathfrak{E}_X^{\circledast}$ and $M \xrightarrow{\mathfrak{e}} L$ an arbitrary deflation. Then we have a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{e}})} & \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{L} \\ {\scriptstyle \mathfrak{s}_2}\downarrow\wr & & {\scriptstyle \mathfrak{s}_1}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & M & \xrightarrow{\mathfrak{e}} & L \end{array} \qquad (1)$$

whose vertical arrows belong to $\mathfrak{E}_X^{\circledast}$. Therefore, the left square of (1) determines isomorphism $Ker(F(\mathfrak{k}(\widetilde{\mathfrak{e}})) \xrightarrow{\phi(\mathfrak{e})} Ker(F(\mathfrak{k}(\mathfrak{e}))$ which is functorial in $\mathfrak{e}$. So that we obtain an isomorphism $\lim Ker(F(\mathfrak{k}(\widetilde{\mathfrak{e}})) \xrightarrow{\sim} \lim Ker(F(\mathfrak{k}(\mathfrak{e})) = S_-F(L)$ whose composition with the canonical arrow $S_-F(\mathcal{L}) \longrightarrow \lim Ker(F(\mathfrak{k}(\widetilde{\mathfrak{e}}))$ coincides with the morphism $S_-F(\mathcal{L}) \xrightarrow{S_-F(\mathfrak{s})} S_-F(L)$ (see the argument of 3.3.2).

On the other hand, for any deflation $\mathcal{M}' \xrightarrow{\gamma} \mathcal{L}$, there is a commutative diagram

$$\begin{array}{ccccc} Ker(\gamma) & \xrightarrow{\mathfrak{k}(\gamma)} & \mathcal{M}_1 & \xrightarrow{\gamma} & \mathcal{L} \\ \downarrow\wr & & id\downarrow & & \downarrow \mathfrak{s} \\ Ker(\mathfrak{s}\gamma) & \xrightarrow{\mathfrak{k}(\mathfrak{s}\gamma)} & \mathcal{M}_1 & \xrightarrow{\mathfrak{s}\gamma} & L \end{array} \qquad (2)$$

Here the left vertical arrow is an isomorphism, because $Ker(\mathfrak{s})$ is an initial object (see 2.3.4.3). The left square of (2) induces an isomorphism $Ker(F(\mathfrak{k}(\mathfrak{s}\gamma)) \xrightarrow{\phi(\gamma)} Ker(F(\mathfrak{k}(\gamma))$ which is functorial in γ. The latter implies that the composition $\varphi(\gamma)$ of $\phi(\gamma)$ with the unique morphism $S_-F(L) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{s}\gamma))$ defines a cone $S_-F(L) \xrightarrow{\varphi(\gamma)} Ker(F(\mathfrak{k}(\gamma))$, hence a unique morphism $S_-F(L) \xrightarrow{\varphi} S_-F(\mathcal{L})$. The claim is that φ is the inverse to the morphism $S_-F(\mathcal{L}) \xrightarrow{S_-F(\mathfrak{s})} S_-F(L)$.

We complete (2) to a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{\mathfrak{s}\gamma}) & \xrightarrow{\mathfrak{k}(\widetilde{\mathfrak{s}\gamma})} & \widetilde{\mathcal{M}}_1 & \xrightarrow{\widetilde{\mathfrak{s}\gamma}} & \mathcal{L} \\ \mathfrak{t}_2\downarrow\wr & & \mathfrak{t}_1\downarrow & & \downarrow id \\ Ker(\gamma) & \xrightarrow{\mathfrak{k}(\gamma)} & \mathcal{M}_1 & \xrightarrow{\gamma} & \mathcal{L} \\ \mathfrak{s}_2\downarrow\wr & & id\downarrow & & \downarrow \mathfrak{s} \\ Ker(\mathfrak{s}\gamma) & \xrightarrow{\mathfrak{k}(\mathfrak{s}\gamma)} & \mathcal{M}_1 & \xrightarrow{\mathfrak{s}\gamma} & L \end{array} \qquad (3)$$

where the square

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_1 & \xrightarrow{\widetilde{\mathfrak{s}\gamma}} & \mathcal{L} \\ \mathfrak{t}_1\downarrow & & \downarrow \mathfrak{s} \\ \mathcal{M}_1 & \xrightarrow{\mathfrak{s}\gamma} & L \end{array}$$

is cartesian. Since $\mathfrak{t}_1 \in \mathfrak{E}_X^{\circledast}$, the diagram (3) induces isomorphisms

$$KerF(\mathfrak{k}(\widetilde{\mathfrak{s}\gamma})) \xrightarrow{\sim} KerF(\mathfrak{k}(\gamma)) \xrightarrow{\sim} KerF(\mathfrak{k}(\mathfrak{s}\gamma))$$

which imply isomorphisms of the lower row of the commutative diagram

$$\begin{array}{ccccccc} S_-F(\mathcal{L}) & \xrightarrow{id} & S_-F(\mathcal{L}) & \xrightarrow{S_-F(\mathfrak{s})} & S_-F(L) & \xleftarrow{id} & S_-F(L) \\ \downarrow & & \downarrow id & & \downarrow \varphi & & \downarrow \\ \lim KerF(\mathfrak{k}(\widetilde{\mathfrak{s}\gamma})) & \xrightarrow{\sim} & \lim KerF(\mathfrak{k}(\gamma)) & \xrightarrow{id} & S_-F(\mathcal{L}) & \xrightarrow{\sim} & \lim KerF(\mathfrak{k}(\mathfrak{s}\gamma)) \end{array}$$

The isomorphness of φ (or, equivalently, the isomorphness of $S_-F(\mathfrak{s})$) follows from the universal property of limits. ∎

3.4. The dual picture: ∂-functors and universal ∂-functors. Let $(C_X, \mathfrak{I}_X)$ be a left exact category, which means by definition that $(C_X^{op}, \mathfrak{I}_X^{op})$ is a right exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is the data which becomes a ∂^*-functor in the dual right exact category. A ∂-functor on $(C_X, \mathfrak{I}_X)$ is *universal* if its dualization is a universal ∂^*-functor. We leave to the reader the reformulation in the context of ∂-functors of all notions and facts about ∂^*-functors. Below, there are two versions – non-linear and linear, of a fundamental example of a universal ∂-functor.

3.4.1. Example: $Ext^\bullet$. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object. For any $L \in ObC_X$, we have the corresponding representable functor

$$C_X^{op} \xrightarrow{h_X(L)} Sets, \quad M \longmapsto C_X^{op}(L, M) = C_X(M, L).$$

Therefore, by (the dual version of) 3.3.2, there exists a universal ∂-functor $Ext_X^\bullet(L) = (Ext_X^i(L) |\ i \geq 0)$, whose zero component, $Ext_X^0(L)$, coincides with $h_X(L)$.

3.4.2. The functors $\mathcal{E}xt_k^\bullet$. Suppose that the category C_X is k-linear. Then for any $L \in ObC_X$, the functor $h_X(L)$ factors through the category $k-mod$ (that is through the forgetful functor $k-mod \longrightarrow Sets$). Therefore, by 3.3.2, there exists a universal ∂-functor $\mathcal{E}xt_X^\bullet(L) = (\mathcal{E}xt_X^i(L) |\ i \geq 0)$, whose zero component, $\mathcal{E}xt_X^0(L)$, coincides with the presheaf of k-modules $C_X(-, L)$.

3.5. Universal ∂^*-functors and 'exactness'.

3.5.1. The properties $(CE5)$ and $(CE5^*)$. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We say that it satisfies $(CE5^*)$ (resp. $(CE5)$) if the limit of a filtered system (resp. the colimit of a cofiltered system) of conflations in $(C_Y, \mathfrak{E}_Y)$ exists and is a conflation.

In particular, if $(C_X, \mathfrak{E}_X)$ satisfies $(CE5^*)$ (resp. $(CE5)$), then the limit of any filtered system (resp. the colimit of any cofiltered system) of deflations is a deflation.

The properties $(CE5)$ and $(CE5^*)$ make sense for left exact categories as well. Notice that a right exact category satisfies $(CE5^*)$ (resp. $(CE5)$) iff the dual left exact category satisfies $(CE5)$ (resp. $(CE5^*)$).

3.5.2. Note. If $(C_X, \mathfrak{E}_X)$ is an abelian category with the canonical exact structure, then the property $(CE5)$ for $(C_X, \mathfrak{E}_X)$ is equivalent to the Grothendieck's property (AB5) and, therefore, the property $(CE5^*)$ is equivalent to $(AB5^*)$ (see [Gr, 1.5]).

The property $(CE5)$ holds for Grothendieck toposes.

In what follows, we use $(CE5^*)$ for right exact categories and the dual property $(CE5)$ for left exact categories.

3.5.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and $(C_Y, \mathfrak{E}_Y)$ satisfy $(CE5^*)$. Let F be a weakly right 'exact' functor $(C_X, \mathfrak{E}_X) \longrightarrow (C_Y, \mathfrak{E}_Y)$ such that $S_-(F)$ exists. Then for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the sequence*

$$S_-(F)(N) \xrightarrow{S_-(F)(\mathfrak{j})} S_-(F)(M) \xrightarrow{S_-(F)(\mathfrak{e})} S_-(F)(L) \xrightarrow{\mathfrak{d}_0(E)} F(N) \xrightarrow{F(\mathfrak{j})} F(M) \qquad (1)$$

is 'exact'. The functor $S_-(F)$ is a weakly right 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.

Proof. Let $C_X \xrightarrow{F} C_Y$ be a right 'exact' functor such that $S_-(F)$ exists.

(a) The claim is that for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, the canonical morphism $S_-(F)(L) \xrightarrow{\widetilde{\mathfrak{d}}_0(E)} Ker(F(\mathfrak{j}))$ is a deflation.

(a1) Let $M \xrightarrow{\mathfrak{e}} L$ and $M' \xrightarrow{\mathfrak{e}'} L$ be deflations of an object L of C_X, and let

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ {\scriptstyle \mathfrak{e}'}\searrow & & \swarrow{\scriptstyle \mathfrak{e}} \\ & L & \end{array}$$

be a commutative diagram (– a morphism of deflations). This diagram extends to a morphism of the corresponding conflations

$$\begin{array}{ccccc} N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id_L} \\ N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L \end{array} \tag{2}$$

Since $\mathfrak{e}' = \mathfrak{e} \circ f$, it follows from 2.3.4.1 that the left square of (2) is cartesian.

For an arbitrary functor $C_X \xrightarrow{F} C_Y$, the diagram (2) gives rise to the commutative diagram

$$\begin{array}{ccccccc} Ker(F(\mathfrak{j}')) & \xrightarrow{\mathfrak{k}'} & F(N') & \xrightarrow{F(\mathfrak{j}')} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\ {\scriptstyle \widetilde{\gamma}}\downarrow & cart & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ Ker(\alpha) & \xrightarrow{\mathfrak{k}(\alpha)} & \mathcal{N} & \xrightarrow{\alpha} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\ {\scriptstyle \widetilde{\phi}}\downarrow\wr & & {\scriptstyle \phi}\downarrow & cart & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle id} \\ Ker(F(\mathfrak{j})) & \xrightarrow{\mathfrak{k}} & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L) \end{array} \tag{3}$$

where the lower middle square is cartesian which implies (by 2.3.3) that $\widetilde{\phi}$ is an isomorphism; the morphism γ is uniquely determined by the equalities $\phi \circ \gamma = F(f')$ and $F(\mathfrak{j}') = \alpha \circ \gamma$, and the left upper square is cartesian due to the latter equality (see 2.3.4.1).

(a2) Suppose now that the morphism $M' \xrightarrow{f} M$ in the diagram (2) (and (3)) is a deflation and the functor F is right 'exact'. Since the left square of the diagram (2) is cartesian, the morphism γ in (3) is a deflation. Therefore, since the left upper square in (3) is cartesian, the arrow $\widetilde{\gamma}$ is a deflation; or, what is the same, the canonical morphism $Ker(F(\mathfrak{j}')) \longrightarrow Ker(F(\mathfrak{j}))$ (equal to the composition $\widetilde{\phi} \circ \widetilde{\gamma}$) is a deflation.

(a3) Notice that $S_-(F)(L)$ is isomorphic to the limit of $Ker(F(\mathfrak{k}(\mathfrak{e}')))$, where $\mathfrak{e}'$ runs through the (filtered) diagram $\mathfrak{E}_X/M$ of *refinements* of the deflation $M \xrightarrow{\mathfrak{e}} L$. That is $S_-(F) = \lim Ker(F(\mathfrak{k}(\mathfrak{t} \circ \mathfrak{e})))$, where $\mathfrak{t}$ runs through the deflations of M (and morphisms of this diagram are also deflations).

Thus, the canonical morphism $S_-(F)(L) \xrightarrow{\widetilde{\mathfrak{d}}_0(E)} Ker(F(\mathfrak{j}))$ is the limit of a filtered system of deflations. Therefore, by hypothesis, it is a deflation.

(b) For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of the right exact category $(C_X, \mathfrak{E}_X)$, the canonical morphism $S_-(F)(M) \longrightarrow Ker(\mathfrak{d}_0(E))$ is a deflation. In fact, let

$$
\begin{array}{ccccc}
N'' & \xrightarrow{\mathfrak{j}''} & M'' & \xrightarrow{\mathfrak{e}''} & M \\
{\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \mathfrak{e}} \\
\widetilde{N}' & \xrightarrow{\widetilde{\mathfrak{j}}'} & M'' & \xrightarrow{\widetilde{\mathfrak{e}}'} & L \\
{\scriptstyle \widetilde{\mathfrak{k}}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}'} & & \downarrow{\scriptstyle id} \\
N' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & L \\
{\scriptstyle \widetilde{\mathfrak{k}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}} & & \downarrow{\scriptstyle id} \\
N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L
\end{array}
\tag{4}
$$

be a commutative diagram whose rows are conflations and the morphisms $\mathfrak{k}$ and $\mathfrak{k}'$ are deflations. By 2.3.4.1 (or 2.3.4.3) that the two lower left squares of (4) are cartesian. In particular, the arrows $\widetilde{\mathfrak{k}}'$ and $\widetilde{\mathfrak{k}}$ are deflations. It follows from 2.3.4.2(b) that the upper two arrows of the left column of (4) form a conflation; i.e. $N'' \xrightarrow{\mathfrak{k}} N'$ is the kernel of $\widetilde{\mathfrak{k}}'$. The diagram (4) yeilds the commutative diagram

$$
\begin{array}{ccccccc}
Ker(F(\mathfrak{j}'')) & \longrightarrow & F(N'') & \xrightarrow{F(\mathfrak{j}'')} & F(M'') & \xrightarrow{F(\mathfrak{e}'')} & F(M) \\
\downarrow & cart & {\scriptstyle F(\mathfrak{k})}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle F(\mathfrak{e})} \\
Ker(F(\widetilde{\mathfrak{j}}')) & \longrightarrow & F(\widetilde{N}') & \xrightarrow{F(\widetilde{\mathfrak{j}}')} & F(M'') & \xrightarrow{F(\widetilde{\mathfrak{e}}')} & F(L) \\
\downarrow & & {\scriptstyle F(\widetilde{\mathfrak{k}}')}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{k}')} & & \downarrow{\scriptstyle id} \\
Ker(F(\mathfrak{j}')) & \longrightarrow & F(N') & \xrightarrow{F(\mathfrak{j}')} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\
\downarrow & & {\scriptstyle F(\widetilde{\mathfrak{k}})}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{k})} & & \downarrow{\scriptstyle id} \\
Ker(F(\mathfrak{j})) & \longrightarrow & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array}
\tag{5}
$$

Since the functor F is weakly right 'exact', the diagram (5) is decomposed into the

diagram

$$
\begin{array}{ccccccc}
Ker(F(\mathfrak{j}'')) & \longrightarrow & F(N'') & & & & \\
\gamma_1\downarrow & cart & \gamma_2\downarrow & & & & \\
Ker(\mathfrak{s}) & \longrightarrow & Ker(F(\widetilde{\mathfrak{k}}')) & \xrightarrow{F(\mathfrak{j}'')} & F(M'') & \xrightarrow{F(\mathfrak{e}'')} & F(M) \\
\mathfrak{k}(\mathfrak{s})\downarrow & cart & \mathfrak{k}'\downarrow & & \downarrow id & & \downarrow F(\mathfrak{e}) \\
Ker(F(\widetilde{\mathfrak{j}}')) & \longrightarrow & F(\widetilde{N}') & \xrightarrow{F(\widetilde{\mathfrak{j}}')} & F(M'') & \xrightarrow{F(\widetilde{\mathfrak{e}}')} & F(L) \\
\mathfrak{s}\downarrow & & F(\widetilde{\mathfrak{k}}')\downarrow & & \downarrow F(\mathfrak{k}') & & \downarrow id \\
Ker(F(\mathfrak{j}')) & \longrightarrow & F(N') & \xrightarrow{F(\mathfrak{j}')} & F(M') & \xrightarrow{F(\mathfrak{e}')} & F(L) \\
\downarrow & & F(\widetilde{\mathfrak{k}})\downarrow & & \downarrow F(\mathfrak{k}) & & \downarrow id \\
Ker(F(\mathfrak{j})) & \longrightarrow & F(N) & \xrightarrow{F(\mathfrak{j})} & F(M) & \xrightarrow{F(\mathfrak{e})} & F(L)
\end{array}
\tag{6}
$$

where γ_1, γ_2 are deflations, $\mathfrak{k}'$ is the kernel (morphism) of $F(\widetilde{\mathfrak{k}}')$; $F(\mathfrak{j}')\circ\gamma_2 = F(\mathfrak{j}'')$, and $\mathfrak{k}'\circ\gamma_2 = F(\mathfrak{k})$. It follows that the two upper left squares of (6) are cartesian. The left column of the diagram (6) induces, via passing to limit, the sequence of arrows

$$S_-F(M) \xrightarrow{\widetilde{\gamma}} Ker(\mathfrak{d}_0(E)) \xrightarrow{\mathfrak{k}_0} S_-F(L) \xrightarrow{\sigma} Ker(F(\mathfrak{j})) \xrightarrow{\mathfrak{k}(F(\mathfrak{j}))} F(N)$$

where $\mathfrak{k}(F(\mathfrak{j}))\circ\sigma = \mathfrak{d}_0(E)$, $\mathfrak{k}_0\circ\widetilde{\gamma} = S_-F(\mathfrak{e})$; σ is a deflation by (a) above, and $\widetilde{\gamma}$ is a deflation by hypothesis, because it is a filtered limit of deflations. ∎

3.6. 'Exact' ∂^*-functors and universal ∂^*-functors. Fix right exact categories $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$, both with initial objects. A ∂^*-functor $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to C_Y is called *'exact'* if for every conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ in $(C_X, \mathfrak{E}_X)$, the complex

$$\ldots \xrightarrow{T_2(\mathfrak{e})} T_2(L) \xrightarrow{\mathfrak{d}_1(E)} T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_i(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'.

3.6.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$ be right exact categories. Suppose that $(C_Y, \mathfrak{E}_Y)$ satisfies $(CE5^*)$. Let $T = (T_i |\ i \geq 0)$ be a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. If the functor T_0 is right 'exact', then the universal ∂^*-functor T is 'exact'.*

Proof. If T_0 is right 'exact', then, by 3.5.3, the functor $T_1 \simeq S_-(T_0)$ is right 'exact' and for any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$, the sequence

$$T_1(N) \xrightarrow{T_1(\mathfrak{j})} T_1(M) \xrightarrow{T_1(e)} T_1(L) \xrightarrow{\mathfrak{d}_0(E)} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'. Since $T_{n+1} = S_-(T_n)$, the assertion follows from 3.5.3 by induction. ∎

3.6.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category. For each object L of C_X, the ∂-functor $Ext^\bullet_X(L) = (Ext^i_X(L) \mid i \geq 0)$ is 'exact'.*

Suppose that the category C_X is k-linear. Then for each $L \in ObC_X$, the ∂-functor $\mathcal{E}xt^\bullet_X(L) = (\mathcal{E}xt^i_X(L) \mid i \geq 0)$ is 'exact'.

Proof. In fact, the ∂-functor $Ext^\bullet_X(L)$ is universal by definition (see 3.4.1), and the functor $Ext^0_X(L) = C_X(-, L)$ is left exact. In particular, it is left 'exact'.

If C_X is a k-linear category, then the universal ∂-functors $\mathcal{E}xt^\bullet_X(L)$, $L \in ObC_X$, with the values in the category of k-modules (defined in 3.4.2) are 'exact' by a similar reason. ∎

3.7. Universal problems for universal ∂^*- and ∂-functors. Fix a right exact svelte category $(C_X, \mathfrak{E}_X)$ with an initial object. Let $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ denote the category whose objects are universal ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to categories C_Y (with initial objects). Let T be a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Y and $\widetilde{T}$ a universal ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a functor from C_Y to C_Z which preserves filtered limits and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. It (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

We denote by $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *complete* (i.e. having limits of small diagrams) categories C_Y and morphisms are pairs (F, ϕ) such that the functor F preserves limits.

Dually, for a left exact category $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ with a final object, we denote by $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ the category whose objects are universal ∂-functors from $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to categories with final object. Given two universal ∂-functors T and T' from $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a functor from C_Y to C_Z preserving filtered colimits and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

We denote by $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ the subcategory of $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ whose objects are ∂-functors with values in cocomplete categories and morphisms are pairs (F, ψ) such that the functor F preserves colimits.

3.7.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with initial objects and $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ a svelte left exact category with final objects. The categories $\partial^*\mathfrak{Un}(X, \mathfrak{E}_X)$, $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$, $\partial\mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$, and $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ have initial objects.*

Proof. (a) We start with the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$. Consider the Yoneda embedding

$$C_\mathfrak{X} \xrightarrow{h_\mathfrak{X}} C^\wedge_X, \quad M \longmapsto C_\mathfrak{X}(-, M).$$

We denote by $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ the universal ∂-functor from $(C_\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$ to $C^\wedge_\mathfrak{X}$ such that $Ext^0_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}} = h_\mathfrak{X}$. The claim is that $Ext^\bullet_{\mathfrak{X},\mathfrak{I}_\mathfrak{X}}$ is an initial object of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_\mathfrak{X})$.

In fact, let C_Y be a cocomplete category. By [GZ, II.1.3], the composition with the Yoneda embedding $C_\mathfrak{X} \xrightarrow{h_\mathfrak{X}} C^\wedge_\mathfrak{X}$ is an equivalence between the category $\mathcal{H}om_{\mathfrak{c}}(C^\wedge_\mathfrak{X}, C_Y)$ of *continuous* (that is having a right adjoint, or, equivalently, preserving colimits) functors $C^\wedge_\mathfrak{X} \longrightarrow C_Y$ and the category $\mathcal{H}om(C_\mathfrak{X}, C_Y)$ of functors from $C_\mathfrak{X}$ to C_Y. Let $C_\mathfrak{X} \xrightarrow{F} C_Y$ be

an arbitrary functor and $C_{\mathfrak{X}}^{\wedge} \xrightarrow{F_{\mathfrak{c}}} C_Y$ the corresponding continuous functor. By definition, $S_+F(L) = \mathrm{colim}(Cok(F(M \longrightarrow Cok(\mathrm{j}))$, where $L \xrightarrow{\mathrm{j}} M$ runs through inflations of L. Since $F_{\mathfrak{c}}$ preserves colimits, it follows from (the dual version of) 3.3.4(a) that $F_{\mathfrak{c}} \circ Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is a universal ∂-functor whose zero component is $F_{\mathfrak{c}} \circ Ext^{0}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}} = F_{\mathfrak{c}} \circ h_{\mathfrak{X}} = F$. Therefore, by (the dual version of the argument of) 3.3.2, the universal ∂-functor $F_{\mathfrak{c}} \circ Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is isomorphic to $S_+^{\bullet}F$. This shows that $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ is an initial object of the category $\partial\mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$.

(b) Let $C_{\mathfrak{X}_{\mathfrak{s}}}$ denote the smallest strictly full subcategory of the category $C_{\mathfrak{X}}^{\wedge}$ containing all presheaves $Ext^{n}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}(L),\ L \in ObC_{\mathfrak{X}},\ n \geq 0$. The category $\partial\mathfrak{Un}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ has an initial object which is the corestriction of the ∂-functor $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ to the subcategory $C_{\mathfrak{X}_{\mathfrak{s}}}$.

Indeed, let C_Y be a category with a final object and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂-functor from $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ to C_Y. Let $C_Y^{\vee^{op}}$ denote the category of presheaves of sets on C_Y^{op} (i.e. functors $C_Y \longrightarrow Sets$) and h_Y^o the (dual) Yoneda functor $C_Y \longrightarrow C_Y^{\vee^{op}},\ L \longmapsto C_Y(L,-)$. The category $C_Y^{\vee^{op}}$ is cocomplete (and complete) and the functor h_Y^o preserves colimits. Therefore, by 3.3.4, the composition $h_Y^o \circ T$ is a universal ∂-functor from $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ to $C_Y^{\vee^{op}}$. By (a) above, the ∂-functor $h_Y^o \circ T$ is the composition of the universal ∂-functor $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ from $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ to $C_{\mathfrak{X}}^{\wedge}$ and the unique continuous functor $C_{\mathfrak{X}}^{\wedge} \xrightarrow{G} C_Y^{\vee^{op}}$ such that $G \circ h_{\mathfrak{X}} = h_Y^o \circ T_0$. Since the functor h_Y^o is fully faithful, this implies that the universal ∂-functor $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ is isomorphic to the composition of the corestriction of $Ext^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ to the subcategory $C_{\mathfrak{X}_{\mathfrak{s}}}$ and a unique functor $C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{G_{\mathfrak{s}}} C_Y$ such that the composition $h_Y^o \circ G_{\mathfrak{s}}$ coincides with the restriction of the functor G to the subcategory $C_{\mathfrak{X}_{\mathfrak{s}}}$.

(c) The assertions about ∂^*-functors are obtained via dualization. Essential details are as follows. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $C_X^{\vee}$ the category of functors $C_X \longrightarrow Sets$ (interpreted as the category of presheaves of sets on C_X^{op}) and by h_X^o the (dual) Yoneda functor $C_X \longrightarrow C_X^{\vee^{op}},\ M \longmapsto C(M,-)$. Let $Ext^{\bullet}_{(X,\mathfrak{E}_X)}$ denote the universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $C_X^{\vee^{op}}$ such that $Ext^{0}_{(X,\mathfrak{E}_X)} = h_X^o$.

Let C_Y be a complete category. By the dual version of [GZ, II.1.3], the composition with the functor h_X^o is a category equivalence between the category $\mathcal{H}om(C_X, C_Y)$ and the category $\mathcal{H}om^{\mathfrak{c}}(C_X^{\vee^{op}}, C_Y)$ of functors $C_X^{\vee^{op}} \longrightarrow C_Y$ which have a left adjoint. Let F be a functor $C_X \longrightarrow C_Y$ and $F^{\mathfrak{c}}$ the corresponding *cocontinuous* (i.e. having a left adjoint) functor from $C_X^{\vee^{op}}$ to C_Y. Since the functor $F^{\mathfrak{c}}$ preserves limits, it follows from 3.3.4 (a), that the composition $F^{\mathfrak{c}} \circ Ext^{\bullet}_{(X,\mathfrak{E}_X)}$ is a universal ∂^*-functor. Its zero component, $F^{\mathfrak{c}} \circ Ext^{0}_{(X,\mathfrak{E}_X)} = F^{\mathfrak{c}} \circ h_X^o$, equals to F. Therefore, by 3.3.2, the universal ∂^*-functor $F^{\mathfrak{c}} \circ Ext^{\bullet}_{(X,\mathfrak{E}_X)}$ is isomorphic to $S_-^{\bullet}F$. This shows that $Ext^{\bullet}_{(X,\mathfrak{E}_X)}$ is an initial object of the category $\partial^*\mathfrak{Un}_{\mathfrak{c}}(X,\mathfrak{E}_X)$.

It follows from (b) (by duality) that the corestriction of the ∂^*-functor $Ext^{\bullet}_{(X,\mathfrak{E}_X)}$ to the smallest subcategory of $C_X^{\vee^{op}}$ containing all representable functors and closed under the endofunctor S_- (that is the full subcategory of $C_X^{\vee^{op}}$ generated by the functors $Ext^{n}_{(X,\mathfrak{E}_X)}(L),\ L \in ObC_X,\ n \geq 0$) is an initial object of the category $\partial^*\mathfrak{Un}(X,\mathfrak{E}_X)$ of universal ∂^*-functors. ∎

3.7.2. The k-linear version. Fix a right exact k-linear additive category $(C_X, \mathfrak{E}_X)$. Let $\partial_k^*\mathfrak{Un}(X,\mathfrak{E}_X)$ denote the category whose objects are universal k-linear ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to k-linear additive categories C_Y. Let T be a universal k-linear ∂^*- functor

from $(C_X, \mathfrak{E}_X)$ to C_Y and $\widetilde{T}$ a universal k-linear ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to C_Z. A morphism from T to T' is a pair (F, ϕ), where F is a k-linear functor from C_Y to C_Z which preserves filtered limits and ϕ is a ∂^*-functor isomorphism $F \circ T \xrightarrow{\sim} T'$. It (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

We denote by $\partial_k^* \mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial_k^* \mathfrak{Un}(X, \mathfrak{E}_X)$ whose objects are k-linear ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *complete* (i.e. having limits of small diagrams) k-linear categories C_Y and morphisms are pairs (F, ϕ) such that the functor F preserves limits.

Dually, for a left exact additive k-linear category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial_k \mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal k-linear ∂-functors from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to k-linear additive categories. Given two universal k-linear ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively C_Y and C_Z, a morphism from T to T' is a pair (F, ψ), where F is a k-linear functor from C_Y to C_Z preserving filtered colimits and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

We denote by $\partial_k \mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial_k \mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are k-linear ∂-functors with values in cocomplete categories and morphisms are pairs (F, ψ) such that the functor F preserves colimits.

3.7.2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte k-linear right exact category and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte k-linear left exact category. The categories $\partial_k^* \mathfrak{Un}(X, \mathfrak{E}_X)$, $\partial_k^* \mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$, $\partial_k \mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$, and $\partial_k \mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. The argument is similar to that of 3.7.1, except for we replace the category $C_{\mathfrak{X}}^{\wedge}$ (resp. $C_{\mathfrak{X}}^{\vee}$) of presheaves of sets on $C_{\mathfrak{X}}$ (resp. on $C_{\mathfrak{X}}^{op}$) by the category $\mathcal{M}_k(\mathfrak{X})$ (resp. $\mathcal{M}_k(\mathfrak{X}^o)$) of presheaves of k-modules on $C_{\mathfrak{X}}$ (resp. on $C_{\mathfrak{X}}^{op} = C_{\mathfrak{X}^o}$).

(a) The initial object of the category $\partial_k \mathfrak{Un}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the universal k-linear ∂-functor $\mathcal{E}xt^{\bullet}_{X, \mathfrak{I}_X}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the category $\mathcal{M}_k(\mathfrak{X})$ of presheaves of k-modules on $C_{\mathfrak{X}}$ whose zero component is the Yoneda embedding $C_{\mathfrak{X}} \longrightarrow \mathcal{M}_k(\mathfrak{X})$, $L \longmapsto C_{\mathfrak{X}}(-, L)$.

(b) The initial object of the category $\partial_k \mathfrak{Un}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the corestriction of $\mathcal{E}xt^{\bullet}_{X, \mathfrak{I}_X}$ to the smallest additive strictly full subcategory of $\mathcal{M}_k(\mathfrak{X})$ which contains all presheaves $\mathcal{E}xt^n_{X, \mathfrak{I}_X}(L)$, $L \in ObC_{\mathfrak{X}}$, $n \geq 0$.

(c) The universal k-linear ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X, \mathfrak{E}_X)}$ from the right exact k-linear category $(C_X, \mathfrak{E}_X)$ to the category $\mathcal{M}_k(X^o)$ of presheaves of k-modules on $C_{X^o} = C_X^{op}$ is an initial object of the category $\partial_k^* \mathfrak{Un}_{\mathfrak{c}}(X, \mathfrak{E}_X)$.

(d) The corestriction of the ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X, \mathfrak{E}_X)}$ to the smallest strictly full additive subcategory of $\mathcal{M}_k(X^o)$ spanned by the presheaves $\mathcal{E}xt^n_{(X, \mathfrak{E}_X)}(L)$, $L \in ObC_X$, $n \geq 0$, is an initial object of the category $\partial_k^* \mathfrak{Un}(X, \mathfrak{E}_X)$.

The argument is similar to that of 3.7.1. Details are left to the reader. ∎

3.8. Universal problems for universal 'exact' ∂^*- and ∂-functors. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with an initial object. Let $\partial^* \mathfrak{UEx}(X, \mathfrak{E}_X)$ denote the category whose objects are universal 'exact' ∂^*-functors $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{E}_X)$ to right exact categories $(C_Y, \mathfrak{E}_Y)$ satisfying $(CE5^*)$ (cf. 3.5.1) such that the functor T_0 maps deflations to deflations. Let T be a universal 'exact' ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ and $\widetilde{T}$ a universal 'exact' ∂^*- functor from $(C_X, \mathfrak{E}_X)$ to $(C_Z, \mathfrak{E}_Z)$. A morphism from T to T' is a pair (F, ϕ), where F is a functor from C_Y to C_Z which preserves filtered

limits and conflations, and ϕ is an isomorphism of ∂^*-functors $F \circ T \xrightarrow{\sim} T'$. It (F', ϕ') is a morphism from T' to T'', then the composition of (F, ϕ) and (F', ϕ') is defined by $(F', \phi') \circ (F, \phi) = (F' \circ F, \phi' \circ F'\phi)$.

We denote by $\partial^*\mathfrak{UEx}_{\mathfrak{c}}(X, \mathfrak{E}_X)$ the subcategory of $\partial^*\mathfrak{UEx}(X, \mathfrak{E}_X)$ whose objects are ∂^*-functors from $(C_X, \mathfrak{E}_X)$ to *complete* right exact categories $(C_Y, \mathfrak{E}_Y)$ satisfying $(CE5^*)$ and morphisms are pairs (F, ϕ) such that the functor F preserves limits.

Dually, for a left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ with a final object, we denote by $\partial\mathfrak{UEx}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal 'exact' ∂-functors $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to left exact categories satisfying $(CE5)$ such that the functor T_0 maps inflations to inflations. Given two universal 'exact' ∂-functors T and T' from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to respectively $(C_Y, \mathfrak{I}_Y)$ and $(C_Z, \mathfrak{I}_Z)$, a morphism from T to T' is a pair (F, ψ), where F is a functor from C_Y to C_Z preserving filtered colimits and conflations and ψ is a functor isomorphism $T' \xrightarrow{\sim} F \circ T$. The composition is defined by $(F', \psi') \circ (F, \psi) = (F' \circ F, F'\psi \circ \psi')$.

We denote by $\partial\mathfrak{UEx}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial\mathfrak{UEx}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ whose objects are ∂-functors with values in cocomplete left exact categories (with final objects) satisfying $(CE5)$ and morphisms are pairs (F, ψ) such that the functor F preserves colimits.

3.8.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category with initial objects and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ a svelte left exact category with final objects. The categories $\partial^*\mathfrak{UEx}(X, \mathfrak{E}_X)$, $\partial^*\mathfrak{UEx}_{\mathfrak{c}}(X, \mathfrak{E}_X)$, $\partial\mathfrak{UEx}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$, and $\partial\mathfrak{UEx}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. (a) The Yoneda embedding

$$C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} C_{\mathfrak{X}}^{\wedge}, \quad L \longmapsto \widehat{L} = C_{\mathfrak{X}}(-, L)$$

is a fully faithful left exact functor. Therefore, it maps strict monomorphisms (in particular, inflations – arrows of $\mathfrak{I}_{\mathfrak{X}}$) to strict monomorphisms of $C_{\mathfrak{X}}^{\wedge}$ which happen to be universally strict. We denote by $\mathfrak{I}_{\mathfrak{X}}^{\wedge}$ the coarsest left exact structure on $C_{\mathfrak{X}}^{\wedge}$ which contains $h_{\mathfrak{X}}(\mathfrak{I}_X)$ and is closed with respect to inductive colimits.

Since $h_{\mathfrak{X}}$ is a left exact functor, it is left 'exact' functor from the left exact category $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the left exact category $(C_{\mathfrak{X}}^{\wedge}, \mathfrak{I}_{\mathfrak{X}}^{\wedge})$. Therefore, by (the dual version of) 3.6.1, the universal ∂-functor $Ext^{\bullet}_{X, \mathfrak{I}_X}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $C_{\mathfrak{X}}^{\wedge}$ whose zero component is the Yoneda embedding $h_{\mathfrak{X}}$ is an 'exact' ∂-functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_{\mathfrak{X}}^{\wedge}, \mathfrak{I}_{\mathfrak{X}}^{\wedge})$.

The claim is that the universal 'exact' ∂-functor $Ext^{\bullet}_{X, \mathfrak{I}_X}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_{\mathfrak{X}}^{\wedge}, \mathfrak{I}_{\mathfrak{X}}^{\wedge})$ is an initial object of the category $\partial\mathfrak{UEx}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$.

In fact, let $(C_Z, \mathfrak{I}_Z)$ be a left exact category such that the category C_Z is cocomplete and F a left 'exact' functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_Z, \mathfrak{I}_Z)$. Then the corresponding continuous functor $C_{\mathfrak{X}}^{\wedge} \xrightarrow{F^*} C_Z$ is an 'exact' functor from $(C_{\mathfrak{X}}^{\wedge}, \mathfrak{I}_{\mathfrak{X}}^{\wedge})$ to $(C_Z, \mathfrak{I}_Z)$.

Since the functor F^* is right exact, it suffices to show that F^* maps inflations to inflations, i.e. $\mathfrak{I}_{\mathfrak{X}}^{\wedge}$ to $\mathfrak{I}_Z$. The arrows of $\mathfrak{I}_{\mathfrak{X}}^{\wedge}$ are obtained from the class of (strict) monomorphisms $h_{\mathfrak{X}}(\mathfrak{I}_{\mathfrak{X}})$ via compositions and push-forwards. The functor F^* preserves push-forwards and (any functor preserves) compositions. Since $F = F^* \circ h_{\mathfrak{X}}$, the class of morphisms $F^*(h_{\mathfrak{X}}(\mathfrak{I}_{\mathfrak{X}}))$ coincides with the class of monomorphisms $F(\mathfrak{I}_{\mathfrak{X}})$. Therefore, it follows from the above description of $\mathfrak{I}_{\mathfrak{X}}^{\wedge}$ (and the fact that F^* preserves push-forwards) that $F^*(\mathfrak{I}_{\mathfrak{X}}^{\wedge})$ is contained in $\mathfrak{I}_Z$.

(b) The initial object of the category $\partial\mathfrak{U}\mathfrak{E}\mathfrak{x}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ is the corestriction of the universal ∂-functor $\mathfrak{E}xt^{\bullet}_{X,\mathfrak{I}_X}$ to the smallest strictly full subcategory of $C_{\mathfrak{X}^o_{\mathfrak{I}}}$ which contains all sheaves $\mathfrak{E}xt^n_{X,\mathfrak{I}_X}(L),\ L\in ObC_{\mathfrak{X}},\ n\geq 0$.

(c) The universal ∂^*-functor $\mathfrak{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ from the right exact category $(C_X,\mathfrak{E}_X)$ to the category $C^{\wedge}_{X^o}$ of presheaves of sets on $C_{X^o}=C_X^{op}$ endowed with the coarsest right exact structure containing the image of $\mathfrak{E}_X$ is an initial object of the category $\partial^*\mathfrak{U}\mathfrak{E}\mathfrak{x}_{\mathfrak{c}}(X,\mathfrak{E}_X)$.

(d) The corestriction of the ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ to the smallest strictly full subcategory of $C^{\wedge}_{X^o}$ spanned by the presheaves $\mathcal{E}xt^n_{(X,\mathfrak{E}_X)}(L),\ L\in ObC_X,\ n\geq 0$, is an initial object of the category $\partial^*\mathfrak{U}\mathfrak{E}\mathfrak{x}(X,\mathfrak{E}_X)$.

The argument is similar to that of 3.7.1. Details are left to the reader. ∎

3.8.2. The k-linear version. Fix a right exact k-linear category $(C_X,\mathfrak{E}_X)$. Let $\partial^*_k\mathfrak{U}\mathfrak{E}\mathfrak{x}(X,\mathfrak{E}_X)$ denote the category whose objects are universal 'exact' k-linear ∂^*-functors $T=(T_i,\mathfrak{d}_i\mid i\geq 0)$ from $(C_X,\mathfrak{E}_X)$ to right exact k-linear categories $(C_Y,\mathfrak{E}_Y)$ satisfying $(CE5^*)$ such that T_0 maps deflations to deflations. Let T be a universal 'exact' k-linear ∂^*- functor from $(C_X,\mathfrak{E}_X)$ to $(C_Y,\mathfrak{E}_Y)$ and $\widetilde{T}$ a universal 'exact' k-linear ∂^*- functor from $(C_X,\mathfrak{E}_X)$ to $(C_Z,\mathfrak{E}_Z)$. A morphism from T to T' is a pair (F,ϕ), where F is a k-linear functor from C_Y to C_Z which preserves filtered limits and conflations, and ϕ is an isomorphism of ∂^*-functors $F\circ T\xrightarrow{\sim}T'$. If (F',ϕ') is a morphism from T' to T'', then the composition of (F,ϕ) and (F',ϕ') is defined by $(F',\phi')\circ(F,\phi)=(F'\circ F,\phi'\circ F'\phi)$.

We denote by $\partial^*_k\mathfrak{U}\mathfrak{E}\mathfrak{x}_{\mathfrak{c}}(X,\mathfrak{E}_X)$ the subcategory of $\partial^*_k\mathfrak{U}\mathfrak{E}\mathfrak{x}(X,\mathfrak{E}_X)$ whose objects are ∂^*-functors from $(C_X,\mathfrak{E}_X)$ to *complete* right exact categories $(C_Y,\mathfrak{E}_Y)$ and morphisms are pairs (F,ϕ) such that the functor F preserves limits.

Dually, for a left exact k-linear category $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$, we denote by $\partial_k\mathfrak{U}\mathfrak{E}\mathfrak{x}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ the category whose objects are universal 'exact' k-linear ∂-functors $T=(T_i,\mathfrak{d}_i\mid i\geq 0)$ from $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ to k-linear left exact categories satisfying $(CE5)$ such that the functor T_0 maps inflations to inflations. Given two universal 'exact' k-linear ∂-functors T and T' from $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ to respectively $(C_Y,\mathfrak{I}_Y)$ and $(C_Z,\mathfrak{I}_Z)$, a morphism from T to T' is a pair (F,ψ), where F is a k-linear functor from C_Y to C_Z preserving filtered colimits and conflations and ψ is a functor isomorphism $T'\xrightarrow{\sim}F\circ T$. The composition is defined by $(F',\psi')\circ(F,\psi)=(F'\circ F,F'\psi\circ\psi')$.

We denote by $\partial_k\mathfrak{U}\mathfrak{E}\mathfrak{x}^{\mathfrak{c}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ the subcategory of $\partial_k\mathfrak{U}\mathfrak{E}\mathfrak{x}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ whose objects are k-linear ∂-functors with values in cocomplete left exact categories and morphisms are pairs (F,ψ) such that the functor F preserves colimits.

3.8.2.1. Proposition. *Let $(C_X,\mathfrak{E}_X)$ be a svelte k-linear right exact category and $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$ a svelte k-linear left exact category. The defined above categories $\partial^*_k\mathfrak{U}\mathfrak{E}\mathfrak{x}(X,\mathfrak{E}_X)$, $\partial^*_k\mathfrak{U}\mathfrak{E}\mathfrak{x}_{\mathfrak{c}}(X,\mathfrak{E}_X)$, $\partial_k\mathfrak{U}\mathfrak{E}\mathfrak{x}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$, and $\partial_k\mathfrak{U}\mathfrak{E}\mathfrak{x}^{\mathfrak{c}}(\mathfrak{X},\mathfrak{I}_{\mathfrak{X}})$ have initial objects.*

Proof. The argument is similar to that of 3.8.1, except for we replace the category $C^{\wedge}_{\mathfrak{X}}$ (resp. $C^{\vee}_{\mathfrak{X}}$) of presheaves of sets on $C_{\mathfrak{X}}$ (resp. on $C^{op}_{\mathfrak{X}}$) by the category $\mathcal{M}_k(\mathfrak{X})$ (resp. $\mathcal{M}_k(\mathfrak{X}^o)$) of presheaves of k-modules on $C_{\mathfrak{X}}$ (resp. on $C^{op}_{\mathfrak{X}}=C_{\mathfrak{X}^o}$).

(a) For a svelte k-linear left exact category $(C_{\mathfrak{X}},\mathfrak{I}_{\mathfrak{X}})$, we denote by $\mathfrak{I}_{\mathfrak{X},k}$ the coarsest left exact structure on the category $\mathcal{M}_k(\mathfrak{X})$ of presheaves of k-modules on $C_{\mathfrak{X}}$ closed under inductive colimits and such that the Yoneda embedding $C_{\mathfrak{X}}\xrightarrow{h_{\mathfrak{X},k}}\mathcal{M}_k(\mathfrak{X})$ maps inflations

to inflations (i.e. $\mathfrak{I}_{\mathfrak{X}}$ to $\mathfrak{I}_{\mathfrak{X},k}$) and is a left exact k-linear functor, hence it is a left 'exact' functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(\mathcal{M}_k(\mathfrak{X}), \mathfrak{I}_{\mathfrak{X},k})$. Therefore, by the k-linear version of 3.6.1, the universal functor $\mathcal{E}xt^{\bullet}_{\mathfrak{X},\mathfrak{I}_{\mathfrak{X}}}$ whose zero component is the Yoneda embedding $h_{\mathfrak{X},k}$ is 'exact'.

If $(C_Z, \mathfrak{I}_Z)$ be a left exact k-linear category such that the category C_Z is cocomplete and F a left 'exact' k-linear functor from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to $(C_Z, \mathfrak{I}_Z)$, then the corresponding continuous functor $C^{\wedge}_{\mathfrak{X}} \xrightarrow{F^*} C_Z$ is an 'exact' functor from $(\mathcal{M}_k, \mathfrak{I}_{\mathfrak{X},k})$ to $(C_Z, \mathfrak{I}_Z)$.

The argument is similar to that of the corresponding part of 3.8.1.

This implies that the universal k-linear ∂-functor $\mathcal{E}xt^{\bullet}_{X,\mathfrak{I}_X}$ from $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ to the left exact category $(\mathcal{M}_k(\mathfrak{X}), \mathfrak{I}_{\mathfrak{X},k})$ is the initial object of the category $\partial_k\mathfrak{UE}\mathfrak{x}^{\mathfrak{c}}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$.

(b) The initial object of the category $\partial_k\mathfrak{UE}\mathfrak{x}(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X}})$ is the corestriction of $\mathcal{E}xt^{\bullet}_{X,\mathfrak{I}_X}$ to the smallest additive strictly full left exact subcategory of $(\mathcal{M}_k(\mathfrak{X}), \mathfrak{I}_{\mathfrak{X},k})$ which contains all presheaves $\mathcal{E}xt^n_{X,\mathfrak{I}_X}(L),\ L \in ObC_{\mathfrak{X}},\ n \geq 0$.

(c) It follows from (a) (by duality) that the universal k-linear ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ from the right exact k-linear category $(C_X, \mathfrak{E}_X)$ to the category $\mathcal{M}_k(X^o)$ of presheaves of k-modules on $C_{X^o} = C_X^{op}$ is an 'exact' universal k-linear ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to the right exact category $(\mathcal{M}_k(X^o), \mathfrak{E}_{\mathfrak{X}^o_k})$, where $\mathfrak{E}_{\mathfrak{X}^o_k}$ is the coarsest right exact structure on $\mathcal{M}_k(X^o)$ such that the Yoneda embedding $C_{X^o} \longrightarrow \mathcal{M}_k(X^o)$ maps $\mathfrak{E}_X$ to $\mathfrak{E}_{\mathfrak{X}^o_k}$. This 'exact' universal k-linear ∂^*-functor is an initial object of the category $\partial^*_k\mathfrak{UE}\mathfrak{x}_{\mathfrak{c}}(X, \mathfrak{E}_X)$.

(d) The corestriction of the ∂^*-functor $\mathcal{E}xt^{\bullet}_{(X,\mathfrak{E}_X)}$ to the smallest strictly full additive right exact subcategory of $\mathcal{M}_k(X^o)$ spanned by the presheaves $\mathcal{E}xt^n_{(X,\mathfrak{E}_X)}(L),\ L \in ObC_X,\ n \geq 0$, is an initial object of the category $\partial^*_k\mathfrak{UE}\mathfrak{x}(X, \mathfrak{E}_X)$.

Details are left to the reader. ∎

3.9. Relative satellites. Fix a right exact category $(C_{\mathfrak{S}}, \mathfrak{E}_{\mathfrak{S}})$. Fix an object $\mathcal{Y}$ of $C_{\mathfrak{S}}$ and consider the right exact category $(\mathcal{Y}\backslash C_{\mathfrak{S}}, \mathfrak{E}_{\mathcal{Y}\backslash\mathfrak{S}})$, where $\mathfrak{E}_{\mathcal{Y}\backslash\mathfrak{S}}$ denote the right exact structure on $\mathcal{Y}\backslash C_{\mathfrak{S}}$ induced by $\mathfrak{E}_{\mathfrak{S}}$.

3.9.1. The ∂^*-functor $\mathcal{F}^{\mathcal{Y}}_{\bullet}$. For a functor $C_{\mathfrak{S}} \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$, let $\mathcal{F}^{\mathcal{Y}}_0$ denote the composition of the canonical functor $\mathcal{Y}\backslash C_{\mathfrak{S}} \longrightarrow C_{\mathfrak{S}}$ and $C_{\mathfrak{S}} \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$. Suppose that the category $C_{\mathcal{Z}}$ has initial objects, kernels of arrows, and limits of filtered systems. Then the functor $\mathcal{F}^{\mathcal{Y}}_0$ extends to a (unique up to isomorphism) ∂^*-functor $\mathcal{F}^{\mathcal{Y}}_{\bullet} = (\mathcal{F}^{\mathcal{Y}}_n, \mathfrak{d}^{\mathcal{Y}}_n \mid n \geq 0)$ from the right exact category $(\mathcal{Y}\backslash C_{\mathfrak{S}}, \mathfrak{E}_{\mathcal{Y}\backslash\mathfrak{S}})$ to $C_{\mathcal{Z}}$. If the category $C_{\mathfrak{S}}$ has initial objects and $\mathcal{Y}$ is one of them, then the category $\mathcal{Y}\backslash C_{\mathfrak{S}}$ is isomorphic to the category $C_{\mathfrak{S}}$ and the functor $\mathcal{F}^{\mathcal{Y}}_{\bullet}$ is the composition of this isomorphism and the universal ∂^*-functor $\mathcal{F}_{\bullet}$, where $\mathcal{F}_0 = \mathcal{F}$.

It follows from the definition of satellites that for every object $(\mathcal{V}, \mathcal{Y} \xrightarrow{\xi_{\mathcal{V}}} \mathcal{V})$ of the category $\mathcal{Y}\backslash C_{\mathfrak{S}}$, we have

$$\mathcal{F}^{\mathcal{Y}}_1(\mathcal{V}, \xi_{\mathcal{V}}) = \mathcal{S}_{\mathcal{Y}}(\mathcal{F})(\mathcal{V}, \xi_{\mathcal{V}}) = \lim Ker(\mathcal{F}(\mathcal{Y} \prod_{\mathfrak{e}, \xi_{\mathcal{V}}} \mathcal{W} \xrightarrow{\mathfrak{p}_{\mathcal{W}}} \mathcal{W})), \tag{1}$$

where $\mathfrak{p}_{\mathcal{W}}$ is the canonical projection and the limit is taken by the filtered system of deflations $(\mathcal{W}, \xi_{\mathcal{W}}) \xrightarrow{\mathfrak{e}} (\mathcal{V}, \xi_{\mathcal{V}})$. By (the argument of) 3.3.2, $\mathcal{F}^{\mathcal{Y}}_n = \mathcal{S}^n_{\mathcal{Y}}(\mathcal{F})$ for all $n \geq 0$.

3.9.2. The ∂^*-functor $\mathcal{F}^{\mathcal{Y},\mathfrak{E}_{\mathfrak{S}}}_{\bullet}$. Let $C_{\mathcal{Z}}$ be a category with final objects and cokernels of arbitrary morphisms. For any functor $C_{\mathfrak{S}} \xrightarrow{\mathcal{F}} C_{\mathcal{Z}}$, let $\mathcal{F}_{\mathcal{Y}}$ denote the func-

tor $\mathcal{Y}\backslash C_{\mathfrak{S}} \longrightarrow C_{\mathcal{Z}}$ which assigns to every object $(\mathcal{W}, \mathcal{Y} \xrightarrow{\xi} \mathcal{W})$ the object $Cok(F(\xi))$ and acts correspondingly on morphisms. Notice that the functor $\mathcal{F}_{\mathcal{Y}}$ maps the initial object $(\mathcal{Y}, id_{\mathcal{Y}})$ of the category $\mathcal{Y}\backslash C_{\mathfrak{S}}$ to a final object of the category $C_{\mathcal{Z}}$. If, in addition, the category $C_{\mathcal{Z}}$ has initial objects (e.g. it is pointed), kernels of arrows and limits of filtered systems, then there exists a (unique up to isomorphism) universal ∂^*-functor $\mathcal{F}_{\bullet}^{\mathcal{Y},\mathfrak{E}_{\mathfrak{S}}} = (\mathcal{F}_n^{\mathcal{Y},\mathfrak{E}_{\mathfrak{S}}}, \mathfrak{d}_n^{\mathcal{Y},\mathfrak{E}_{\mathfrak{S}}} \mid n \geq 0)$ such that $\mathcal{F}_0^{\mathcal{Y},\mathfrak{E}_{\mathfrak{S}}} = \mathcal{F}_{\mathcal{Y}}$.

4. Stable categories of left exact categories. Suspended categories and cohomological functors.

4.1. Observations. Let $(C_X, \mathfrak{I}_X)$ be a svelte left exact category with a final object x and C_Y a category with a final object y and arbitrary colimits. Then, by the (dual version of the) argument of 3.3.2, we have an endofunctor S_+ of the category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y, together with a cone $\mathfrak{y} \xrightarrow{\lambda} S_+$, where $\mathfrak{y}$ is the constant functor with the values in the final object y of the category C_Y. For any conflation $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ of $(C_X, \mathfrak{I}_X)$ and any functor $C_X \xrightarrow{F} C_Y$, we have a commutative diagram

$$\begin{array}{ccccc} F(N) & \xrightarrow{F\mathfrak{j}} & F(M) & \xrightarrow{F\mathfrak{e}} & F(L) \\ & & \downarrow & & \downarrow \mathfrak{d}_0(E) \\ & & y & \xrightarrow{\lambda(N)} & S_+F(N) \end{array} \tag{1}$$

Here $S_+F(N) = \mathrm{colim}(Cok(F(M' \xrightarrow{\mathfrak{e}'} N)))$, where the colimit is taken by the diagram of all conflations $N \xrightarrow{\mathfrak{j}'} M' \xrightarrow{\mathfrak{e}'} L$ (see the argument of 3.3.2). By [GZ, II.1.3], there is a natural equivalence between the category $\mathcal{H}om(C_X, C_Y)$ and the category of functors from C_X and C_Y and the category $\mathcal{H}om_{\mathfrak{c}}(C_X, C_Y)$ of continuous (i.e. having a right adjoint, or, equivalently, preserving colimits) functors from $C_X^{\wedge}$ to C_Y. Let F^* denote a (determined uniquely up to isomorphism) continuous functor corresponding to F, i.e. $F = F^* \circ h_X$, where h_X is the Yoneda embedding $C_X \longrightarrow C_X^{\wedge}$, $M \longmapsto \widehat{M} = C_X(-, M)$. Since the functor F^* preserves colimits, the formula for $S_+F(N)$ can be rewritten as follows:

$$\begin{aligned} S_+F(N) &= \mathrm{colim}(Cok(F(M' \xrightarrow{\mathfrak{e}'} N))) = \mathrm{colim}(Cok(F^*(\widehat{M}' \xrightarrow{\widehat{\mathfrak{e}}'} \widehat{N}))) \\ &= F^*(\mathrm{colim}(Cok(\widehat{M}' \xrightarrow{\widehat{\mathfrak{e}}'} \widehat{N}))) = F^*S_+h_X(N) = F^*Ext_X^1(N). \end{aligned} \tag{2}$$

where colimit is taken by the diagram of all conflations $N \xrightarrow{\mathfrak{j}'} M' \xrightarrow{\mathfrak{e}'} L$.

For any presheaf of sets $\mathcal{G}$ on C_X, we set

$$\widehat{\Theta}_{X*}(\mathcal{G})(-) = C_X^{\wedge}(Ext_X^1(-), \mathcal{G}). \tag{3}$$

The map $\mathcal{G} \longmapsto \widehat{\Theta}_{X*}(\mathcal{G})$ extends to an endofunctor $C_X^{\wedge} \xrightarrow{\widehat{\Theta}_{X*}} C_X^{\wedge}$. It follows from the definition of $\widehat{\Theta}_{X*}$ (and the Yoneda's formula) that

$$C_X^{\wedge}(Ext_X^1(-), \mathcal{G}) = \widehat{\Theta}_{X*}(\mathcal{G})(-) \simeq C_X^{\wedge}(-, \widehat{\Theta}_{X*}(\mathcal{G})). \tag{4}$$

Let Ext^{1*}_X denote the continuous functor $C^\wedge_X \longrightarrow C^\wedge_X$ corresponding to Ext^1_X. It follows from (4) that $C^\wedge_X(Ext^{1*}_X(-),\mathcal{G}) \simeq C^\wedge_X(-,\widehat{\Theta}_{X*}(\mathcal{G}))$, that is the functor $\widehat{\Theta}_{X*}$ is a right adjoint to Ext^{1*}_X. For convenience, we shall write $\widehat{\Theta}^*_X$ instead of Ext^{1*}_X.

Taking as F the Yoneda functor h_X (and setting $\widehat{N} = h_X(N)$), we obtain from the diagram (1) the diagram

$$\begin{array}{ccccc} \widehat{N} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{L} \\ & & \downarrow & & \downarrow \mathfrak{d}_0(E) \\ & & \widehat{x} & \xrightarrow{\lambda(\widehat{N})} & \widehat{\Theta}^*_X(\widehat{N}) \end{array} \tag{5}$$

Once the functor $\widehat{\Theta}^*_X$ is given, all the information about the universal ∂-functor $Ext^\bullet_X = (Ext^i_X, \mathfrak{d}_i \mid i \geq 0)$, and, therefore, due to the universality of $Ext^\bullet_X$, all the information about all universal ∂-functors from the left exact category $(C_X, \mathfrak{I}_X)$, is contained in the diagrams (5), where $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ runs through the class of conflations of $(C_X, \mathfrak{I}_X)$.

In fact, the universal ∂-functor $Ext^\bullet_X$ is of the form $(\widehat{\Theta}^{*n}_X \circ h_X, \widehat{\Theta}^{*n}_X(\mathfrak{d}_0) | n \geq 0)$; and for any functor F from C_X to a category C_Y with colimits and final objects, the universal ∂-functor $(T_i, \mathfrak{d}_i \mid i \geq 0)$ from $(C_X, \mathfrak{I}_X)$ to C_Y with $T_0 = F$ is isomorphic to

$$F^* \circ Ext^\bullet_X = (F^*\widehat{\Theta}^{*n}_X \circ h_X, F^*\widehat{\Theta}^{*n}_X(\mathfrak{d}_0) \mid n \geq 0). \tag{6}$$

4.1.1. Note. If C_X is a pointed category, then the presheaf $\widehat{x} = C_X(-,x)$ is both a final and an initial object of the category $C^\wedge_X$. In particular, the morphism $\widehat{x} \xrightarrow{\lambda(\widehat{N})} \widehat{\Theta}^*_X(\widehat{N})$ in (5) is uniquely defined, hence is not a part of the data. In this case, the data consists of diagrams

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}^*_X(\widehat{N}),$$

where $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ runs through conflations of $(C_X, \mathfrak{I}_X)$.

4.2. Stable category of a left exact category. Consider the full subcategory $C_{X_\mathfrak{p}}$ of the category $C^\wedge_X$ whose objects are $\widehat{\Theta}^{*n}_X(\mathcal{M})$, where $\mathcal{M}$ runs through representable presheaves and n through nonnegative integers. We denote by $\theta_{X_\mathfrak{p}}$ the endofunctor $C_{X_\mathfrak{p}} \longrightarrow C_{X_\mathfrak{p}}$ induced by $\widehat{\Theta}^*_X$. It follows that $C_{X_\mathfrak{p}}$ is the smallest $\widehat{\Theta}^*_X$-stable strictly full subcategory of the category $C^\wedge_X$ containing all presheaves $\widehat{M} = C_X(-,M),\ M \in ObC_X$.

4.2.1. Triangles. We call the diagram

$$\widehat{N} \xrightarrow{\widehat{\mathfrak{j}}} \widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{L} \xrightarrow{\mathfrak{d}_0(E)} \widehat{\Theta}^*_X(\widehat{N}), \tag{1}$$

where $E = (N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L)$ is a conflation in $(C_X, \mathfrak{I}_X)$, a *standard triangle*.

A *triangle* is any diagram in $C_{X_\mathfrak{p}}$ of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{X_\mathfrak{p}}(\mathcal{N}), \tag{2}$$

which is isomorphic to a standard triangle. It follows that for every triangle, the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \downarrow & & \downarrow \mathfrak{d} \\ \widehat{x} & \xrightarrow{\lambda(\mathcal{N})} & \widehat{\Theta}^*_X(\mathcal{N}) \end{array}$$

commutes. Triangles form a category $\mathfrak{Tr}_{X_\mathfrak{p}}$: morphisms from

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

are given by commutative diagrams

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

The composition is obvious.

4.2.2. The prestable category of a left exact category. Thus, we have obtained a data $(C_{X_\mathfrak{p}}, (\theta_{X_\mathfrak{p}}, \lambda), \mathfrak{Tr}_{X_\mathfrak{p}})$. We call this data the *prestable* category of the left exact category $(C_X, \mathfrak{I}_X)$.

4.2.3. The stable category of a left exact category with final objects. Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x and $(C_{X_\mathfrak{p}}, \theta_{X_\mathfrak{p}}, \lambda; \mathfrak{Tr}_{X_\mathfrak{p}})$ the associated with $(C_X, \mathfrak{I}_X)$ presuspended category. Let $\Sigma = \Sigma_{\theta_{X_\mathfrak{p}}}$ be the class of all arrows $\mathfrak{t}$ of $C_{X_\mathfrak{p}}$ such that $\theta_{X_\mathfrak{p}}(\mathfrak{t})$ is an isomorphism.

We call the quotient category $C_{X_\mathfrak{s}} = \Sigma^{-1} C_{X_\mathfrak{p}}$ the *stable* category of the left exact category $(C_X, \mathfrak{I}_X)$. The endofunctor $\theta_{X_\mathfrak{p}}$ determines a conservative endofunctor $\theta_{X_\mathfrak{s}}$ of the stable category $C_{X_\mathfrak{s}}$. The localization functor $C_{X_\mathfrak{p}} \xrightarrow{\mathfrak{q}^*_\Sigma} C_{X_\mathfrak{s}}$ maps final objects to final objects. Let $\lambda_\mathfrak{s}$ denote the image $\widetilde{x} = \mathfrak{q}^*_\Sigma(\widehat{x}) \longrightarrow \theta_{X_\mathfrak{s}}$ of the cone $\widehat{x} \xrightarrow{\lambda} \theta_{X_\mathfrak{p}}$.

Finally, we denote by $\mathfrak{Tr}_{X_\mathfrak{s}}$ the strictly full subcategory of the category of diagrams of the form $\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow \theta_{X_\mathfrak{s}}(\mathcal{N})$ generated by $\mathfrak{q}^*_\Sigma(\mathfrak{Tr}_{X_\mathfrak{s}})$.

The data $(C_{X_\mathfrak{s}}, \theta_{X_\mathfrak{s}}, \lambda_\mathfrak{s}; \mathfrak{Tr}_{X_\mathfrak{s}})$ will be called the *stable* category of $(C_X, \mathfrak{I}_X)$.

4.2.4. Dual notions. If $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$ is a right exact category with an initial object, one obtains, dualizing the definitions of 4.2.2 and 4.4.3, the notions of the *precostable* and *costable* category of $(C_\mathfrak{X}, \mathfrak{E}_\mathfrak{X})$.

4.3. Presuspended categories. Fix a category $C_{\mathfrak{X}}$ with a final object x and a functor $C_{\mathfrak{X}} \xrightarrow{\widetilde{\theta}_{\mathfrak{X}}} x\backslash C_{\mathfrak{X}}$, or, what is the same, a pair $(\theta_{\mathfrak{X}}, \lambda)$, where $\theta_{\mathfrak{X}}$ is an endofunctor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}}$ and λ is a cone $x \longrightarrow \theta_{\mathfrak{X}}$. We denote by $\widetilde{\mathfrak{Tr}_{\mathfrak{X}}}$ the category whose objects are all diagrams of the form

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$$

such that the square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ \downarrow & & \downarrow \mathfrak{d} \\ x & \xrightarrow{\lambda(\mathcal{N})} & \theta_{\mathfrak{X}}(\mathcal{N}) \end{array}$$

commutes. Morphisms from

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$$

to

$$\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

are triples $(\mathcal{N} \xrightarrow{f} \mathcal{N}', \mathcal{M} \xrightarrow{g} \mathcal{M}', \mathcal{L} \xrightarrow{h} \mathcal{L}')$ such that the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow \theta_{\mathfrak{X}}(f) \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

commutes. The composition of morphisms is natural.

4.3.1. Definition. A *presuspended* category is a triple $(C_{\mathfrak{X}}, \widetilde{\theta}_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ and $\widetilde{\theta}_{\mathfrak{X}} = (\theta_{\mathfrak{X}}, \lambda)$ are as above and $\mathfrak{Tr}_{\mathfrak{X}}$ is a strictly full subcategory of the category $\widetilde{\mathfrak{Tr}_{\mathfrak{X}}}$ whose objects are called *triangles*, which satisfies the following conditions:

(PS1) Let $C_{\mathfrak{X}_0}$ denote the full subcategory of $C_{\mathfrak{X}}$ generated by objects $\mathcal{N}$ such that there exists a triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$. For every $\mathcal{N} \in ObC_{\mathfrak{X}_0}$, the diagram

$$\mathcal{N} \xrightarrow{id_{\mathcal{N}}} \mathcal{N} \longrightarrow x \xrightarrow{\lambda(\mathcal{N})} \theta_{\mathfrak{X}}(\mathcal{N})$$

is a triangle.

(PS2) For any triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$ and any morphism $\mathcal{N} \xrightarrow{f} \mathcal{N}'$ with $\mathcal{N}' \in ObC_{\mathfrak{X}_0}$, there is a triangle $\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$ such that f extends to a morphism of triangles

$$(\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})) \xrightarrow{(f,g,h)} (\mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')).$$

(PS3) For any pair of triangles

$$\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{N}' \xrightarrow{\mathfrak{j}'} \mathcal{M}' \xrightarrow{\mathfrak{e}'} \mathcal{L}' \xrightarrow{\mathfrak{d}'} \theta_{\mathfrak{X}}(\mathcal{N}')$$

and any commutative square

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' \end{array}$$

there exists a morphism $\mathcal{L} \xrightarrow{h} \mathcal{L}'$ such that (f, g, h) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} & \xrightarrow{\mathfrak{d}} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \theta_{\mathfrak{X}}(f)} \\ \mathcal{N}' & \xrightarrow{\mathfrak{j}'} & \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' & \xrightarrow{\mathfrak{d}'} & \theta_{\mathfrak{X}}(\mathcal{N}') \end{array}$$

commutes.

(PS4) For any pair of triangles

$$\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N}) \quad \text{and} \quad \mathcal{M} \xrightarrow{x} \mathcal{M}' \xrightarrow{s} \widetilde{\mathcal{M}} \xrightarrow{r} \theta_{\mathfrak{X}}(M),$$

there exists a commutative diagram

$$\begin{array}{ccccccc} \mathcal{N} & \xrightarrow{u} & \mathcal{M} & \xrightarrow{v} & \mathcal{L} & \xrightarrow{w} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ {\scriptstyle id}\downarrow & & {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle id} \\ \mathcal{N} & \xrightarrow{u'} & \mathcal{M}' & \xrightarrow{v'} & \mathcal{L}' & \xrightarrow{w'} & \theta_{\mathfrak{X}}(\mathcal{N}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \theta_{\mathfrak{X}}(u)} & & \\ \widetilde{\mathcal{M}} & \xrightarrow{id} & \widetilde{\mathcal{M}} & \xrightarrow{r} & \theta_{\mathfrak{X}}(\mathcal{M}) & & \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} & & & & \\ \theta_{\mathfrak{X}}(\mathcal{M}) & \xrightarrow{\theta_{\mathfrak{X}}(v)} & \theta_{\mathfrak{X}}(\mathcal{L}) & & & & \end{array} \tag{2}$$

whose two upper rows and two central columns are triangles.

(PS5) For every triangle $\mathcal{N} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \xrightarrow{\mathfrak{d}} \theta_{\mathfrak{X}}(\mathcal{N})$, the sequence

$$\ldots \longrightarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(\mathcal{N}), -) \longrightarrow C_{\mathfrak{X}}(\mathcal{L}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{M}, -) \longrightarrow C_{\mathfrak{X}}(\mathcal{N}, -)$$

is exact.

4.3.1.1. Remarks. (a) If $C_{\mathfrak{X}}$ is an additive category, then three of the axioms above coincide with the corresponding Verdier's axioms of triangulated category (under condition

that $C_{\mathfrak{X}_0} = C_{\mathfrak{X}}$). Namely, (PS1) coincides with the first half of the axiom (TRI), the axiom (PS3) coincides with the axiom (TRIII), and (PS4) with (TRIV) (see [Ve2, Ch.II]).

(b) It follows from (PS4) that if $\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow \theta_{\mathfrak{X}}(\mathcal{N})$ is a triangle, then all three objects, $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{L}$, belong to the subcategory $C_{\mathfrak{X}_0}$.

(c) The axiom (PS2) can be regarded as a base change property, and axiom (PS4) expresses the stability of triangles under composition. So that the axioms (PS1), (PS2) and (PS4) say that triangles form a 'pretopology' on the subcategory $C_{\mathfrak{X}_0}$. The axiom (PS5) says that this pretopology is *subcanonical*: the representable presheaves are sheaves.

These interpretations (as well as the axioms themselves) come from the main examples: prestable and stable categories of a left exact category.

4.3.2. The category of presuspended categories. Let $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda_{\mathfrak{X}}; Tr_{\mathfrak{X}})$ and $\mathfrak{T}_+C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, \lambda_{\mathfrak{Y}}; Tr_{\mathfrak{Y}})$ be presuspended categories. A *triangle* functor from $\mathfrak{T}_+C_{\mathfrak{X}}$ to $\mathfrak{T}_+C_{\mathfrak{Y}}$ is a pair (F, ϕ), where F is a functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ which maps initial object to an initial object and ϕ is a functor isomorphism $F \circ \theta_{\mathfrak{X}} \longrightarrow \theta_{\mathfrak{Y}} \circ F$ such that for every triangle $\mathcal{N} \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{L} \xrightarrow{w} \theta_{\mathfrak{X}}(\mathcal{N})$ of $\mathfrak{T}_+C_{\mathfrak{X}}$, the sequence

$$F(\mathcal{N}) \xrightarrow{F(u)} F(\mathcal{M}) \xrightarrow{F(v)} F(\mathcal{L}) \xrightarrow{\phi(\mathcal{N})F(w)} \theta_{\mathfrak{Y}}(F(\mathcal{N}))$$

is a triangle of $\mathfrak{T}_+C_{\mathfrak{Y}}$. The composition of triangle functors is defined naturally:

$$(G, \psi) \circ (F, \phi) = (G \circ F, \psi F \circ G\phi).$$

Let (F, ϕ) and (F', ϕ') be triangle functors from $\mathfrak{T}_-C_{\mathfrak{X}}$ to $\mathfrak{T}_-C_{\mathfrak{Y}}$. A morphism from (F, ϕ) to (F', ϕ') is given by a functor morphism $F \xrightarrow{\lambda} F'$ such that the diagram

$$\begin{array}{ccc} \theta_{\mathfrak{Y}} \circ F & \xrightarrow{\phi} & F \circ \theta_{\mathfrak{X}} \\ {\scriptstyle \theta_{\mathfrak{Y}}\lambda}\downarrow & & \downarrow{\scriptstyle \lambda\theta_{\mathfrak{X}}} \\ \theta_{\mathfrak{Y}} \circ F' & \xrightarrow{\phi'} & F' \circ \theta_{\mathfrak{X}} \end{array}$$

commutes. The composition is the compsition of the functor morphisms.

Altogether gives the definition of a bicategory $\mathfrak{PCat}$ formed by svelte presuspended categories, triangle functors as 1-morphisms and morphisms between them as 2-morphisms.

As usual, the term "category $\mathfrak{PCat}$" means that we forget about 2-morphisms.

Dualizing (i.e. inverting all arrows in the constructions above), we obtain the bicategory $\mathfrak{P}^o\mathfrak{Cat}$ formed by *precosuspended* svelte categories as objects, triangular functors as 1-morphisms, and morphisms between them as 2-morphisms.

4.4. Quasi-suspended categories. We call a presuspended category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ *quasi-suspended* if the functor $\theta_{\mathfrak{X}}$ is conservative. We denote by $\mathfrak{SCat}$ the full subcategory of the category $\mathfrak{PCat}$ whose objects are quasi-suspended svelte categories.

Let $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}})$ be a presuspended category and $\Sigma = \Sigma_{\theta_{\mathfrak{X}}}$ the class of all arrows s of the category $C_{\mathfrak{X}}$ such that $\theta_{\mathfrak{X}}(s)$ is an isomorphism. Let $\Theta_{\mathfrak{X}}$ denote the endofunctor of the quotient category $\Sigma^{-1}C_{\mathfrak{X}}$ uniquely determined by the equality $\Theta_{\mathfrak{X}} \circ \mathfrak{q}^*_{\Sigma} = \mathfrak{q}^*_{\Sigma} \circ \theta_{\mathfrak{X}}$,

where $\mathfrak{q}^*_\Sigma$ denotes the localization functor $C_{\mathfrak{X}} \longrightarrow \Sigma^{-1}C_{\mathfrak{X}}$. Notice that the functor $\mathfrak{q}^*_\Sigma$ maps final objects to final objects. Let $\widetilde{\lambda}$ denote the morphism $\mathfrak{q}^*_\Sigma(x) \longrightarrow \Theta_{\mathfrak{X}}$ induced by $x \xrightarrow{\lambda} \theta_{\mathfrak{X}}$ (that is by $\mathfrak{q}^*_\Sigma(x) \xrightarrow{\mathfrak{q}^*_\Sigma(\lambda)} \mathfrak{q}^*_\Sigma \circ \theta_{\mathfrak{X}} = \Theta_{\mathfrak{X}} \circ \mathfrak{q}^*_\Sigma$) and $\widetilde{\mathfrak{Tr}_{\mathfrak{X}}}$ the essential image of $\mathfrak{Tr}_{\mathfrak{X}}$. Then the data $(\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}_{\mathfrak{X}}})$ is a quasi-suspended category.

The constructed above map

$$(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \longmapsto (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}_{\mathfrak{X}}})$$

extends to a functor $\mathfrak{PCat} \xrightarrow{\mathfrak{J}^*} \mathfrak{SCat}$ which is a left adjoint to the inclusion functor $\mathfrak{SCat} \xrightarrow{\mathfrak{J}_*} \mathfrak{PCat}$. The natural triangle (localization) functors

$$(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \lambda; \mathfrak{Tr}_{\mathfrak{X}}) \xrightarrow{\mathfrak{q}^*_\Sigma} (\Sigma^{-1}C_{\mathfrak{X}}, \Theta_{\mathfrak{X}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}_{\mathfrak{X}}})$$

form an adjunction arrow $Id_{\mathfrak{PCat}} \longrightarrow \mathfrak{J}_*\mathfrak{J}^*$. The other adjunction arrow is identical.

4.5. The stable category of a left exact category with final objects. Let $(C_X, \mathfrak{I}_X)$ be a left exact category with a final object x and $(C_{X_\mathfrak{p}}, \theta_{X_\mathfrak{p}}, \lambda; \mathfrak{Tr}_{X_\mathfrak{p}})$ the associated with $(C_X, \mathfrak{I}_X)$ presuspended category. We call the category $\Sigma^{-1}C_{X_\mathfrak{p}}$ the *stable* category of the left exact category $(C_X, \mathfrak{I}_X)$. The corresponding quasi-suspended category $(\Sigma^{-1}C_{X_\mathfrak{p}}, \Theta_{X_\mathfrak{p}}, \widetilde{\lambda}; \widetilde{\mathfrak{Tr}_{X_\mathfrak{p}}})$ will be called the *stable quasi-suspended* category of $(C_X, \mathfrak{I}_X)$.

4.5.1. Proposition. *Let $(C_X, \mathfrak{I}_X)$ be a left exact category with final objects. Suppose that $(C_X, \mathfrak{I}_X)$ has enough pointed (i.e. having a morphism from a final object) injectives. Then the stable quasi-suspended category of $(C_X, \mathfrak{I}_X)$ is naturally equivalent to its weak stable category.*

Proof. It is easy to see that the natural functor $C_X \longrightarrow \Sigma^{-1}C_X$ factors through the weak stable category of $(C_X, \mathfrak{I}_X)$. The claim is that the corresponding (unique) functor from the weak stable category of $(C_X, \mathfrak{I}_X)$ to $\Sigma^{-1}C_X$ is a category equivalence. ∎

4.6. Homology and homotopy of 'spaces'.

4.6.1. Homology of 'spaces' with coefficients in a right exact category. Let C_X be a svelte category and $(C_\mathcal{Z}, \mathfrak{E}_\mathcal{Z})$ a svelte right exact category with colimits and initial objects. We denote by $C_{\mathfrak{H}(\mathcal{Z},X)}$ the category of functors $C_X \longrightarrow C_\mathcal{Z}$.

We define the zero homology group of a 'space' X with coefficients in $C_X \xrightarrow{\mathcal{F}} C_\mathcal{Z}$ by $H_0(X, \mathcal{F}) = \operatorname{colim} \mathcal{F}$. The higher homology groups, $H_n(X, \mathcal{F})$, are values at $\mathcal{F}$ of satellites of the functor $C_{\mathfrak{H}(\mathcal{Z},X)} \xrightarrow{H_0(X,-)} C_\mathcal{Z}$ with respect to the (object-wise) right exact structure $\mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)}$ induced by $\mathfrak{E}_\mathcal{Z}$ (cf. K.2.6).

Thus, we have a universal ∂^*-functor

$$H_\bullet(X, -) = (H_n(X, -),\ \mathfrak{d}_n \mid n \geq 0)$$

from the right exact category *of coefficients* $(C_{\mathfrak{H}(\mathcal{Z},X)}, \mathfrak{E}_{\mathfrak{H}(\mathcal{Z},X)})$ to $(C_\mathcal{Z}, \mathfrak{E}_\mathcal{Z})$.

4.6.1.1. Proposition. *Suppose that the right exact category* $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ *satisfies* $(CE5^*)$. *Then the universal* ∂^**-functor* $H_\bullet(X, -)$ *is 'exact'.*

Proof. Let $\mathfrak{J}_*$ denote the canonical embedding of the category $C_{\mathcal{Z}}$ into the category $C_{\mathfrak{H}(\mathcal{Z},X)} = \mathcal{H}om(C_X, C_{\mathcal{Z}})$ which assigns to every object M of the category $C_{\mathcal{Z}}$ the constant functor mapping all arrows of C_X to id_M. The functor $\mathfrak{J}_*$ has a left adjoint, $\mathfrak{J}^*$, which assigns to every functor $C_X \longrightarrow C_{\mathcal{Z}}$ its colimit and to every functor morphism the corresponding morphism of colimits. The composition $\mathfrak{J}^*\mathfrak{J}_*$ is (isomorphic to) the identical functor; i.e. $\mathfrak{J}^*$ is a (continuous) localization functor. The functor $\mathfrak{J}_*$ is exact, hence 'exact', for any category C_X. The functor $\mathfrak{J}^*$ is right exact (in particular right 'exact'), because it has a right adjoint. The assertion follows now from 3.6.1. ∎

4.6.1.2. Note. There is a natural equivalence between the category of local systems of abelian groups on the classifying topological space $\mathcal{B}(X)$ of a category C_X and the morphism inverting functors from C_X to $\mathbb{Z} - mod$. If $\mathcal{F}$ is a morphism inverting functor $C_X \longrightarrow \mathbb{Z} - mod$ and $\mathcal{L}_{\mathcal{F}}$ the corresponding local system, then the homology groups $H_n(X, \mathcal{F})$ are naturally isomorphic to the homology groups $H_n(\mathcal{B}(X), \mathfrak{L}_{\mathcal{F}})$ of the classifying space $\mathcal{B}(X)$ with coefficients in the local coefficient system $\mathcal{L}_{\mathcal{F}}$ (cf. [Q, Section 1].

4.6.2. The 'space' of paths of a 'space'. Let $\mathcal{P}\mathfrak{a}_*$ be the functor from Cat to the category of diagrams of sets of the form $A \rightrightarrows B$ which assigns to each category C_X the diagram $HomC_X \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} ObC_X$, where $\mathfrak{s}$ maps an arrow to its source and $\mathfrak{t}$ to its target. The functor $\mathcal{P}\mathfrak{a}_*$ has a left adjoint, $\mathcal{P}\mathfrak{a}^*$, which assigns to each diagram $T = (T_1 \rightrightarrows T_0)$ the category $\mathcal{P}\mathfrak{a}^*(T)$ of paths of T. The adjunction morphism $\mathcal{P}\mathfrak{a}^*\mathcal{P}\mathfrak{a}_*(C_X) \xrightarrow{\varepsilon(X)^*} C_X$ is a functor which is identical on objects and mapping each path of arrows

$$M_1 \longrightarrow M_2 \longrightarrow \ \ldots \ \longrightarrow M_n$$

to its composition $M_1 \longrightarrow M_n$.

We denote by $\mathfrak{P}\mathfrak{a}(X)$ the 'space' represented by the category $\mathcal{P}\mathfrak{a}^*\mathcal{P}\mathfrak{a}_*(C_X)$ and call it the *'space' of paths* of the 'space' X. The map $X \longmapsto \mathfrak{P}\mathfrak{a}(X)$ extends to an endofunctor, $\mathfrak{P}\mathfrak{a}$, of the category $|Cat|^o$. The adjunction morphism $\mathcal{P}\mathfrak{a}^*\mathcal{P}\mathfrak{a}_*(C_X) \xrightarrow{\varepsilon(X)^*} C_X$ is interpreted as an inverse image functor of a morphism of 'spaces' $X \xrightarrow{\varepsilon(X)} \mathfrak{P}\mathfrak{a}(X)$. The morphisms $\varepsilon = (\varepsilon(X) \mid X \in Ob|Cat|^o)$ form a functor morphism $Id_{|Cat|^o} \longrightarrow \mathfrak{P}\mathfrak{a}$.

4.6.2.1. The 'space' of paths and the loop 'space' of a pointed 'space'. Consider the pointed category $|Cat|^o/x$ associated with the category of 'spaces' $|Cat|^o$; Here x is the initial object of $|Cat|^o$ represented by the category with one (identical) morphism. By C1.5, a choice of a pseudo-functor

$$|Cat|^o \longrightarrow Cat^{op}, \quad X \longmapsto C_X, \ f \longmapsto f^*; \ (gf)^* \xrightarrow{\mathfrak{c}_{f,g}} f^*g^*,$$

induces an equivalence between the category $|Cat|^o/x$ and the category $|Cat|^o_x$ whose objects are pairs $(X, \mathfrak{O}_X)$, where $\mathfrak{O}_X \in ObC_X$; morphisms from $(X, \mathfrak{O}_X)$ to $(Y, \mathfrak{O}_Y)$

are pairs (f,ϕ), where f is a morphism of 'spaces' $X \longrightarrow Y$ and ϕ is an isomorphism $f^*(\mathfrak{O}_Y) \longrightarrow \mathfrak{O}_X$. The composition of $(X,\mathfrak{O}_X) \xrightarrow{(f,\phi)} (Y,\mathfrak{O}_Y) \xrightarrow{(g,\psi)} (Z,\mathfrak{O}_Z)$ is the morphism $(g\circ f, \phi\circ f^*(\psi)\circ \mathfrak{c}_{f,g})$.

The endofunctor $\mathfrak{Pa}$ of $|Cat|^o$ induces an endofunctor $\mathfrak{Pa}_x$ of $|Cat|^o_x$ which assigns to each pointed 'space' $(X,\mathfrak{O}_X)$ the pointed 'space' $(\mathfrak{Pa}(X),\mathfrak{O}_X)$ of paths of $(X,\mathfrak{O}_X)$. It follows that the canonical morphism $X \xrightarrow{\varepsilon(X)} \mathfrak{Pa}(X)$ is a morphism of pointed 'spaces' $(X,\mathfrak{O}_X) \longrightarrow \mathfrak{Pa}_x(X,\mathfrak{O}_X) = (\mathfrak{Pa}(X),\mathfrak{O}_X)$.

It follows from C1.5.1 that the category representing the cokernel of the canonical morphism $(X,\mathfrak{O}_X) \xrightarrow{\varepsilon(x)} \mathfrak{Pa}_x(X,\mathfrak{O}_X)$ is the subcategory of the category $C_{\mathfrak{Pa}(X)}$ whose objects are isomorphic to $\mathfrak{O}_X$ and morphisms are paths of arrows $M_1 \longrightarrow \ldots \longrightarrow M_n$ whose composition is an isomorphism. This category is equivalent to its full subcategory $C_{\Omega(X,\mathfrak{O}_X)}$ of $C_{\mathfrak{Pa}(X)}$ which has one object, $\mathfrak{O}_X$.

We call the 'space' $\Omega(X,\mathfrak{O}_X)$ represented by the latter category the *loop 'space'* of the pointed 'space' $(X,\mathfrak{O}_X)$.

4.6.2.2. Left exact structures on the category of pointed 'spaces'. Let $\mathfrak{E}^{spl}_x$ be the class of all split epimorphisms of diagrams $A \rightrightarrows B$. By 2.6.3.2, the class $\mathcal{P}\mathfrak{a}_*^{-1}(\mathfrak{E}^{spl}_x)$ is a right exact structure on the category of swelte pointed categories. This right exact structure determines a left exact structure $\mathfrak{J}_0$ on the category $|Cat|^o_x$ of pointed 'spaces', so that $(|Cat|^o_x,\mathfrak{J}_0)$ is a Karoubian left exact category. Each path 'space' $(\mathfrak{Pa}(X),\mathfrak{O}_X)$ is an injective object of $(|Cat|^o_x,\mathfrak{J}_0)$, and the canonical morphism $(X,\mathfrak{O}_X) \xrightarrow{\varepsilon(X)} (\mathfrak{Pa}(X),\mathfrak{O}_X)$ belongs to $\mathfrak{J}_0$. The fact that every epimorphism of diagrams of the form $A \rightrightarrows B$ splits implies that the class $\mathfrak{J}_0$ consists of all morphisms $(X,\mathfrak{O}_X) \xrightarrow{\mathfrak{j}} (Y,\mathfrak{O}_Y)$ of the pointed 'spaces' such that the image of $\mathfrak{j}^*$ is naturally equivalent to the category C_X.

4.6.3. The first homotopy group of a pointed 'space'. Given a svelte category C_X, we denote by $C_{\mathcal{G}(X)}$ the groupoid obtained from C_X by localization at $Hom(C_X)$. The map $\mathcal{G}$ which assigns to each object $(X,\mathfrak{O}_X)$ of the category $|Cat|^o_x$ the pair $(\mathcal{G}(X),\mathfrak{O}_X)$ (we identify objects of $C_{\mathcal{G}(X)}$ with objects of C_X) is naturally extended to a functor from $|Cat|^o_x$ to its full subcategory $\mathcal{G}\mathfrak{r}^o_x$ generated by objects $(Y,\mathfrak{O}_Y)$ such that C_Y is a groupoid. This functor is a left adjoint to the inclusion functor $\mathcal{G}\mathfrak{r}^o_x \longrightarrow |Cat|^o_x$.

4.6.3.1. Definition. The *fundamental group* $\pi_1(X,\mathfrak{O}_X)$ of the pointed 'space' $(X,\mathfrak{O}_X)$ is the group $C_{\mathcal{G}(X)}(\mathfrak{O}_X,\mathfrak{O}_X)$ of the automorphisms of the object $\mathfrak{O}_X$ of the groupoid $C_{\mathcal{G}(X)}$ associated with the category C_X. (see [GZ, II.6.2]).

By [Q, Proposition 1], the group $\pi_1(X,\mathfrak{O}_X)$ is isomorphic to the fundamental group $\pi_1(\mathcal{B}(X),\mathfrak{O}_X)$ of the pointed classifying space $(\mathcal{B}(X),\mathfrak{O}_X)$ of the category C_X.

4.6.4. Higher homotopy groups of a pointed 'space'. The map which assigns to every pointed 'space; $(X,\mathfrak{O}_X)$ its fundamental group $\pi_1(X,\mathfrak{O}_X)$ is a functor from $(|Cat|^o_x)^{op}$ to the category $\mathbb{Z}-mod$ of abelian groups. The functor π_1 maps every inflation to an epimorphism and every conflation $(X,\mathfrak{O}_X) \longrightarrow (Y,\mathfrak{O}_Y) \longrightarrow (Z,\mathfrak{O}_Z)$ to an exact sequence of abelian groups $\pi_1(Z,\mathfrak{O}_Z) \longrightarrow \pi_1(Y,\mathfrak{O}_Y) \longrightarrow \pi_1(X,\mathfrak{O}_X)$. Therefore, by 3.3.2, the universal ∂^*-functor $\pi_\bullet = (\pi_n,\mathfrak{d}_n \mid n \geq 1)$ from $(|Cat|^o_x,\mathfrak{J}_0)^{op}$ to $\mathbb{Z}-mod$ is 'exact'. We call $\pi_n(X,\mathfrak{O}_X)$ the *n-th homotopy group* of the pointed 'space' $(X,\mathfrak{O}_X)$.

4.6.4.1. Proposition. *For any pointed 'space' $(X, \mathfrak{O}_X)$ and any $n \geq 1$, there is a natural isomorphism $\pi_{n+1}(X, \mathfrak{O}_X) \simeq \pi_n(\Omega(X, \mathfrak{O}_X))$.*

Proof. This follows from the long exact sequence

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \pi_{n+1}(\mathfrak{Pa}(X, \mathfrak{O}_X)) & & \pi_n(\mathfrak{Pa}(X, \mathfrak{O}_X)) & \longrightarrow & \ldots \\ & & \downarrow & & \uparrow & & \\ & & \pi_{n+1}(X, \mathfrak{O}_X) & \longrightarrow & \pi_n(\Omega(X, \mathfrak{O}_X)) & & \end{array}$$

corresponding to the (functorial) conflation $(X, \mathfrak{O}_X) \longrightarrow \mathfrak{Pa}(X, \mathfrak{O}_X) \longrightarrow \Omega(X, \mathfrak{O}_X)$ of pointed 'spaces' and the fact that the 'space' $(\mathfrak{Pa}(X), \mathfrak{O}_X)$ is an injective object of the pointed left exact category $(|Cat|^o_x, \mathfrak{J}_0)^{op}$, hence $\pi_n(\mathfrak{Pa}(X), \mathfrak{O}_X) = 0$ for $n \geq 1$. ∎

5. Projectives and injectives.

Fix a right exact category $(C_X, \mathfrak{E}_X)$.

5.1. Lemma. *The following conditions on an object P of C_X are equivalent:*
(a) Every deflation $M \longrightarrow P$ splits.
(b) For every deflation $M \xrightarrow{\mathfrak{e}} N$ and a morphism $P \xrightarrow{f} N$, there exists a morphism $P \xrightarrow{g} M$ such that $f = \mathfrak{e} \circ g$.

Proof. Obviously, $(b) \Rightarrow (a)$: it suffices to take $f = id_P$.
$(a) \Rightarrow (b)$. Since deflations are stable under any base change, there is a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{f'} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ P & \xrightarrow{f} & N \end{array}$$

whose left vertical arrow is a deflation. By (a), it splits; i.e. there is a morphism $P \xrightarrow{g} \widetilde{M}$ such that $\mathfrak{e}' \circ g = id_P$. Therefore, $\mathfrak{e} \circ (f' \circ g) = (\mathfrak{e} \circ f') \circ g = (f \circ \mathfrak{e}') \circ g = f$. ∎

5.2. Projectives. Let $(C_X, \mathfrak{E}_X)$ be a right exact category. We call an object P of C_X a *projective* object of $(C_X, \mathfrak{E}_X)$, if it satisfies the equivalent conditions of 5.1. We denote by $\mathcal{P}_{\mathfrak{E}_X}$ the full subcategory of C_X generated by projective objects.

5.2.1. Example. Let $(C_X, \mathfrak{E}_X)$ be a right exact category whose deflations are split. Then every object of C_X is a projective object of $(C_X, \mathfrak{E}_X)$.

5.3. Right exact categories with enough projectives. We say that $(C_X, \mathfrak{E}_X)$ has *enough projectives* if for every object N of C_X there exists a deflation $P \longrightarrow N$, where P is a projective object.

5.3.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and let $C_Y \xrightarrow{f^*} C_X$ be a functor having a right adjoint, f_*. If the functor f_* maps deflations to deflations (e.g. it is right weakly 'exact'), then f^* maps projectives to projectives.*

Proof. Let P be a projective object of $(C_Y, \mathfrak{E}_Y)$ and $M \xrightarrow{\mathfrak{e}} f^*(P)$ a deflation. Then, by hypothesis, $f_*(M) \xrightarrow{f_*(\mathfrak{e})} f_*f^*(P)$ is a deflation. Since P is a projective object, there exists an arrow $P \xrightarrow{\mathfrak{t}} f^*(M)$ such that the diagram

$$\begin{array}{ccc} & P & \\ {\scriptstyle \mathfrak{t}}\swarrow & & \searrow{\scriptstyle \eta(P)} \\ f_*(M) & \xrightarrow{f_*(\mathfrak{e})} & f_*f^*(P) \end{array} \tag{1}$$

commutes (here $\eta(P)$ is an adjunction arrow). Then the composition $f^*(P) \xrightarrow{\mathfrak{t}'} M$ of $f^*(P) \xrightarrow{f^*(\mathfrak{t})} f^*f_*(M)$ and the adjunction morphism $f^*f_*(M) \xrightarrow{\varepsilon(M)} M$ splits the deflation $M \xrightarrow{\mathfrak{e}} f^*(P)$. This follows from the commutativity of the diagram

$$\begin{array}{ccc} & f^*(P) & \\ {\scriptstyle f^*(\mathfrak{t})}\swarrow & & \searrow{\scriptstyle f^*\eta(P)} \\ f^*f_*(M) & \xrightarrow{f^*f_*(\mathfrak{e})} & f^*f_*f^*(P) \\ {\scriptstyle \varepsilon(M)}\downarrow & & \downarrow{\scriptstyle \varepsilon f^*(P)} \\ M & \xrightarrow{\mathfrak{e}} & f^*(P) \end{array} \tag{2}$$

and the equality $\varepsilon f^* \circ f^*\eta = Id_{f^*}$. ∎

5.3.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and let $C_Y \xrightarrow{f^*} C_X$ be a functor having a right adjoint, f_*. Suppose that $\mathfrak{E}_Y$ consists of all split epimorphisms of C_Y and the functor f_* maps deflations to deflations (that is split epimorphisms) and reflects deflations (i.e. if $f_*(\mathfrak{t})$ is a split epimorphism, then $\mathfrak{t}$ is a deflation). Then $(C_X, \mathfrak{E}_X)$ has enough projectives.*

Proof. Since $\mathfrak{E}_Y$ consists of split epimorphisms, all objects of C_Y are projective. Therefore, by 5.3.1, every object of the form $f^*(N)$, $N \in ObC_Y$, is projective. For every object M of C_X the adjunction morphism $f^*f_*(M) \xrightarrow{\varepsilon(M)} M$ is a deflation, because the morphism $f_*\varepsilon(M)$ is a split epimorphism, hence, by hypothesis, it belongs to $\mathfrak{E}_Y$. ∎

5.4. Right exact structure with a given class of projectives. Let C_X be a category and $\mathfrak{P}$ a class of objects of C_X. Let $\mathfrak{E}(\mathfrak{P})$ denote the class of all arrows $M \xrightarrow{f} L$ of C_X such that $C_X(P, M) \xrightarrow{C_X(f,P)} C_X(P, L)$ is surjective and for any morphism $N \xrightarrow{g} L$, there exists a pull-back of f along g.

5.4.1. Lemma. *The class $\mathfrak{E}(\mathfrak{P})$ is the class of covers of a Grothendieck pretopology.*

Proof. Obviously, the class $\mathfrak{E}(\mathfrak{P})$ contains all isomorphisms and is closed under compositions. By assumption, for any morphism $M \xrightarrow{\mathfrak{t}} L$ of $\mathfrak{E}(\mathfrak{P})$ and an arbitrary morphism $N \xrightarrow{g} L$ of C_X, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{g'} & M \\ {\scriptstyle \widetilde{\mathfrak{t}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}} \\ N & \xrightarrow{g} & L \end{array} \tag{3}$$

The functor $C_X(P,-)$ preserves cartesian squares for any object P of C_X. In particular, the image

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{g'} & M \\ {\scriptstyle\widetilde{\mathfrak{t}}}\downarrow & cart & \downarrow{\scriptstyle\mathfrak{t}} \\ N & \xrightarrow{g} & L \end{array} \tag{4}$$

of (3) is a cartesian square. If P belongs to $\mathfrak{P}$, then its right vertical of (4) is surjective, hence its left vertical arrow is surjective too. Tbis shows the pull-back $\widetilde{M} \xrightarrow{\widetilde{\mathfrak{t}}} N$ of the morphism $\mathfrak{t}$ belongs to $\mathfrak{E}(\mathfrak{P})$. ∎

5.4.2. Proposition. *Let C_X be a category. For any class of objects $\mathfrak{P}$ of the category C_X, the class of morphisms $\mathfrak{E}^{\mathfrak{st}}_X(\mathfrak{P}) \stackrel{\text{def}}{=} \mathfrak{E}^{\mathfrak{st}}_X \bigcap \mathfrak{E}(\mathfrak{P})$ is the finest among the right exact structures $\mathfrak{E}_X$ on C_X such that all objects of $\mathfrak{P}$ are projectives of $(C_X, \mathfrak{E}_X)$.*

Proof. Recall that $\mathfrak{E}^{\mathfrak{st}}_X$ is the finest right exact structure on C_X; it consists of all universal strict epimorphisms of C_X. The intersection of Grothendieck pretopologies is a Grothendieck pretopology. Since it is contained in $\mathfrak{E}^{\mathfrak{st}}_X$, it is a left exact structure. Evidently, any right exact structure $\mathfrak{E}_X$ such that all objects of $\mathfrak{P}$ are projectives of $(C_X, \mathfrak{E}_X)$, is coarser than $\mathfrak{E}^{\mathfrak{st}}_X(\mathfrak{P})$. ∎

5.5. Right exact categories of modules over monads. Fix a category C_Y such that the class $\mathfrak{E}^{spl}_Y$ of split epimorphisms of C_Y is stable under base change. Equivalently, for each split epimorphism $M \xrightarrow{\mathfrak{t}} L$ and for an arbitrary morphism $N \xrightarrow{f} L$ of the category C_Y, there exists a cartesian square

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{f'} & M \\ {\scriptstyle\widetilde{\mathfrak{t}}}\downarrow & cart & \downarrow{\scriptstyle\mathfrak{t}} \\ N & \xrightarrow{f} & L \end{array}$$

(whose left vertical arrow is split, because $\mathfrak{t}$ is split). So that $(C_Y, \mathfrak{E}^{spl}_Y)$ is a right exact category. Let $\mathcal{F} = (F, \mu)$ be a monad on the category C_Y. Set $C_X = \mathcal{F} - mod$ (i.e. $X = \mathbf{Sp}(\mathcal{F}/Y)$ – the spectrum of the monad $\mathcal{F}$) and denote by f_* the forgetful functor $C_X \longrightarrow C_Y$. We set $\mathfrak{E}_X = f_*^{-1}(\mathfrak{E}^{spl}_Y)$. Since f_* preserves and reflects limits (in particular, pullbacks), the arrows of $\mathfrak{E}_X$ are covers of a pretopology, i.e. $(C_X, \mathfrak{E}_X)$ is a right exact category. The functor f_* has a left adjoint, $V \stackrel{f^*}{\longmapsto} (F(V), \mu(V))$, and all together satisfy the conditions of 5.3.2. Therefore, $(C_X, \mathfrak{E}_X)$ has enough projectives. Explicitely, it follows from (the argument of) 5.3.2 that objects $f^*(V) = (F(V), \mu(V))$ are projectives of $(C_X, \mathfrak{E}_X)$ for all $V \in ObC_Y$, and for every $\mathcal{F}$-module $\mathcal{M} = (M, \xi)$, the action $F(M) \xrightarrow{\xi} M$ can be regarded as a canonical deflation from a projective object:

$$f^* f_*(\mathcal{M}) = (F(M), \mu(M)) \xrightarrow{\xi} \mathcal{M}.$$

5.5.1. Proposition. *Suppose that $(C_Y, \mathfrak{E}^{spl}_Y)$ is a Karoubian right exact category (i.e. C_Y is a Karoubian category and split epimorphisms are stable under base change).*

Then for every monad $\mathcal{F} = (F, \mu)$ on C_Y, the right exact category $(\mathcal{F} - mod, \mathfrak{E}_X)$, where $\mathfrak{E}_X$ is the induced by $\mathfrak{E}_Y^{spl}$ right exact structure, is Karoubian.

Proof. (a) The forgetful functor $\mathcal{F} - mod \xrightarrow{f_*} C_Y$ reflects and preserves limits; in particular, it reflects and preserves pull-backs. Therefore, the stability of split epimorphisms of C_Y under base change implies the same property of split epimorphisms of $\mathcal{F} - mod$.

(b) It remains to show that $\mathcal{F} - mod$ is a Karoubian category. Let $\mathcal{M} = (M, \xi)$ be an $\mathcal{F}$-module and $\mathfrak{p}$ an idempotent $\mathcal{M} \longrightarrow \mathcal{M}$. Since C_Y is a Karoubian category, the idempotent $f_*(\mathcal{M}) = M \xrightarrow{f_*(\mathfrak{p})} M$ splits. By 1.5.1, the latter is equivalent to the existence of the kernel of the pair of arrows $M \underset{f_*(\mathfrak{p})}{\overset{id_M}{\rightrightarrows}} M$. Since the forgetful functor f_* reflects and preserves limits, in particular kernels of pairs of arrows, there exists the kernel of pair of arrows $\mathcal{M} \underset{\mathfrak{p}}{\overset{id_M}{\rightrightarrows}} \mathcal{M}$; i.e. the idempotent $\mathfrak{p}$ splits. ∎

5.5.2. Corollary. *Let $\mathcal{G} = (G, \delta)$ be a comonad on a Karoubian category C_X. Suppose that class $\mathfrak{I}_X^{spl}$ of split monomorphisms in C_X is stable under cobase change (i.e. $\mathfrak{I}_X^{spl}$ is a left exact structure on C_X). Let $\mathfrak{I}_\mathfrak{Y}$ be the preimage of $\mathfrak{I}_X^{spl}$ in the category $C_\mathfrak{Y} = \mathcal{G} - comod$ of $\mathcal{G}$-comodules. Then $(C_\mathfrak{Y}, \mathfrak{I}_\mathfrak{Y})$ is a Karoubian left exact category having enough injectives.*

Proof. The assertion is dual to that of 5.5.1. Futher on, we need details which are as follows. Let $C_\mathfrak{Y}$ be the category $\mathcal{G} - comod$ of $\mathcal{G}$-comodules with the exact structure induced by the forgetful functor

$$C_\mathfrak{Y} = \mathcal{G} - comod \xrightarrow{g^*} C_X.$$

Its right adjoint

$$C_X \xrightarrow{g_*} C_\mathfrak{Y} = \mathcal{G} - comod, \quad M \longmapsto (G(M), \delta(M)), \tag{1}$$

maps every object M of the category C_X to an $\mathcal{E}_\mathfrak{Y}$-injective object. If the category C_X is Karoubian, then, for every object $\mathcal{M} = (M, M \xrightarrow{\nu} G(M))$, the adjunction morphism

$$\mathcal{M} \xrightarrow{\nu} g_* g^*(\mathcal{M}) = (G(M), \delta(M)) \tag{2}$$

is an inflation of $\mathcal{G}$-comodules (see the argument of the dual assertion 5.5.1). ∎

5.5.3. Corollary. *Under the conditions of 5.5.2, an object $\mathcal{M} = (M, \nu)$ of the category $C_\mathfrak{Y}$ of $\mathcal{G}$-comodules is $\mathfrak{I}_\mathfrak{Y}$-injective iff the adjunction morphism $\mathcal{M} \xrightarrow{\nu} (G(M), \delta(M))$ splits (as a morphism of $\mathcal{G}$-comodules).*

5.5.4. Proposition. *Suppose that C_X is a Karoubian category whose split epimorphisms (resp. split monomorphisms) are stable under base (resp. cobase) change. Let $\mathcal{F} = (F, \mu)$ be a continuous monad on C_X (i.e. the functor F has a right adjoint) and*

f_* the forgetful functor $\mathcal{F}-mod \longrightarrow C_X$. Set $C_{\mathfrak{X}} = \mathcal{F}-mod$, $\mathfrak{E}_{\mathfrak{X}} = f_*^{-1}(\mathfrak{E}_X^{spl})$, and $\mathfrak{I}_{\mathfrak{X}} = f_*^{-1}(\mathfrak{I}_X^{spl})$. Then $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ is a right exact category with enough projectives and $(C_{\mathfrak{X}}, \mathfrak{I}_{\mathfrak{X}})$ is a left exact category with enough injectives.

Proof. If the monad $\mathcal{F} = (F, \mu)$ is continuous, i.e. the functor F has a right adjoint, $F^!$, then (and only then) the forgetful functor $\mathcal{F}-mod = C_{\mathfrak{X}} \xrightarrow{f_*} C_X$ has a right adjoint, $f^!$, such that $F^! = f_* f^!$. Thus, we have the comonad $\mathcal{F}^! = (F^!, \delta)$ corresponding to the pair of adjoint functors f_*, $f^!$ and an isomorphism of categories

$$\mathcal{F}-mod \xrightarrow{\Phi} \mathcal{F}^!-comod$$

which assigns to every $\mathcal{F}$-module $(M, F(M) \xrightarrow{\xi} M)$ the $\mathcal{F}^!$-comodule $(M, M \xrightarrow{\widehat{\xi}} F^!(M))$ determined (uniquely up to isomorphism) by adjunction. It follows that the diagram

$$\begin{array}{ccc} \mathcal{F}-mod & \xrightarrow{\Phi} & \mathcal{F}^!-comod \\ {\scriptstyle f_*}\searrow & & \swarrow{\scriptstyle g^*} \\ & C_X & \end{array} \tag{3}$$

commutes. By 5.5.1, the category $C_{\mathfrak{X}} = \mathcal{F}-mod$ has enough $\mathfrak{E}_{\mathfrak{X}}$-injectives. By 5.5.2, the category $C_{\mathfrak{Y}} = \mathcal{F}^!-comod$ has enough $\mathfrak{I}_{\mathfrak{Y}}$-injectives. The functor Φ in (3) is an isomorphism of exact categories, hence the assertion. ∎

5.6. Coeffaceable functors, universal ∂^*-functors, and projectives. Let $(C_X, \mathfrak{E}_X)$ be a right exact category and C_Y a category with an initial object. We call a functor $C_X \xrightarrow{F} C_Y$ *coeffaceable*, or $\mathfrak{E}_X$-*coeffaceable*, if for any object L of C_X, there exists a deflation $M \xrightarrow{\mathfrak{t}} L$ such that $F(\mathfrak{t})$ is a trivial morphism.

5.6.1. Coeffaceble functors and projectives. If a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable, then the morphism $F(\mathfrak{t})$ is trivial for any projective deflation $\mathfrak{t}$, and F maps every projective object of $(C_X, \mathfrak{E}_X)$ to an initial object of C_Y.

In fact, a projective deflation $M \xrightarrow{\mathfrak{t}} L$ factors through any other deflation of L; and, by hypothesis, there exists a deflation $M \xrightarrow{\mathfrak{e}} L$ such that $F(\mathfrak{e})$ is trivial. Therefore, the morphism $F(\mathfrak{t})$ is trivial. An object M is projective iff id_M is a projective deflation; and the triviality of $F(id_M)$ means precisely that $F(M)$ is an initial object.

So that if the right exact category $(C_X, \mathfrak{E}_X)$ has enough projective deflations (resp. enough projectives), then a functor $C_X \xrightarrow{F} C_Y$ is $\mathfrak{E}_X$-coeffaceable iff $F(\mathfrak{e})$ is trivial for any projective deflation $\mathfrak{e}$ (resp. $F(M)$ is an initial object for every projective object M).

5.6.2. Universal ∂^*-functors and pointed projectives. Let C_Z be a category with initial objects. We call an object M of C_Z *pointed* if there are morphisms from M to initial objects, or, equivalently, a unique morphism from an initial object to M is split.

5.6.2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. Then $T_i(P)$ is an initial object for any pointed projective object P and for all $i \geq 1$.*

Proof. Let F denote the functor $T_i,\ i \geq 0$. By 3.3.2, $T_{i+1} \simeq S_-(F)(P)$. Let x be an initial object of C_X and P a projective object of $(C_X, \mathfrak{E}_X)$ such that there exists a morphism $P \longrightarrow x$, then $T_{i+1}(P) \simeq S_-(F)(P)$ is an initial object.

In fact, consider the conflation $x \xrightarrow{i_P} P \xrightarrow{id_P} P$. If there exists a morphism $P \longrightarrow x$, then the unique arrow $x \xrightarrow{i_P} P$ is a split monomorphism. Therefore $F(i_P)$ is a (split) monomorphism. By 2.1.1, the latter implies that $Ker(F(i_P))$ is an initial object. Since the object P is projective, any deflation $M \xrightarrow{\mathfrak{e}} P$ is split; i.e. there exists a morphism of deflations $(P \xrightarrow{id_P} P) \xrightarrow{\mathfrak{u}} (M \xrightarrow{\mathfrak{e}} P)$. This implies that the canonical morphism $S_-(F)(L) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{e}))$ factors through the morphism $Ker(F(i_P)) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{e}))$ determined by the morphism of deflations $\mathfrak{u}$. Since $Ker(F(i_P)) = y$ is an initial object of the category C_Y, it follows that the morphism $Ker(F(i_P)) \longrightarrow Ker(F(\mathfrak{k}(\mathfrak{e}))$ is unique (in particular, it does not depend on the choice of the section $P \xrightarrow{\mathfrak{u}} M$). Therefore, the canonical morphism $S_-(F)(L) \longrightarrow Ker(F(i_P)) = y$ is an isomorphism. ∎

5.6.2.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ a universal ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to C_Y. Suppose that $(C_X, \mathfrak{E}_X)$ has enough projectives and projectives of $(C_X, \mathfrak{E}_X)$ are pointed objects. Then the functors T_i are coeffaceable for all $i \geq 1$.*

Proof. The assertion follows from 5.6.2.1 and 5.6.1. ∎

5.6.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Z, \mathfrak{E}_Z)$ be right exact categories and $C_Z \xrightarrow{f^*} C_X$ a functor having a right adjoint f_*. Suppose that f^* maps deflations of the form $N \longrightarrow f_*(M)$ to deflations and the adjunction arrow $f^*f_*(M) \xrightarrow{\epsilon(M)} M$ is a deflation for all M (which is the case if any morphism $\mathfrak{t}$ of C_X such that $f_*(\mathfrak{t})$ is a split epimorphism belongs to $\mathfrak{E}_X$). Let $(C_Z, \mathfrak{E}_Z)$ have enough projectives, and all projectives are pointed objects. Then each projective of $(C_X, \mathfrak{E}_X)$ is a pointed object.*

If, in addition, f_* maps deflations to deflations, then $(C_X, \mathfrak{E}_X)$ has enough projectives.

Proof. (a) Let M be an object of C_X. Since $(C_Z, \mathfrak{E}_Z)$ has enough projectives, there exists a deflation $\widetilde{P} \xrightarrow{\mathfrak{t}} f_*(M)$, where $\widetilde{P}$ is a projective object. By hypothesis, the morphisms

$$f^*(\widetilde{P}) \xrightarrow{f^*(\mathfrak{t})} f^*f_*(M) \xrightarrow{\epsilon(M)} M \tag{1}$$

are deflations. If the object M is projective, their composition is a split epimorphism; i.e. it has a section $M \xrightarrow{\zeta} f^*(\widetilde{P})$. By hypothesis, there exists a morphism $\widetilde{P} \xrightarrow{\gamma} z$, where z is an initial object of C_Z. Since the functor f^* has a right adjoint, $f^*(z)$ is an initial object of the category C_X, and we have a morphism $M \xrightarrow{f^*(\gamma)\circ\zeta} f^*(z)$.

(b) If, in addition, the functor f_* maps deflations to deflations, then, by 5.3.1, its left adjoint f^* maps projectives to projectives. So that the composition of the arrows (1) is a deflation with a projective domain. ∎

5.6.4. Note. The conditions of 5.6.3 can be replaced by the requirement that if $N \longrightarrow f_*(M)$ is a deflation, then the corresponding morphism $f^*(N) \longrightarrow M$ is a deflation.

This requirement follows from the conditions of 5.6.3, because the morphism $f^*(N) \longrightarrow M$ corresponding to $N \xrightarrow{\mathfrak{t}} f_*(M)$ is the composition of $f^*(\mathfrak{t})$ and the adjunction arrow $f^*f_*(M) \xrightarrow{\epsilon(M)} M$.

5.6.5. Example. Let $(C_X, \mathfrak{E}_X)$ be the category Alg_k of associative k-algebras endowed with the canonical (that is the finest) right exact structure. This means that class $\mathfrak{E}_X$ of deflations coincides with the class of all are strict epimorphisms of k-algebras. Let $(C_Y, \mathfrak{E}_Y)$ be the category of k-modules with the canonical exact structure, and f_* the forgetful functor $Alg_k \longrightarrow k-mod$. Its left adjoint, f^* preserves strict epimorphisms, and the functor f_* preserves and reflects deflations; i.e. a k-algebra morphism t is a strict epimorphism iff $f_*(t)$ is an epimorphism. In particular, the adjunction arrow $f^*f_*(A) \longrightarrow A$ is a strict epimorphism for all A. By 5.6.3, $(C_X, \mathfrak{E}_X)$ has enough projectives and each projective has a morphism to the initial object k; that is each projective has a structure of an augmented k-algebra.

5.6.6. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects; and let $T = (T_i, \mathfrak{d}_i |\ i \geq 0)$ be an 'exact' ∂^*-functor from $(C_X, \mathcal{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$.*

If the functors T_i are $\mathfrak{E}_X$-coeffaceable for $i \geq 1$, then T is a universal ∂^*-functor.

Proof. Let $T' = (T'_i, \mathfrak{d}'_i |\ i \geq 0)$ be another ∂^*-functor from $(C_X, \mathcal{E}_X)$ to C_Y and f_0 a functor morphism $T'_0 \longrightarrow T_0$. Fix an object L of C_X. Let $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ be a conflation such that $T_1(\mathfrak{e})$ factors through the initial object y of C_Y. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
T'_1(M) & \xrightarrow{T'_1(\mathfrak{e})} & T'_1(L) & \xrightarrow{\mathfrak{d}'} & T'_0(N) & \xrightarrow{T'_0(\mathfrak{j})} & T'_0(M) & \xrightarrow{T'_0(\mathfrak{e})} & T'_0(L) \\
 & & & & {\scriptstyle f_0(N)}\downarrow & & \downarrow{\scriptstyle f_0(M)} & & \downarrow{\scriptstyle f_0(L)} \\
T_1(M) & \xrightarrow{T_1(\mathfrak{e})} & T_1(L) & \xrightarrow{\mathfrak{d}} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M) & \xrightarrow{T_0(\mathfrak{e})} & T_0(L)
\end{array} \qquad (1)$$

Since the lower row of (1) is an 'exact' sequence and $T_1(\mathfrak{e})$ factors through the initial object y, the sequence

$$y \longrightarrow T_1(L) \xrightarrow{\mathfrak{d}} T_0(N) \xrightarrow{T_0(\mathfrak{j})} T_0(M)$$

is 'exact'. Therefore, there exists a unique morphism $T'_1(L) \xrightarrow{f_1(L)} T_1(L)$ such that the diagram

$$\begin{array}{ccccc}
T'_1(L) & \xrightarrow{\mathfrak{d}'} & T'_0(N) & \xrightarrow{T'_0(\mathfrak{j})} & T'_0(M) \\
{\scriptstyle f_1(L)}\downarrow & & {\scriptstyle f_0(N)}\downarrow & & \downarrow{\scriptstyle f_0(M)} \\
T_1(L) & \xrightarrow{\mathfrak{d}} & T_0(N) & \xrightarrow{T_0(\mathfrak{j})} & T_0(M)
\end{array}$$

commutes. By a standard argument, it follows from the uniqueness of $f_1(L)$ and the fact that the family of the deflations of L is filtered (since pull-backs of deflations are deflations) that the morphism $f_1(L)$ does not depend on a choice of the conflation and the family $f_1 = (f_1(L) \mid L \in ObC_X)$ is a functor morphism $T'_1 \longrightarrow T_1$ compatible with the connecting morphisms $\mathfrak{d}_0,\ \mathfrak{d}'_0$. ∎

5.6.6.1. Note. If a right exact category $(C_X, \mathfrak{E}_X)$ has enough projectives and each projective is a pointed object, then, by 5.6.2.2, for any universal ∂^*-functor T, the functors T_i are $\mathfrak{E}_X$-coeffaceable for $i \geq 1$.

5.6.7. Proposition. *Let $(C_X, \mathfrak{E}_X)$, $(C_Y, \mathfrak{E}_Y)$, and $(C_Z, \mathfrak{E}_Z)$ be right exact categories. Suppose that $(C_X, \mathfrak{E}_X)$ has enough projectives and C_Y has kernels of all morphisms. If $T = (T_i |\ i \geq 0)$ is a universal, 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ and F a functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_Z, \mathfrak{E}_Z)$ which respects conflations, then the composition $F \circ T = (F \circ T_i |\ i \geq 0)$ is a universal 'exact' ∂^*-functor.*

Proof. Since T is a universal ∂^*-functor, it follows from 5.6.2.2 that the functors T_i are $\mathfrak{E}_X$-coeffaceable for all $i \geq 1$. Moreover, because C_X has enough projectives, the coeffaceability of T_i means precisely that $T_i(L) = 0$ for any projective object L of $(C_X, \mathfrak{E}_X)$. Therefore, $F \circ T_i(L) = 0$ for all $i \geq 1$, i.e. the ∂^*-functor $F \circ T$ is $\mathfrak{E}_X$-effaceable. Since by hypothesis, T is an 'exact' ∂^*-functor and F is an 'exact' functor, their composition, $F \circ T$, is an 'exact' ∂^*-functor. By 5.6.6(a), it is universal. ∎

5.6.8. A remark about (co)effaceable functors. Let C_X be a category with initial objects and $\mathcal{B}$ its subcategory. We say that an object M of C_X is *right* (resp. *left*) *orthogonal* to $\mathcal{B}$ if for every $N \in Ob\mathcal{B}$, there are only trivial morphisms from N to M (resp. from M to N). We denote by $\mathcal{B}^{\perp}$ (resp. ${}^{\perp}\mathcal{B}$) the full subcategory of C_X generated by objects right (resp. left) orthogonal to $\mathcal{B}$.

Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories, and let y be an initial object of the category C_Y. The category $\mathcal{H}om(C_X, C_Y)$ of functors from C_X to C_Y has an initial object, which is the constant functor with values in y. Let $\mathfrak{R}ex((C_X, \mathfrak{E}_X), (C_Y, \mathfrak{E}_Y))$ be the full subcategory of $\mathcal{H}om(C_X, C_Y)$ whose objects are right 'exact' functors. And let $\mathcal{E}ff^o((C_X, \mathfrak{E}_X), C_Y)$ denote the full subcategory of $\mathcal{H}om(C_X, C_Y)$ generated by coeffaceable functors from $(C_X, \mathfrak{E}_X)$ to C_Y.

5.6.8.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelt right exact category with enough projectives and $(C_Y, \mathfrak{E}_Y)$ a right exact category with an initial object y. Suppose that C_Y is a category with kernels of morphisms and the unique morphism from the initial object y of C_Y are monomorphisms. Then $\mathcal{E}ff^o((C_X, \mathfrak{E}_X), C_Y)$ is right orthogonal to the subcategory generated by all functors $C_X \longrightarrow C_Y$ which map deflations to strict epimorphisms.*

Proof. Let $F \in Ob\mathcal{E}ff^o((C_X, \mathfrak{E}_X), C_Y)$; and let $G \xrightarrow{\phi} F$ be a functor morphism, where G is a functor which maps deflations to strict epimorphisms. Since $(C_X, \mathfrak{E}_X)$ has enough projectives, for each object L of C_X, there exists a deflation $P \xrightarrow{\mathfrak{e}} L$ such that P is a projective object. Then we have a commutative diagram

$$\begin{array}{ccc} G(P) & \xrightarrow{\phi(P)} & F(P) \\ {\scriptstyle G(\mathfrak{e})}\downarrow & & \downarrow{\scriptstyle F(\mathfrak{e})} \\ G(L) & \xrightarrow{\phi(L)} & F(L) \end{array}$$

with $F(P)$ being an initial object of C_Y; so that the composition $\phi(L) \circ G(\mathfrak{e})$ is a trivial morphism. By 2.1.2, all morphisms from initial objects are monomorphisms iff for any

morphism $M \xrightarrow{f} N$ the kernel morphism $Ker(f) \longrightarrow M$ is a monomorphism. Since $G(\mathfrak{e})$ is a strict epimorphism and the kernel morphism $Ker(\phi(L)) \longrightarrow G(L)$ is a monomorphism, it follows from 2.3.4.4 that $\phi(L)$ is a trivial morphism. ∎

5.6.8.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelt right exact pointed category with enough projectives, and let $(C_Y, \mathfrak{E}_Y)$ be the category of pointed sets with the canonical exact structure. Then a functor $C_X \xrightarrow{F} C_Y$ is coeffaceable iff it is a right orthogonal to the subcategory $\mathfrak{R}ex((C_X, \mathfrak{E}_X), (C_Y, \mathfrak{E}_Y))$ of right 'exact' functors from $(C_X, \mathfrak{E}_X)$ to C_Y.*

Proof. The fact that coeffaceable functors to C_Y are right orthogonal to right 'exact' functors follows from 5.6.8.1, because right 'exact' functors map deflations to deflations, and deflations are strict epimorphisms.

Conversely, let a functor $C_X \xrightarrow{F} C_Y$ be right orthogonal to all right 'exact' functors from $(C_X, \mathfrak{E}_X)$ to C_Y. Notice that for any projective object P of $(C_X, \mathfrak{E}_X)$, the functor $\check{P} = C_X(P, -)$ is 'exact', in particular it is right 'exact'. By the (dual version of) Yoneda lemma, $Hom(\check{P}, F) \simeq F(P)$. Since, by hypothesis, $Hom(\check{P}, F)$ consists of the trivial morphism, $F(P)$ is trivial for all projective objects P of $(C_X, \mathfrak{E}_X)$. Since $(C_X, \mathfrak{E}_X)$ has enough projectives, this means precisely that F is a coeffaceable functor. ∎

The k-linear version of 5.6.8.2 is as follows.

5.6.8.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelt right exact k-linear category with enough projectives. A k-linear functor $C_X \xrightarrow{F} k-mod$ is coeffaceable iff it is right orthogonal to the subcategory $\mathfrak{R}ex_k((C_X, \mathfrak{E}_X), k-mod)$ of right 'exact' k-linear functors from $(C_X, \mathfrak{E}_X)$ to $k-mod$.*

Proof. The argument is similar to that of 5.6.8.2. ∎

6. Left exact categories of 'spaces'.

We start with studying left exact stuctures formed by localizations of 'spaces' represented by svelte categories. Then the obtained facts are used to define natural left exact structures on the category of 'spaces' represented by right exact categories.

The following proposition is a refinement of [R1, 1.4.1].

6.1. Proposition. *Let $Z \xleftarrow{f} X \xrightarrow{q} Y$ be morphisms of 'spaces' such that q (i.e. its inverse image functor $C_Y \xrightarrow{q^*} C_X$) is a localization. Then*

(a) The canonical morphism $Z \xrightarrow{\widetilde{q}} Z \coprod\limits_{f,q} Y$ is a localization.

(b) If q is a continuous localization, then $\widetilde{q}$ is a continuous localization.

(c) If $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s)$ is invertible$\}$ is a left (resp. right) multiplicative system, then $\Sigma_{\widetilde{q^*}}$ has the same property.

Proof. Let $\mathfrak{X}$ denote the 'space' $Z \coprod\limits_{f,q} Y$. The category $C_{\mathfrak{X}}$ is $C_Z \prod\limits_{f^*,g^*} C_Y$. Recall that objects of $C_Z \prod\limits_{f^*,g^*} C_Y$ are triples $(L, M; \phi)$, where $L \in ObC_Z$, $M \in ObC_Y$, and ϕ is an

isomorphism $f^*(L) \xrightarrow{\sim} q^*(M)$. A morphism $(L, M; \phi) \longrightarrow (L', M'; \phi')$ is given by a pair of arrows, $L \xrightarrow{\alpha} L'$ and $M \xrightarrow{\beta} M'$, such that the diagram

$$\begin{array}{ccc} f^*(L) & \xrightarrow{f^*(\alpha)} & f^*(L') \\ \phi \downarrow \wr & & \wr \downarrow \phi' \\ q^*(M) & \xrightarrow{q^*(\beta)} & q^*(M') \end{array}$$

commutes. The composition of morphisms is defined naturally.

The (canonical) inverse image $C_{\mathfrak{X}} \xrightarrow{\widetilde{q}^*} C_Z$ of the coprojection $Z \xrightarrow{\widetilde{q}} \mathfrak{X}$ maps each object $(L, M; \phi)$ to L and each morphism $(L, M; \phi) \xrightarrow{(\mathfrak{s}, \mathfrak{t})} (L', M'; \phi')$ to $L \xrightarrow{\mathfrak{s}} L'$. It follows that the class $\Sigma_{\widetilde{q}^*}$ consists of all morphisms $(L, M; \phi) \xrightarrow{(\mathfrak{s}, \mathfrak{t})} (L', M'; \phi')$ such that $\mathfrak{s}$ is an isomorphism, hence $\mathfrak{t} \in \Sigma_{q^*}$.

(a) Since q^* is a localication, for any $L \in ObC_Z$, there exists $M \in ObC_Y$ such that there is an isomorphism $f^*(L) \xrightarrow{\phi} q^*(M)$. The map $L \longmapsto (L, M; \phi)$ (– a choice for each L of an object M and isomorphism ϕ) extends uniquely to a functor $C_Z \longrightarrow \Sigma^{-1}_{\widetilde{q}^*} C_{\mathfrak{X}}$ which is quasi-inverse to the canonical functor $\Sigma^{-1}_{\widetilde{q}^*} C_{\mathfrak{X}} \longrightarrow C_Z,\ (L, M; \phi) \longmapsto L$.

(b) Suppose that q is a continuous localization; i.e. the localization functor q^* has a right adjoint, q_*. Fix adjunction arrows $Id_{C_Y} \xrightarrow{\eta} q_* q^*$ and $q^* q_* \xrightarrow{\epsilon} Id_{C_X}$. Since q^* is a localization, ϵ is an isomorphism. Therefore, we have a functor $C_Z \xrightarrow{\widetilde{q}_*} C_{\mathfrak{X}}$ which maps any object L of C_Z to the object $(L, q_* f^*(L); \epsilon f^*(L))$ of the category $C_{\mathfrak{X}}$ and any morphism $L \xrightarrow{\xi} L'$ to the morphism $(\xi, q_* f^*(\xi))$ of $C_{\mathfrak{X}}$.

The functor $\widetilde{q}_*$ is a right adjoint to the projection $\widetilde{q}^*$. The adjunction morphism $Id_{C_{\mathfrak{X}}} \longrightarrow \widetilde{q}_* \widetilde{q}^*$ assigns to each object $(L, M; \phi)$ of the category $C_{\mathfrak{X}}$ the morphism

$$(L, M; \phi) \xrightarrow{(id_L, \widehat{\phi})} (L, q_* f^*(L); \epsilon f^*(L)),$$

where $M \xrightarrow{\widehat{\phi}} q_* f^*(L)$ denote the morphism conjugate to $q^*(M) \xrightarrow{\phi^{-1}} f^*(L)$. The adjunction arrow $\widetilde{q}^* \widetilde{q}_* \longrightarrow Id_{C_Z}$ is the identical morphism. The latter implies that $\widetilde{q}^*$ is a localization functor.

(c) Suppose that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a left multiplicative system. Let $(L, M; \phi) \xrightarrow{(\mathfrak{s}, \mathfrak{t})} (L', M'; \phi')$ be a morphism of $\Sigma_{\widetilde{q}^*}$ (that is $L \xrightarrow{\mathfrak{s}} L'$ is an isomorphism) and $(L, M; \phi) \xrightarrow{(\xi, \gamma)} (L'', M''; \phi'')$ an arbitrary morphism of $C_{\mathfrak{X}}$. The claim is that there exists a commutative diagram

$$\begin{array}{ccc} (L, M; \phi) & \xrightarrow{(\xi, \gamma)} & (L'', M''; \phi'') \\ (\mathfrak{s}, \mathfrak{t}) \downarrow & & \downarrow (\mathfrak{s}', \mathfrak{t}') \\ (L', M'; \phi') & \xrightarrow{(\xi', \gamma')} & (\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \end{array} \qquad (1)$$

in $C_{\mathfrak{X}}$ whose right vertical arrow belongs to $\Sigma_{\widetilde{q^*}}$.

In fact, since $M \xrightarrow{\mathfrak{t}} M'$ belongs to Σ_{q^*} and Σ_{q^*} is a left multiplicative system, there exists a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & M'' \\ {\scriptstyle \mathfrak{t}}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}'} \\ M' & \xrightarrow{\gamma'} & \widetilde{M} \end{array}$$

in C_Y such that $\mathfrak{t}' \in \Sigma_{q^*}$. Setting $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}) \circ \phi''$, we obtain the required commutative diagram (1).

(c1) Suppose that $(L, M; \phi) \xrightarrow{(\mathfrak{s},\mathfrak{t})} (L', M'; \phi')$ is a morphism of $\Sigma_{\widetilde{q^*}}$ which equalizes a pair of arrows $(L', M'; \phi') \underset{(\xi,\gamma)}{\overset{(\alpha,\beta)}{\rightrightarrows}} (L'', M''; \phi'')$. Then there exists a morphism $(L'', M''; \phi'') \xrightarrow{(\mathfrak{s}',\mathfrak{t}')} (\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ of $\Sigma_{\widetilde{q^*}}$ which equalizes this pair of arrows.

In fact, since $\mathfrak{s}$ is an isomorphism, the equality $(\alpha, \beta) \circ (\mathfrak{s}, \mathfrak{t}) = (\xi, \gamma) \circ (\mathfrak{s}, \mathfrak{t})$ implies that $\alpha = \xi$. Since Σ_{q^*} is a left multiplicative system, the equality $\beta \circ \mathfrak{t} = \gamma \circ \mathfrak{t}$ (and the fact that $\mathfrak{t} \in \Sigma_{q^*}$) implies the existence of a morphism $M'' \xrightarrow{\mathfrak{t}'} \widetilde{M}$ in Σ_{q^*} such that $\mathfrak{t}' \circ \beta = \mathfrak{t}' \circ \gamma$. Taking $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}') \circ \phi''$, we obtain the required morphism of $\Sigma_{\widetilde{q^*}}$.

(c') Suppose that Σ_{q^*} is stable under the base change. Then $\Sigma_{\widetilde{q^*}}$ has the same property.

In fact, let a morphism $(L', M'; \phi') \xrightarrow{(\mathfrak{s},\mathfrak{t})} (L, M; \phi)$ of $C_{\mathfrak{X}}$ belong to $\Sigma_{\widetilde{q^*}}$, and let $(L'', M''; \phi'') \xrightarrow{(\xi,\gamma)} (L, M; \phi)$ be an arbitrary morphism of $C_{\mathfrak{X}}$. Then there exists a commutative diagram

$$\begin{array}{ccc} (\widetilde{L}, \widetilde{M}; \widetilde{\phi}) & \xrightarrow{(\xi',\gamma')} & (L', M'; \phi') \\ {\scriptstyle (\mathfrak{s}',\mathfrak{t}')}\downarrow & & \downarrow{\scriptstyle (\mathfrak{s},\mathfrak{t})} \\ (L'', M''; \phi'') & \xrightarrow{(\xi,\gamma)} & (L, M; \phi) \end{array} \qquad (2)$$

in $C_{\mathfrak{X}}$ whose left vertical arrow belongs to $\Sigma_{\widetilde{q^*}}$.

Since $M' \xrightarrow{\mathfrak{t}} M$ belongs to Σ_{q^*} and Σ_{q^*} is a left multiplicative system, there exists a commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma'} & M' \\ {\scriptstyle \mathfrak{t}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} \\ M'' & \xrightarrow{\gamma} & \widetilde{M} \end{array}$$

in C_Y such that $\mathfrak{t}' \in \Sigma_{q^*}$. Setting $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}')^{-1} \circ \phi''$, we obtain a morphism $(\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \xrightarrow{(\mathfrak{s}',\mathfrak{t}')} (L'', M''; \phi'')$ which belongs to $\Sigma_{\widetilde{q^*}}$. Set $\xi' = \mathfrak{s}^{-1} \circ \xi$. The claim is that the pair (ξ', γ') is a morphism from $(\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ to $(L', M'; \phi')$ which makes

the diagram (2) commute. By definition, (ξ', γ') being a morphism from $(\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ to $(L', M'; \phi')$ means the commutativity of the diagram

$$\begin{array}{ccc} f^*(\widetilde{L}) & \xrightarrow{f^*(\xi')} & f^*(L') \\ \widetilde{\phi}\ \big\downarrow\wr & & \wr\big\downarrow\ \phi' \\ q^*(\widetilde{M}) & \xrightarrow{q^*(\gamma')} & q^*(M') \end{array}$$

which amounts to the equalities

$$q^*(\gamma') \circ q^*(\mathfrak{t}')^{-1} \circ \phi'' = q^*(\gamma') \circ \widetilde{\phi} = \phi' \circ f^*(\xi') = \phi' \circ f^*(\mathfrak{s})^{-1} \circ f^*(\xi). \tag{3}$$

It follows from the equality $\mathfrak{t} \circ \gamma' = \gamma \circ \mathfrak{t}'$ that $q^*(\gamma') \circ q^*(\mathfrak{t}')^{-1} = q^*(\mathfrak{t})^{-1} \circ q^*(\gamma)$. On the other hand, the fact that $(\mathfrak{s}, \mathfrak{t})$ is a morphism from $(L', M'; \phi')$ to $(L, M; \phi)$ means that $q^*(\mathfrak{t}) \circ \phi' = \phi \circ f^*(\mathfrak{s})$, or, equivalently, $\phi' \circ f^*(\mathfrak{s})^{-1} = q^*(\mathfrak{t})^{-1} \circ \phi$. Therefore, (3) is equivalent to the equality $q^*(\mathfrak{t})^{-1} \circ q^*(\gamma) \circ \phi'' = q^*(\mathfrak{t})^{-1} \circ \phi \circ f^*(\xi)$, or $q^*(\gamma) \circ \phi'' = \phi \circ f^*(\xi)$. The latter equality expresses the fact that (ξ, γ) is a morphism from $(L'', M''; \phi'')$ to $(L, M; \phi)$; hence (3) holds. The commutativity of the diagram (2) follows directly from the definition of the morphism (ξ', γ').

(c") Let Σ_{q^*} have the property:

(#) if an arrow $M' \xrightarrow{\mathfrak{t}} M$ belongs to Σ_{q^*} and equalizes a pair of arrows $M'' \rightrightarrows M'$, then there exists a morphism $M'' \xrightarrow{\mathfrak{t}'} M'$ in Σ_{q^*} which equalizes this pair of arrows.

Then $\Sigma_{\widetilde{q^*}}$ has the same property; that is if $(L', M'; \phi') \xrightarrow{(\mathfrak{s}, \mathfrak{t})} (L, M; \phi)$ is a morphism of $\Sigma_{\widetilde{q^*}}$ which equalizes a pair of arrows $(L'', M''; \phi'') \underset{(\xi, \gamma)}{\overset{(\alpha, \beta)}{\rightrightarrows}} (L', M'; \phi')$, then there exists a morphism $(\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \xrightarrow{(\mathfrak{s}', \mathfrak{t}')} (L'', M''; \phi'')$ of $\Sigma_{\widetilde{q^*}}$ which equalizes this pair of arrows.

In fact, since $\mathfrak{s}$ is an isomorphism, the equality $(\mathfrak{s}, \mathfrak{t}) \circ (\alpha, \beta) = (\mathfrak{s}, \mathfrak{t}) \circ (\xi, \gamma)$ implies that $\alpha = \xi$. Since Σ_{q^*} is a right multiplicative system, the equality $\mathfrak{t} \circ \beta = \mathfrak{t} \circ \gamma$ (and the fact that $\mathfrak{t} \in \Sigma_{q^*}$) implies the existence of a morphism $M'' \xrightarrow{\mathfrak{t}'} \widetilde{M}$ in Σ_{q^*} such that $\mathfrak{t}' \circ \beta = \mathfrak{t}' \circ \gamma$. Taking $\widetilde{L} = L''$, $\mathfrak{s}' = id_{L''}$, and $\widetilde{\phi} = q^*(\mathfrak{t}')^{-1} \circ \phi''$, we obtain an object $(\widetilde{L}, \widetilde{M}; \widetilde{\phi})$ and a morphism $(\widetilde{L}, \widetilde{M}; \widetilde{\phi}) \xrightarrow{(\mathfrak{s}', \mathfrak{t}')} (L'', M''; \phi'')$ which belongs to $\Sigma_{\widetilde{q^*}}$ and equalizes the pair of arrows $(L'', M''; \phi'') \underset{(\xi, \gamma)}{\overset{(\alpha, \beta)}{\rightrightarrows}} (L', M'; \phi')$.

If follows from (c') and (c") above that $\Sigma_{\widetilde{q^*}}$ is a right multiplicative system if Σ_{q^*} is a right multiplicative system. ∎

6.2. Corollary. *Let $Z \xleftarrow{f} X \xrightarrow{q} Y$ be morphisms of 'spaces' such that q is a localization, and let $Z \xrightarrow{\widetilde{q}} Z \coprod\limits_{f,q} Y$ be a canonical morphism. Suppose the category C_Y has*

finite limits (resp. finite colimits). Then $\widetilde{q}^*$ is a left (resp. right) exact localization, if the localization q^* is left (resp. right) exact.

Proof. By 6.1(a), $\widetilde{q}^*$ is a localization functor.

Suppose that the category C_Y has finite limits and the localization functor $C_Y \xrightarrow{q^*} C_X$ is left exact. Then it follows from [GZ, I.3.4] that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a right multiplicative system. The latter implies, by 6.1(c), that $\Sigma_{\widetilde{q^*}}$ is a right multiplicative system. Therefore, by [GZ, I.3.1], the localization functor $\widetilde{q}^*$ is left exact. ∎

The following assertion is a refinement of [R1, 1.4.2].

6.3. Proposition. *Let $X \xleftarrow{p} Z \xrightarrow{q} Y$ be morphisms of 'spaces' such that p^* and q^* are localization functors. Then the square*

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{q_1} & X \coprod\limits_{p,q} Y \end{array}$$

is cartesian.

Proof. Let $X \xleftarrow{u} W \xrightarrow{v} Y$ be morphisms of 'spaces' such that $q_1 \circ u = p_1 \circ v$. In other words, there exists an isomorphism $u^* \circ q_1^* \xrightarrow{\psi} v^* \circ p_1^*$. Let $M \xrightarrow{s} M'$ be any morphism of Σ_{q^*}. Since p^* is a localization functor, there exists $L \in ObC_X$ and an isomorphism $p^*(L) \xrightarrow{\phi} q^*(M)$. We have a morphism $(L, M; \phi) \xrightarrow{(id_L, s)} (L, M'; \phi')$ of the category $C_{\mathfrak{X}}$, where $\phi' = q^*(s)\phi$ and $\mathfrak{X}$ denotes the 'space' $X \coprod\limits_{p,q} Y$ represented by the category $C_{\mathfrak{X}} = C_X \prod C_Y$. By the definition of the canonical functors q_1^* and p_1^*, we have $q_1^*(id_L, s) = \overset{p^*,q^*}{id_L}$ and $p_1^*(id_L, s) = s$. Therefore, $v^*(s) = v^* \circ p_1^*(id_L, s)$ and $u^* \circ q_1^*(id_L, s) = u^*(id_L) = id_{u^*(L)}$. Since there is an isomorphism, $u^* \circ q_1^* \xrightarrow{\psi} v^* \circ p_1^*$, we have a commutative diagram

$$\begin{array}{ccc} u^*(L) & \xrightarrow{\psi(L,M;\phi)} & v^*(M) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle v^*(s)} \\ u^*(L) & \xrightarrow{\psi(L,M';\phi')} & v^*(M') \end{array}$$

whose horizontal arrows are isomorphisms, hence $v^*(s)$ is an isomorphism. Thus, v^* maps arrows of Σ_{q^*} to isomorphisms. Since q^* is a localization, there exists a unique functor $C_Y \xrightarrow{\widetilde{v}^*} C_W$ such that $v^* = \widetilde{v}^* \circ q^*$; that is the morphism v is uniquely represented as the composition $q \circ w$. Similarly, the morphism u is represented as the composition $p \circ \widetilde{u}$ for a unique $\widetilde{u}$. The equality $q_1 \circ u = p_1 \circ v$ can be now rewritten as $(q_1 \circ p) \circ \widetilde{u} = (p_1 \circ q) \circ \widetilde{v} =$

$(q_1 \circ p) \circ \widetilde{v}$, which means that $\widetilde{u}^* \circ (q_1 \circ p)^* \simeq \widetilde{v}^* \circ (q_1 \circ p)^*$. By 6.1(a), the functors q_1^* and p_1^* are localizations, hence $(q_1 \circ p)^* = (p_1 \circ q)^* \simeq q^* \circ p_1^*$ is a localization. Therefore the isomorphism $\widetilde{u}^* \circ (q_1 \circ p)^* \simeq \widetilde{v}^* \circ (q_1 \circ p)^*$ implies (is equivalent to) that $\widetilde{v}^*$ is isomorphic to $\widetilde{u}^*$, that is $\widetilde{u} = \widetilde{v}$. ∎

6.4. Left exact structures on the category of 'spaces'. Let $\mathfrak{L}$ denote the class of all localizations of 'spaces' (i.e. morphisms whose inverse image functors are localizations). We denote by $\mathfrak{L}_\ell$ (resp. $\mathfrak{L}_\mathfrak{r}$) the class of localizations $X \xrightarrow{q} Y$ of 'spaces' such that $\Sigma_{q^*} = \{s \in HomC_Y \mid q^*(s) \text{ is invertible}\}$ is a left (resp. right) multiplicative system. We denote by $\mathfrak{L}_\mathfrak{e}$ the intersection of $\mathfrak{L}_\ell$ and $\mathfrak{L}_\mathfrak{r}$ (i.e. the class of localizations q such that Σ_{q^*} is a multiplicative system) and by $\mathfrak{L}^\mathfrak{c}$ the class of continuous (i.e. having a direct image functor) localizations of 'spaces'. Finally, we set $\mathfrak{L}^\mathfrak{c}_\mathfrak{e} = \mathfrak{L}^\mathfrak{c} \cap \mathfrak{L}_\mathfrak{e}$; i.e. $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ is the class of continuous localizations $X \xrightarrow{q} Y$ such that Σ_{q^*} is a multiplicative system.

6.4.1. Proposition. *Each of the classes of morphisms* $\mathfrak{L}$, $\mathfrak{L}_\ell$, $\mathfrak{L}_\mathfrak{r}$, $\mathfrak{L}_\mathfrak{e}$, $\mathfrak{L}^\mathfrak{c}$, *and* $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ *are structures of a left exact category on the category* $|Cat|^o$ *of 'spaces'.*

Proof. It is immediate that each of these classes is closed under composition and contains all isomorphisms of the category $|Cat|^o$. It follows from 6.1 that each of the classes is stable under cobase change. In other words, the arrows of each class can be regarded as cocovers of a copretopology. It remains to show that these copretopologies are subcanonical. Since $\mathfrak{L}$ is the finest copretopology, it suffices to show that $\mathfrak{L}$ is subcanonical.

The copretopology $\mathfrak{L}$ being subcanonical means precisely that for any localization $X \xrightarrow{q} Y$, the square

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q_1} \\ Y & \xrightarrow{q_2} & Y \coprod\limits_{q,q} Y \end{array}$$

is cartesian. But, this follows from 6.3. ∎

6.5. Observation. Each object of the left exact category $(|Cat|^o, \mathfrak{L}^\mathfrak{c})$ is injective.

In fact, a 'space' X is an injective object of $(|Cat|^o, \mathfrak{L}^\mathfrak{c})$ iff each morphism $X \xrightarrow{q} Y$ is split; i.e. there is a morphism $Y \xrightarrow{\mathfrak{t}} X$ such that $\mathfrak{t} \circ q = id_X$. Since the morphism q is continuous, it has a direct image functor, q_*, which is fully faithful, because q^* is a localization functor. The latter means precisely that the adjunction arrow $q^*q_* \longrightarrow Id_{C_X}$ is an isomorphism. Therefore, the morphism $Y \xrightarrow{\mathfrak{t}} X$ whose inverse image functor coincides with q_* satisfies the equality $\mathfrak{t} \circ q = id_X$.

6.6. Relative 'spaces'. The category $|Cat|^o$ has canonical initial object represented by the category with one object and one morphism, but does not have final objects (since we do not allow empty categories). In particular, the notion of the cokernel of a morphism is not defined in $|Cat|^o$. So that we cannot apply to $|Cat|^o$ the theory of derived functors (satellites) sketched in Section 3. The category of *relative* 'spaces' (i.e. 'spaces' over a given 'space') has both final objects and cokernels of arbitrary morphisms.

Fix a 'space' S. The category $|Cat|^o/S$ has a natural final object – (S, id_S), and cokernels of morphisms. The cokernel of a morphism $(X,g) \xrightarrow{f} (Y,h)$ of 'spaces' over S is the pair $(Y \coprod\limits_{f,g} S, \widetilde{h})$, where $Y \coprod\limits_{f,g} S \xrightarrow{\widetilde{h}} S$ is the unique arrow determined by the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ S & \xrightarrow{id_S} & S \end{array}$$

The canonical inverse image functor $\widetilde{h}^*$ of the morphism $\widetilde{h}$ maps every object M of the category C_S to the object $(h^*(M), M; f^*h^*(M) \xrightarrow{\sim} g^*(M))$ of the category $C_Y \coprod\limits_{f^*,g^*} C_S$ representing the 'space' $Y \coprod\limits_{f,g} S$.

6.6.1. Lemma. *Let C_X be a category and V its object. Any left exact structure $\mathfrak{I}_X$ on C_X induces a left exact structure, $\mathfrak{I}_X/V$ on the category C_X/V.*

Proof. By the definition of $\mathfrak{I}_X/V$, a morphism $(L,\xi) \xrightarrow{f} (L',\xi')$ of C_X/V belongs to $\mathfrak{I}_X/V$ iff the morphism $L \xrightarrow{f} L'$ belongs to $\mathfrak{I}_X$. We leave to the reader the verifying that $\mathfrak{I}_X/V$ is a left exact structure on C_X/V. ∎

In particular, each left exact structure from the list of 7.4.1 induces a left exact structure on the category $|Cat|^o/S$.

6.7. Left exact structures on the category of k-'spaces'. Fix a commutative associative unital ring k. Recall that *k-'spaces'* are 'spaces' represented by k-linear additive categories. They are objects of the category $|Cat_k|^o$ whose arrows $X \longrightarrow Y$ are represented by isomorphism classes of k-linear functors $C_Y \longrightarrow C_X$. The category $|Cat_k|^o$ is pointed: its zero object is represented by the zero category. Every morphism $X \xrightarrow{f} Y$ of $|Cat_k|^o$ has a canonical cokernel $Y \xrightarrow{\mathfrak{c}} Cok(f)$, where $C_{Cok(f)}$ is the subcategory $Ker(f^*)$ of C_Y (– the full subcategory generated by all objects L such that $f^*(L) = 0$) and $\mathfrak{c}^*$ is the inclusion functor $Ker(f^*) \longrightarrow C_Y$.

Each of the left exact structures $\mathfrak{L}$, $\mathfrak{L}_\ell$, $\mathfrak{L}_\mathfrak{r}$, $\mathfrak{L}_\mathfrak{e}$, $\mathfrak{L}^\mathfrak{c}$, and $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}$ on the category $|Cat|^o$ of 'spaces' (see 6.4) induces a left exact structure on the category $|Cat_k|^o$ of k-spaces. Thus, we have left exact structures $\mathfrak{L}(k)$, $\mathfrak{L}_\ell(k)$, $\mathfrak{L}_\mathfrak{r}(k)$, $\mathfrak{L}_\mathfrak{e}(k)$, $\mathfrak{L}^\mathfrak{c}(k)$, and $\mathfrak{L}^\mathfrak{c}_\mathfrak{e}(k)$ on $|Cat_k|^o$.

6.8. Left exact structures on the category of right (or left) exact 'spaces'. A *right exact 'space'* is a pair $(X, \mathfrak{E}_X)$, where X is a 'space' and $\mathfrak{E}_X$ is a right exact structure on the category C_X. We denote by $\mathfrak{Esp}_\mathfrak{r}$ the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ and morphisms from $(X, \mathfrak{E}_X)$ to $(Y, \mathfrak{E}_Y)$ are given by morphisms $X \xrightarrow{f} Y$ of 'spaces' whose inverse image functor, f^*, is *'exact'*; i.e. f^* maps deflations to deflations and preserves pull-backs of deflations.

Dually, a *left exact 'space'* is a pair $(Y, \mathfrak{I}_Y)$, where $(C_Y, \mathfrak{I}_Y)$ is a left exact category. We denote by $\mathfrak{Esp}_\ell$ the category whose objects are left exact 'spaces' $(Y, \mathfrak{I}_Y)$ and morphisms

$(Y, \mathfrak{I}_Y) \longrightarrow (Z, \mathfrak{I}_Z)$ are given by morphisms $Y \longrightarrow Z$ whose inverse image functors are *'coexact'*, which means that they preserve arbitrary push-forwards of inflations.

6.8.1. Note. The categories $\mathfrak{Esp}_{\mathfrak{r}}$ and $\mathfrak{Esp}_{\ell}$ are naturally isomorphic to each other: the isomorphism is given by the dualization functor $(X, \mathfrak{E}_X) \longmapsto (X^o, \mathfrak{E}_X^{op})$. Therefore every assertion about the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' translates into an assertion about the category $\mathfrak{Esp}_{\ell}$ of left exact 'spaces' and vice versa.

6.8.2. Proposition. *The category $\mathfrak{Esp}_{\mathfrak{r}}$ has fibered coproducts.*

Proof. Let $(X, \mathfrak{E}_X) \xleftarrow{f} (Z, \mathfrak{E}_Z) \xrightarrow{g} (Y, \mathfrak{E}_Y)$ be morphisms of $\mathfrak{Esp}_{\mathfrak{r}}$; and let $\mathfrak{X}$ denote the 'space' $X \coprod\limits_{f,g} Y$, i.e. $C_{\mathfrak{X}} = C_X \prod\limits_{f^*,g^*} C_Y$. Let $\mathfrak{E}_{\mathfrak{X}}$ denote the class of all morphisms $(L, M; \phi) \xrightarrow{(\xi,\gamma)} (L', M'; \phi')$ of $C_{\mathfrak{X}}$ such that $L \xrightarrow{\xi} L'$ belongs to $\mathfrak{E}_X$ and $M \xrightarrow{\gamma} M'$ is an arrow of $\mathfrak{E}_Y$. The claim is that $\mathfrak{E}_{\mathfrak{X}}$ is a right exact structure on $C_{\mathfrak{X}}$ and $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ is a coproduct $(X, \mathfrak{E}_X) \coprod\limits_{f,g} (Y, \mathfrak{E}_Y)$ of right exact 'spaces'.

It is immediate that $\mathfrak{E}_{\mathfrak{X}}$ contains all isomorphisms and is closed under composition. Let $(L, M; \phi) \xrightarrow{(\xi,\gamma)} (L', M'; \phi')$ be a morphism of $\mathfrak{E}_{\mathfrak{X}}$, and let $(L'', M''; \phi'') \xrightarrow{(\alpha,\beta)} (L', M'; \phi')$ be an arbitrary morphism of $C_{\mathfrak{X}}$. Since the inverse image functors f^* and g^* preserve corresponding deflations and their pull-backs and $\xi,\ \gamma$ are deflations, the isomorphisms $\phi,\ \phi'$, and ϕ'' induce an isomorphism $f^*(\widetilde{L}) \xrightarrow{\widetilde{\phi}} g^*(\widetilde{M})$, where $\widetilde{L} = L \coprod\limits_{\xi,\alpha} L''$ and $\widetilde{M} = M \coprod\limits_{\gamma,\beta} M''$. It is easy to see that the square

$$\begin{array}{ccc} (\widetilde{L}, \widetilde{M}; \widetilde{\phi}) & \xrightarrow{(\alpha',\beta')} & (L, M; \phi) \\ {\scriptstyle (\widetilde{\xi},\widetilde{\gamma})}\downarrow & & \downarrow{\scriptstyle (\xi,\gamma)} \\ (L'', M''; \phi'') & \xrightarrow{(\alpha,\beta)} & (L', M'; \phi') \end{array} \qquad (1)$$

is cartesian, $\widetilde{\xi} \in \mathfrak{E}_X$, and $\widetilde{\gamma} \in \mathfrak{E}_Y$. Therefore, $(\widetilde{\xi}, \widetilde{\gamma}) \in \mathfrak{E}_{\mathfrak{X}}$.

If $(\alpha, \beta) = (\xi, \gamma)$, then the square (1) is cocartesian, because the squares

$$\begin{array}{ccc} \widetilde{L} & \xrightarrow{\xi'} & L \\ {\scriptstyle \widetilde{\xi}}\downarrow & & \downarrow{\scriptstyle \xi} \\ L & \xrightarrow{\xi} & L' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma'} & M \\ {\scriptstyle \widetilde{\gamma}}\downarrow & & \downarrow{\scriptstyle \gamma} \\ M & \xrightarrow{\gamma} & M' \end{array}$$

are (both cartesian and) cocartesian. Altogether shows that the arrows of $\mathfrak{E}_{\mathfrak{X}}$ are covers of a subcanonical pretopology; i.e. $\mathfrak{E}_{\mathfrak{X}}$ is a structure of a right exact category on $C_{\mathfrak{X}}$. ∎

6.8.3. Canonical left exact structures on the category $\mathfrak{Esp}_{\mathfrak{r}}$. Let $\mathfrak{L}_{\mathfrak{es}}$ denote the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that q^* is a localization functor and each arrow of $\mathfrak{E}_X$ is isomorphic to an arrow $q^*(\mathfrak{e})$ for some $\mathfrak{e} \in \mathfrak{E}_Y$.

If Σ_{q^*} is a left or right multiplicative system, then this condition means that $\mathfrak{E}_X$ is the smallest right exact structure containing $q^*(\mathfrak{E}_Y)$.

6.8.3.1. Proposition. *The class $\mathfrak{L}_{\mathfrak{es}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'.*

Proof. The class $\mathfrak{L}_{\mathfrak{es}}$ contains, obviously, all isomorphisms, and it is easy to see that it is closed under composition. It remains to show that $\mathfrak{L}_{\mathfrak{es}}$ is stable under cobase change and its arrows are cocovers of a subcanonical copretopology.

Let $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{L}_{\mathfrak{es}}$ and $(X, \mathfrak{E}_X) \xrightarrow{f} (Z, \mathfrak{E}_Z)$ an arbitrary morphism. The claim is that the canonical morphism $Z \xrightarrow{\widetilde{q}} Z \coprod\limits_{f,q} Y$ belongs to $\mathfrak{L}_{\mathfrak{es}}$.

Consider the corresponding cartesian (in pseudo-categorical sense) square of right exact categories:

$$\begin{array}{ccc} (C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}) & \xrightarrow{p^*} & (C_Y, \mathfrak{E}_Y) \\ {\scriptstyle \widetilde{q}^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ (C_Z, \mathfrak{E}_Z) & \xrightarrow{f^*} & (C_X, \mathfrak{E}_X) \end{array} \tag{2}$$

where $\mathfrak{X} = Z \coprod\limits_{f,q} Y$; that is $C_{\mathfrak{X}} = C_Z \prod\limits_{f^*,q^*} C_Y$. Recall that the functor $\widetilde{q}^*$ maps each object $(L, M; \phi)$ of the category $C_{\mathfrak{X}}$ to the object L of C_Z and each morphism (ξ, γ) to ξ. By 6.1(a), $\widetilde{q}^*$ is a localization functor (because q^* is a localization functor).

Let $L \xrightarrow{\mathfrak{e}} L'$ be an arrow of $\mathfrak{E}_Z$. Then $f^*(\mathfrak{e})$ is a morphism of $\mathfrak{E}_X$. Since $X \xrightarrow{q} Y$ is a morphism of $\mathfrak{L}_{\mathfrak{es}}$, there exists a morphism $M \xrightarrow{\mathfrak{t}} M'$ of $\mathfrak{E}_Y$ and a commutative diagram

$$\begin{array}{ccc} f^*(L) & \xrightarrow{f^*(\mathfrak{e})} & f^*(L') \\ {\scriptstyle \psi}\downarrow\wr & & \wr\downarrow{\scriptstyle \psi'} \\ q^*(M) & \xrightarrow{q^*(\mathfrak{t})} & q^*(M') \end{array}$$

whose vertical arrows are isomorphisms. By the definition of the right exact category $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, this means that $(\mathfrak{e}, \mathfrak{t})$ is a morphism $(L, M; \psi) \longrightarrow (L', M'; \psi')$ of $C_{\mathfrak{X}}$ which belongs to $\mathfrak{E}_{\mathfrak{X}}$. The localization functor $\widetilde{q}^*$ maps it to $\mathfrak{e}$. Thus, $\mathfrak{E}_Z = \widetilde{q}^*(\mathfrak{E}_{\mathfrak{X}})$, hence $\widetilde{q} \in \mathfrak{E}_{\mathfrak{X}}$. This shows that $\mathfrak{L}_{\mathfrak{es}}$ is stable under cobase change.

It remains to verify that for every morphism $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ of $\mathfrak{L}_{\mathfrak{es}}$ the square

$$\begin{array}{ccc} (C_{\mathfrak{Y}}, \mathfrak{E}_{\mathfrak{Y}}) & \xrightarrow{p_1^*} & (C_Y, \mathfrak{E}_Y) \\ {\scriptstyle p_2^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ (C_Y, \mathfrak{E}_Y) & \xrightarrow{q^*} & (C_X, \mathfrak{E}_X) \end{array} \tag{3}$$

is cocartesian. Here $C_{\mathfrak{Y}} = C_Y \prod\limits_{q^*,q^*} C_Y$.

Consider a quasi-commutative diagram

$$\begin{array}{ccc} (C_{\mathfrak{Y}}, \mathfrak{E}_{\mathfrak{Y}}) & \xrightarrow{p_1^*} & (C_Y, \mathfrak{E}_Y) \\ {\scriptstyle p_2^*}\downarrow & & \downarrow{\scriptstyle v^*} \\ (C_Y, \mathfrak{E}_Y) & \xrightarrow{u^*} & (C_W, \mathfrak{E}_W) \end{array} \tag{4}$$

of 'exact' functors. Since, by 6.3, that the square

$$\begin{array}{ccc} C_{\mathfrak{Y}} & \xrightarrow{p_1^*} & C_Y \\ {\scriptstyle p_2^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ C_Y & \xrightarrow{q^*} & C_X \end{array}$$

is cocartesian, there exists a unique up to isomorphism functor $C_X \xrightarrow{w^*} C_W$ such that $v^* \simeq w^*q^* \simeq u^*$. The claim is that w^* is an 'exact' functor from $(C_X, \mathfrak{E}_X)$ to $(C_W, \mathfrak{E}_W)$. Since $q \in \mathfrak{L}_{\mathfrak{es}}$, every morphism of $\mathfrak{E}_X$ is isomorphic to a morphism of $q^*(\mathfrak{E}_Y)$ and v^* maps $\mathfrak{E}_Y$ to $\mathfrak{E}_W$. Therefore w^* maps $\mathfrak{E}_X$ to $\mathfrak{E}_W$. The fact that q^* and $v^* \simeq w^*q^*$ are 'exact' functors implies that the functor w^* is 'exact'. ∎

6.8.3.2. Corollary. *Each of the classes of morphisms of 'spaces'* $\mathfrak{L}_\ell$, $\mathfrak{L}_{\mathfrak{r}}$, $\mathfrak{L}_{\mathfrak{e}}$, $\mathfrak{L}^{\mathfrak{c}}$, *and* $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{e}}$ *(cf. 6.4, 6.4.1) induces a structure of a left exact category on the category* $\mathfrak{Esp}_{\mathfrak{r}}$ *of right exact 'spaces'.*

Proof. The class $\mathfrak{L}_\ell$ induces the class $\mathfrak{L}^{\mathfrak{es}}_\ell$ of morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}$ formed by all arrows $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ from $\mathfrak{L}_{\mathfrak{es}}$ such that the morphism of 'spaces' $X \xrightarrow{q} Y$ belongs to $\mathfrak{L}_\ell$. Similarly, we define the classes $\mathfrak{L}^{\mathfrak{es}}_\ell$, $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}$, and $\mathfrak{L}^{\mathfrak{e},\mathfrak{c}}_{\mathfrak{es}}$. ∎

6.8.3.3. The left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$. For a right exact 'space' $(X, \mathfrak{E}_X)$, let $Sq(X, \mathfrak{E}_X)$ denote the class of all cartesian squares in the category C_X some of the arrows of which (at least two) belong to $\mathfrak{E}_X$.

The class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ consists of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ of right exact 'spaces' such that its inverse image functor, $\mathfrak{q}^*$, is equivalent to a localization functor and each square of $Sq(X, \mathfrak{E}_X)$ is isomorphic to some square of $\mathfrak{q}^*(Sq(Y, \mathfrak{E}_Y))$.

6.8.3.4. Proposition. *The class* $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ *is a left exact structure on the category* $\mathfrak{Esp}_{\mathfrak{r}}$ *of right exact 'spaces' which is coarser than* $\mathfrak{L}_{\mathfrak{es}}$ *and finer than* $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}$.

Proof. The argument is left to the reader. ∎

6.9. Relative right exact 'spaces'. The category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' has initial objects and no final object. Final objects appear if we fix a right exact 'space' $\mathcal{S} = (S, \mathfrak{E}_{\mathcal{S}})$ and consider the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{S}$ instead of $\mathfrak{Esp}_{\mathfrak{r}}$. The category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{S}$ has a natural final object and cokernels of all morphisms. It also inherits left exact structures from $\mathfrak{Esp}_{\mathfrak{r}}$, in particular those defined above (see 6.8.3.2). Therefore, our theory of derived functors (satellites) can be applied to functors from $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{S}$.

6.10. The category of right exact k-'spaces'. For a commutative unital ring k, we denote by $\mathfrak{Esp}^{\mathfrak{r}}_k$ the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ such that C_X is a k-linear additive category and morphisms are morphisms of right exact 'spaces' whose inverse image functors are k-linear.

Each of the left exact structures $\mathfrak{L}_{\mathfrak{es}}$, $\mathfrak{L}^{\mathfrak{es}}_{\ell}$, $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}$, $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}$, and $\mathfrak{L}^{\mathfrak{e},\mathfrak{c}}_{\mathfrak{es}}$ induces a left exact structure on the category $\mathfrak{Esp}^{\mathfrak{r}}_k$ of right exact k-'spaces'. We denote them by respectively $\mathfrak{L}_{\mathfrak{es}}(k)$, $\mathfrak{L}^{\mathfrak{es}}_{\ell}(k)$, $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}(k)$, $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}(k)$, $\mathfrak{L}^{\mathfrak{c}}_{\mathfrak{es}}(k)$, and $\mathfrak{L}^{\mathfrak{e},\mathfrak{c}}_{\mathfrak{es}}(k)$.

6.11. The path 'space' of a right exact 'space'. Fix a right exact svelte category $(C_X, \mathfrak{E}_X)$. Let $C_{\mathfrak{X}}$ be the quotient of the category $C_{\mathfrak{Pa}(X)}$ of paths of the category C_X by the relations $\mathfrak{s} \circ \widetilde{\mathfrak{f}} = \mathfrak{f} \circ \mathfrak{t}$, where

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{\mathfrak{f}}} & M \\ {\scriptstyle \mathfrak{t}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{s}} \\ N & \xrightarrow{\mathfrak{f}} & L \end{array}$$

runs through cartesian squares in C_X whose vertical arrows belong to $\mathfrak{E}_X$. In particular, $ObC_{\mathfrak{X}} = ObC_X$. We denote by $\mathfrak{E}_{\mathfrak{X}}$ the image in $C_{\mathfrak{X}}$ of all paths of morphisms of $\mathfrak{E}_X$ and by $\mathfrak{Pa}(X, \mathfrak{E}_X)$ the pair $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$.

6.11.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a svelte right exact category and $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) = \mathfrak{Pa}(X, \mathfrak{E}_X)$ (see above).*

(a) The class of morphisms $\mathfrak{E}_{\mathfrak{X}}$ is a right exact structure on the category $C_{\mathfrak{X}}$.

(b) The canonical functor $C_{\mathfrak{Pa}(X)} \xrightarrow{\varepsilon^*_X} C_X$ (identical on objects and mapping paths of arrows to their composition) factors uniquely through a functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{p}^*_X} C_X$ which is an inverse image functor of a morphism $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{p}_X} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ that belongs to $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$.

(c) The right exact 'space' $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ is an injective object of a left exact category $(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}})$.

Proof. (a) It follows (from the fact that the composition of cartesian squares is a cartesian square) that $\mathfrak{E}_{\mathfrak{X}}$ is a right exact structure on $C_{\mathfrak{X}}$.

(b) The functor $C_{\mathfrak{Pa}(X)} \xrightarrow{\varepsilon^*_X} C_X$ is (equivalent to) a localization functor which factors uniquely through $C_{\mathfrak{X}} \xrightarrow{\mathfrak{p}^*_X} C_X$. Therefore, $\mathfrak{p}^*_X$ is (equivalent to) a localization functor. It follows from definitions that $\mathfrak{p}^*_X$ maps cartesian squares with deflations among their arrows to cartesian squares of the same type. Moreover, all cartesian squares with this property are obtained this way. Therefore, the morphism $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{p}_X} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ belongs to the class $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$.

(c) Let $(\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{r}}$ and $(\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}) \xrightarrow{\mathfrak{f}} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ an arbitrary morphism.The claim is that there exists a morphism $(Y, \mathfrak{E}_Y) \xrightarrow{\gamma} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$ of right exact 'spaces' such that $\mathfrak{f} = \gamma \circ \mathfrak{q}$.

(c1) Let $(C_Y, \mathfrak{E}_Y) \xrightarrow{\mathfrak{q}^*} (C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ and $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}) \xrightarrow{\mathfrak{f}^*} (C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ be inverse image functors

of respectively $\mathfrak{q}$ and $\mathfrak{f}$. Consider the standard cartesian square

$$\begin{array}{ccc} (C_{\mathfrak{Y}}, \mathfrak{E}_{\mathfrak{Y}}) & \xrightarrow{\widetilde{\mathfrak{q}}^*} & (C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}) \\ {\scriptstyle \mathfrak{f}_1^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}^*} \\ (C_Y, \mathfrak{E}_Y) & \xrightarrow{\mathfrak{q}^*} & (C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}}) \end{array}$$

of right exact categories. By 6.8.3.4, the morphism $(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) \xrightarrow{\widetilde{\mathfrak{q}}} (\mathfrak{Y}, \mathfrak{E}_{\mathfrak{Y}})$ represented by the functor $\widetilde{\mathfrak{q}}^*$ belongs to $\mathfrak{L}^{\mathfrak{ES}}_{\mathfrak{Sq}}$. Moreover, the functor $C_{\mathfrak{Y}} \xrightarrow{\widetilde{\mathfrak{q}}^*} C_{\mathfrak{X}}$ is surjective.

In fact, $\mathfrak{q}$ being a morphism of $\mathfrak{L}^{\mathfrak{ES}}_{\mathfrak{Sq}}$ implies that every square $\square$ of $Sq(\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}})$ is isomorphic to $\mathfrak{q}^*(\widetilde{\square})$ for some $\widetilde{\square} \in Sq(Y, \mathfrak{E}_Y)$. In particular, every morphism of $C_{\mathcal{Z}}$ is isomorphic to the image of some arrow of C_Y. Thus, for any morphism $M \xrightarrow{\xi} L$ of $C_{\mathfrak{X}}$, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{f}^*(M) & \xrightarrow{\mathfrak{f}^*(\xi)} & \mathfrak{f}^*(L) \\ {\scriptstyle \phi}\downarrow\wr & & \wr\downarrow{\scriptstyle \psi} \\ \mathfrak{q}^*(V) & \xrightarrow{\mathfrak{q}^*(\gamma)} & \mathfrak{q}^*(W) \end{array}$$

whose vertical arrows are isomorphisms. In other words, the pair (ξ, γ) is a morphism $(M, V; \phi) \longrightarrow (L, W; \psi)$ of the category $C_{\mathfrak{Y}}$, and $\xi = \widetilde{\mathfrak{q}}^*(\xi, \gamma)$.

(c2) The functor $\widetilde{\mathfrak{q}}^*$ maps $Sq(\mathfrak{Y}, \mathfrak{E}_{\mathfrak{Y}})$ onto $Sq(\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$.

(c3) Given a class of arrows $\mathcal{S}$ in $HomC_Y$, we denote by $\mathfrak{R}_{\mathcal{S}}$ the class of all pairs of arrows $M \rightrightarrows L$ which are equalized by an arrow $N \longrightarrow M$ from $\mathcal{S}$. If $\mathcal{S}$ contains all identical morphisms, closed under composition, and *filtered*, i.e. every pair of arrows $L \longrightarrow M \longleftarrow N$ of $\mathcal{S}$ can be completed to a commutative square

$$\begin{array}{ccc} \widetilde{L} & \longrightarrow & L \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

whose arrows belong to $\mathcal{S}$, then $\mathfrak{R}_{\mathcal{S}}$ is an equivalence relation.

(c4) In particular, $\mathfrak{R}_{\Sigma_{\mathfrak{q}^*}}$ is an equivalence relation, because $\Sigma_{\mathfrak{q}^*}$ is a right multiplicative system. For every $L \in ObC_Y$, we denote by $\mathfrak{E}_{Y,L}$ the class of all deflations of L. For each $L \in ObC_Y$, we choose representatives of equivalence classes with respect to $\mathfrak{R}_{\Sigma_{\mathfrak{q}^*}}$ of arrows to L. ∎

6.12. Complements.

6.12.1. The left exact structure $\mathfrak{L}^{\mathfrak{E}}_{\mathfrak{r}}$. Fix a right exact category $(C_X, \mathfrak{E}_X)$. We say that a class Σ of deflations is $\mathfrak{E}_X$*-saturated* if it is the intersection of a saturated system of arrows of C_X and $\mathfrak{E}_X$.

6.12.1.1. Lemma. *Let Σ be an $\mathfrak{E}_X$-saturated class of deflations. Then Σ is a right multiplicative system iff it is stable under base change.*

Proof. Let Σ be an $\mathfrak{E}_X$-saturated system of deflations. In particular, it contains all isomorphisms of C_X and is closed under compositions.

If Σ is stable under base change, it is a right multiplicative system.

Conversely, if Σ is a right multiplicative system, then, by [GZ, I.3.1], the localization functor $C_X \xrightarrow{\mathfrak{q}^*} \Sigma^{-1}C_X$ is right exact. In particular, it maps all cartesian squares of C_X to cartesian squares of $\Sigma^{-1}C_X$. Since $\mathfrak{E}_X$ is stable under base change, every diagram $M \xrightarrow{\mathfrak{s}} L \xrightarrow{f} N$ with $\mathfrak{s} \in \mathfrak{E}_X$ can be completed to a cartesian square

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle\mathfrak{t}}\downarrow & & \downarrow{\scriptstyle\mathfrak{s}} \\ N & \xrightarrow{f} & L \end{array} \qquad (1)$$

and $\mathfrak{t} \in \mathfrak{E}_X$. If $\mathfrak{s} \in \Sigma$, then the localization $\mathfrak{q}^*$ maps (1) to a cartesian square whose right vertical arrow, $\mathfrak{q}^*(\mathfrak{s})$, is an isomorphism. Therefore its left vertical arrow is an isomorphism. Since Σ is $\mathfrak{E}_X$-saturated, this implies that $\mathfrak{t} \in \Sigma$. ∎

We denote by $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$ the preorder (under the inclusion) of all $\mathfrak{E}_X$-saturated right multiplicative systems Σ of $\mathfrak{E}_X$ having the following property:

(#) If the right horizontal arrows in the commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \underset{\mathfrak{p}'_2}{\overset{\mathfrak{p}'_1}{\rightrightarrows}} & \mathcal{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{L} \\ {\scriptstyle\widetilde{\mathfrak{t}}}\downarrow & & \downarrow{\scriptstyle\mathfrak{s}} & & \downarrow{\scriptstyle\mathfrak{s}'} \\ M \times_L M & \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

are deflations, the pairs of arrows are kernel pairs of these deflations and two left vertical arrows belong to Σ, then the remaining vertical arrow belongs to Σ.

6.12.1.2. Proposition. *(a) For any morphism $(Y, \mathfrak{E}_Y) \xrightarrow{\mathfrak{q}} (X, \mathfrak{E}_X)$ of the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces', the intersection $\Sigma_{\mathfrak{q}^*} \bigcap \mathfrak{E}_X = \{\mathfrak{t} \in \mathfrak{E}_X \mid \mathfrak{q}^*(\mathfrak{t})$ is invertible$\}$ belongs to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$.*

(b) For any $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$, the localization functor $C_X \xrightarrow{\mathfrak{q}^*} \Sigma^{-1}C_X = C_{\mathfrak{X}}$ is an inverse image functor of a morphism $(\mathfrak{X}, \mathfrak{E}^{\mathfrak{st}}_{\mathfrak{X}}) \xrightarrow{\mathfrak{q}} (X, \mathfrak{E}_X)$ of $\mathfrak{Esp}_{\mathfrak{r}}$. As usual, $\mathfrak{E}^{\mathfrak{st}}_{\mathfrak{X}}$ denote the finest right exact structure on $C_{\mathfrak{X}}$.

Proof. (a) By definition of morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}$, its inverse image functor maps pull-backs of deflations to pull-backs of deflations. Therefore the intersection $\Sigma_{\mathfrak{q}^*} \bigcap \mathfrak{E}_X = \{\mathfrak{t} \in \mathfrak{E}_X \mid \mathfrak{q}^*(\mathfrak{t})$ is invertible $\}$ is (by definition) saturated and stable under base change. The property (#) follows from the 'exactness' of the localization functor $\mathfrak{q}^*$.

(b) Let $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$. Since Σ is a right multiplicative system, the localization functor $C_X \xrightarrow{\mathfrak{q}^*} \Sigma^{-1}C_X = C_{\mathfrak{X}}$ is left exact. In particular, it maps all cartesian squares to cartesian squares. It remains to show that it maps deflations to strict epimorphisms.

Let $M \xrightarrow{\mathfrak{e}} L$ be a morphism of $\mathfrak{E}_X$ and $M \times_L M \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} M$ its kernel pair. Let $\mathfrak{q}^*(M) \xrightarrow{\xi'} \mathfrak{q}^*(N)$ be a morphism which equalizes the pair $\mathfrak{q}^*\bigl(M \times_L M \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} M\bigr)$. Since Σ is a right multiplicative system, the morphism ξ' is the composition $\mathfrak{q}^*(\xi)\mathfrak{q}^*(\mathfrak{s})^{-1}$ for some morphisms $M \xleftarrow{\mathfrak{s}} \mathcal{M} \xrightarrow{\xi} N$, where $\mathfrak{s} \in \Sigma$. Thus we have a diagram

$$\begin{array}{ccccc}
\mathcal{M}_1 & \xrightarrow{\mathfrak{u}_1} & M \times_L M & \xleftarrow{\mathfrak{u}_2} & \mathcal{M}_2 \\
{\scriptstyle \mathfrak{t}_1}\downarrow & cart & {\scriptstyle \mathfrak{p}_1}\downarrow\downarrow{\scriptstyle \mathfrak{p}_2} & cart & \downarrow{\scriptstyle \mathfrak{t}_2} \\
\mathcal{M} & \xrightarrow{\mathfrak{s}} & M & \xleftarrow{\mathfrak{s}} & \mathcal{M}
\end{array}$$

whose both squares are cartesian, all arrows are deflations, and all horizontal arrows belong to Σ. Therefore, there exists a cartesian square

$$\begin{array}{ccc}
\widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{v}_1'} & \mathcal{M}_2 \\
{\scriptstyle \mathfrak{v}_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{u}_2} \\
\mathcal{M}_1 & \xrightarrow{\mathfrak{u}_1} & M \times_L M
\end{array}$$

whose all arrows belong to Σ. Altogether leads to a commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{M}} & \overset{\mathfrak{p}_1'}{\underset{\mathfrak{p}_2'}{\rightrightarrows}} & \mathcal{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \mathcal{L} \\
{\scriptstyle \widetilde{\mathfrak{t}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} & & \downarrow{\scriptstyle \mathfrak{s}'} \\
M \times_L M & \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & L
\end{array}$$

whose rows are exact diagrams and two (left) vertical arrows belong to Σ. Therefore, the remaining vertical arrow belongs to Σ. The localization functor $\mathfrak{q}^*$ maps the compositions $\xi \circ \mathfrak{p}_1'$ and $\xi \circ \mathfrak{p}_2'$ to the same arrow. This means precisely that there exists a morphism $\lambda \in \Sigma$ such that $\xi \circ \mathfrak{p}_1' \circ \lambda = \xi \circ \mathfrak{p}_2' \circ \lambda$ (cf. [GZ, I.2.2]). Since all morphisms of Σ are epimorphisms, the latter equality implies that the morphism ξ equalizes the pair $\widetilde{\mathcal{M}} \overset{\mathfrak{p}_1'}{\underset{\mathfrak{p}_2'}{\rightrightarrows}} \mathcal{M}$. Therefore, it factors uniquely through the morphism $\mathcal{M} \xrightarrow{\widetilde{\mathfrak{e}}} \mathcal{L}$; i.e. $\xi = \widetilde{\xi} \circ \widetilde{\mathfrak{e}}$. The pair of arrows $L \xleftarrow{\mathfrak{s}'} \mathcal{L} \xrightarrow{\widetilde{\xi}} N$ determines a unique morphism $\mathfrak{q}^*(L) \longrightarrow \mathfrak{q}^*(N)$ whose composition with $q^*(\mathfrak{e})$ equals to ξ'. ∎

We denote by $\mathfrak{L}_{\mathfrak{r}}^{\mathfrak{E}}$ the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ whose inverse image functor is equivalent to the localization functor at a system which belongs to $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(X, \mathfrak{E}_X)$.

6.12.1.3. Proposition. *The class of morphisms $\mathfrak{L}_{\mathfrak{r}}^{\mathfrak{E}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'.*

Proof. The class of morphisms $\mathfrak{L}_{\mathfrak{r}}^{\mathfrak{E}}$ contains all isomorphisms and is closed under compositions and cobase change. ∎

7. K-theory of right exact 'spaces'.

7.1. The functor K_0.

7.1.1. The group $\mathbb{Z}_0|C_X|$. For a svelte category C_X, we denote by $|C_X|$ the set of isomorphism classes of objects of C_X, by $\mathbb{Z}|C_X|$ the free abelian group generated by $|C_X|$, and by $\mathbb{Z}_0(C_X)$ the subgroup of $\mathbb{Z}|C_X|$ generated by differences $[M]-[N]$ for all arrows $M \longrightarrow N$ of the category C_X. Here $[M]$ denotes the isomorphism class of an object M.

7.1.2. Proposition. *(a) The maps $X \longmapsto \mathbb{Z}|C_X|$ and $X \longmapsto \mathbb{Z}_0(C_X)$ extend naturally to presheaves of $\mathbb{Z}$-modules on the category of 'spaces' $|Cat|^o$ (i.e. to functors from $(|Cat|^o)^{op}$ to $\mathbb{Z}-mod$).*

(b) If the category C_X has an initial (resp. final) object x, then $\mathbb{Z}_0(C_X)$ is the subgroup of $\mathbb{Z}|C_X|$ generated by differences $[M]-[x]$, where $[M]$ runs through the set $|C_X|$ of isomorphism classes of objects of C_X.

Proof. The argument is left to the reader. ∎

7.1.3. Remarks. (a) Evidently, there are natural isomorphisms $\mathbb{Z}|C_X| \simeq \mathbb{Z}|C_X^{op}|$ and $\mathbb{Z}_0(C_X) \simeq \mathbb{Z}_0(C_X^{op})$.

(b) Let $\mathbb{Z}_0(C_X)$ be regarded as a groupoid with one object, $\bullet$. Then the map which assigns to every object of C_X the object $\bullet$ and to any morphism $M \longrightarrow N$ of C_X the difference $[M]-[N]$ is a functor from C_X to the groupoid $\mathbb{Z}_0(C_X)$.

7.1.4. The group K_0 of a right exact 'space'. Let $(X, \mathfrak{E}_X)$ be a right exact 'space'. We denote by $K_0(X, \mathfrak{E}_X)$ the quotient of the group $\mathbb{Z}_0|C_X|$ by the subgroup generated by the expressions $[M']-[M]+[L]-[N]$ for all cartesian squares

$$\begin{array}{ccc} M' & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ L' & \xrightarrow{f} & L \end{array}$$

whose vertical arrows are deflations.

We call $K_0(X, \mathfrak{E}_X)$ the *group* K_0 of the right exact 'space' $(X, \mathfrak{E}_X)$.

7.1.4.1. Example: the group K_0 of a 'space'. Any 'space' X is identified with the *trivial* right exact 'space' $(X, Iso(C_X))$. We set $K_0(X) = K_0(X, Iso(C_X))$. That is $K_0(X)$ coincides with the group $\mathbb{Z}_0(C_X)$.

7.1.5. Proposition. *(a) The map $(X, \mathfrak{E}_X) \longmapsto K_0(X, \mathfrak{E}_X)$ extends to a contravariant functor, K_0, from the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' (cf. 6.8) to the category $\mathbb{Z}-mod$ of abelian groups.*

(b) Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{Esp}_{\mathfrak{r}}$ having the following property:

(†) if M' and L' are non-isomorphic objects of C_X which can be connected by non-oriented sequence of arrows (i.e. they belong to one connected component of the associated groupoid), then there exist objects M and L of C_Y which have the same property and such that $f^*(M) \simeq M'$, $f^*(L) \simeq L'$.

Then

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(f)} K_0(X, \mathfrak{E}_X)$$

is a group epimorphism. In particular, the functor K_0 maps 'exact' localizations to epimorphisms.

Proof. (a) Let $(X, \mathfrak{E}_X)$ and $(Y, \mathfrak{E}_Y)$ be right exact 'spaces' and $(C_Y, \mathfrak{E}_Y) \xrightarrow{f^*} (C_X, \mathfrak{E}_X)$ an 'exact' functor. Then f^* induces a morphism

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(f)} K_0(X, \mathfrak{E}_X)$$

uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Z}_0(C_Y) & \xrightarrow{\mathbb{Z}_0(f^*)} & \mathbb{Z}_0(C_X) \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_X} \\ K_0(Y, \mathfrak{E}_Y) & \xrightarrow{K_0(f)} & K_0(X, \mathfrak{E}_X) \end{array} \tag{1}$$

of $\mathbb{Z}$-modules. Here $\mathbb{Z}_0(f^*)$ denotes the morphism of abelian groups induced by the functor f^*. The vertical arrows, p_Y and p_X, are natural epimorphisms.

(b) Suppose that $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ is a morphism of $\mathfrak{Esp}_{\mathfrak{r}}$ having the property (†). Then $\mathbb{Z}_0(C_Y) \xrightarrow{\mathbb{Z}_0(f^*)} \mathbb{Z}_0(C_X)$ is a group epimorphism. Thus, $K_0(f) \circ p_Y = p_X \circ \mathbb{Z}_0(f^*)$ is an epimorphism, which implies that $K_0(f)$ is an epimorphism. ∎

7.1.5.1. Corollary. *Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{Esp}_{\mathfrak{r}}$ whose inverse image functor, f^*, induces a surjective map $|C_Y| \longrightarrow |C_X|$ of isomorphism classes of objects. If the groupoid associated with the category C_Y is connected, then $K_0(f)$ is a surjective map. In particular, $K_0(f)$ is surjective if the category C_Y has initial or final objects.*

Proof. The assertion follows from 7.1.5(b). ∎

7.1.5.2. Corollary. *For any 'exact' localization $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ (i.e. $\mathfrak{q}^*$ is equivalent to a localization functor), the map $K_0(\mathfrak{q})$ is an epimorphism.*

Proof. If $\mathfrak{q}^*$ is equivalent to a localization functor, then each object of C_X is isomorphic to an object of $\mathfrak{q}^*(C_Y)$ and any morphism $\mathfrak{q}^*(M) \longrightarrow \mathfrak{q}^*(L)$ is the composition of the form $\mathfrak{q}^*(\mathfrak{s}_n)^{-1} \circ \mathfrak{q}^*(f_n) \circ \ \cdots \ \circ \mathfrak{q}^*(\mathfrak{s}_1)^{-1} \circ \mathfrak{q}^*(f_1)$ for some chain of arrows

$$M \xrightarrow{f_1} \widetilde{M}_1 \xleftarrow{\mathfrak{s}_1} M_1 \xrightarrow{f_1} \ \ldots \ \xrightarrow{f_n} \widetilde{M}_n \xleftarrow{\mathfrak{s}_n} M_n = L.$$

In particular, the condition (†) of 7.1.5(b) holds. ∎

7.1.6. Proposition. *Let $(X, \mathfrak{E}_X)$ be a right exact 'space' such that the category C_X has initial objects. Then the group $K_0(X, \mathfrak{E}_X)$ is the quotient of the free abelian group $\mathbb{Z}|C_X|$ generated by the isomorphism classes of objects of C_X by the subgroup generated by*

$[M]-[L]-[N]$ *for all conflations* $N \longrightarrow M \longrightarrow L$ *and the isomorphism class of initial objects of* C_X.

Proof. (a) The expressions $[M]-[L]-[N]$, where $N \xrightarrow{\mathfrak{k}} M \xrightarrow{\mathfrak{e}} L$ runs through conflations of $(C_X, \mathfrak{E}_X)$, are among the relations because each of them corresponds to a cartesian square

$$\begin{array}{ccc} N & \longrightarrow & x \\ {\scriptstyle \mathfrak{k}}\downarrow & cart & \downarrow \\ M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

where x is an initial object of C_X.

(b) On the other hand, let

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{L} \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\mathfrak{e}} & L \end{array} \tag{1}$$

be a cartesian square whose horizontal arrows are deflations. Therefore we have a commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{\widetilde{\mathfrak{k}}} & \widetilde{M} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{L} \\ {\scriptstyle id}\downarrow & & {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\mathfrak{k}} & M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

whose rows are conflations. The rows give relations $[\widetilde{M}]-[\widetilde{L}]-[N]$ and $[M]-[L]-[N]$. Their difference, $[\widetilde{M}]-[M]+[L]-[\widetilde{L}]$, is the relation corresponding to the cartesian square (1). Hence the assertion. ∎

7.1.7. The categories $\mathfrak{Esp}^w_{\mathfrak{t}}$ and $\mathfrak{Esp}^*_{\mathfrak{t}}$. Let $\mathfrak{Esp}^w_{\mathfrak{t}}$ denote the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ such that C_X has initial objects; and morphisms $(X, \mathfrak{E}_X) \longrightarrow (Y, \mathfrak{E}_Y)$ are given by morphisms of 'spaces' $X \xrightarrow{f} Y$ whose inverse image functors preserve conflations. In particular, they map initial objects to initial objects.

We denote by $\mathfrak{Esp}^*_{\mathfrak{t}}$ the subcategory of $\mathfrak{Esp}_{\mathfrak{t}}$ whose objects are right exact 'spaces' $(C_X, \mathfrak{E}_X)$ such that the category C_X has initial objects and morphisms are defined by the requirement that their inverse image functor maps initial objects to initial objects.

It follows that $\mathfrak{Esp}^*_{\mathfrak{t}}$ is a subcategory of the category $\mathfrak{Esp}^w_{\mathfrak{t}}$. The k-linear versions of these categories coincide.

7.1.8. Proposition. *(a) The map* $(X, \mathfrak{E}_X) \longmapsto K_0(X, \mathfrak{E}_X)$ *extends to a contravariant functor,* K_0^w, *from the category* $\mathfrak{Esp}^w_{\mathfrak{t}}$ *to the category* $\mathbb{Z}-mod$ *of abelian groups.*

(b) Let $(X, \mathfrak{E}_X) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ *be a morphism of* $\mathfrak{Esp}^w_{\mathfrak{t}}$ *such that* f^* *induces a surjective map* $|C_Y| \longrightarrow |C_X|$ *of the isomorphism classes of objects. Then*

$$K_0^w(Y, \mathfrak{E}_Y) \xrightarrow{K_0^w(f)} K_0^w(X, \mathfrak{E}_X)$$

is a group epimorphism. In particular, the functor K_0 maps 'exact' localizations to epimorphisms.

Proof. The assertions follow from 7.1.6. ∎

7.2. The relative functors K_0 and their derived functors. Fix a right exact 'space' $\mathcal{Y} = (Y, \mathfrak{E}_{\mathcal{Y}})$. The functor $(\mathfrak{Esp}_{\mathfrak{t}})^{op} \xrightarrow{K_0} \mathbb{Z} - mod$ induces a functor

$$(\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y})^{op} \xrightarrow{K_0^{\mathcal{Y}}} \mathbb{Z} - mod$$

defined by

$$K_0^{\mathcal{Y}}(\mathcal{X}, \xi) = K_0^{\mathcal{Y}}(\mathcal{X}, \mathcal{X} \xrightarrow{\xi} \mathcal{Y}) = Cok(K_0(\mathcal{Y}) \xrightarrow{K_0(\xi)} K_0(\mathcal{X}))$$

and acting correspondingly on morphisms.

The main advantage of the functor $K_0^{\mathcal{Y}}$ is that its domain, the category $\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y}$ has a final object, cokernels of morphisms, and natural left exact structures induced by left exact structures on $\mathfrak{Esp}_{\mathfrak{t}}$. Fix a left exact structure $\mathfrak{I}$ on $\mathfrak{Esp}_{\mathfrak{t}}$ (say, one of those defined in 6.8.3.2) and denote by $\mathfrak{I}_{\mathcal{Y}}$ the left exact structure on $\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y}$ induced by $\mathfrak{I}$. Notice that, since the category $\mathbb{Z} - mod$ is complete (and cocomplete), there is a well defined satellite endofunctor of $\mathcal{H}om((\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y})^{op}, \mathbb{Z} - mod)$, $F \longmapsto \mathcal{S}_{\mathfrak{I}_{\mathcal{Y}}} F$. So that for every functor F from $(\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y})^{op}$ to $\mathbb{Z} - mod$, there is a unique up to isomorphism universal ∂^*-functor $(\mathcal{S}^i_{\mathfrak{I}_{\mathcal{Y}}} F, \mathfrak{d}_i \mid i \geq 0)$.

In particular, there is a universal contravariant ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}} = (K_i^{\mathcal{Y},\mathfrak{I}}, \mathfrak{d}_i \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ of right exact 'spaces' over $\mathcal{Y}$ to the category $\mathbb{Z} - mod$ of abelian groups; that is $K_i^{\mathcal{Y},\mathfrak{I}} = \mathcal{S}^i_{\mathfrak{I}_{\mathcal{Y}}} K_0^{\mathcal{Y},\mathfrak{I}}$ for all $i \geq 0$.

We call the groups $K_i^{\mathcal{Y},\mathfrak{I}}(\mathcal{X}, \xi)$ *universal* K-groups of the right exact 'space' $(\mathcal{X}, \xi)$ over $\mathcal{Y}$ with respect to the left exact structure $\mathfrak{I}$.

7.3. 'Exactness' properties. In general, the ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}}$ is not 'exact'. The purpose of this section is to find some natural left exact structures $\mathfrak{I}$ on the category $\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y}$ of right exact 'spaces' over $\mathcal{Y}$ and its subcategory $\mathfrak{Esp}^*_{\mathfrak{t}}/\mathcal{Y}$ (cf. 7.1.7) for which the ∂^*-functor $K_\bullet^{\mathcal{Y},\mathfrak{I}}$ is 'exact'.

7.3.1. Proposition. *Let $(X, \xi) \xrightarrow{q} (X', \xi')$ be a morphism of the category $\mathfrak{Esp}_{\mathfrak{t}}/\mathcal{Y}$ such that $X \xrightarrow{q} X'$ belongs to $\mathfrak{L}_{\mathfrak{es}}$ (cf. 6.8.3) and has the following property:*

(#) if $M \xrightarrow{\mathfrak{s}} L$ is a morphism of $C_{X'}$ such that $q^*(\mathfrak{s})$ is invertible, then the element $[M] - [L]$ of the group $K_0(X')$ belongs to the image of the map $K_0(X'') \xrightarrow{K_0(\mathfrak{c}_q)} K_0(X')$, where $(X', \xi') \xrightarrow{\mathfrak{c}_q} (X'', \xi'')$ is the cokernel of the morphism $(X, \xi) \xrightarrow{q} (X', \xi')$.

Suppose, in addition, that one of the following two conditions holds:

(i) the category $C_{X'}$ has an initial object;

(ii) for any pair of arrows $N \xrightarrow{f} L \xleftarrow{\mathfrak{s}} M$, of the category $C_{X'}$ such that $\mathfrak{q}^*(\mathfrak{s})$ is

invertible, there exists a commutative square

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle \mathfrak{t}}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}} \\ N & \xrightarrow{f} & L \end{array}$$

such that $\mathfrak{q}^*(\mathfrak{t})$ *is invertible.*

Then for every conflation $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_\mathfrak{q}} (X'',\xi'')$ *of the left exact category* $(\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}, \mathfrak{I}_\mathcal{Y})$ *the sequence*

$$K_0^\mathcal{Y}(X'',\xi'') \xrightarrow{K_0^\mathcal{Y}(\mathfrak{c}_\mathfrak{q})} K_0^\mathcal{Y}(X',\xi') \xrightarrow{K_0^\mathcal{Y}(\mathfrak{q})} K_0^\mathcal{Y}(X,\xi) \longrightarrow 0$$

of morphisms of abelian groups is exact.

Proof. (a) The map $K_0^\mathcal{Y}(X',\xi') \xrightarrow{K_0^\mathcal{Y}(\mathfrak{q})} K_0^\mathcal{Y}(X,\xi)$ is surjective, because, by 7.1.5.2, the map $K_0(X',\mathfrak{E}_{X'}) \xrightarrow{K_0(\mathfrak{q})} K_0(X,\mathfrak{E}_X)$ is surjective.

(b) Fix a cokernel $(X',\xi') \xrightarrow{\mathfrak{c}_\mathfrak{q}} (X'',\xi'')$ of $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi')$ and its inverse image functor $C_{X''} \xrightarrow{\mathfrak{c}_\mathfrak{q}^*} C_{X'}$. Notice that the condition (#) is equivalent to the condition

(#') If M and L are objects of $C_{X'}$ such that $q^*(M) \simeq q^*(L)$, then $[M]-[L]$ belongs to $Im(K_0(\mathfrak{c}_\mathfrak{q}))$.

Obviously, (#') implies (#). On the other hand, since q^* is a localization functor, the existence of an isomorphism between $q^*(M)$ and $q^*(L)$ is equivalent to the existence of a diagram

$$M \longleftarrow M_1 \longrightarrow M_2 \longleftarrow \ \ldots \ \longleftarrow M_n \longrightarrow L$$

whose arrows belong to $\Sigma_{q^*} = \{s \in HomC_{X'} \mid q^*(s) \text{ is invertible}\}$, which shows that (#) implies the condition (#').

(b1) The condition (#') is equivalent to the condition

(#") The kernel of the morphism $\mathbb{Z}_0(C_{X'}) \xrightarrow{\mathbb{Z}_0(q^*)} \mathbb{Z}_0(C_X)$ is contained in the subgroup $Im(\mathbb{Z}_0(\mathfrak{c}_q^*)) + Ker(p_{X'})$ of the abelian group $\mathbb{Z}_0 C_{X'})$.

Here $p_{X'}$ is the canonical epimorphism $\mathbb{Z}_0(C_{X'}) \longrightarrow K_0(X')$.

In fact, the condition (#') follows from (#"): it suffices to apply (#") to the elements of $Ker(\mathbb{Z}_0(q^*))$ of the form $[M]-[L]$.

An element $z = \sum_{[M]\in|C_{X'}|} \lambda_{[M]}[M]$ of the abelian group $\mathbb{Z}_0(C_{X'})$ can be written as $\sum_{N\in|C_X|} \sum_{\{[M]|[q^*(M)]=N\}} \lambda_{[M]}[M]$. It follows that the element z belongs to $Ker(\mathbb{Z}_0(q^*))$ iff $\sum_{\{[M]|[q^*(M)]=N\}} \lambda_{[M]} = 0$ for each $N \in |C_X|$. It follows from the condition (#') that $\sum_{\{[M]|[q^*(M)]=N\}} \lambda_{[M]}[M]$ belongs to the subgroup $Im(\mathbb{Z}_0(\mathfrak{c}_q^*)) + Ker(p_{X'})$ of the group

$\mathbb{Z}_0(C_{X'})$ whenever $\sum_{\{[M]|[q^*(M)]=N\}} \lambda_{[M]} = 0$. Therefore, each element of $Ker(\mathbb{Z}_0(q^*))$ belongs to the subgroup $Im(\mathbb{Z}_0(\mathfrak{c}_q^*)) + Ker(p_{X'})$.

(c) Consider the commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
Ker(p_{X''}) & \xrightarrow{\mathfrak{K}(\mathfrak{c}_\mathfrak{q})} & Ker(p_{X'}) & \xrightarrow{\mathfrak{K}(q)} & Ker(p_X) \\
\mathfrak{j}_{X''}\downarrow & & \mathfrak{j}_{X'}\downarrow & & \downarrow \mathfrak{j}_X \\
\mathbb{Z}_0(C_{X''}) & \xrightarrow{\mathbb{Z}_0(c_q^*)} & \mathbb{Z}_0(C_{X'}) & \xrightarrow{\mathbb{Z}_0(q^*)} & \mathbb{Z}_0(C_X) \\
p_{X''}\downarrow & & p_{X'}\downarrow & & \downarrow p_X \\
K_0(X'') & \xrightarrow{K_0(c_q)} & K_0(X') & \xrightarrow{K_0(q)} & K_0(X) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array} \tag{1}$$

with exact columns.

(c1) The map $Ker(p_{X'}) \xrightarrow{\mathfrak{K}(q)} Ker(p_X)$ is surjective.

In fact, by hypothesis, the localization $X' \xrightarrow{q} X$ belongs to $\mathfrak{L}_{\mathfrak{es}}$; that is each morphism of $\mathfrak{E}_X$ is isomorphic to an arrow of $q^*(\mathfrak{E}_{X'})$.

(i) Suppose that the category $C_{X'}$ has initial objects. Since q is a morphism of the category $\mathfrak{Esp}_\mathfrak{t}$, its inverse image functor, q^*, maps initial objects to initial objects (in particular, the category C_X has initial objects, which is also a consequence of q^* being a localization functor) and conflations to conflations. Therefore, any conflation of the right exact category $(C_X, \mathfrak{E}_X)$ is isomorphic to $q^*(N \longrightarrow M \longrightarrow L)$ for some conflation $N \longrightarrow M \longrightarrow L$ of the right exact category $(C_{X'}, \mathfrak{E}_{X'})$. So that the subgroup $Ker(p_X)$ is generated by the elements $\mathfrak{K}(q)([M] - [N] - [L])$, where $N \longrightarrow M \longrightarrow L$ runs through conflations of $(C_{X'}, \mathfrak{E}_{X'})$, whence the surjectivity of $Ker(p_{X'}) \xrightarrow{\mathfrak{K}(q)} Ker(p_X)$.

(ii) Suppose now that the condition (ii) holds. The claim is that, in this case, every cartesian square in C_X whose vertical arrows are deflations is isomorphic to to the image of a cartesian square in $C_{X'}$ with the same property.

Since each arrow of $\mathfrak{E}_X$ is isomorphic to an arrow of $q^*(\mathfrak{E}_X)$ and q^* is a localization functor, every cartesian square in C_X is isomorphic to the cartesian square of the form

$$\begin{array}{ccc}
q^*(\widetilde{N}) & \xrightarrow{\widetilde{f}} & q^*(M) \\
\mathfrak{u}\downarrow & cart & \downarrow q^*(\mathfrak{e}) \\
q^*(N) & \xrightarrow{f} & q^*(L)
\end{array} \tag{2}$$

where $\mathfrak{e} \in \mathfrak{E}_{X'}$. The functor q^* being a localization implies that the morphism f is the composition

$$q^*(f_n)q^*(s_n)^{-1} \ \ldots \ q^*(f_2)q^*(s_2)^{-1}q^*(f_1)q^*(s_1)^{-1}.$$

By the condition (2), there exists a commutatative square

$$\begin{array}{ccc} \widetilde{N}_1 & \xrightarrow{\widetilde{f_1}} & L \\ {\scriptstyle \mathfrak{t}_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{s}_1} \\ N_1 & \xrightarrow{f_1} & L_1 \end{array}$$

such that $q^*(\mathfrak{t}_1)$ is an isomorphism. Therefore, $q^*(f_1)q^*(s_1)^{-1} = q^*(\mathfrak{t}_1)^{-1}\widetilde{f_1}$, which implies that $f = q^*(f_n)q^*(s_n)^{-1} \ \ldots \ q^*(f_2)q^*(\mathfrak{t}_1 s_2)^{-1}q^*(\widetilde{f_1})$. Continuing this process, we obtain morphisms $N \xleftarrow{\mathfrak{t}} \mathcal{N} \xrightarrow{f'} L$ such that $q^*(\mathfrak{t})$ is an isomorphism and $f = q^*(\mathfrak{t})^{-1}q^*(f')$. Since the morphism $\mathfrak{e}$ in the diagram (2) is a deflation, there is a cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f''} & M \\ {\scriptstyle \widetilde{\mathfrak{e}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}} \\ \mathcal{N} & \xrightarrow{f''} & L \end{array}$$

Since the functor q^* is 'exact' it maps this cartesian square to a cartesian square which is (thanks to the iniversal property of cartesian squares) isomorphic to the square (2).

This shows that the subgroup $Ker(p_X)$ is generated by the elements $\mathfrak{K}(q)([M'] - [M] + [L] - [N])$ for all cartesian squares

$$\begin{array}{ccc} M' & \xrightarrow{\widetilde{f}} & M \\ {\scriptstyle \mathfrak{e}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{e}} \\ L' & \xrightarrow{f} & L \end{array}$$

whose vertical arrows are deflations. Hence the surjectivity of $Ker(p_{X'}) \xrightarrow{\mathfrak{K}(q)} Ker(p_X)$.

(c2) There is the inclusion $Ker(K_0(q)) \subseteq Im(K_0(\mathfrak{c}_\mathfrak{q}))$.

Indeed, let $z \in K_0(X')$, and let z' be an element of $\mathbb{Z}_0(C_{X'})$ such that $p_{X'}(z') = z$. The element z belongs to $Ker(K_0(q))$ iff the element $\mathbb{Z}_0(q^*)(z')$ belongs to $Ker(p_X)$. Thanks to the surjectivity of $\mathfrak{K}(q)$ (argued in (c1)), there is an element z'' in $Ker(p_{X'})$ such that $\mathfrak{K}(q)(z'') = \mathbb{Z}_0(q^*)(z')$. Therefore, $\mathbb{Z}_0(q^*)(z' - z'') = 0$.

By the property (#"), which is equivalent to the property (#) of the proposition (see (b1) above), the kernel of the morphism $\mathbb{Z}_0(C_{X'}) \xrightarrow{\mathbb{Z}_0(q^*)} \mathbb{Z}_0(C_X)$ is contained in the subgroup $Im(\mathbb{Z}_0(\mathfrak{c}_q^*)) + Ker(p_{X'})$ of the abelian group $\mathbb{Z}_0(C_{X'})$. So that

$$z' \in z'' + Im(\mathbb{Z}_0(\mathfrak{c}_q^*)) + Ker(p_{X'}) \subseteq Im(\mathbb{Z}_0(\mathfrak{c}_q^*)) + Ker(p_{X'}).$$

Therefore, $z = p_{X'}(z') \in p_{X'}(Im(\mathbb{Z}_0(\mathfrak{c}_q^*))) \subseteq Im(K_0(\mathfrak{c}_\mathfrak{q}))$.

(c3) It follows from the commutative diagram

$$\begin{array}{ccccc} K_0(X'') & \xrightarrow{K_0(\widetilde{q})} & K_0(Y) & & \\ {\scriptstyle K_0(\mathfrak{c}_\mathfrak{q})}\downarrow & & \downarrow{\scriptstyle K_0(\xi)} & & \\ K_0(X') & \xrightarrow{K_0(q)} & K_0(X) & \xrightarrow{\lambda(q)} & K^{\mathcal{Y}}(X,\xi) = Cok(K_0(\xi)) \end{array}$$

that the inclusion $Ker(K_0(q)) \subseteq Im(K_0(\mathfrak{c}_\mathfrak{q}))$ implies the exactness of the sequence

$$K_0(X'') \xrightarrow{K_0(\mathfrak{c}_\mathfrak{q})} K_0(X') \xrightarrow{\lambda(q)K_0(q)} K^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

which, in turn, implies the exactness of the sequence

$$K_0^{\mathcal{Y}}(X'',\xi'') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{c}_\mathfrak{q})} K_0^{\mathcal{Y}}(X',\xi') \xrightarrow{K_0^{\mathcal{Y}}(\mathfrak{q})} K_0^{\mathcal{Y}}(X,\xi) \longrightarrow 0$$

as claimed. ∎

7.3.2. Proposition. *The class $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$ of all morphisms $(X,\xi) \xrightarrow{q} (X',\xi')$ of $\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}$ such that $X \xrightarrow{q} X'$ belongs to $\mathfrak{L}_{\mathfrak{es}}$ and satisfies the condition (#) of 7.3.1, is a left exact structure on the category $\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}$.*

Proof. It is clear that $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$ contains all isomorphisms. We need to show that it is stable under cobase change and closed under compositions.

(i) Let $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi')$ be a morphism of $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$ and $(X,\xi) \xrightarrow{\mathfrak{f}} (Z,\zeta)$ an arbitrary morphism of $\mathfrak{Esp}_\mathfrak{r}/\mathcal{Y}$. We have a quasi-commutative diagram

$$\begin{array}{ccccc} C_{Z''} & \xrightarrow{\lambda^*_\mathfrak{p}} & C_Y & & \\ {\scriptstyle \mathfrak{c}^*_\mathfrak{p}}\downarrow & cart & \downarrow{\scriptstyle \zeta^*} & & \\ C_{Z''} & \xrightarrow{\mathfrak{c}^*_\mathfrak{p}} & C_{Z'} & \xrightarrow{\mathfrak{p}^*} & C_Z \\ {\scriptstyle \mathfrak{f}''^*}\downarrow\wr & & {\scriptstyle \mathfrak{f}'^*}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}^*} \\ C_{X''} & \xrightarrow{\mathfrak{c}^*_\mathfrak{q}} & C_{X'} & \xrightarrow{\mathfrak{q}^*} & C_X \\ & & {\scriptstyle \mathfrak{c}^*_\mathfrak{q}}\uparrow & cart & \uparrow{\scriptstyle \xi^*} \\ & & C_{X''} & \xrightarrow{\lambda^*_\mathfrak{q}} & C_Y \end{array} \qquad (3)$$

whose right squares are cartesian and (therefore) the left vertical arrow is a category equivalence. A morphism $(M,L;\phi) \xrightarrow{(\mathfrak{s},\mathfrak{t})} (M',L';\phi')$ belongs to $\Sigma_{\mathfrak{p}^*}$ iff $\mathfrak{t}$ is an isomorphism and, therefore, $M \xrightarrow{\mathfrak{s}} M'$ belongs to $\Sigma_{\mathfrak{q}^*}$. Since we are looking at isomorphism classes of objects, we can and will assume that $L = L'$ and $\mathfrak{t}$ is the identity morphism.

By the condition (#) of 7.3.1, the fact that $\mathfrak{s} \in \Sigma_{\mathfrak{q}^*}$ implies that $[M] - [M'] \in \mathbb{Z}_0(\mathfrak{c}^*_\mathfrak{q})$. The composition of $C_{Z''} \xrightarrow{\mathfrak{c}^*_\mathfrak{p}} C_{Z'}$ with the isomorphism $C_{X''} \xrightarrow{\mathfrak{f}^*} C_{Z''}$ maps

each object $(\mathcal{M}, \mathcal{K}; \mathfrak{q}^*(\mathcal{M}) \xrightarrow{\psi} \xi^*(\mathcal{K}))$ of $C_{X''}$ to the object $(\mathcal{M}, \zeta^*(\mathcal{K}); \mathfrak{q}^*(\mathcal{M}) \xrightarrow{\sim} \mathfrak{f}^*\zeta^*(\mathcal{K}))$ of the category $C_{Z'}$. Therefore, the inclusion $[M] - [M'] \in \mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{q}})$ implies the inclusion $[(M, L; \phi)] - [(M', L; \phi')] \in \mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{p}})$.

(ii) Let $(X, \xi) \xrightarrow{\mathfrak{p}} (\mathfrak{X}, \xi_{\mathfrak{X}})$ and $(\mathfrak{X}, \xi_{\mathfrak{X}}) \xrightarrow{\mathfrak{q}} (Z, \xi_Z)$ be morphisms of $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$. The claim is that their composition, $(X, \xi) \xrightarrow{\mathfrak{qp}} (Z, \xi_Z)$, belongs to $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$ too; that is the localization $X \xrightarrow{\mathfrak{qp}} Z$ (which belongs to $\mathfrak{L}_{\mathfrak{es}}$) satisfies the property (#) of 7.3.1. We have the quasi-commutative diagram

$$
\begin{array}{ccccccc}
 & & C_{Z'''} & \longrightarrow & C_Y & & \\
 & & {\scriptstyle \mathfrak{c}^*_{\gamma_{\mathfrak{q}}}}\downarrow & cart & \downarrow{\scriptstyle \xi_{\mathfrak{X}''}} & & \\
C_{Z'''} & \xrightarrow{\mathfrak{c}^*_{\gamma_{\mathfrak{q}}}} & C_{Z''} & \xrightarrow{\gamma^*_{\mathfrak{q}}} & C_{\mathfrak{X}''} & \xrightarrow{\lambda^*_{\mathfrak{p}}} & C_Y \\
\wr\downarrow & & {\scriptstyle \mathfrak{c}^*_{\mathfrak{qp}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{c}^*_{\mathfrak{p}}} & cart & \downarrow{\scriptstyle \zeta^*} \\
C_{X''} & \xrightarrow{\mathfrak{c}^*_{\mathfrak{q}}} & C_Z & \xrightarrow{\mathfrak{q}^*} & C_{\mathfrak{X}} & \xrightarrow{\mathfrak{p}^*} & C_X \\
 & & {\scriptstyle \mathfrak{c}^*_{\mathfrak{q}}}\uparrow & cart & \uparrow{\scriptstyle \xi_{\mathfrak{X}}} & & \\
 & & C_{X''} & \longrightarrow & C_Y & &
\end{array}
\tag{4}
$$

with cartesian squares as indicated. To this diagram, there corresponds the commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Z}_0(C_{X''}) & \xrightarrow{id} & \mathbb{Z}_0(C_{X''}) & & \\
{\scriptstyle \mathbb{Z}_0(\beta^*)}\downarrow & & {\scriptstyle \mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{q}})}\downarrow & & \\
\mathbb{Z}_0(C_{Z''}) & \xrightarrow{\mathbb{Z}_0(c^*_{\mathfrak{qp}})} & \mathbb{Z}_0(C_Z) & \xrightarrow{\mathbb{Z}_0((\mathfrak{qp})^*)} & \mathbb{Z}_0(C_X) \\
{\scriptstyle \mathbb{Z}_0(\gamma^*_{\mathfrak{q}})}\downarrow & & {\scriptstyle \mathbb{Z}_0(\mathfrak{q}^*)}\downarrow & & \downarrow{\scriptstyle id} \\
\mathbb{Z}_0(C_{\mathfrak{X}''}) & \xrightarrow{\mathbb{Z}_0(c^*_{\mathfrak{p}})} & \mathbb{Z}_0(C_{\mathfrak{X}}) & \xrightarrow{\mathbb{Z}_0(\mathfrak{p}^*)} & \mathbb{Z}_0(C_X)
\end{array}
\tag{5}
$$

of abelian groups with surjective vertical arrows $\mathbb{Z}_0(\gamma^*_{\mathfrak{q}})$ and $\mathbb{Z}_0(\mathfrak{q}^*)$. Here the functor β^* is the composition of the equivalence $C_{X''} \longrightarrow C_{Z'''}$ and the functor $C_{Z'''} \xrightarrow{\mathfrak{c}^*_{\gamma_{\mathfrak{q}}}} C_{Z''}$ (see the diagram (4)). Let $z \in Ker(\mathbb{Z}_0(c^*_{\mathfrak{qp}}))$, or, equivalently, $\mathbb{Z}_0(\mathfrak{q}^*)(z) \in Ker(\mathbb{Z}_0(\mathfrak{p}^*))$. Since $(X, \xi) \xrightarrow{\mathfrak{p}} (\mathfrak{X}, \xi_{\mathfrak{X}})$ belongs to $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$, it follows from the condition (#) of 7.3.1 that $Ker(\mathbb{Z}_0(\mathfrak{p}^*)) \subseteq Im(\mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{p}}))$ (see the argument of 7.3.1). Since $\mathbb{Z}_0(\gamma^*_{\mathfrak{q}})$ is an epimorphism, there exists an element $b \in \mathbb{Z}_0(C_{Z''})$ such that $z - \mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{qp}})(b) \in Ker(\mathbb{Z}_0(\mathfrak{q}^*)$. Since $\mathfrak{q}$ satisfies the condition (#) of 7.3.1, $Ker(\mathbb{Z}_0(\mathfrak{q}^*)) \subseteq Im(\mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{q}}))$, and it follows from the upper square of (5) that $Im(\mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{q}})) \subseteq Im(\mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{qp}}))$. All together implies that z is an element of $Im(\mathbb{Z}_0(\mathfrak{c}^*_{\mathfrak{qp}}))$; i.e. the condition (#) holds. This shows that the composition $(X, \xi) \xrightarrow{\mathfrak{q}\circ\mathfrak{p}} (Z, \xi_Z)$ belongs to the class $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$.

(iii) The argument above proves that $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$ is the class of covers of a copretopology. This copretopology is subcanonical (i.e. it is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$), because the copretopology $\mathfrak{L}_{\mathfrak{es}}$ on $\mathfrak{Esp}_{\mathfrak{r}}$ is subcanonical. ∎

7.3.2.1. Proposition. *The class* $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es},\mathfrak{r}}$ *of all morphisms* $(X,\xi) \xrightarrow{q} (X',\xi')$ *og* $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$ *such that the functor* $C_{X'} \xrightarrow{q^*} C_X$ *satisfies the condition (ii) of 7.3.1, is a left exact structure on the category* $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$.

Proof. The assertion follows from 7.3.2 and the part (c) of the argument of 6.1. ∎

7.3.3. Proposition. *Let* $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ *be a right exact 'space', and let* $\mathfrak{I}$ *be a left exact structure on the category* $\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ *which is coarser than* $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es},\mathfrak{r}}$ *(cf. 7.3.2). Then the universal* ∂^**-functor* $K^{\mathcal{Y}}_\bullet = (K^{\mathcal{Y}}_i, \mathfrak{d}_i \mid i \geq 0)$ *from the left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ *to the category* $\mathbb{Z}-mod$ *of abelian groups is 'exact'; i.e. for any conflation* $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_\mathfrak{q}} (X'',\xi'')$, *the associated long sequence*

$$\ldots \xrightarrow{K^{\mathcal{Y}}_1(\mathfrak{q})} K^{\mathcal{Y}}_1(X,\xi) \xrightarrow{\mathfrak{d}_0} K^{\mathcal{Y}}_0(X'',\xi'') \xrightarrow{K^{\mathcal{Y}}_0(\mathfrak{c}_\mathfrak{q})} K^{\mathcal{Y}}_0(X',\xi') \xrightarrow{K^{\mathcal{Y}}_0(\mathfrak{q})} K^{\mathcal{Y}}_0(X,\xi) \longrightarrow 0$$

is exact.

Proof. Since the left exact structure $\mathfrak{I}_{\mathcal{Y}}$ is coarser than $\mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es}}$, it satisfies the condition (#) of 7.3.1. Therefore, by 7.3.1, for any conflation $(X,\xi) \xrightarrow{\mathfrak{q}} (X',\xi') \xrightarrow{\mathfrak{c}_\mathfrak{q}} (X'',\xi'')$ of the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$, the sequence

$$K^{\mathcal{Y}}_0(X'',\xi'') \xrightarrow{K^{\mathcal{Y}}_0(\mathfrak{c}_\mathfrak{q})} K^{\mathcal{Y}}_0(X',\xi') \xrightarrow{K^{\mathcal{Y}}_0(\mathfrak{q})} K^{\mathcal{Y}}_0(X,\xi) \longrightarrow 0$$

of $\mathbb{Z}$-modules is exact. Therefore, by 3.6.1, the universal ∂^*-functor $K^{\mathcal{Y}}_\bullet = (K^{\mathcal{Y}}_i, \mathfrak{d}_i \mid i \geq 0)$ from $(\mathfrak{Esp}^*_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{I}_{\mathcal{Y}})$ to $\mathbb{Z}-mod$ is 'exact'. ∎

It is convenient to have the following generalization of the previous assertion.

7.3.4. Proposition. *Let* $\mathcal{Y} = (Y, \mathfrak{E}_Y)$ *be a right exact 'space',* $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ *be a left exact category, and* $\mathfrak{F}$ *a functor* $C_\mathfrak{S} \longrightarrow \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}$ *which maps conflations of* $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ *to conflations of the left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}, \mathfrak{L}^{\mathcal{Y}}_{\mathfrak{es},\mathfrak{r}})$. *Then there exists a (unique up to isomorphism) universal* ∂^**-functor* $K^{\mathfrak{S},\mathfrak{F}}_\bullet = (K^{\mathfrak{S},\mathfrak{F}}_i, \mathfrak{d}_i \mid i \geq 0)$ *from the right exact category* $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ *to* $\mathbb{Z}-mod$ *whose zero component,* $K^{\mathfrak{S},\mathfrak{F}}_0$, *is the composition of the functor* $\mathcal{C}^{op}_\mathfrak{S} \xrightarrow{\mathfrak{F}^{op}} \mathfrak{Esp}_{\mathfrak{r}}/\mathcal{Y}^{op}$ *and the functor* $K^{\mathcal{Y}}_0$.
The ∂^**-functor* $K^{\mathfrak{S},\mathfrak{F}}_\bullet$ *is 'exact'.*

Proof. The existence of the ∂^*-functor $K^{\mathfrak{S},\mathfrak{F}}_\bullet$ follows, by 3.3.2, from the completeness (– existence of limits of small diagrams) of the category $\mathbb{Z}-mod$ of abelian groups. The main thrust of the proposition is in the 'exactness' of $K^{\mathfrak{S},\mathfrak{F}}_\bullet$.

By hypothesis, the functor $\mathfrak{F}$ maps conflations to conflations. Therefore, it follows from 7.3.1 that for any conflation $\mathfrak{X} \longrightarrow \mathfrak{X}' \longrightarrow \mathfrak{X}''$ of the left exact category $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$, the sequence of abelian groups $K^{\mathfrak{S},\mathfrak{F}}_0(\mathfrak{X}'') \longrightarrow K^{\mathfrak{S},\mathfrak{F}}_0(\mathfrak{X}') \longrightarrow K^{\mathfrak{S},\mathfrak{F}}_0(\mathfrak{X}) \longrightarrow 0$ is exact. By 3.6.1, this implies the 'exactness' of the ∂^*-functor $K^{\mathcal{Y}}_\bullet$. ∎

7.4. The 'absolute' case. Let $|Cat_*|^o$ denote the subcategory of the category $|Cat|^o$ of 'spaces' whose objects are 'spaces' represented by 'spaces' with initial objects

and morphisms are those morphisms of 'spaces' whose inverse image functor maps initial objects to initial objects. The category $|Cat_*|^o$ is pointed: it has a canonical zero (that is both initial and final) object, x, which is represented by the category with one (identical) morphism. Thus, the final objects of the category $|Cat|^o$ of all 'spaces' are zero objects of the subcategory $|Cat_*|^o$.

Each morphism $X \xrightarrow{\mathfrak{f}} Y$ of the category $|Cat_*|^o$ has a cokernel, $Y \xrightarrow{\mathfrak{c}_\mathfrak{f}} \mathcal{C}(\mathfrak{f})$, where the category $C_{\mathcal{C}(\mathfrak{f})}$ representing the 'space' $\mathcal{C}(\mathfrak{f})$ is the kernel $Ker(\mathfrak{f}^*)$ of the functor $\mathfrak{f}^*$. By definition, $Ker(\mathfrak{f}^*)$ is the full subcategory of the category C_Y generated by all objects of C_Y which the functor $\mathfrak{f}^*$ maps to initial objects. The inverse image functor $\mathfrak{c}_\mathfrak{f}^*$ of the canonical morphism $\mathfrak{c}_\mathfrak{f}$ is the natural embedding $Ker(\mathfrak{f}^*) \longrightarrow C_Y$.

The category $\mathfrak{Esp}_\mathfrak{t}^*$ formed by right exact 'spaces' with initial objects and morphisms whose inverse image functor is 'exact' and maps initial objects to initial objects (cf. 7.1.7), is pointed and the forgetfull functor

$$\mathfrak{Esp}_\mathfrak{t}^* \xrightarrow{\mathfrak{I}^*} |Cat_*|^o, \quad (X, \mathfrak{E}_X) \longmapsto X,$$

is a left adjoint to the canonical full embedding $|Cat_*|^o \xrightarrow{\mathfrak{I}_*} \mathfrak{Esp}_\mathfrak{t}^*$ which assigns to every 'space' X the right exact category $(X, Iso(C_X))$. Both functors, $\mathfrak{I}^*$ and $\mathfrak{I}_*$, map zero objects to zero objects.

Let x be a zero object of the category $\mathfrak{Esp}_\mathfrak{t}^*$. Then $\mathfrak{Esp}_\mathfrak{t}^*/x$ is naturally isomorphic to $\mathfrak{Esp}_\mathfrak{t}^*$ and $K_0^x = K_0$.

7.4.1. The left exact structure $\mathfrak{L}_{\mathfrak{es}}^*$. We denote by $\mathfrak{L}_{\mathfrak{es}}^*$ the canonical left exact structure $\mathfrak{L}_{\mathfrak{es}}^x$; it does not depend on the choice of the zero object x. It follows from the definitions above that $\mathfrak{L}_{\mathfrak{es}}^*$ consists of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ having the following properties:

(a) $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ is a localization functor (which is 'exact' and maps initial objects to initial objects), and every arrow of $\mathfrak{E}_X$ is isomorphic to an arrow of $\mathfrak{q}^*(\mathfrak{E}_Y)$.

(b) If $M \xrightarrow{s} M'$ is an arrow of C_Y such that $\mathfrak{q}^*(s)$ is an isomorphism, then $[M]-[M']$ is an element of $KerK_0(\mathfrak{q}^*)|$.

7.4.2. Proposition. *Let $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ be a left exact category, and let $\mathfrak{F}$ be a functor $C_\mathfrak{S} \longrightarrow \mathfrak{Esp}_\mathfrak{t}^*$ which maps conflations of $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})$ to conflations of the left exact category $(\mathfrak{Esp}_\mathfrak{t}^*, \mathfrak{L}_{\mathfrak{es}}^*)$. Let $\mathcal{G}$ be a functor from $(\mathfrak{Esp}_\mathfrak{t}^*)^{op}$ to a category C_Z with limits of 'small' filtered systems and initial objects. Then*

(a) There exists a universal ∂^*-functor $\mathcal{G}_\bullet^{\mathfrak{S},\mathfrak{F}} = (\mathcal{G}_i^{\mathfrak{S},\mathfrak{F}}, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ from $(\mathcal{C}_\mathfrak{S}, \mathfrak{I}_\mathfrak{S})^{op}$ to C_Z whose zero component, $K_0^{\mathfrak{S},\mathfrak{F}}$, is the composition of the functor $\mathcal{C}_\mathfrak{S}^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_\mathfrak{t}^*)^{op}$ and the functor $\mathcal{G}$.

(b) If $(C_Z, \mathfrak{E}_Z)$ is a right exact category and the functor $\mathcal{G}$ is left 'exact', then the ∂^*-functor $\mathcal{G}_\bullet^{\mathfrak{S},\mathfrak{F}}$ is 'exact'. In particular, the ∂^*-functor $\mathcal{G}_\bullet = (\mathcal{G}_i, \mathfrak{d}_i \mid i \geq 0)$ from $(\mathfrak{Esp}_\mathfrak{t}^*, \mathfrak{L}_{\mathfrak{es}}^*)$ to $(C_Z, \mathfrak{E}_Z)$ is 'exact'.

Proof. The assertion is a special case of 7.3.4. ∎

7.4.2.1. Corollary. *Let $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ be a left exact category, and $C_{\mathfrak{S}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{r}}$ a functor which maps conflations of $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ to conflations of the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$. Then there exists a (unique up to isomorphism) universal ∂^*-functor $K^{\mathfrak{S},\mathfrak{F}}_\bullet = (K^{\mathfrak{S},\mathfrak{F}}_i, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ from $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ to $\mathbb{Z}-mod$ whose zero component, $K^{\mathfrak{S},\mathfrak{F}}_0$, is the composition of the functor $\mathcal{C}^{op}_{\mathfrak{S}} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}^*_{\mathfrak{r}})^{op}$ and the functor K_0.*

The ∂^*-functor $K^{\mathfrak{S},\mathfrak{F}}_\bullet$ is 'exact'. In particular, the ∂^*-functor $K_\bullet = (K_i, \mathfrak{d}_i \mid i \geq 0)$ from $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$ to $\mathbb{Z}-mod$ is 'exact'.

7.4.3. The class of morphisms $\mathfrak{L}^{\circledast}_{\mathfrak{es}}$. We denote by $\mathfrak{L}^{\circledast}_{\mathfrak{es}}$ the class of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ of $\mathfrak{L}^*_{\mathfrak{es}}$ such that $Cok(\mathfrak{q})$ is a zero object, or, equivalently, $Ker(\mathfrak{q}^*)$ is a trivial category. It follows that $\mathfrak{L}^{\circledast}_{\mathfrak{es}}$ consists of all morphisms $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ such that

(a) $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ is an 'exact' localization functor with a trivial kernel, and every arrow of $\mathfrak{E}_X$ is isomorphic to an arrow of $\mathfrak{q}^*(\mathfrak{E}_Y)$.

(b) If $\mathfrak{q}^*(M \xrightarrow{s} M')$ is an isomorphism, then $[M] = [M']$ in $K_0(Y)$.

7.4.4. Proposition. *The class $\mathfrak{L}^{\circledast}_{\mathfrak{es}}$ is a left exact structure on the category $\mathfrak{Esp}^*_{\mathfrak{r}}$ of right exact 'spaces' with initial objects.*

Proof. The assertion is a special (dual) case of 5.3.7.1. ∎

7.4.5. Proposition. *Let $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ be a left exact category, $\mathfrak{F}$ a functor $C_{\mathfrak{S}} \longrightarrow \mathfrak{Esp}^*_{\mathfrak{r}}$ which maps conflations of $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})$ to conflations of the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$, and $K^{\mathfrak{S},\mathfrak{F}}_\bullet = (K^{\mathfrak{S},\mathfrak{F}}_i, \widetilde{\mathfrak{d}}_i \mid i \geq 0)$ a universal ∂^*-functor from $(\mathcal{C}_{\mathfrak{S}}, \mathfrak{I}_{\mathfrak{S}})^{op}$ to $\mathbb{Z}-mod$ whose zero component, $K^{\mathfrak{S},\mathfrak{F}}_0$, is the composition of $\mathcal{C}^{op}_{\mathfrak{S}} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}^*_{\mathfrak{r}})^{op}$ and K_0 (cf. 7.4.2.1).*

If $\mathcal{X} \xrightarrow{\mathfrak{q}} \mathcal{Y}$ is a morphism of $\mathfrak{I}_{\mathfrak{S}}$ with trivial cokernel, then the morphisms

$$K^{\mathfrak{S},\mathfrak{F}}_i(\mathcal{Y}) \xrightarrow{K^{\mathfrak{S},\mathfrak{F}}_i} K^{\mathfrak{S},\mathfrak{F}}_i(\mathcal{X})$$

are isomorphisms for all $i \geq 0$.

Proof. Let $\mathfrak{I}^{\circledast}_{\mathfrak{S}}$ denote the class of all morphisms of $\mathfrak{I}_{\mathfrak{S}}$ having a trivial cokernel. By (the dual version of) 3.3.7.1, the class $\mathfrak{I}^{\circledast}_{\mathfrak{S}}$ is a left exact structure on the category $C_{\mathfrak{S}}$.

Since the functor $C_{\mathfrak{S}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{r}}$ maps conflations to conflations, it maps final objects of the category $C_{\mathfrak{S}}$ to zero objects of $\mathfrak{Esp}^*_{\mathfrak{r}}$. In particular, $\mathfrak{F}$ maps morphisms of $\mathfrak{I}^{\circledast}_{\mathfrak{S}}$ to morphisms of $\mathfrak{L}^{\circledast}_{\mathfrak{es}}$. By 7.4.2.1, the ∂^*-functor is 'exact', so that for any conflation $\mathcal{X} \xrightarrow{\mathfrak{q}} \mathcal{X}' \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} \mathcal{X}''$, the sequence

$$K^{\mathfrak{S},\mathfrak{F}}_0(\mathcal{X}'') \xrightarrow{K^{\mathfrak{S},\mathfrak{F}}_0(\mathfrak{c}_{\mathfrak{q}})} K^{\mathfrak{S},\mathfrak{F}}_0(\mathcal{X}') \xrightarrow{K^{\mathfrak{S},\mathfrak{F}}_0(\mathfrak{q})} K^{\mathfrak{S},\mathfrak{F}}_0(\mathcal{X}) \longrightarrow 0$$

is exact. If $\mathfrak{q} \in \mathfrak{I}^{\circledast}_{\mathfrak{S}}$, then $K^{\mathfrak{S},\mathfrak{F}}_0(\mathcal{X}'') = K_0(\mathfrak{F}(\mathcal{X}'')) = 0$. So that in this case the morphism $K^{\mathfrak{S},\mathfrak{F}}_0(\mathcal{X}') \xrightarrow{K^{\mathfrak{S},\mathfrak{F}}_0(\mathfrak{q})} K^{\mathfrak{S},\mathfrak{F}}_0(\mathcal{X})$ is an isomorphism. The assertion follows now from 5.3.7.2. ∎

7.4.6. Corollary. *For every morphism* $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (X', \mathfrak{E}_{X'})$ *of* $\mathfrak{L}^{\circledast}_{\mathfrak{es}}$ *the corresponding map*

$$K_i(X', \mathfrak{E}_{X'}) \xrightarrow{K_i(\mathfrak{q})} K_i(X, \mathfrak{E}_X)$$

is an isomorphism for all $i \geq 0$.

7.5. Universal K-theory of abelian categories. Let $\mathfrak{Esp}^{\mathfrak{a}}_k$ denote the category whose objects are 'spaces' X represented by k-linear abelian categories and morphisms $X \xrightarrow{f} Y$ are represented by k-linear exact functors.

There is a natural functor

$$\mathfrak{Esp}^{\mathfrak{a}}_k \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^{*}_{\mathfrak{r}} \tag{1}$$

which assigns to each object X of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ the right exact (actually, exact) 'space' $(X, \mathfrak{E}^{st}_X)$, where $\mathfrak{E}^{st}_X$ is the *canonical* (i.e. the finest) right exact structure on the category C_X, and maps each morphism $X \xrightarrow{f} Y$ to the morphism $(X, \mathfrak{E}^{st}_X) \xrightarrow{f} (Y, \mathfrak{E}^{st}_Y)$ of right exact 'spaces'. One can see that the functor $\mathfrak{F}$ maps the zero object of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ (represented by the zero category) to a zero object of the category $\mathfrak{Esp}^{*}_{\mathfrak{r}}$.

7.5.1. Proposition. *Let* C_X *and* C_Y *be* k*-linear abelian categories endowed with the canonical exact structure. Any exact localization functor* $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ *satisfies the conditions (a) and (b) of 7.4.1.*

Proof. In fact, each morphism $\mathfrak{q}^*(M) \xrightarrow{\widetilde{h}} \mathfrak{q}^*(N)$ is of the form $\mathfrak{q}^*(h)\mathfrak{q}^*(s)^{-1}$ for some morphisms $M' \xrightarrow{h} N$ and $M' \xrightarrow{s} M$ such that $\mathfrak{q}^*(s)$ is invertible. The morphism h is a (unique) composition $\mathfrak{j} \circ \mathfrak{e}$, where $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism. Since the functor $\mathfrak{q}^*$ is exact, $\mathfrak{q}^*(\mathfrak{j})$ is a monomorphism and $\mathfrak{q}^*(\mathfrak{e})$ is an epimorphism. Therefore, $\widetilde{h}$ is an epimorphism iff $\mathfrak{q}^*(\mathfrak{j})$ is an isomorphism. This shows that the condition (a) holds.

Let $M \xrightarrow{s} M'$ be a morphism and

$$0 \longrightarrow Ker(s) \longrightarrow M \xrightarrow{s} M' \longrightarrow Cok(s) \longrightarrow 0$$

the associated with s exact sequence. Representing s as the composition, $\mathfrak{j} \circ \mathfrak{e}$, of a monomorphism $\mathfrak{j}$ and an epimorphism $\mathfrak{e}$, we obtain two short exact sequences,

$$0 \longrightarrow Ker(s) \longrightarrow M \xrightarrow{\mathfrak{e}} N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N \xrightarrow{\mathfrak{j}} M' \longrightarrow Cok(s) \longrightarrow 0,$$

hence $[M] = [Ker(s)] + [N]$ and $[M'] = [N] + [Cok(s)]$, or $[M'] = [M] + [Ker(s)] - [Cok(s)]$ in $K_0(Y)$. It follows from the exactness of the functor $\mathfrak{q}^*$ that the morphism $\mathfrak{q}^*(s)$ is an isomorphism iff $Ker(s)$ and $Cok(s)$ are objects of the category $Ker(\mathfrak{q}^*)$. Therefore, in this case, it follows that $[M'] = [M]$ modulo $\mathbb{Z}|Ker(\mathfrak{q}^*)|$ in $K_0(Y)$. ∎

7.5.2. Proposition. *(a) The class* $\mathfrak{L}^{\mathfrak{a}}$ *of all morphisms* $X \xrightarrow{q} Y$ *of the category* $\mathfrak{Esp}^{\mathfrak{a}}_k$ *such that* $C_Y \xrightarrow{q^*} C_X$ *is a localization functor is a left exact structure on* $\mathfrak{Esp}^{\mathfrak{a}}_k$.

(b) The functor $\mathfrak{Esp}_k^{\mathfrak{a}} \xrightarrow{\mathfrak{F}} \mathfrak{Esp}_{\mathfrak{r}}^*$ *is an 'exact' functor from the left exact category* $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{L}^{\mathfrak{a}})$ *to the left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{L}_{\mathfrak{es}}^*)$*. Moreover,* $\mathfrak{L}^{\mathfrak{a}} = \mathfrak{F}^{-1}(\mathfrak{L}_{\mathfrak{es}}^*)$*, that is the left exact structure* $\mathfrak{L}^{\mathfrak{a}}$ *is induced by the left exact structure* $\mathfrak{L}_{\mathfrak{es}}^*$ *via the functor* $\mathfrak{F}$*.*

Proof. (i) The category $\mathfrak{Esp}_k^{\mathfrak{a}}$ has push-forwards.

In fact, for any pair of arrows $X \xleftarrow{f} Z \xrightarrow{g} Y$ of $\mathfrak{Esp}_k^{\mathfrak{a}}$, consider the cartesian (in the pseudo-categorical sense) square

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{p_1^*} & C_X \\ {\scriptstyle p_2^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ C_Y & \xrightarrow{g^*} & C_Z \end{array} \qquad (2)$$

of inverse image functors. One can see from the description of the category $C_{\mathfrak{X}}$ and functors p_1^* and p_2^* that $C_{\mathfrak{X}}$ is a k-linear abelian category and the functors p_1^* and p_2^* are k-linear and exact, because f^* and g^* have this property. Since the square (2) is cartesian in pseudo-categorical sense, it is cartesian in the category formed by k-linear abelian categories and k-linear exact functors. Therefore, the corresponding commutative square of 'spaces'

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p_1} \\ Y & \xrightarrow{p_2} & \mathfrak{X} \end{array}$$

is cocartesian.

(ii) It follows from the construction of push-forwards in $\mathfrak{Esp}_k^{\mathfrak{a}}$ that the functor $\mathfrak{F}$ preserves cocartesian squares. It is clear that $\mathfrak{F}^{-1}(\mathfrak{L}_{\mathfrak{es}}^*) \subseteq \mathfrak{L}^{\mathfrak{a}}$. On the other hand, by 7.5.1, the functor $\mathfrak{F}$ maps morphisms of $\mathfrak{L}^{\mathfrak{a}}$ (– exact localizations) to morphisms of $\mathfrak{L}_{\mathfrak{es}}^*$. Therefore $\mathfrak{L}^{\mathfrak{a}} = \mathfrak{F}^{-1}(\mathfrak{L}_{\mathfrak{es}}^*)$. Since the functor $\mathfrak{F}$ maps cocartesian squares to cocartesian squares, it follows that $\mathfrak{L}^{\mathfrak{a}}$ is a left exact structure on the category $\mathfrak{Esp}_k^{\mathfrak{a}}$. ∎

7.5.3. The Grothendieck functor. The composition $K_0^{\mathfrak{a}}$ of the functor

$$(\mathfrak{Esp}_k^{\mathfrak{a}})^{op} \xrightarrow{\mathfrak{F}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^*)^{op}$$

and the functor $(\mathfrak{Esp}_{\mathfrak{r}}^*)^{op} \xrightarrow{K_0^*} \mathbb{Z} - mod$ assigns to each object X of the category $\mathfrak{Esp}_k^{\mathfrak{a}}$ the abelian group $K_0^*(X, \mathfrak{E}_X^{st})$ which coincides with the Grothendieck group of the abelian category C_X. We call $K_0^{\mathfrak{a}}$ the *Grothendieck functor*.

7.5.4. Proposition. *There exists a universal* ∂^**-functor* $K_\bullet^{\mathfrak{a}} = (K_i^{\mathfrak{a}}, \mathfrak{d}_i^{\mathfrak{a}} \mid i \geq 0)$ *from the right exact category* $(\mathfrak{Esp}_k^{\mathfrak{a}}, \mathfrak{L}^{\mathfrak{a}})^{op}$ *to the category* $\mathbb{Z} - mod$ *whose zero component is the Grothendieck functor* K_0*. The universal* ∂^**-functor* $K_\bullet^{\mathfrak{a}}$ *is 'exact'; that is for any exact localization* $X \xrightarrow{\mathfrak{q}} X'$*, the canonical long sequence*

$$\ldots \xrightarrow{K_1^{\mathfrak{a}}(\mathfrak{q})} K_1^{\mathfrak{a}}(X) \xrightarrow{\mathfrak{d}_0^{\mathfrak{a}}(\mathfrak{q})} K_0^{\mathfrak{a}}(X'') \xrightarrow{K_0^{\mathfrak{a}}(\mathfrak{c}_{\mathfrak{q}})} K_0^{\mathfrak{a}}(X') \xrightarrow{K_0^{\mathfrak{a}}(\mathfrak{q})} K_0^{\mathfrak{a}}(X) \longrightarrow 0 \qquad (3)$$

is exact.

Proof. By 7.5.2(b), the functor $\mathfrak{Esp}^{\mathfrak{a}}_k \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{t}}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})$ to the left exact category $(\mathfrak{Esp}^*_{\mathfrak{t}}, \mathfrak{L}^*_{\mathfrak{cs}})$ which maps the zero object of the category $\mathfrak{Esp}^{\mathfrak{a}}_k$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}^*_{\mathfrak{t}}$. Therefore, $\mathfrak{F}$ maps conflations to conflations.

The assertion follows now from 7.4.2.1 applied to the functor $\mathfrak{F}$. ∎

7.5.5. The universal ∂^*-functor $K^{\mathfrak{a}}_\bullet$ and the Quillen's K-theory. For a 'space' X represented by a svelte k-linear abelian category C_X, we denote by $K^{\mathfrak{Q}}_i(X)$ the i-th Quillen's K-group of the category C_X. For each $i \geq 0$, the map $X \longmapsto K^{\mathfrak{Q}}_i(X)$ extends naturally to a functor

$$(\mathfrak{Esp}^{\mathfrak{a}}_k)^{op} \xrightarrow{K^{\mathfrak{Q}}_i} \mathbb{Z} - mod$$

It follows from the Quillen's localization theorem [Q, 5.5] that for any exact localization $X \xrightarrow{\mathfrak{q}} X'$ and each $i \geq 0$, there exists a *connecting morphism* $K^{\mathfrak{Q}}_{i+1}(X) \xrightarrow{\mathfrak{d}^{\mathfrak{Q}}_i(\mathfrak{q})} K^{\mathfrak{Q}}_0(X'')$, where $C_{X''} = Ker(\mathfrak{q}^*)$, such that the sequence

$$\ldots \xrightarrow{K^{\mathfrak{Q}}_1(\mathfrak{q})} K^{\mathfrak{Q}}_1(X) \xrightarrow{\mathfrak{d}^{\mathfrak{Q}}_0(\mathfrak{q})} K^{\mathfrak{Q}}_0(X'') \xrightarrow{K^{\mathfrak{Q}}_0(\mathfrak{c}_{\mathfrak{q}})} K^{\mathfrak{Q}}_0(X') \xrightarrow{K^{\mathfrak{Q}}_0(\mathfrak{q})} K^{\mathfrak{Q}}_0(X) \longrightarrow 0 \qquad (4)$$

is exact. It follows (from the proof of the Quillen's localization theorem) that the connecting morphisms $\mathfrak{d}^{\mathfrak{Q}}_i(\mathfrak{q})$, $i \geq 0$, depend functorially on the localization morphism $\mathfrak{q}$. In other words, $K^{\mathfrak{Q}}_\bullet = (K^{\mathfrak{Q}}_i, \mathfrak{d}^{\mathfrak{Q}}_i \mid i \geq 0)$ is an 'exact' ∂^*-functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to the category $\mathbb{Z} - mod$ of abelian groups.

Naturally, we call the ∂^*-functor $K^{\mathfrak{Q}}_\bullet$ the *Quillen's K-functor.*

Since $K^{\mathfrak{a}}_\bullet = (K^{\mathfrak{a}}_i, \mathfrak{d}^{\mathfrak{a}}_i \mid i \geq 0)$ is a universal ∂^*-functor from $(\mathfrak{Esp}^{\mathfrak{a}}_k, \mathfrak{L}^{\mathfrak{a}})^{op}$ to $\mathbb{Z} - mod$, the identical isomorphism $K^{\mathfrak{Q}}_0 \longrightarrow K^{\mathfrak{a}}_0$ extends uniquely to a ∂^*-functor morphism

$$K^{\mathfrak{Q}}_\bullet \xrightarrow{\varphi^{\mathfrak{Q}}_\bullet} K^{\mathfrak{a}}_\bullet. \qquad (5)$$

7.5.6. Remark. There is a canonical functorial morphism of the universal determinant group $K^{det}_1(X)$ (introduced by Bass [Ba, p. 389]) to the Quillen's $K^{\mathfrak{Q}}_1(X)$. If X is *affine*, i.e. C_X is the category of modules over a ring, this morphism is an isomorphism. It is known [Ger, 5.2] that if C_X is the category of coherent sheaves on the complete non-singular curve of genus 1 over $\mathbb{C}$, then $K^{det}_1(X) \longrightarrow K^{\mathfrak{Q}}_1(X)$ is not a monomorphism. In particular, the composition $K^{det}_1(X) \longrightarrow K^{\mathfrak{a}}_1(X)$ of the morphism $K^{det}_1(X) \longrightarrow K^{\mathfrak{Q}}_1(X)$ and the canonical morphism $K^{\mathfrak{Q}}_1(X) \xrightarrow{\varphi^{\mathfrak{Q}}_1(X)} K^{\mathfrak{a}}_1(X)$ is not a monomorphism.

7.6. Universal K-theory of k-linear right exact categories. Let $\mathfrak{Esp}^{\mathfrak{r}}_k$ denote the category whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$, where the 'space' X is represented by a k-linear svelte additive category and morphisms $(X, \mathfrak{E}_X) \longrightarrow (Y, \mathfrak{E}_Y)$ are given by morphisms of 'spaces' $X \xrightarrow{f} Y$ whose inverse image functors are k-linear 'exact' functors.

By 1.4, the 'exactness' of a morphism f means precisely that its inverse image functor, f^*, maps conflations to conflations.

There is a natural functor

$$\mathfrak{Esp}_k^{\mathfrak{r}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^* \tag{1}$$

which maps objects and morphisms of the category $\mathfrak{Esp}_k^{\mathfrak{r}}$ to the corresponding objects and morphisms of the category $\mathfrak{Esp}_{\mathfrak{r}}^*$.

7.6.1. Proposition. *The functor* $\mathfrak{Esp}_k^{\mathfrak{r}} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}} \mathfrak{Esp}_{\mathfrak{r}}^*$ *preserves cocartesian squares and maps the zero object of the category* $\mathfrak{Esp}_k^{\mathfrak{r}}$ *to the zero object of the category* $\mathfrak{Esp}_{\mathfrak{r}}^*$.

Proof. The argument is similar to that of 7.5.2(b). Details are left to the reader. ∎

7.6.2. Corollary. *The class of morphisms* $\mathfrak{L}_k^{\mathfrak{r}} = \mathfrak{F}_{\mathfrak{r}}^{-1}(\mathfrak{L}_{\mathfrak{es}}^*)$ *is a left exact structure on the category* $\mathfrak{Esp}_k^{\mathfrak{r}}$ *and* $\mathfrak{F}_r$ *is an 'exact' functor from the left exact category* $(\mathfrak{Esp}_k^{\mathfrak{r}}, \mathfrak{L}_k^{\mathfrak{r}})$ *to the left exact category* $(\mathfrak{Esp}_{\mathfrak{r}}^*, \mathfrak{L}_{\mathfrak{es}}^*)$.

Proof. Since the functor $\mathfrak{F}_{\mathfrak{r}}$ preserves cocartesian squares, the preimage $\mathfrak{F}_{\mathfrak{r}}^{-1}(\tau)$ of any copretopology τ on $\mathfrak{Esp}_{\mathfrak{r}}^*$ is a copretopology on the category $\mathfrak{Esp}_k^{\mathfrak{r}}$. In particular, $\mathfrak{L}_k^{\mathfrak{r}} = \mathfrak{F}_{\mathfrak{r}}^{-1}(\mathfrak{L}_{\mathfrak{es}}^*)$ is the class of cocovers of a copretopology. The copretopology $\mathfrak{L}_k^{\mathfrak{r}}$ is subcanonical, i.e. $\mathfrak{L}_k^{\mathfrak{r}}$ is a left exact structure on the category $\mathfrak{Esp}_k^{\mathfrak{r}}$.

It follows from the definition of the functor $\mathfrak{F}_{\mathfrak{r}}$ that a morphism $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ of $\mathfrak{Esp}_k^{\mathfrak{r}}$ is a localization iff $\mathfrak{F}_{\mathfrak{r}}(\mathfrak{q})$ is a localization. In particular, $\mathfrak{L}_k^{\mathfrak{r}}$ consists of localizations. The copretopology $\mathfrak{L}_k^{\mathfrak{r}}$ is subcanonical iff for any morphism $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ the cocartesian square

$$\begin{array}{ccc} (X, \mathfrak{E}_X) & \xrightarrow{\mathfrak{q}} & (Y, \mathfrak{E}_Y) \\ {\scriptstyle \mathfrak{q}}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_1} \\ (Y, \mathfrak{E}_Y) & \xrightarrow{\mathfrak{p}_2} & (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) \end{array} \tag{2}$$

is cartesian, or, equivalently, the diagram

$$(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y) \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}) \tag{3}$$

is exact. The claim is that, indeed, the diagram (3) is exact.

In fact, let $(Z, \mathfrak{E}_Z) \xrightarrow{f} (Y, \mathfrak{E}_Y)$ be a morphism which equalizes the pair of arrows $(Y, \mathfrak{E}_Y) \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} (\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}})$. Since the functor $\mathfrak{F}_{\mathfrak{r}}$ transforms (2) into a cartesian square, there exists a unique morphism $\mathfrak{F}_{\mathfrak{r}}(Z, \mathfrak{E}_Z) \xrightarrow{h} \mathfrak{F}_{\mathfrak{r}}(X, \mathfrak{E}_X)$ such that $\mathfrak{F}_{\mathfrak{r}}(\mathfrak{q}) \circ h = \mathfrak{F}_{\mathfrak{r}}(f)$. It follows that the inverse image h^* of h is a k-linear functor $C_X \longrightarrow C_Z$. Therefore h is the image of (a uniquely determined) morphism $(Z, \mathfrak{E}_Z) \longrightarrow (X, \mathfrak{E}_X)$, hence the morphism f factors uniquely through $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$. ∎

7.6.3. The functor $K_0^{\mathfrak{r}}$**.** We denote by $K_0^{\mathfrak{r}}$ the composition of the functor

$$(\mathfrak{Esp}_k^{\mathfrak{r}})^{op} \xrightarrow{\mathfrak{F}_{\mathfrak{r}}^{op}} (\mathfrak{Esp}_{\mathfrak{r}}^*)^{op}$$

and the functor $(\mathfrak{Esp}^*_{\mathfrak{r}})^{op} \xrightarrow{K_0^*} \mathbb{Z}-mod$.

7.6.4. Proposition. *There exists a universal ∂^*-functor $K^{\mathfrak{r}}_\bullet = (K^{\mathfrak{r}}_i, \mathfrak{d}^{\mathfrak{r}}_i \mid i \geq 0)$ from the right exact category $(\mathfrak{Esp}^{\mathfrak{r}}_k, \mathfrak{L}^{\mathfrak{r}})^{op}$ to the category $\mathbb{Z}-mod$ whose zero component is the functor $K^{\mathfrak{r}}_0$. The universal ∂^*-functor $K^{\mathfrak{r}}_\bullet$ is 'exact'; that is for any exact localization $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (X', \mathfrak{E}_{X'})$ which belongs to $\mathfrak{L}^{\mathfrak{r}}$, the canonical long sequence*

$$\begin{array}{ccccccc} K^{\mathfrak{r}}_1(X.\mathfrak{E}_X) & \xleftarrow{K^{\mathfrak{r}}_1(\mathfrak{q})} & K^{\mathfrak{r}}_1(X', \mathfrak{E}_{X'}) & \xleftarrow{K^{\mathfrak{r}}_1(\mathfrak{c}_{\mathfrak{q}})} & K^{\mathfrak{r}}_1(X'', \mathfrak{E}_{X''}) & \xleftarrow{\mathfrak{d}^{\mathfrak{r}}_1(\mathfrak{q})} & \dots \\ \mathfrak{d}^{\mathfrak{r}}_0(\mathfrak{q}) \downarrow & & & & & & \\ K^{\mathfrak{r}}_0(X'', \mathfrak{E}_{X''}) & \xrightarrow{K^{\mathfrak{r}}_0(\mathfrak{c}_{\mathfrak{q}})} & K^{\mathfrak{r}}_0(X', \mathfrak{E}_{X'}) & \xrightarrow{K^{\mathfrak{r}}_0(\mathfrak{q})} & K^{\mathfrak{r}}_0(X, \mathfrak{E}_X) & \longrightarrow & 0 \end{array} \tag{4}$$

is exact.

Proof. The functor $\mathfrak{Esp}^{\mathfrak{r}}_k \xrightarrow{\mathfrak{F}} \mathfrak{Esp}^*_{\mathfrak{r}}$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_k, \mathfrak{L}^{\mathfrak{r}})$ to the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$ which maps the zero object of the category $\mathfrak{Esp}^{\mathfrak{r}}_k$ (– the 'space' represented by the zero category) to a zero object of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$. Therefore, $\mathfrak{F}$ maps conflations to conflations. It remains to apply 7.4.2.1. ∎

7.6.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact k-linear additive category, $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ the associated exact k-linear category, and $(C_X, \mathfrak{E}_X) \xrightarrow{\gamma_X^*} (C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ the canonical fully faithful 'exact' universal functor (see 2.6.1) regarded as an inverse image functor of a morphism $(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}}) \xrightarrow{\gamma_X} (X, \mathfrak{E}_X)$.*

The map $K_0(X, \mathfrak{E}_X) \xrightarrow{K_0(\gamma_X)} K_0(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ is a group epimorphism.

Proof. The assertion follows from the description of the exact category $(C_{X_{\mathfrak{e}}}, \mathfrak{E}_{X_{\mathfrak{e}}})$ (see the argument of 2.6.1). Details are left to the reader. ∎

7.6.6. The category of exact k-'spaces' and Grothendieck-Quillen functor. Let $\mathfrak{Esp}^{\mathfrak{e}}_k$ denote the full subcategory of the category $\mathfrak{Esp}^{\mathfrak{r}}_k$ whose objects are pairs $(X, \mathfrak{E}_X)$ such that $(C_X, \mathfrak{E}_X)$ is an exact k-linear category.

It follows from 2.6.1 that the inclusion functor, $\mathfrak{Esp}^{\mathfrak{e}}_k \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}^{\mathfrak{r}}_k$ has a right adjoint, $\mathfrak{J}_*$ which assigns to each right exact k-space $(X, \mathfrak{E}_X)$ the associated exact k-space $(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}})$. The adjunction arrow $\mathfrak{J}^*\mathfrak{J}_* \longrightarrow Id_{\mathfrak{Esp}^{\mathfrak{r}}_k}$ assigns to each object $(X, \mathfrak{E}_X)$ of $\mathfrak{Esp}^{\mathfrak{r}}_k$ the morphism $(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}}) \xrightarrow{\gamma_X} (X, \mathfrak{E}_X)$ (see 7.6.5). The adjunction morphism $Id_{\mathfrak{Esp}^{\mathfrak{e}}_k} \longrightarrow \mathfrak{J}_*\mathfrak{J}^*$ is the identity morphism.

Thus, $\mathfrak{Esp}^{\mathfrak{r}}_k \xrightarrow{\mathfrak{J}_*} \mathfrak{Esp}^{\mathfrak{e}}_k$ is a localization functor. According to 7.6.5, the functor $(\mathfrak{Esp}^{\mathfrak{r}}_k)^{op} \xrightarrow{K_0^{\mathfrak{r}}} \mathbb{Z}-mod$ factors through the localization functor

$$(\mathfrak{Esp}^{\mathfrak{r}}_k)^{op} \xrightarrow{\mathfrak{J}_*^{op}} (\mathfrak{Esp}^{\mathfrak{e}}_k)^{op}.$$

That is the functor $K^{\mathfrak{r}}_0$ is isomorphic to the composition $K^{\mathfrak{e}}_0 \circ \mathfrak{J}_*^{op}$, where $K^{\mathfrak{e}}_0$ denote the restriction of $K^{\mathfrak{r}}_0$ to the subcategory $(\mathfrak{Esp}^{\mathfrak{e}}_k)^{op}$, i.e. the composition $K^{\mathfrak{r}}_0 \circ \mathfrak{J}^{*^{op}}$.

For each exact k-space $(X, \mathfrak{E}_X)$, the group $K_0^{\mathfrak{e}}(X, \mathfrak{E}_X)$ coincides with the Grothendieck group K_0 of the exact category $(C_X, \mathfrak{E}_X)$ as it was defined by Quillen [Q].

7.6.7. Proposition. *The restriction $\mathfrak{L}^{\mathfrak{e}}$ of the left exact structure $\mathfrak{L}^{\mathfrak{r}}$ on $\mathfrak{Esp}_k^{\mathfrak{r}}$ to the subcategory $\mathfrak{Esp}_k^{\mathfrak{e}}$ is a left exact structure on $\mathfrak{Esp}_k^{\mathfrak{e}}$.*

Proof. The inclusion functor $\mathfrak{Esp}_k^{\mathfrak{e}} \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}_k^{\mathfrak{r}}$ preserves all colimits; in particular, it preserves cocartesian squares. The latter implies that $\mathfrak{L}^{\mathfrak{e}} = \mathfrak{J}^{*^{-1}}(\mathfrak{L}^{\mathfrak{r}})$ is a left exact structure on $\mathfrak{Esp}_k^{\mathfrak{r}}$. ∎

In particular, we have a universal ∂^*-functor $K_\bullet^{\mathfrak{e}} = (K_i^{\mathfrak{e}}, \mathfrak{d}_i^{\mathfrak{e}} \mid i \geq 0)$ from $(\mathfrak{Esp}_k^{\mathfrak{e}}, \mathfrak{L}^{\mathfrak{e}})^{op}$ to $\mathbb{Z} - mod$.

7.6.8. Remarks on K-theory of k-linear exact categories. The category $\mathfrak{Esp}_k^{\mathfrak{e}}$ of exact k-spaces has an automorphism $\mathfrak{D}$ which assigns to each 'space' $(X, \mathfrak{E}_X)$ the dual 'space' $(X, \mathfrak{E}_X)^o$ represented by the opposite exact category $(C_X, \mathfrak{E}_X)^{op}$.

7.6.8.1. Proposition. *Let F be a contravariant functor from the category $\mathfrak{Esp}_k^{\mathfrak{e}}$ of exact k-'spaces' to a category $C_{\mathcal{Z}}$ with filtered limits. If for each 'space' $(X, \mathfrak{E}_X)$, there is an isomorphism $F(X, \mathfrak{E}_X) \simeq F((X, \mathfrak{E}_X)^o)$ functorial in $(X, \mathfrak{E}_X)$, then the universal ∂^*-functor $S_\bullet^{\mathfrak{L}^{\mathfrak{e}}} F$ is isomorphic to its composition with the duality automorphism $\mathfrak{D}$ of the category $\mathfrak{Esp}_k^{\mathfrak{e}}$.*

Proof. The argument is left to the reader. ∎

7.6.8.2. Corollary. *There is a natural isomorphism of universal ∂^*-functors*

$$K_\bullet^{\mathfrak{e}} \simeq K_\bullet^{\mathfrak{e}} \circ \mathfrak{D}.$$

Proof. In fact, $K_0(X, \mathfrak{E}_X)$ is naturally isomorphic to $K_0((X, \mathfrak{E}_X)^o)$, because the (identical) isomorphism $ObC_X \xrightarrow{\sim} Ob(C_X^{op})$ implies a canonical isomorphism $\mathbb{Z}|C_X| \simeq \mathbb{Z}|C_X^{op}|$, relations defining K_0 correspond to conflations, and the dualization functor $\mathfrak{D}$ induces an isomorphism between the corresponding categories of conflations. ∎

7.7. Digression: non-additive exact categories.

7.7.1. Definition. We call a right exact category $(C_X, \mathfrak{E}_X)$ (and the corresponding right exact 'space' $(X, \mathfrak{E}_X)$) an *exact* category (resp. an *exact* 'space'), if the Yoneda embedding induces an equivalence of $(C_X, \mathfrak{E}_X)$ with a fully exact subcategory of the right exact category $(C_{X_{\mathfrak{E}}}, \mathfrak{E}^{\mathfrak{st}})$ of sheaves on $(C_X, \mathfrak{E}_X)$.

Let $\mathfrak{Esp}_{\mathfrak{e}}$ denote the full subcategory of the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces' generated by exact 'spaces'.

7.7.2. Proposition. *The inclusion functor $\mathfrak{Esp}_{\mathfrak{e}} \xrightarrow{\mathfrak{J}^*} \mathfrak{Esp}_{\mathfrak{r}}$ has a right adjoint.*

Proof. This right adjoint, $\mathfrak{J}_*$, assigns to each right exact 'space' $(X, \mathfrak{E}_X)$ the 'space' $(X_{\mathfrak{e}}, \mathfrak{E}_{X_{\mathfrak{e}}})$, where $C_{X_{\mathfrak{e}}}$ is the smallest fully exact subcategory of the right exact category of sheaves on $(C_X, \mathfrak{E}_X)$ endowed with the induced right exact structure. ∎

7.8. Reduction by resolution.

7.8.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and C_Y its fully exact subcategory such that*

(a) If $M' \longrightarrow M \longrightarrow M''$ is a conflation with $M \in ObC_Y$, then $M' \in ObC_Y$.

(b) For any $M'' \in ObC_X$, there exists a deflation $M \longrightarrow M''$ with $M \in ObC_Y$.

Then the morphism $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.

Proof. Let $(C_Z, \mathfrak{E}_Z) \xrightarrow{q^*} (C_Y, \mathfrak{E}_Y)$ be an 'exact' functor. Then we have a commutative diagram

$$\begin{array}{ccc} C_Z & \xrightarrow{\widetilde{\mathfrak{j}^*}} & C_{\mathcal{Z}} \\ {\scriptstyle q^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}^*} \\ C_Y & \longrightarrow & C_X \end{array} \tag{1}$$

where $C_Y \xrightarrow{\mathfrak{j}^*} C_X$ is the inclusion functor and the category $C_{\mathcal{Z}}$ is defined as follows. Its objects are triples $(M'', M; \mathfrak{t})$, where $M'' \in ObC_X$, $M \in ObC_Z$, and $\mathfrak{t}$ is a deflation $\mathfrak{j}^*q^*(M) \longrightarrow M''$). Morphisms are defined naturally. The functor $\widetilde{\mathfrak{j}^*}$ maps each object L of C_Z to $(\mathfrak{j}^*q^*(L), L; id)$ and acts correspondingly on morphisms. The functor $\mathfrak{p}^*$ is the projection $(M'', M; \mathfrak{t}) \longmapsto M''$. The right exact structures on C_X and C_Z induce a right exact structure $\mathfrak{E}_{\mathcal{Z}}$ on $C_{\mathcal{Z}}$ such that all functors of the diagram (1) become 'exact'.

It follows from the condition (b) that if the functor q^* is *essentially* surjective on objects, then the functor $C_{\mathcal{Z}} \xrightarrow{\mathfrak{p}^*} C_X$ has the same property. If $\mathfrak{q}^*$ is an inverse image functor of a morphism of $\mathfrak{L}^\star_{\mathfrak{es}}$, then same holds for $\mathfrak{p}^*$.

It follows from the conditions (a) and (b) that the map $K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(\mathfrak{j})} K_0(X, \mathfrak{E}_X)$ is surjective. In fact, let $N \longrightarrow M \longrightarrow L$ be conflation in C_X. Thanks to the condition (b), it can be inserted into a commutative diagram

$$\begin{array}{ccccc} \mathcal{M} & \longrightarrow & M' & \longrightarrow & \mathcal{L} \\ \downarrow & & cart & & \downarrow \\ M & \longrightarrow & & & L \end{array}$$

where all arrows are deflations, the square is cartesian, and $\mathcal{L}$, $\mathcal{M}$ are objects of the subcategory C_Y. Therefore, we obtain a commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{N}} & \longrightarrow & \widetilde{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{L}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & M & \longrightarrow & L \end{array} \tag{2}$$

whose rows and columns are conflations. Therefore,

$$[M] - [L] - [N] = ([\mathcal{M}] - [\mathcal{L}] - [\mathcal{M}]) - ([\widetilde{\mathcal{M}}] - [\widetilde{\mathcal{L}}] - [\widetilde{\mathcal{N}}]).$$

It follows from the condition (a) (applied to the columns of the diagram (2)) that two upper rows of (2) are conflations in $(C_Y, \mathfrak{E}_Y)$. Since the kernel of the map $K_0(\mathrm{j})$ consists of combinations (with coefficients in $\mathbb{Z}$) of the expressions $[M]-[L]-[N]$, where $N \longrightarrow M \longrightarrow L$ runs through conflations of $(C_X, \mathfrak{E}_X)$, it follows that these combinations are equal to zero. ∎

7.8.1.1. Note. The first part of the argument of 7.8.1 shows that if C_Y is a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ satisfying the condition (b) and F_0 is a functor from $\mathfrak{Esp}_{\mathfrak{r}}^{op}$ to a category with filtered limits such that $F_0(Y, \mathfrak{E}_Y) \longrightarrow F_0(X, \mathfrak{E}_X)$ is an isomorphism, then $S_-^n F_0(Y, \mathfrak{E}_Y) \longrightarrow S_-^n F_0(X, \mathfrak{E}_X)$ is an isomorphism for all $n \geq 0$.

The condition (a) was used only in the proof that $K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(X, \mathfrak{E}_X)$ is an isomorphism.

7.8.2. Proposition. *Let $(C_X, \mathfrak{E}_X)$ and $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$ be right exact categories with initial objects and $T = (T_i, \mathfrak{d}_i \mid i \geq 0)$ an 'exact' ∂^*-functor from $(C_X, \mathfrak{E}_X)$ to $(C_{\mathcal{Z}}, \mathfrak{E}_{\mathcal{Z}})$. Let C_Y be the full subcategory of C_X generated by T-acyclic objects (that is objects V such that $T_i(V)$ is an initial object of $C_{\mathcal{Z}}$ for $i \geq 1$). Assume that for every $M \in ObC_X$, there is a deflation $P \longrightarrow M$ with $P \in ObC_Y$, and that $T_n(M)$ is an initial object of $C_{\mathcal{Z}}$ for n sufficiently large. Then the natural map $K_\bullet(Y, \mathfrak{E}_Y) \longrightarrow K_\bullet(X, \mathfrak{E}_X)$ is an isomorphism.*

Proof. Let C_{Y_n} denote the full subcategory of the category C_X generated by all objects M such that $T_i(M)$ is an initial object of $C_{\mathcal{Z}}$ for $i \geq n$.

(i) All the subcategories C_{Y_n} are fully exact.

Indeed, if $N \longrightarrow M \longrightarrow L$ is a conflation in $(C_X, \mathfrak{E}_X)$ such that N and L are objects of the subcategory C_{Y_n}, then, thanks to the 'exactness' of the ∂^*-functor T, we have an exact sequence

$$\ldots \longrightarrow T_{m+1}(L) \longrightarrow T_m(N) \longrightarrow T_m(M) \longrightarrow T_m(L) \longrightarrow \ldots$$

If $m \geq n$, then the objects $T_m(N)$ and $T_m(L)$ are initial. Since the kernel of a morphism of an object $\mathcal{M}$ to an initial object is isomorphic to $\mathcal{M}$, it follows that $T_m(M)$ is an initial object.

(ii) Let $N \longrightarrow M \longrightarrow L$ be a conflation in $(C_X, \mathfrak{E}_X)$ such that $M \in ObC_{Y_n}$ and $L \in ObC_{Y_{n+1}}$. Then N is an object of C_{Y_n}.

In fact, we have an 'exact' sequence

which yields the 'exact' sequence $z \longrightarrow T_m(N) \longrightarrow z$ for all $m \geq n$, where z is an initial object of the category $C_{\mathcal{Z}}$. Therefore, $T_m(N)$ is an initial object for $m \geq n$.

(iii) This shows that the subcategory C_{Y_n} of the right exact category $(C_{Y_{n+1}}, \mathfrak{E}_{Y_{n+1}})$ satisfies the condition (a) of 7.8.1. The condition (b) of 7.8.1 holds, because $C_Y = C_{Y_1} \subseteq C_{Y_n}$ and, by hypothesis, for every $M \in ObC_X$, there exists a deflation $P \longrightarrow M$ with $P \in ObC_Y$. Applying 7.8.1, we obtain that the natural map $K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \longrightarrow K_\bullet(Y_{n+1}, \mathfrak{E}_{Y_{n+1}})$ is an isomorphism for all $n \geq 1$. Since, by hypothesis, $C_X = \bigcup_{n \geq 1} C_{Y_n}$, the isomorphisms $K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \xrightarrow{\sim} K_\bullet(Y_{n+1}, \mathfrak{E}_{Y_{n+1}})$ imply that the natural map $K_0(Y, \mathfrak{E}_Y) \longrightarrow K_0(X, \mathfrak{E}_X)$ is an isomorphism. ∎

7.8.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects; and let*

$$\begin{array}{ccccc} Ker(f') & \xrightarrow{\beta'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\ {\scriptstyle \mathfrak{k}'}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\ Ker(\alpha_1) & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \end{array} \qquad (3)$$

be a commutative diagram (determined by its lower right square) such that $Ker(\mathfrak{k}'')$ and $Ker(\beta_2)$ are trivial. Then

(a) The upper row of (3) is 'exact', and the morphism β'_1 is the kernel of α'_1.

(b) Suppose, in addition, that the arrows f', α_1 and α_2 in (3) are deflations and $(C_X, \mathfrak{E}_X)$ has the following property:

(#) If $M \xrightarrow{\mathfrak{e}} N$ is a deflation and $M \xrightarrow{p} M$ an idempotent morphism (i.e. $p^2 = p$) which has a kernel and such that the composition $\mathfrak{e} \circ p$ is a trivial morphism, then the composition of the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and $M \xrightarrow{\mathfrak{e}} N$ is a deflation.

Then the upper row of (3) is a conflation.

Proof. (a) It follows from C1.5.1 that the upper row of (3) is 'exact'. It follows from the argument of C1.5.1 that the morphism $Ker(f') \xrightarrow{\beta'_1} Ker(f)$ is the kernel morphism of $Ker(f) \xrightarrow{\alpha'_1} Ker(f'')$.

(b) The following argument is an appropriate modification of the proof the 'snake' lemma C1.5.2.

(b1) We have a commutative diagram

$$\begin{array}{ccccccc} & & \widetilde{A}_1 & \xrightarrow{id} & Ker(f''\alpha_1) & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\ Ker(\alpha_1) & \xrightarrow{\psi'} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\ {\scriptstyle id}\downarrow & & {\scriptstyle h}\downarrow & cart & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ Ker(\alpha_1) & \xrightarrow{f'} & Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \end{array} \qquad (4)$$

with cartesian squares as indicated. It follows (from the left lower cartesian square of (4)) that $Ker(h)$ is naturally isomorphic to $Ker(f)$.

(b2) Since the upper right square of (4) is cartesian, we have a commutative diagram

$$\begin{array}{ccccc} Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(f''\alpha_1)=\widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\ id\downarrow & & \widetilde{\mathfrak{k}}''\downarrow & cart & \downarrow\mathfrak{k}'' \\ Ker(\alpha_1) & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\ f'\downarrow & & f\downarrow & & \downarrow f'' \\ Ker(\alpha_2) & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \end{array} \qquad (5)$$

(b3) Since $Ker(\alpha_1) \xrightarrow{f'} Ker(\alpha_2)$ is a deflation, there exists a cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\gamma} & \widetilde{A}_1 \\ \mathfrak{p}\downarrow & cart & \downarrow h \\ Ker(\alpha_1) & \xrightarrow{f'} & Ker(\alpha_2) \end{array} \qquad (6)$$

whose upper horizontal arrow, γ, is also a deflation.

The commutative diagram (5) shows, among other things, that the arrow f' factors through h (see the diagram (4)), there exists a splitting, $Ker(\alpha_1) \xrightarrow{\mathfrak{s}} \mathcal{M}$, of the morphism $\mathfrak{p}$. Set $p = \mathfrak{s}\circ\mathfrak{p}$. The morphism $\mathcal{M} \xrightarrow{p} \mathcal{M}$ is an idempotent which has the same kernel as $\mathfrak{p}$, because $\mathfrak{s}$ is a monomorphism.

(b4) Let $\mathcal{M} \xrightarrow{\varphi} Ker(f'')$ denote the composition of the deflations $\mathcal{M} \xrightarrow{\gamma} \widetilde{A}_1$ and $\widetilde{A}_1 \xrightarrow{\widetilde{\alpha}_1} Ker(f'')$. The composition $\varphi\circ p$ is trivial.

In fact, $\varphi\circ p = \widetilde{\alpha}_1\circ\gamma\circ\mathfrak{s}\circ\mathfrak{p}$, and, by the origin of the morphism $\mathfrak{s}$, the composition $\gamma\circ\mathfrak{s}$ coincides with $\mathfrak{k}(\widetilde{\alpha}_1)$; so that $\varphi\circ p = (\widetilde{\alpha}_1\circ\mathfrak{k}(\widetilde{\alpha}_1))\circ\mathfrak{p}$ which shows the triviality of $\varphi\circ p$.

(b5) Suppose that the condition (#) holds. Then the triviality of $\varphi\circ p$ implies that the composition φ with the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} \mathcal{M}$ is a deflation. It follows from the commutative diagram

$$\begin{array}{ccccc} Ker(p) & \xrightarrow{id} & Ker(h) & \xrightarrow{\sim} & Ker(f) \\ \downarrow & & \downarrow & & \downarrow\alpha_1' \\ \mathcal{M} & \xrightarrow{\gamma} & \widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\ \mathfrak{p}\downarrow & cart & \downarrow h & & \\ Ker(\alpha_1) & \xrightarrow{f'} & Ker(\alpha_2) & & \end{array} \qquad (7)$$

that the composition of $Ker(p) \xrightarrow{\mathfrak{k}(p)} \mathcal{M}$ with $\mathcal{M} \xrightarrow{\varphi} Ker(f'')$ equals to the composition of $Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$ with an isomorphism $Ker(p) \xrightarrow{\sim} Ker(f)$. Therefore, the morphism $Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$ is a deflation. Together with (a) above, this means that the upper row of the diagram (3) is a conflation. ∎

7.8.4. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 7.8.3. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ which has the following properties:*

(a) If $N \longrightarrow M \longrightarrow L$ is a conflation in $(C_X, \mathfrak{E}_X)$ and N, M are objects of C_Y, then L belongs to C_Y too.

(b) For any deflation $M \longrightarrow \mathcal{L}$ with $\mathcal{L} \in ObC_Y$, there exist a deflation $\mathcal{M} \longrightarrow \mathcal{L}$ with $\mathcal{M} \in ObC_Y$ and a morphism $\mathcal{M} \longrightarrow M$ such that the diagram

$$\begin{array}{ccc} & & \mathcal{M} \\ & \swarrow & \searrow \\ M & \longrightarrow & \mathcal{L} \end{array}$$

commutes.

(c) If P, $\mathcal{M}$ are objects of C_Y and $P \longrightarrow x$ is a morphism to initial object, then $P \coprod \mathcal{M}$ exists (in C_X) and the sequence $P \longrightarrow P \coprod \mathcal{M} \longrightarrow \mathcal{M}$ (where the left arrow is the canonical coprojection and the right arrow corresponds to the $\mathcal{M} \xrightarrow{id} \mathcal{M}$ and the composition of $P \longrightarrow x \longrightarrow \mathcal{M}$) is a conflation.

Let C_{Y_n} be a full subcategory of C_X generated by all objects L having a C_Y-resolution of the length $\leq n$. And set $C_{Y_\infty} = \bigcup_{n \geq 0} C_{Y_n}$. Then C_{Y_n} is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$ for all $n \leq \infty$ and the natural morphisms

$$K_\bullet(Y, \mathfrak{E}_Y) \xrightarrow{\sim} K_\bullet(Y_1, \mathfrak{E}_{Y_1}) \xrightarrow{\sim} \ \ldots \ \xrightarrow{\sim} K_\bullet(Y_n, \mathfrak{E}_{Y_n}) \xrightarrow{\sim} \ K_\bullet(Y_\infty, \mathfrak{E}_{Y_\infty})$$

are isomorphisms for all $n \geq 0$.

Proof. Let $N \longrightarrow M \longrightarrow L$ be a conflation in $(C_X, \mathfrak{E}_X)$. Then for any integer $n \geq 0$, we have

(i) If $L \in ObC_{Y_{n+1}}$ and $M \in ObC_{Y_n}$, then $N \in ObC_{Y_n}$.

(ii) If N and L are objects of $C_{Y_{n+1}}$, then M is an object of $C_{Y_{n+1}}$.

(iii) If M and L are objects of $C_{Y_{n+1}}$, then N is an object of $C_{Y_{n+1}}$.

It suffices to prove the assertion for $n = 0$.

(i) Since $L \in ObC_{Y_1}$, there exists a conflation $P' \longrightarrow P \longrightarrow L$, where P and P' are objects of C_Y. Thus, we have a commutative diagram

$$\begin{array}{ccccc} x & \longrightarrow & P' & \xrightarrow{id} & P' \\ \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & \widetilde{P} & \longrightarrow & P \\ id \downarrow & & \downarrow & cart & \downarrow \\ N & \longrightarrow & M & \longrightarrow & L \end{array} \qquad (8)$$

whose rows and columns are conflations. Here x is an initial object of the category C_X. Since M and P' belong to C_Y and C_Y is a fully exact subcategory of $(C_X, \mathfrak{E}_X)$, in particular, it is closed under extensions, the object $\widetilde{P}$ belongs to C_Y. Since $\widetilde{P}$ and P are objects of C_Y, it follows from the condition (a) that $N \in ObC_Y$.

(ii) Since $L \in ObC_{Y_1}$, there exists a deflation $P \longrightarrow L$ with $P \in ObC_Y$. Applying (b) to the deflation $\widetilde{P} \longrightarrow P$ in (3), we obtain a deflation $\mathcal{M} \longrightarrow P$ such that $\mathcal{M} \in ObC_Y$ and the composition $\mathcal{M} \longrightarrow L$ factors through the deflation $M \longrightarrow L$ (see (8)). Since $N \in ObC_{Y_1}$, there exists a conflation $\widetilde{\mathcal{P}}' \longrightarrow \mathcal{P} \longrightarrow N$ where $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ are objects of C_Y. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{P}}' & \longrightarrow & \widetilde{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{P}}'' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P} & \longrightarrow & \mathcal{P} \coprod \mathcal{M} & \longrightarrow & \mathcal{M} \\
\downarrow & & \downarrow & & \downarrow \\
N & \longrightarrow & M & \longrightarrow & L
\end{array} \tag{9}$$

whose two lower rows and the left and the right columns are conflations. By 7.8.3(b), the upper row of (9) is a conflation too. Applying (i) to the right column of (9), we obtain that $\widetilde{\mathcal{P}}'' \in ObC_Y$. This implies that $\widetilde{\mathcal{M}} \in ObC_Y$, whence $M \in ObC_{Y_1}$.

(iii) Since $M \in ObC_{Y_1}$, there is a commutative diagram

$$\begin{array}{ccccc}
P' & \overset{id}{\longrightarrow} & P' & \overset{\lambda}{\longrightarrow} & x \\
\downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & \widetilde{P} & \longrightarrow & L \\
\downarrow & cart & \downarrow & & \downarrow id \\
N & \longrightarrow & M & \longrightarrow & L
\end{array} \tag{10}$$

whose rows and columns are conflations. Here x is an initial object of C_X and λ is a unique morphism $P' \longrightarrow x$ determined by the fact that $P' \longrightarrow K$ is the kernel of $K \longrightarrow N$. Since $L \in ObC_{Y_1}$, applying (i) to the middle row, we obtain that $K \in ObC_Y$. So, $N \in ObC_Y$. ∎

7.8.5. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects having the property (#) of 7.8.3. Let C_Y be a fully exact subcategory of a right exact category $(C_X, \mathfrak{E}_X)$ satisfying the conditions (a) and (c) of 7.8.4. Let $M' \longrightarrow M \longrightarrow M''$ be a conflation in $(C_X, \mathfrak{E}_X)$, and let $\mathcal{P}' \longrightarrow M'$, $\mathcal{P}'' \longrightarrow M''$ be C_Y-resolutions of the length $n \geq 1$. Suppose that resolution $\mathcal{P}'' \longrightarrow M'$ is projective. Then there exists a C_Y-resolution $\mathcal{P} \longrightarrow M$ of the length n such that $\mathcal{P}_i = \mathcal{P}'_i \coprod \mathcal{P}''_i$ for all $i \geq 1$ and the splitting 'exact' sequence $\mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}''$ is an 'exact' sequence of complexes.*

Proof. We have the diagram

$$\begin{array}{ccccc}
\mathcal{P}'_0 & & & & \mathcal{P}''_0 \\
\downarrow & & & & \downarrow \\
M' & \longrightarrow & M & \longrightarrow & M''
\end{array}$$

whose row is a conflation and vertical arrows are deflations. Since, by hypothesis, $\mathcal{P}''_0$ is a projective object of $(C_X, \mathfrak{E}_X)$ and $M \longrightarrow M''$ is a deflation, the right vertical arrow,

$\mathcal{P}''_0 \longrightarrow M''$, factors through $M \longrightarrow M''$. Therefore (like in the argument 7.8.4(ii)), we obtain a commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{e}') & \longrightarrow & Ker(\mathfrak{e}) & \longrightarrow & Ker(\mathfrak{e}'') \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}'_0 & \longrightarrow & \mathcal{P}'_0 \coprod \mathcal{P}''_0 & \longrightarrow & \mathcal{P}''_0 \\ {\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\ M' & \longrightarrow & M & \longrightarrow & M'' \end{array}$$

By 7.8.3(b), the upper row of this diagram is a conflation, which allows to repeat the step with the diagram

$$\begin{array}{ccccc} \mathcal{P}'_1 & & & & \mathcal{P}''_1 \\ \downarrow & & & & \downarrow \\ Ker(\mathfrak{e}') & \longrightarrow & Ker(\mathfrak{e}) & \longrightarrow & Ker(\mathfrak{e}'') \end{array}$$

whose vertical arrows are deflations; etc.. ■

7.9. Characteristic 'exact' filtrations and sequences.

7.9.1. The right exact 'spaces' $(X_n, \mathfrak{E}_{X_n})$**.** For a right exact exact 'space' $(X, \mathfrak{E}_X)$, let C_{X_n} be the category whose objects are sequences $M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_0$ of n morphisms of $\mathfrak{E}_X$, $n \geq 1$, and morphisms between sequences are commutative diagrams

$$\begin{array}{ccccccc} M_n & \longrightarrow & M_{n-1} & \longrightarrow & \ldots & \longrightarrow & M_0 \\ {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & \ldots & & \downarrow{\scriptstyle f_0} \\ M'_n & \longrightarrow & M'_{n-1} & \longrightarrow & \ldots & \longrightarrow & M'_0 \end{array}$$

Notice that if x is an initial object of the category C_X, then $x \longrightarrow \ldots \longrightarrow x$ is an initial object of C_{X_n}.

We denote by $\mathfrak{E}_{X_n}$ the class of all morphisms (f_i) of the category C_{X_n} such that $f_i \in \mathfrak{E}_X$ for all $0 \leq i \leq n$.

7.9.1.1. Proposition. *(a) The pair $(C_{X_n}, \mathfrak{E}_{X_n})$ is a right exact category.*

(b) The map which assigns to each right exact 'space' $(X, \mathfrak{E}_X)$ the right exact 'space' $(X_n, \mathfrak{E}_{X_n})$ extends naturally to an 'exact' endofunctor of the left exact category $(\mathfrak{Esp}_r, \mathfrak{L}_{\mathfrak{es}})$ of right 'exact' 'spaces' which induces an 'exact' endofunctor $\mathcal{P}_n$ of its exact subcategory $(\mathfrak{Esp}^*_r, \mathfrak{L}^*_{\mathfrak{es}})$.

Proof. The argument is left to the reader. ■

7.9.2. Proposition. (Additivity of 'characteristic' filtrations) *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and $f^*_n \xrightarrow{\mathfrak{t}_n} f^*_{n-1} \xrightarrow{\mathfrak{t}_{n-1}} \ldots \xrightarrow{\mathfrak{t}_1} f^*_0$ a sequence of deflations of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ such that the functors $\mathfrak{k}^*_i = Ker(\mathfrak{t}^*_i)$ are 'exact' for all $1 \leq i \leq n$. Then $K_\bullet(f_n) = K_\bullet(f_0) + \sum_{1 \leq i \leq n} K_\bullet(\mathfrak{k}_i)$.*

Proof. (a) For $1 \leq i \leq n$, let $\mathfrak{p}^*_{Y,i}$ denote the functor $C_{Y_n} \longrightarrow C_Y$ which assigns to every object $\mathcal{M} = (M_n \xrightarrow{\gamma_n} \ldots \xrightarrow{\gamma_1} M_0)$ of C_{Y_n} the object M_i and to every morphism

$f = (f_m)$ the morphism f_i. The assignment to any object $\mathcal{M} = (M_n \xrightarrow{\gamma_n} \dots \xrightarrow{\gamma_1} M_0)$ of C_{Y_n} of C_{Y_n} the deflation $M_i \xrightarrow{\gamma_i} M_{i-1}$ is a functor morphism $\mathfrak{p}^*_{Y,i} \xrightarrow{\mathfrak{t}_i^Y} \mathfrak{p}^*_{Y,i-1}$. Let $\mathfrak{k}^*_{Y,i}$ denote the kernel of $\mathfrak{t}_i^Y$, i.e. is the functor $C_{Y_n} \longrightarrow C_Y$ that assigns to an object $\mathcal{M} = (M_n \xrightarrow{\gamma_n} \dots \xrightarrow{\gamma_1} M_0)$ the kernel of $M_i \xrightarrow{\gamma_i} M_{i-1}$. Thus, we obtain a diagram

$$\begin{array}{ccccccccc} \mathfrak{k}^*_{Y,n} & & \mathfrak{k}^*_{Y,n-1} & & \cdots & & \mathfrak{k}^*_{Y,1} & & \\ \downarrow & & \downarrow & & \cdots & & \downarrow & & \\ \mathfrak{p}^*_{Y,n} & \xrightarrow{\mathfrak{t}_n^Y} & \mathfrak{p}^*_{Y,n-1} & \longrightarrow & \dots & \longrightarrow & \mathfrak{p}^*_{Y,1} & \xrightarrow{\mathfrak{t}_1^Y} & \mathfrak{p}^*_{Y,0} \end{array} \tag{1}$$

of functors from C_{Y_n} to C_Y whose horizontal arrows are deflations.

The functors $\mathfrak{p}^*_{Y,i-1}$ and $\mathfrak{k}^*_{Y,i}$ map initial objects to initial objects and pull-backs of deflations to pull-backs of deflations; i.e. they are inverse image functors of morphisms of the category $\mathfrak{Esp}^*_{\mathfrak{t}}$. These morphisms depend functorially on the right exact 'space' $(Y, \mathfrak{E}_Y)$, that is they form functor morphisms

$$\mathcal{P}_n \xrightarrow{\mathfrak{p}_i} Id_{\mathfrak{Esp}^*_{\mathfrak{t}}}, \quad 0 \leq i \leq n, \quad \text{and} \quad \mathcal{P}_n \xrightarrow{\mathfrak{k}_i} Id_{\mathfrak{Esp}^*_{\mathfrak{t}}}, \quad 1 \leq i \leq n.$$

These morphisms induce morphisms

$$K_\bullet \xleftarrow{K_\bullet(\mathfrak{k}_i)} K_\bullet \circ \mathcal{P}_n \xrightarrow{K_\bullet(\mathfrak{p}_i)} K_\bullet$$

of ∂^*-functors. The claim is that the morphism $K_\bullet(\mathfrak{p}_n)$ coincides with the morphism $K_\bullet(\mathfrak{p}_0) + \sum_{1 \leq i \leq n} K_\bullet(\mathfrak{k}_i)$.

In fact, the zero components of these morphisms coincide. Since $K_\bullet$ is a universal ∂^*-functor, this implies that the entire morphisms coincide with each other.

(b) The argument above proves, in a functorial way, the assertion 7.9.2 for the special case of the sequence of deflations $\mathfrak{p}^*_{Y,n} \xrightarrow{\mathfrak{t}_n^Y} \mathfrak{p}^*_{Y,n-1} \longrightarrow \dots \longrightarrow \mathfrak{p}^*_{Y,1} \xrightarrow{\mathfrak{t}_1^Y} \mathfrak{p}^*_{Y,0}$. of 'exact' functors from C_{Y_n} to C_Y. That is

$$K_\bullet(\mathfrak{p}_{Y,n}) = K_\bullet(\mathfrak{p}_{Y,0}) + \sum_{1 \leq i \leq n} K_\bullet(\mathfrak{k}_{Y,i}). \tag{2}$$

Consider now the general case.

A sequence of deflations $f_n^* \xrightarrow{\mathfrak{t}_n} f_{n-1}^* \xrightarrow{\mathfrak{t}_{n-1}} \dots \xrightarrow{\mathfrak{t}_1} f_0^*$ of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ defines an 'exact' functor $(C_X, \mathfrak{E}_X) \xrightarrow{\widetilde{\mathfrak{f}}_n^*} (C_{Y_n}, \mathfrak{E}_{Y_n})$. The kernels $\mathfrak{k}_i^* = Ker(\mathfrak{t}_i)$ map initial objects to initial objects. The fact that they are 'exact' (which is equivalent to the condition that they map deflations to deflations) means that they are inverse image functors of morphisms of $\mathfrak{Esp}^*_{\mathfrak{t}}$, hence the morphisms $K_\bullet(\mathfrak{k}_i)$ are well

defined. Therefore, the morphism $K_\bullet(f_0) + \sum_{1\le i\le n} K_\bullet(\mathfrak{k}_i)$ from $K_\bullet(X, \mathfrak{E}_X)$ to $K_\bullet(Y, \mathfrak{E}_Y)$ is well defined. One can see that

$$K_\bullet(f_n) = K_\bullet(\mathfrak{p}_{Y,n}) \circ K_\bullet(\widetilde{\mathfrak{f}}_n) \quad \text{and}$$
$$K_\bullet(f_0) + \sum_{1\le i\le n} K_\bullet(\mathfrak{k}_i) = \big(K_\bullet(\mathfrak{p}_{Y,0}) + \sum_{1\le i\le n} K_\bullet(\mathfrak{k}_{Y,i})\big) \circ K_\bullet(\widetilde{\mathfrak{f}}_n)$$

So that the assertion follows from the equality (2). ∎

7.9.3. Corollary. *Let $(C_X, \mathfrak{E}_X)$ and $(C_Y, \mathfrak{E}_Y)$ be right exact categories with initial objects and $g^* \longrightarrow f^* \longrightarrow h^*$ a conflation of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$. Then $K_\bullet(f) = K_\bullet(g) + K_\bullet(h)$.*

7.9.4. Corollary. (Additivity for 'characteristic' 'exact' sequences) *Let*

$$\mathfrak{f}_n^* \longrightarrow \mathfrak{f}_{n-1}^* \longrightarrow \ldots \longrightarrow \mathfrak{f}_1^* \longrightarrow \mathfrak{f}_0^*$$

be an 'exact' sequence of 'exact' functors from $(C_X, \mathfrak{E}_X)$ to $(C_Y, \mathfrak{E}_Y)$ which map initial objects to initial objects. Suppose that $\mathfrak{f}_1^* \longrightarrow \mathfrak{f}_0^*$ is a deflation and $\mathfrak{f}_n^* \longrightarrow \mathfrak{f}_{n-1}^*$ is the kernel of $\mathfrak{f}_{n-1}^* \longrightarrow \mathfrak{f}_{n-2}^*$. Then the morphism $\sum_{0\le i\le n} (-1)^i K_\bullet(\mathfrak{f}_i)$ from $K_\bullet(X, \mathfrak{E}_X)$ to $K_\bullet(Y, \mathfrak{E}_Y)$ is equal to zero.

Proof. The assertion follows from 7.9.3 by induction.

A more conceptual proof goes along the lines of the argument of 7.9.2. Namely, we assign to each right exact category $(C_Y, \mathfrak{E}_Y)$ the right exact category $(C_{Y_n^{\mathfrak{e}}}, \mathfrak{E}_{Y_n^{\mathfrak{e}}})$ whose objects are 'exact' sequences $\mathcal{L} = (L_n \longrightarrow L_{n-1} \longrightarrow \ldots \longrightarrow L_1 \longrightarrow L_0)$, where $L_1 \longrightarrow L_0$ is a deflation and $L_n \longrightarrow L_{n-1}$ is the kernel of $L_{n-1} \longrightarrow L_{n-2}$. This assignment defines an endofunctor $\mathfrak{P}_n^{\mathfrak{e}}$ of the category $\mathfrak{Esp}_{\mathfrak{t}}^*$ of right exact 'spaces' with initial objects, and maps $\mathcal{L} \longmapsto L_i$ determine morphisms $\mathfrak{P}_n^{\mathfrak{e}} \longrightarrow Id_{\mathfrak{Esp}_{\mathfrak{t}}^*}$. The rest of the argument is left to the reader. ∎

7.10. Complements.

7.10.1. Another description of the functor K_0. Fix a right exact category $(C_X, \mathfrak{E}_X)$. Let $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ denote the category having the same objects as C_X and with morphisms defined as follows. For any pair M, L of objects, consider all diagrams (if any) of the form $M \xleftarrow{\mathfrak{e}} \widetilde{M} \xrightarrow{f} L$, where $\mathfrak{e}$ is a deflation and f an arbitrary morphism of C_X. We consider isomorphisms between such diagrams of the form (id_M, ϕ, id_L) and define morphisms from M to L as isomorphism classes of these diagrams. The composition of the morphisms $N \xleftarrow{\mathfrak{t}} \widetilde{N} \xrightarrow{g} M$ and $M \xleftarrow{\mathfrak{e}} \widetilde{M} \xrightarrow{f} L$ is the morphism represented by the pair $(\mathfrak{t} \circ \widetilde{\mathfrak{e}}, f \circ g')$ in the diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{N}} & \xrightarrow{g'} & \widetilde{M} & \xrightarrow{f} & L \\
{\scriptstyle\widetilde{\mathfrak{e}}}\downarrow & cart & \downarrow{\scriptstyle\mathfrak{e}} & & \\
\widetilde{N} & \xrightarrow{g} & M & & \\
{\scriptstyle\mathfrak{t}}\downarrow & & & & \\
M & & & &
\end{array}$$

with cartesian square. If the category C_X is svelte (i.e. it represents a 'space'), then $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ is a well defined svelte category.

There is a canonical functor $C_X \xrightarrow{\mathfrak{r}_X^*} C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ which is identical on objects and maps each morphism $M \xrightarrow{f} L$ to the morphism represented by the diagram $M \xleftarrow{id} M \xrightarrow{f} L$.

Let $C_{|\mathfrak{E}_X|}$ denote the subcategory of C_X formed by all deflations. The map which assigns to every morphism $M \xrightarrow{\mathfrak{e}} N$ of $\mathfrak{E}_X$ the morphism of $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ represented by the diagram $N \xleftarrow{\mathfrak{e}} M \xrightarrow{id} M$ is a functor $C^{op}_{|\mathfrak{E}_X|} \xrightarrow{\mathfrak{l}_X^*} C_{\mathfrak{L}(X,\mathfrak{E}_X)}$.

Let $\mathcal{G}(X)$ denote the group $\mathbb{Z}|C_X|$ which is identified with the corresponding groupoid with one object. Let $\mathfrak{p}_X$ denote the map $HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} \longrightarrow \mathcal{G}(X)$ which assigns to a morphism $[N \xleftarrow{\mathfrak{e}} M \xrightarrow{f} L]$ represented by the diagram $N \xleftarrow{\mathfrak{e}} M \xrightarrow{f} L$ the element $[M]-[N]$ of the group $\mathcal{G}(X)$. We have a (non-commutative) diagram

$$\begin{array}{ccc} Hom^2C_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xrightarrow{\mathfrak{p}_X\times\mathfrak{p}_X} & \mathcal{G}(X)\times\mathcal{G}(X) \\ \mathfrak{c}\downarrow & & \downarrow + \\ HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xrightarrow{\mathfrak{p}_X} & \mathcal{G}(X) \end{array} \tag{1}$$

where Hom^2C_Z stands for the class of composable morphisms of the category C_Z and the vertical arrows are compositions. Taking the compositions in the diagram (1), we obtain a pair of arrows

$$Hom^2C_{\mathfrak{L}(X,\mathfrak{E}_X)} \underset{\mathfrak{v}_X}{\overset{\mathfrak{u}_X}{\rightrightarrows}} \mathcal{G}(X). \tag{2}$$

7.10.1.1. Proposition. *The cokernel of the pair (2) is (isomorphic to) the group $K_0(X,\mathfrak{E}_X)$ defined in 7.1.*

Proof. The fact follows from the definitions. ∎

7.10.1.2. Note. The map $HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} \xrightarrow{\mathfrak{p}_X} \mathcal{G}(X)$ is the composition of the map $HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} \xrightarrow{\pi_X} \mathfrak{E}_X$ and the map $\mathfrak{E}_X \xrightarrow{\lambda_X} \mathcal{G}(X)$ which assigns to each deflation $M \longrightarrow L$ the element $[M]-[L]$ of $\mathcal{G}(X)$. One can see that $\pi_X \circ \mathfrak{l}_X^*$ is the identical map, and the map λ_X is a functor $C^{op}_{|\mathfrak{E}_X|} \longrightarrow \mathcal{G}(X)$.

7.10.1.3. Functorialities. Any 'exact' functor $(C_X,\mathfrak{E}_X) \xrightarrow{\mathfrak{f}^*} (C_Y,\mathfrak{E}_Y)$ between right exact categories induces a functor $C_{\mathfrak{L}(X,\mathfrak{E}_X)} \xrightarrow{\mathfrak{L}(\mathfrak{f})^*} C_{\mathfrak{L}(Y,\mathfrak{E}_Y)}$ such that the diagram

$$\begin{array}{ccccccccc} K_0(X) & \longleftarrow & \mathcal{G}(X) & \xleftarrow{\lambda_X} & C^{op}_{|\mathfrak{E}_X|} & \xrightarrow{\mathfrak{l}_X^*} & C_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xleftarrow{\mathfrak{r}_X^*} & C_X \\ K_0(\mathfrak{f})\downarrow & & \mathcal{G}(\mathfrak{f})\downarrow & & \downarrow & & \mathfrak{L}(\mathfrak{f})^*\downarrow & & \downarrow \mathfrak{f}^* \\ K_0(Y) & \longleftarrow & \mathcal{G}(Y) & \xleftarrow{\lambda_Y} & C^{op}_{|\mathfrak{E}_Y|} & \xrightarrow{\mathfrak{l}_Y^*} & C_{\mathfrak{L}(Y,\mathfrak{E}_Y)} & \xleftarrow{\mathfrak{r}_Y^*} & C_Y \end{array} \tag{3}$$

commutes, as well as the diagram

$$\begin{array}{ccc} HomC_{\mathfrak{L}(X,\mathfrak{E}_X)} & \xrightarrow{\pi_X} & \mathfrak{E}_X \\ \mathfrak{f}^* \downarrow & & \downarrow \mathfrak{E}(\mathfrak{f})^* \\ HomC_{\mathfrak{L}(Y,\mathfrak{E}_Y)} & \xrightarrow{\pi_Y} & \mathfrak{E}_Y \end{array} \tag{4}$$

7.10.2. The Q-construction for right exact categories with initial objects. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $\mathfrak{I}_X$ the class of all inflations of $(C_X, \mathfrak{E}_X)$ (i.e. morphisms which are kernels of deflations) and by $\mathfrak{I}_X^\infty$ the smallest subcategory of C_X containing $\mathfrak{I}_X$.

We denote by $C_{\mathcal{Q}(X,\mathfrak{E}_X)}$ the subcategory of the category $C_{\mathfrak{L}(X,\mathfrak{E}_X)}$ formed by all morphisms $M \xleftarrow{\mathfrak{e}} \widetilde{M} \xrightarrow{\mathfrak{j}} L$, where ($\mathfrak{e}$ is a deflation and) $\mathfrak{j} \in \mathfrak{I}_X^\infty$.

7.10.2.1. Note. If $(C_X, \mathfrak{E}_X)$ is an exact k-linear category, then $\mathfrak{I}_X^\infty = \mathfrak{I}_X$ and the category $C_{\mathcal{Q}(X,\mathfrak{E}_X)}$ coincides with the Quillen's category $\mathcal{Q}C_X$ associated with the exact category $(C_X, \mathfrak{E}_X)$ (see [Q, p. 102]).

Let

$$Hom^2C_{\mathcal{Q}(X,\mathfrak{E}_X)} \underset{\mathfrak{b}_X}{\overset{\mathfrak{a}_X}{\rightrightarrows}} \mathcal{G}(X). \tag{1}$$

be the composition of the pair of maps 7.10.1(2) with the embedding

$$Hom^2C_{\mathcal{Q}(X,\mathfrak{E}_X)} \longrightarrow Hom^2C_{\mathfrak{L}(X,\mathfrak{E}_X)}.$$

7.10.2.2. Proposition. *The unique map $Cok(\mathfrak{a}_X, \mathfrak{b}_X) \longrightarrow K_0(X, \mathfrak{E}_X)$ making commute the diagram*

$$\begin{array}{ccccc} Hom^2C_{\mathcal{Q}(X,\mathfrak{E}_X)} & \underset{\mathfrak{b}_X}{\overset{\mathfrak{a}_X}{\rightrightarrows}} & \mathcal{G}(X) & \longrightarrow & Cok(\mathfrak{a}_X, \mathfrak{b}_X) \\ \downarrow & & id \downarrow & & \downarrow \\ Hom^2C_{\mathfrak{L}(X,\mathfrak{E}_X)} & \underset{\mathfrak{v}_X}{\overset{\mathfrak{u}_X}{\rightrightarrows}} & \mathcal{G}(X) & \longrightarrow & K_0(X, \mathfrak{E}_X) \end{array}$$

is a group isomorphism.

Proof. The assertion is a consequence of 7.1.6. ∎

8. Infinitesimal 'spaces'.

8.1. The Gabriel multiplication in right exact categories. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. Let $\mathbb{T}$ and $\mathbb{S}$ be subcategories of the category C_X. The *Gabriel product* $\mathbb{S}\bullet\mathbb{T}$ is the full subcategory of C_X whose objects M fit into a conflation $L \stackrel{g}{\longrightarrow} M \stackrel{h}{\longrightarrow} N$ such that $L \in Ob\mathbb{S}$ and $N \in Ob\mathbb{T}$.

8.1.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ of the category C_X, there is the inclusion*

$$\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D}) \subseteq (\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}.$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ be subcategories of C_X. Let M be an object of $\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D})$; i.e. there is a conflation $L \longrightarrow M \longrightarrow N$ such that $L \in Ob\mathcal{A}$ and $N \in Ob\mathcal{B} \bullet \mathcal{D}$. The latter means that there is a conflation $N_1 \longrightarrow N \longrightarrow N_2$ with $N_1 \in Ob\mathcal{B}$ and $N_2 \in Ob\mathcal{D}$. Thus, we have a commutative diagram

$$\begin{array}{ccccc} L & \longrightarrow & M_1 & \longrightarrow & N_1 \\ {\scriptstyle id}\downarrow & & \downarrow & {\scriptstyle cart} & \downarrow \\ L & \longrightarrow & M & \longrightarrow & N \\ & & \downarrow & & \downarrow \\ & & N_2 & \stackrel{id}{\longrightarrow} & N_2 \end{array}$$

whose two upper right square is cartesian, and two upper rows and two right columns are conflations. So, we have a conflation $M_1 \longrightarrow M \longrightarrow N_2$ with $N_2 \in Ob\mathcal{D}$ and $M_1 \in Ob\mathcal{A}\bullet\mathcal{B}$, hence M is an object of the subcategory $(\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}$. ∎

8.1.2. Corollary. *Let $(C_X, \mathfrak{E}_X)$ be an exact category. Then the Gabriel multiplication is associative.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ be subcategories of C_X. By 8.1.1, we have the inclusion $\mathcal{A} \bullet (\mathcal{B} \bullet \mathcal{D}) \subseteq (\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{D}$. The opposite inclusion holds by duality, because $(\mathcal{A} \bullet \mathcal{B})^{op} = \mathcal{B}^{op} \bullet \mathcal{A}^{op}$. ∎

8.2. The infinitesimal neighborhoods of a subcategory. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects. We denote by $\mathbb{O}_X$ the full subcategory of C_X generated by all initial objects of C_X. For any subcategory $\mathcal{B}$ of C_X, we define subcategories $\mathcal{B}^{(n)}$ and $\mathcal{B}_{(n)}$, $0 \leq n \leq \infty$, by setting $\mathcal{B}^{(0)} = \mathbb{O}_X = \mathcal{B}_{(0)}$, $\mathcal{B}^{(1)} = \mathcal{B} = \mathcal{B}_{(1)}$, and

$$\mathcal{B}^{(n)} = \mathcal{B}^{(n-1)} \bullet \mathcal{B} \quad \text{for} \quad 2 \leq n < \infty; \quad \text{and} \quad \mathcal{B}^{(\infty)} = \bigcup_{n \geq 1} \mathcal{B}^{(n)};$$

$$\mathcal{B}_{(n)} = \mathcal{B} \bullet \mathcal{B}_{(n-1)} \quad \text{for} \quad 2 \leq n < \infty; \quad \text{and} \quad \mathcal{B}_{(\infty)} = \bigcup_{n \geq 1} \mathcal{B}_{(n)}$$

It follows that $\mathcal{B}^{(n)} = \mathcal{B}_{(n)}$ for $n \leq 2$ and, by 8.1.1, $\mathcal{B}_{(n)} \subseteq \mathcal{B}^{(n)}$ for $3 \leq n \leq \infty$.

We call the subcategory $\mathcal{B}^{(n+1)}$ the *upper* n^{th} *infinitesimal neighborhood* of $\mathcal{B}$ and the subcategory $\mathcal{B}_{(n+1)}$ the *lower* n^{th} *infinitesimal neighborhood* of $\mathcal{B}$. It follows that $\mathcal{B}^{(n+1)}$ is the strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of arrows

$$M_0 \xrightarrow{\mathfrak{j}_1} M_1 \xrightarrow{\mathfrak{j}_2} \ldots \xrightarrow{\mathfrak{j}_n} M_n = M$$

with the property: $M_0 \in Ob\mathcal{B}$, and for each $n \geq i \geq 1$, there exists a deflation $M_i \xrightarrow{\mathfrak{e}_i} N_i$ with $N_i \in Ob\mathcal{B}$ such that $M_{i-1} \xrightarrow{\mathfrak{j}_i} M_i \xrightarrow{\mathfrak{e}_i} N_i$ is a conflation.

Similarly, $\mathcal{B}_{(n+1)}$ is a strictly full subcategory of C_X generated by all $M \in ObC_X$ such that there exists a sequence of deflations

$$M = M_n \xrightarrow{\mathfrak{e}_n} \ldots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$$

such that M_0 and $Ker(\mathfrak{e}_i)$ are objects of $\mathcal{B}$ for $1 \leq i \leq n$.

8.2.1. Note. It follows that $\mathcal{B}^{(n)} \subseteq \mathcal{B}^{(n+1)}$ for all $n \geq 0$, if $\mathcal{B}$ contains an initial object of the category C_X.

8.3. Fully exact subcategories of a right exact category. Fix a right exact category $(C_X, \mathcal{E}_X)$. A subcategory $\mathcal{A}$ of C_X is a *fully exact* subcategory of $(C_X, \mathcal{E}_X)$ if $\mathcal{A} \bullet \mathcal{A} = \mathcal{A}$.

8.3.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be a right exact category with initial objects. For any subcategory $\mathcal{B}$ of C_X, the subcategory $\mathcal{B}^{(\infty)}$ is the smallest fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$.*

Proof. Let $\mathcal{A}$ be a fully exact subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, i.e. $\mathcal{A} = \mathcal{A} \bullet \mathcal{A}$. Then $\mathcal{B}^{(\infty)} \subseteq \mathcal{A}$, iff $\mathcal{B}$ is a subcategory of $\mathcal{A}$.

On the other hand, it follows from 8.1.1 and the definition of the subcategories $\mathcal{B}^{(n)}$ (see 8.2) that $\mathcal{B}^{(n)} \bullet \mathcal{B}^{(m)} \subseteq \mathcal{B}^{(m+n)}$ for any nonnegative integers n and m. In particular, $\mathcal{B}^{(\infty)} = \mathcal{B}^{(\infty)} \bullet \mathcal{B}^{(\infty)}$, that is $\mathcal{B}^{(\infty)}$ is a fully exact subcategory of $(C_X, \mathcal{E}_X)$ containing $\mathcal{B}$. ∎

8.4. Cofiltrations. Fix a right exact category $(C_X, \mathfrak{E}_X)$ with initial objects. A *cofiltration of the length* n+1 of an object M is a sequence of deflations

$$M = M_n \xrightarrow{\mathfrak{e}_n} \ldots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0. \tag{1}$$

The cofiltration (1) is said to be *equivalent* to a cofiltration

$$M = \widetilde{M}_m \xrightarrow{\widetilde{\mathfrak{e}}_n} \ldots \xrightarrow{\widetilde{\mathfrak{e}}_2} \widetilde{M}_1 \xrightarrow{\widetilde{\mathfrak{e}}_1} \widetilde{M}_0$$

if $m = n$ and there exists a permutation σ of $\{0, \ldots, n\}$ such that $Ker(\mathfrak{e}_i) \simeq Ker(\widetilde{\mathfrak{e}}_{\sigma(i)})$ for $1 \leq i \leq n$ and $M_0 \simeq \widetilde{M}_0$.

The following assertion is a version (and a generalization) of Zassenhouse's lemma.

8.4.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ have the following property:*

(‡) *for any pair of deflations* $M_1 \xleftarrow{\mathfrak{t}_1} M \xrightarrow{\mathfrak{t}_2} M_2$*, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle \mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M \longrightarrow M_1 \times_{M_3} M_2$ *is a deflation.*

Then any two cofiltrations of an object M *have equivalent refinements.*

Proof. Let

$$\begin{array}{lllllll} M = M_n & \xrightarrow{\mathfrak{e}_n} & \dots & \xrightarrow{\mathfrak{e}_2} & M_1 & \xrightarrow{\mathfrak{e}_1} & M_0 \quad \text{and} \\ M = \widetilde{M}_m & \xrightarrow{\widetilde{\mathfrak{e}}_n} & \dots & \xrightarrow{\widetilde{\mathfrak{e}}_2} & \widetilde{M}_1 & \xrightarrow{\widetilde{\mathfrak{e}}_1} & \widetilde{M}_0 \end{array}$$

be cofiltrations. If $n = 0$, then the second cofiltration is a refinement of the first one.

(a) Suppose that $n = 1 = m$; that is we have a pair of deflations $\widetilde{M}_1 \xleftarrow{\widetilde{\mathfrak{e}}_1} M \xrightarrow{\mathfrak{e}_1} M_1$. Thanks to the property (‡), there exists a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}_1} & M_1 \\ {\scriptstyle \widetilde{\mathfrak{e}}_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_1} \\ \widetilde{M}_1 & \xrightarrow{\mathfrak{p}'_1} & N \end{array}$$

whose all arrows are deflations, and the unique arrow $M \xrightarrow{\mathfrak{e}_3} M_2 = M_1 \times_N \widetilde{M}_1$ is a deflation too. Since the right lower square in the commutative diagram

$$\begin{array}{ccccc} & & Ker(\widetilde{\mathfrak{e}}_2) & \xrightarrow{\sim} & Ker(\mathfrak{p}_1) \\ & & {\scriptstyle \widetilde{\mathfrak{k}}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}_1} \\ Ker(\mathfrak{e}_2) & \xrightarrow{\mathfrak{k}_2} & M_2 & \xrightarrow{\mathfrak{e}_2} & M_1 \\ \wr\downarrow & & {\scriptstyle \widetilde{\mathfrak{e}}_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{p}_1} \\ Ker(\mathfrak{p}'_1) & \xrightarrow{\mathfrak{k}'_1} & \widetilde{M}_1 & \xrightarrow{\mathfrak{p}'_1} & N \end{array}$$

is cartesian, its upper horizontal and left vertical arrows are isomorphisms. This shows that the cofiltrations

$$\begin{array}{lllllll} M & \xrightarrow{\mathfrak{e}_3} & M_2 & \xrightarrow{\mathfrak{e}_2} & M_1 & \xrightarrow{\mathfrak{p}_1} & N \quad \text{and} \\ M & \xrightarrow{\mathfrak{e}_3} & M_2 & \xrightarrow{\widetilde{\mathfrak{e}}_2} & \widetilde{M}_1 & \xrightarrow{\mathfrak{p}'_1} & N \end{array}$$

are equivalent to each other.

(b) Let $n > 1$ and $m = 1$. Then, applying (a) to the deflations $\widetilde{M}_0 \overset{\widetilde{\mathfrak{e}}_1}{\longleftarrow} M \overset{\mathfrak{e}_n}{\longrightarrow} M_{n-1}$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} M & \overset{\mathfrak{e}'}{\longrightarrow} & M' & \overset{\mathfrak{e}_n}{\longrightarrow} & M_{n-1} & \overset{\mathfrak{e}_n}{\longrightarrow} & M_{n-2} & \longrightarrow \ldots \longrightarrow & M_0 \\ & & \widetilde{\mathfrak{e}}_1 \downarrow & cart & \downarrow \mathfrak{p}_1 & & & & \\ & & \widetilde{M}_1 & \overset{\mathfrak{p}'_1}{\longrightarrow} & N & & & & \end{array}$$

which provides an induction argument.

(c) Finally, (b) provides the main induction step in the general case. Details are left to the reader. ∎

8.5. Semitopologizing, topologizing, and thick subcategories of a right exact category. Fix a right exact category $(C_X, \mathcal{E}_X)$ with initial objects.

8.5.1. Definitions. (a) We call a full subcategory $\mathcal{T}$ of the category C_X *semitopologizing* if the following condition holds:

If $M \overset{\mathfrak{e}}{\longrightarrow} L$ is an arrow of $\mathfrak{E}_X$ and $M \in Ob\mathcal{T}$, then L and $Ker(\mathfrak{e})$ are objects of $\mathcal{T}$.

(b) We call a semitopologizing subcategory $\mathcal{T}$ of the category C_X *topologizing* if it is a right exact subcategory of $(C_X, \mathfrak{E}_X)$, that is if

$$\begin{array}{ccc} \widetilde{N} & \overset{f'}{\longrightarrow} & M \\ \widetilde{\mathfrak{e}} \downarrow & & \downarrow \mathfrak{e} \\ N & \overset{f}{\longrightarrow} & L \end{array}$$

is a cartesian square in C_X and the objects M, L, and N belong to the subcategory $\mathcal{T}$, then $\widetilde{N}$ is an object of $\mathcal{T}$.

(c) We call a subcategory $\mathcal{T}$ of C_X a *thick* subcategory of $(C_X, \mathcal{E}_X)$ if it is topologizing and fully exact, i.e. $\mathcal{T} \bullet \mathcal{T} = \mathcal{T}$.

8.5.2. Proposition. *(a) Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial and final objects such that all morphisms to final objects are deflations. Than any topologizing subcategory of $(C_X, \mathfrak{E}_X)$ is closed under finite products.*

(b) If C_X is an abelian category and $\mathfrak{E}_X$ is the canonical exact structure on C_X, then topologizing subcategories of $(C_X, \mathfrak{E}_X)$ are topologizing subcategories of the abelian category C_X in the sense of Gabriel [Gab].

Proof. (a) Let $\mathcal{T}$ be a topologizing subcategory of $(C_X, \mathfrak{E}_X)$. Then, under the assumptions, it is a strictly full subcategory of C_X containing all final objects of C_X. Let M, N be objects of $\mathcal{T}$ and $x_\bullet$ a final object of C_X. By hypothesis, the unique morphisms $M \longrightarrow x_\bullet$ and $N \longrightarrow x_\bullet$ are deflations. Therefore, the cartesian square

$$\begin{array}{ccc} M \prod N & \overset{p_M}{\longrightarrow} & M \\ p_N \downarrow & & \downarrow \\ N & \longrightarrow & x_\bullet \end{array}$$

is contained in $\mathcal{T}$.

(b) If $(C_X, \mathfrak{E}_X)$ is an abelian category with the canonical structure, then it follows from (a) above that any topologizing subcategory $\mathcal{T}$ of $(C_X, \mathfrak{E}_X)$ is closed under finite (co)products, and if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence with $M \in Ob\mathcal{T}$, then M' and M'' are objects of $\mathcal{T}$. This means that $\mathcal{T}$ is a topologizing subcategory of the abelian category C_X in the sense of Gabriel. On the other hand any topologizing subcategory of C_X in the sense of Gabriel is closed under any finite limits and colimits (taken in C_X), in particular, it is closed under arbitrary pull-backs. ∎

8.5.3. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a k-linear additive right exact category such that all morphisms to zero objects are deflations.*

(a) Any topologizing subcategory of $(C_X, \mathfrak{E}_X)$ is closed under finite products.

(b) If $(C_X, \mathfrak{E}_X)$ is an exact category, then any topologizing subcategory of $(C_X, \mathfrak{E}_X)$ is an exact (sub)category.

Proof. (a) This follows from 8.5.2(a).

(b) Fix a topologizing subcategory $\mathcal{T}$ of an exact k-linear category $(C_X, \mathfrak{E}_X)$. Let $M \xrightarrow{\mathfrak{j}} M' \xrightarrow{\mathfrak{e}} M''$ be a conflation in $\mathcal{T}$ and $M \xrightarrow{f} L$ an arbitrary morphism of $\mathcal{T}$. Since $(C_X, \mathfrak{E}_X)$ is an exact category, there is cocartesian square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{j}} & M' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ N & \xrightarrow{\widetilde{\mathfrak{j}}} & N' \end{array} \tag{1}$$

whose horizontal arrows are inflations. Notice that the pair of morphisms

$$M \xrightarrow{(f,\mathfrak{j})} N \times M' = N \oplus M' \xrightarrow{\widetilde{\mathfrak{j}}+f'} N' \tag{2}$$

is a conflation. In fact, the Gabriel-Quillen embedding is 'exact', hence it sends the cocartesian square (1) to a cocartesian square of the abelian category of sheaves of k-modules on $(C_X, \mathfrak{E}_X)$. And for abelian categories the fact is easy to check. Since the Gabriel-Quillen embedding reflects conflations, it follows that (2) is a conflation.

By (a) above, $N \oplus M' \in Ob\mathcal{T}$, because N and M' are objects of $\mathcal{T}$. Therefore, the object N' belongs to $\mathcal{T}$. ∎

8.5.4. Proposition. *Let $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{f}} (Y, \mathfrak{E}_Y)$ be a morphism of the category $\mathfrak{Esp}^*_{\mathfrak{t}}$. If $\mathcal{T}$ is a semitopologizing (resp. topologizing, resp. thick) subcategory of the right exact category $(C_X, \mathfrak{E}_X)$, then $\mathfrak{f}^{*^{-1}}(\mathcal{T})$ is a semitopologizing (resp. topologizing, resp. thick) subcategory of $(C_Y, \mathfrak{E}_Y)$.*

Proof. By the definition of morphisms of $\mathfrak{Esp}^*_{\mathfrak{t}}$, the inverse image functor $\mathfrak{f}^*$ is an 'exact' (that is preserving pull-backs of deflations) functor from $(C_Y, \mathfrak{E}_Y)$ to $(C_X, \mathfrak{E}_X)$ which maps initial objects to initial objects. The assertion follows from definitions. ∎

8.5.5. Proposition. *Let*

$$\begin{array}{ccc} (Z,\mathfrak{E}_Z) & \xrightarrow{g} & (Y,\mathfrak{E}_Y) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_1} \\ (X,\mathfrak{E}_X) & \xrightarrow{p_2} & (\mathfrak{X},\mathfrak{E}_{\mathfrak{X}}) \end{array}$$

be a cocartesian square in the category $\mathfrak{Esp}^*_{\mathfrak{r}}$, and let C_{X_0}, C_{Y_0} be semitopologizing subcategories of resp. $(C_X,\mathfrak{E}_X)$ and $(C_Y,\mathfrak{E}_Y)$. Then $C_{\mathfrak{X}_0} = C_{X_0}\prod\limits_{f_0^*,g_0^*} C_{Y_0}$ is a semitopologizing subcategory of $(C_{\mathfrak{X}},\mathfrak{E}_{\mathfrak{X}})$. If the subcategories C_{X_0} and C_{Y_0} are topologizing, then $C_{\mathfrak{X}_0}$ is a topologizing subcategory of $(C_{\mathfrak{X}},\mathfrak{E}_{\mathfrak{X}})$.

Proof. (a) By 6.8.2, $\mathfrak{E}_{\mathfrak{X}}$ consists of all morphisms $(M,L;\phi)\xrightarrow{(\xi,\gamma)}(M',L';\phi')$ of the category $C_{\mathfrak{X}}$ such that $\xi\in\mathfrak{E}_X$ and $\gamma\in\mathfrak{E}_Y$. And $Ker(\xi,\gamma)=(Ker(\xi),Ker(\gamma);\phi'')$, where ϕ'' is a uniquely determined (once $Ker(\xi)$ and $Ker(\gamma)$ are fixed) isomorphism. Therefore, if $(M,L;\phi)$ is an object of $C_{\mathfrak{X}_0}$ and both categories C_{X_0} and C_{Y_0} are semitopological, then $(M',L';\phi')$ and $Ker(\xi,\gamma)$ are objects of $C_{\mathfrak{X}_0}$, which shows that $C_{\mathfrak{X}_0}$ is a semitopological subcategory of the category $C_{\mathfrak{X}}$.

(b) Suppose now that C_{X_0} and C_{Y_0} are topologizing subcategories of respectively $(C_X,\mathfrak{E}_X)$ and $(C_Y,\mathfrak{E}_Y)$. By definition of morphisms of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$, the inverse image functors $C_X\xrightarrow{f^*}C_Z$ and $C_Y\xrightarrow{g^*}C_Z$ are 'exact'; i.e. they preserve pull-backs of deflations. This implies that for any deflation $(M,L;\phi)\xrightarrow{(\xi,\gamma)}(M',L';\phi')$ and an arbitary morphism $(M'',L'';\phi'')\xrightarrow{(\alpha,\beta)}(M',L';\phi')$ of the category $C_{\mathfrak{X}}$, there exists a cartesian square

$$\begin{array}{ccc} (\widetilde{M},\widetilde{L};\widetilde{\phi}) & \xrightarrow{(p_2,p_2')} & (M,L;\phi) \\ {\scriptstyle(p_1,p_1')}\downarrow & & \downarrow{\scriptstyle(\xi,\gamma)} \\ (M'',L'';\phi'') & \xrightarrow{(\alpha,\beta)} & (M',L';\phi') \end{array}$$

determined uniquely up to isomorphism by the fact that the squares

$$\begin{array}{ccc} \widetilde{L} & \xrightarrow{p_2} & L \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle\xi} \\ L'' & \xrightarrow{\xi} & L' \end{array} \quad\text{and}\quad \begin{array}{ccc} \widetilde{M} & \xrightarrow{p_2'} & M \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle\gamma} \\ M'' & \xrightarrow{\gamma} & M' \end{array}$$

are both cartesian. Therefore, if L and L'' are objects of the topologizing subcategory C_{Y_0}, then $\widetilde{L}\in ObC_{Y_0}$. Similarly, $\widetilde{M}\in ObC_{X_0}$ if M and M'' are objects of C_{X_0}. This shows that $C_{\mathfrak{X}_0}$ is a topologizing subcategory of $(C_{\mathfrak{X}},\mathfrak{E}_{\mathfrak{X}})$. ∎

8.6. Another left exact structure $\mathfrak{L}_{\mathfrak{esp}}$ on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'. We denote by $\mathfrak{L}_{\mathfrak{esp}}$ the class of all morphisms $(X,\mathfrak{E}_X)\xrightarrow{q}(Y,\mathfrak{E}_Y)$ of right exact

'spaces' such that q^* is a localization functor and for each arrow $q^*(L) \xrightarrow{\mathfrak{e}'} q^*(L')$ of $\mathfrak{E}_X$, there exists an arrow $L \xrightarrow{\mathfrak{e}} L''$ of $\mathfrak{E}_Y$ and an isomorphism $q^*(L'') \xrightarrow{\mathfrak{s}} q^*(L')$ such that $\mathfrak{e}' = \mathfrak{s} \circ q^*(\mathfrak{e})$.

It follows from this definition that the class $\mathfrak{L}_{\mathfrak{esp}}$ is contained in $\mathfrak{L}_{\mathfrak{es}}$.

8.6.1. Proposition. *The class $\mathfrak{L}_{\mathfrak{esp}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'.*

Proof. The class $\mathfrak{L}_{\mathfrak{esp}}$ contains, obviously, all isomorphisms, and it is easy to see that it is closed under composition. It remains to show that $\mathfrak{L}_{\mathfrak{esp}}$ is stable under cobase change and its arrows are cocovers of a subcanonical copretopology.

Let $(X, \mathfrak{E}_X) \xrightarrow{q} (Y, \mathfrak{E}_Y)$ be a morphism of $\mathfrak{L}_{\mathfrak{esp}}$ and $(X, \mathfrak{E}_X) \xrightarrow{f} (Z, \mathfrak{E}_Z)$ an arbitrary morphism. The claim is that the canonical morphism $Z \xrightarrow{\widetilde{q}} Z \coprod\limits_{f,q} Y$ belongs to $\mathfrak{L}_{\mathfrak{esp}}$.

Consider the corresponding cartesian (in pseudo-categorical sense) square of right exact categories:

$$\begin{array}{ccc} (C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}}) & \xrightarrow{p^*} & (C_Y, \mathfrak{E}_Y) \\ {\scriptstyle \widetilde{q}^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ (C_Z, \mathfrak{E}_Z) & \xrightarrow{f^*} & (C_X, \mathfrak{E}_X) \end{array} \tag{2}$$

where $\mathfrak{X} = Z \coprod\limits_{f,q} Y$; that is $C_{\mathfrak{X}} = C_Z \prod\limits_{f^*, q^*} C_Y$. Recall that the functor $\widetilde{q}^*$ maps each object $(L, M; \phi)$ of the category $C_{\mathfrak{X}}$ to the object L of C_Z and each morphism (ξ, γ) to ξ. By 6.1(a), $\widetilde{q}^*$ is a localization functor (because q^* is a localization functor).

Let $(L, M; \phi)$ and $(L', M'; \phi')$ be objects of the category $C_{\mathfrak{X}}$; and let

$$\widetilde{q}^*(L, M; \phi) = L \xrightarrow{\mathfrak{e}'} L' = \widetilde{q}^*(L', M'; \phi') \tag{3}$$

be an arrow of $\mathfrak{E}_Z$. Then $f^*(L) \xrightarrow{f^*(\mathfrak{e})} f^*(L')$ is a morphism of $\mathfrak{E}_X$. Since the localization $X \xrightarrow{q} Y$ belongs to $\mathfrak{L}_{\mathfrak{esp}}$, there exists a morphism $M \xrightarrow{\mathfrak{e}} M''$ of $\mathfrak{E}_Y$ and an isomorphism $q^*(M') \xrightarrow{\psi} q^*(M'')$ such that the diagram

$$\begin{array}{ccc} f^*(L) & \xrightarrow{f^*(\mathfrak{e})} & f^*(L') \\ {\scriptstyle \phi}\downarrow\wr & & \wr\downarrow{\scriptstyle \psi \circ \phi'} \\ q^*(M) & \xrightarrow{q^*(\mathfrak{e})} & q^*(M'') \end{array}$$

commutes. This expresses the fact that the pair $(\mathfrak{e}', \mathfrak{e})$ is a morphism from the object $(L, M; \phi)$ to the object $(L', M''; \psi\phi')$ of the category $C_{\mathfrak{X}}$. The morphism

$$(L, M; \phi) \xrightarrow{(\mathfrak{e}, \mathfrak{e}')} (L', M''; \psi\phi')$$

is a deflation, because both $\mathfrak{e}'$ and $\mathfrak{e}$ are deflations. Finally, $\widetilde{q}^*(\mathfrak{e}', \mathfrak{e}) = \mathfrak{e}'$.

This shows that the class of localizations $\mathfrak{L}_{\mathfrak{esp}}$ is stable under cobase change; i.e. $\mathfrak{L}_{\mathfrak{esp}}$ is the class of cocovers of a copretopology. Since the copretopology $\mathfrak{L}_{\mathfrak{esp}}$ is coarser than $\mathfrak{L}_{\mathfrak{es}}$ and $\mathfrak{L}_{\mathfrak{es}}$ is subcanonical, it follows that $\mathfrak{L}_{\mathfrak{esp}}$ is subcanonical too. In other words, $\mathfrak{L}_{\mathfrak{esp}}$ is a left exact structure on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'. ∎

8.6.2. Proposition. *Let* $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ *be a morphism of* $\mathfrak{L}_{\mathfrak{es}}$ *satisfying the following conditions:*

(i) Every pair of arrows $N \xleftarrow{\mathfrak{s}} M \xrightarrow{\mathfrak{e}} L$, *where* $\mathfrak{e} \in \mathfrak{E}_Y$ *and* $\mathfrak{s} \in \Sigma_{\mathfrak{q}^*} = \{\mathfrak{t} \in HomC_Y \mid \mathfrak{q}^*(\mathfrak{t})$ *is invertible*$\}$, *can be completed to a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}} & L \\ {\scriptstyle\mathfrak{s}}\downarrow & & \downarrow{\scriptstyle\mathfrak{t}} \\ N & \xrightarrow{\mathfrak{e}'} & L' \end{array}$$

with $\mathfrak{e}' \in \mathfrak{E}_Y$ *and* $\mathfrak{t} \in \Sigma_{\mathfrak{q}^*}$.

(ii) Every pair of arrows $N \xrightarrow{\mathfrak{s}} M \xrightarrow{\mathfrak{e}} L$, *where* $\mathfrak{e} \in \mathfrak{E}_Y$ *and* $\mathfrak{s} \in \Sigma_{\mathfrak{q}^*}$, *can be completed to a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}} & L \\ {\scriptstyle\mathfrak{s}}\uparrow & & \uparrow{\scriptstyle\mathfrak{t}} \\ N & \xrightarrow{\mathfrak{e}'} & L' \end{array}$$

with $\mathfrak{e}' \in \mathfrak{E}_Y$ *and* $\mathfrak{t} \in \Sigma_{\mathfrak{q}^*}$.

Then $\mathfrak{q}$ *belongs to the class* $\mathfrak{L}_{\mathfrak{esp}}$.

Proof. Since $\mathfrak{q} \in \mathfrak{L}_{\mathfrak{es}}$, for each morphism $\mathfrak{q}^*(M) \xrightarrow{\mathfrak{e}'} \mathfrak{q}^*(L)$, there exists a commutative diagram

$$\begin{array}{ccc} \mathfrak{q}^*(M) & \xrightarrow{\mathfrak{e}'} & \mathfrak{q}^*(L) \\ \wr\downarrow & & \downarrow\wr \\ \mathfrak{q}^*(\widetilde{M}) & \xrightarrow{\mathfrak{q}^*(\mathfrak{t})} & \mathfrak{q}^*(\widetilde{L}) \end{array}$$

whose vertical arrows are isomorphisms and $\mathfrak{t} \in \mathfrak{E}_Y$. Any isomorphism $\mathfrak{q}^*(M) \xrightarrow{\sim} \mathfrak{q}^*(\widetilde{M})$ is the composition of morphisms $\mathfrak{q}^*(s)^{\pm 1}$, where $s \in \Sigma_{\mathfrak{q}^*}$. Therefore, applying the conditions (i) and (ii), we obtain, in a finite number of steps, a commutative diagram of the form

$$\begin{array}{ccc} \mathfrak{q}^*(M) & \xrightarrow{\mathfrak{e}'} & \mathfrak{q}^*(L) \\ id\downarrow & & \downarrow\wr \\ \mathfrak{q}^*(M) & \xrightarrow{\mathfrak{q}^*(\mathfrak{t}')} & \mathfrak{q}^*(L') \end{array}$$

in which $\mathfrak{t}' \in \mathfrak{E}_{\mathfrak{X}}$. ∎

8.6.3. Proposition. *Let C_X be a category with kernel pairs and finite colimits, whose canonical (i.e. the finest) right exact structure $\mathfrak{E}_X^{st}$ consists of all strict epimorphisms (e.g. C_X is a quasi-abelian category). Then any right exact and 'exact' localization functor $C_X \xrightarrow{\mathfrak{q}^*} C_Z$ is an inverse image functor of a morphism $(Z, \mathfrak{E}_Z^{st}) \xrightarrow{\mathfrak{q}} (X, \mathfrak{E}_X^{st})$ which belongs to $\mathfrak{L}_{\mathfrak{esp}}$.*

Proof. It follows that each morphism $L \xrightarrow{f} M$ is the composition of a strict epimorphism $L \xrightarrow{\mathfrak{t}_f} M_e$ (– the cokernel of the kernel pair of f) and a monomorphism $M_e \xrightarrow{\mathfrak{j}_f} M$. Being right exact, the localization functor $\mathfrak{q}^*$ maps the strict epimorphism $\mathfrak{t}_f$ to (the cokernel of a pair of arrows, hence) a strict epimorphism and the monomorphism $\mathfrak{j}_f$ to an isomorphism. This implies, in particular, the condition (ii). Since the category C_X has finite colimits and the localization functor $\mathfrak{q}^*$ is right exact, the class of morphisms $\Sigma_{\mathfrak{q}^*}$ is a left multiplicative system [GZ, 8.3.4]. In particular, the condition (i) holds. ∎

8.6.4. Note. One can show that the quotient category C_Z satisfies the same property: the class of strict epimorphisms is stable under base change.

8.6.5. The left exact structure $\mathfrak{L}^*_{\mathfrak{esp}}$ on the category $\mathfrak{Esp}^*_{\mathfrak{r}}$. We denote by $\mathfrak{L}^*_{\mathfrak{esp}}$ the intersection $\mathfrak{L}_{\mathfrak{esp}} \bigcap \mathfrak{L}^*_{\mathfrak{es}}$.

8.7. The K-functor $\mathcal{K}^\star_\bullet$. Applying 8.4.2.1 to the identical functor

$$(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{esp}})^{op} \longrightarrow (\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})^{op}$$

(i.e. restricting to a coarser left exact structure $\mathfrak{L}^*_{\mathfrak{esp}}$) we obtain the universal ∂^*-functor $\mathcal{K}^\star_\bullet = (\mathcal{K}^\star_i, \mathfrak{d}^\star_i \mid i \geq 0)$ from $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{esp}})^{op}$ to $\mathbb{Z} - mod$ whose zero component coincides with the functor $(\mathfrak{Esp}^*_{\mathfrak{r}})^{op} \xrightarrow{K_0^*} \mathbb{Z} - mod$.

8.8. The left exact category of right exact infinitesimal 'spaces'. We define a *right exact infinitesimal 'space'* as a pair $((X, \mathfrak{E}_X), Y)$, where $(C_X, \mathfrak{E}_X)$ is a right exact category with initial objects and C_Y is a topologizing subcategory of $(C_X, \mathfrak{E}_X)$ such that $C_X = (C_Y)_{(\infty)}$. A morphism $((X, \mathfrak{E}_X), Y) \longrightarrow ((\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}), \mathfrak{Y})$ of right exact infinitesimal 'spaces' is given by a morphism $X \xrightarrow{f} \mathfrak{X}$ of 'spaces' whose inverse image functor is 'exact', maps initial objects and the subcategory $C_{\mathfrak{Y}}$ to the subcategory C_Y. The composition of morphisms is given by the composition of the corresponding morphisms of 'spaces'. This defines the category which we denote by $\mathfrak{Esp}^{\mathfrak{r}}_\infty$.

It follows from this definition that the maps

$$((X, \mathfrak{E}_X), Y) \longmapsto (X, \mathfrak{E}_X) \quad \text{and} \quad ((X, \mathfrak{E}_X), Y) \longmapsto (Y, \mathfrak{E}_Y)$$

(where $\mathfrak{E}_Y$ is the induced right exact structure on C_Y) extend naturally to functors respectively

$$\mathfrak{Esp}^{\mathfrak{r}}_\infty \xrightarrow{\mathfrak{F}^!} \mathfrak{Esp}^*_{\mathfrak{r}} \quad \text{and} \quad \mathfrak{Esp}^{\mathfrak{r}}_\infty \xrightarrow{\mathfrak{F}^*} \mathfrak{Esp}^*_{\mathfrak{r}}.$$

8.8.1. Proposition. *The functor $\mathfrak{Esp}^*_{\mathfrak{r}} \xrightarrow{\mathfrak{F}_*} \mathfrak{Esp}^{\mathfrak{r}}_\infty$ which assigns to each right exact 'space' $(X, \mathfrak{E}_X)$ the corresponding 'trivial' infinitesimal 'space', $((X, \mathfrak{E}_X), X)$ and acts accordingly on morphisms is left adjoint to the functor $\mathfrak{F}^!$ and right adjoint to $\mathfrak{F}^*$.*

Proof. The adjunction morphism $\mathfrak{F}_* \circ \mathfrak{F}^! \xrightarrow{\varepsilon^!_{\mathfrak{F}}} Id_{\mathfrak{Esp}^{\mathfrak{r}}_\infty}$ assigns to each object $((X,\mathfrak{E}_X),Y)$ the morphism $((X,\mathfrak{E}_X),X) \xrightarrow{id_X} ((X,\mathfrak{E}_X),Y)$. The adjunction arrow $Id_{\mathfrak{Esp}^*_{\mathfrak{r}}} \longrightarrow \mathfrak{F}^! \circ \mathfrak{F}_*$ is the identical morphism.

The adjunction morphism $Id_{\mathfrak{Esp}^{\mathfrak{r}}_\infty} \xrightarrow{\eta_{\mathfrak{F}}} \mathfrak{F}_*\mathfrak{F}^*$ assigns to each object $((X,\mathfrak{E}_X),Y)$ the morphism $((X,\mathfrak{E}_X),Y) \xrightarrow{\mathfrak{j}} ((Y,\mathfrak{E}_X),Y)$, where $\mathfrak{j} = \mathfrak{j}_Y$ is the morphism $X \longrightarrow Y$ whose inverse image functor is the inclusion functor $C_Y \longrightarrow C_X$. The adjunction arrow $\mathfrak{F}^*\mathfrak{F}_* \longrightarrow Id_{\mathfrak{Esp}^*_{\mathfrak{r}}}$ is identical. ∎

8.8.2. Note. Thanks to the full faithfulness of the functor $\mathfrak{F}_*$, there is a canonical morphism $\mathfrak{F}^! \xrightarrow{\rho_{\mathfrak{F}}} \mathfrak{F}^*$ defined as the composition of

$$\mathfrak{F}^! \xrightarrow{\mathfrak{F}^!\eta_{\mathfrak{F}}} \mathfrak{F}^!\mathfrak{F}_*\mathfrak{F}^* \quad \text{and} \quad \mathfrak{F}^!\mathfrak{F}_*\mathfrak{F}^* \xrightarrow{\eta^{!\,-1}_{\mathfrak{F}}\mathfrak{F}^*} \mathfrak{F}^*.$$

The morphism $\rho_{\mathfrak{F}}$ assigns to each infinitesimal right exact 'space' $((X,\mathfrak{E}_X),Y)$ the natural morphism $(X,\mathfrak{E}_X) \xrightarrow{\mathfrak{j}} (Y,\mathfrak{E}_Y)$ whose inverse image functor is the embedding $C_Y \longrightarrow C_X$.

8.8.3. The class of morphisms $\mathfrak{L}^\infty_{\mathfrak{esp}}$. The class $\mathfrak{L}^\infty_{\mathfrak{esp}}$ consists of all morphisms $((X,\mathfrak{E}_X),X_0) \xrightarrow{\mathfrak{q}} ((Y,\mathfrak{E}_Y),Y_0)$ such that $(X,\mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (Y,\mathfrak{E}_Y)$ belongs to $\mathfrak{L}^*_{\mathfrak{esp}}$ (in particular, $C_Y \xrightarrow{\mathfrak{q}^*} C_X$ is an 'exact' localization) and $C_{Y_0} = \mathfrak{q}^{*^{-1}}(C_{X_0})$.

8.8.3.1. Note. For any morphism $((X,\mathfrak{E}_X),X_0) \xrightarrow{\mathfrak{q}} ((Y,\mathfrak{E}_Y),Y_0)$ of $\mathfrak{L}^\infty_{\mathfrak{esp}}$, we have the inclusion $Ker(\mathfrak{q}^*) \subseteq C_{Y_0}$.

In fact, any semitopologizing subcategory of C_X, in particular C_{X_0}, contains the initial objects of C_X (they are, for instance, kernels of identical morphisms). Therefore, the equality $C_{Y_0} = \mathfrak{q}^{*^{-1}}(C_{X_0})$ implies that $Ker(\mathfrak{q}^*) \subseteq C_{Y_0}$.

8.8.4. Proposition. *The class of morphisms $\mathfrak{L}^\infty_{\mathfrak{esp}}$ is a left exact structure on the category $\mathfrak{Esp}^{\mathfrak{r}}_\infty$ of right exact infinitesimal 'spaces'.*

The functors $\mathfrak{F}^!$ and $\mathfrak{F}^*$ are 'exact' functors from the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_\infty, \mathfrak{L}^\infty_{\mathfrak{esp}})$ to the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{esp}})$.

Proof. (a) Let $((X,\mathfrak{E}_X),X_0) \xleftarrow{f} ((Z,\mathfrak{E}_Z),Z_0) \xrightarrow{\mathfrak{q}} ((Y,\mathfrak{E}_Y),Y_0)$ be a pair of morphisms of the category $\mathfrak{Esp}^{\mathfrak{r}}_\infty$. Suppose that the morphism $(Z,\mathfrak{E}_Z) \xrightarrow{\mathfrak{q}} (Y,\mathfrak{E}_Y)$ belongs to $\mathfrak{L}^*_{\mathfrak{esp}}$. Then there exists a canonical cocartesian square

$$\begin{array}{ccc} ((Z,\mathfrak{E}_Z),Z_0) & \xrightarrow{\mathfrak{q}} & ((Y,\mathfrak{E}_Y),Y_0) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_1} \\ ((X,\mathfrak{E}_X),X_0) & \xrightarrow{p_2} & ((\mathfrak{X},\mathfrak{E}_{\mathfrak{X}}),\mathfrak{X}_0) \end{array}$$

where $(\mathfrak{X},\mathfrak{E}_{\mathfrak{X}}) = (X,\mathfrak{E}_X)\coprod_{f,\mathfrak{q}}(Y,\mathfrak{E}_Y)$ and $\mathfrak{X}_0 = X_0 \coprod_{f_0,\mathfrak{q}_0} Y_0$. Here $X_0 \xleftarrow{f_0} Z_0 \xrightarrow{\mathfrak{q}_0} Y_0$ are morphisms induced by resp. f and $\mathfrak{q}$.

It follows from 8.5.5 that $C_{\mathfrak{X}_0}$ is a topologizing subcategory of the right exact category $(C_{\mathfrak{X}}.\mathfrak{E}_{\mathfrak{X}})$. The claim is that, under the assumptions, $(C_{\mathfrak{X}_0})_{(\infty)} = C_{\mathfrak{X}}$; i.e. $((\mathfrak{X}, \mathfrak{E}_{\mathfrak{X}}), \mathfrak{X}_0)$ is an object of the category $\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$. In other words, we need to show that each object $(M, L; \phi)$ of the category $C_{\mathfrak{X}} = C_X \prod_{f^*, \mathfrak{q}^*} C_Y$ has a $C_{\mathfrak{X}_0}$-cofiltration.

Let $M = M_n \xrightarrow{\mathfrak{e}_n} \ldots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$ be a C_{X_0}-cofiltration of the object M. The functor f^* maps this cofiltration to a C_{Z_0}-cofiltration

$$f^*(M) = f^*(M_n) \xrightarrow{f^*(\mathfrak{e}_n)} f^*(M_{n-1}) \xrightarrow{f^*(\mathfrak{e}_{n-1})} \ldots \xrightarrow{f^*(\mathfrak{e}_2)} f^*(M_1) \xrightarrow{f^*(\mathfrak{e}_1)} f^*(M_0)$$

of the object $f^*(M)$. Since $(Z, \mathfrak{E}_Z) \xrightarrow{\mathfrak{q}} (Y, \mathfrak{E}_Y)$ is a morphism of $\mathfrak{L}_{\mathfrak{esp}}$ and we have an isomorphism $f^*(M) \xrightarrow{\phi} \mathfrak{q}^*(L)$, there exists a deflation $L \xrightarrow{\mathfrak{t}_n} L_{n-1}$ and an isomorphism $f^*(M_{n-1}) \xrightarrow{\phi_{n-1}} \mathfrak{q}^*(L_{n-1})$ such that the diagram

$$\begin{array}{ccc} f^*(M) & \xrightarrow{f^*(\mathfrak{e}_n)} & f^*(M_{n-1}) \\ \phi \downarrow \wr & & \wr \downarrow \phi_{n-1} \\ \mathfrak{q}^*(L) & \xrightarrow{\mathfrak{q}^*(\mathfrak{t}_n)} & \mathfrak{q}^*(L_{n-1}) \end{array}$$

commutes. Continuing this process, we obtain a commutative diagram

$$\begin{array}{ccccccccc} f^*(M) & \xrightarrow{f^*(\mathfrak{e}_n)} & f^*(M_{n-1}) & \xrightarrow{f^*(\mathfrak{e}_{n-1})} & \ldots & \xrightarrow{f^*(\mathfrak{e}_2)} & f^*(M_1) & \xrightarrow{f^*(\mathfrak{e}_1)} & f^*(M_0) \\ \phi \downarrow \wr & & \wr \downarrow \phi_{n-1} & & & & \wr \downarrow \phi_1 & & \wr \downarrow \phi_0 \\ \mathfrak{q}^*(L) & \xrightarrow{\mathfrak{q}^*(\mathfrak{t}_n)} & \mathfrak{q}^*(L_{n-1}) & \xrightarrow{\mathfrak{q}^*(\mathfrak{t}_{n-1})} & \ldots & \xrightarrow{\mathfrak{q}^*(\mathfrak{t}_2)} & \mathfrak{q}^*(L_1) & \xrightarrow{\mathfrak{q}^*(\mathfrak{t}_1)} & \mathfrak{q}^*(L_0) \end{array} \qquad (1)$$

whose vertical arrows are isomorphisms and horizontal arrows are images of deflations. Therefore, the diagram (1) encodes a cofiltration

$$(M, L; \phi) \xrightarrow{(\mathfrak{e}_n, \mathfrak{t}_n)} (M_{n-1}, L_{n-1}; \phi_{n-1}) \xrightarrow{(\mathfrak{e}_{n-1}, \mathfrak{t}_{n-1})} \ldots \xrightarrow{(\mathfrak{e}_2, \mathfrak{t}_2)} (M_1, L_1; \phi_1) \xrightarrow{(\mathfrak{e}_1, \mathfrak{t}_1)} (M_0, L_0; \phi_0)$$

of the object $(M, L; \phi)$. This cofiltration is a $C_{\mathfrak{X}_0}$-cofiltration.

In fact, the kernel of the morphism $(M_m, L_m; \phi_m) \xrightarrow{(\mathfrak{e}_m, \mathfrak{t}_m)} (M_{m-1}, L_{m-1}; \phi_{m-1})$ is isomorphic to $(Ker(\mathfrak{e}_m), Ker(\mathfrak{t}_m); \psi_m)$, where $f^*(Ker(\mathfrak{e}_m)) \xrightarrow{\psi_m} \mathfrak{q}^*(Ker(\mathfrak{t}_m))$ is a uniquely determined isomorphism. The kernel $Ker(\mathfrak{e}_m)$ is an object of C_{X_0} by choice, and $Ker(\mathfrak{t}_m)$ is an object of C_{Y_0}, because $\mathfrak{q}^*(Ker(\mathfrak{t}_m)) \in ObC_{Z_0}$ and $C_{Y_0} = \mathfrak{q}^{*^{-1}}(C_{Z_0})$, because $\mathfrak{q}$ is a morphism of $\mathfrak{L}^{\infty}_{\mathfrak{esp}}$ (cf. 8.8.3).

(b) It follows from the description of cocartesian squares in the category $\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$ (given above) that the functor $\mathfrak{F}^!$ preserves push-forwards of morphisms of $\mathfrak{L}^{\infty}_{\mathfrak{esp}}$; that is $\mathfrak{F}^!$ is an 'exact' functor from the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_{\infty}, \mathfrak{L}^{\infty}_{\mathfrak{es}})$ to the left exact category $(\mathfrak{Esp}^{*}_{\mathfrak{r}}, \mathfrak{L}^{*}_{\mathfrak{es}})$.

(c) By 8.8.1, the functor $\mathfrak{F}^*$ has a right adjoint, $\mathfrak{F}_*$, hence it preserves all colimits, in particular, cocartesian squares; and $\mathfrak{F}^*$ maps inflations to inflations (that is $\mathfrak{L}^\infty_{\mathfrak{esp}}$ to $\mathfrak{L}^*_{\mathfrak{esp}}$). Therefore, $\mathfrak{F}^*$ is an 'exact' functor from $(\mathfrak{Esp}^{\mathfrak{r}}_\infty, \mathfrak{L}^\infty_{\mathfrak{es}})$ to $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{es}})$. ∎

8.9. Right exact infinitesimal 'spaces' and derived functors. By 8.8.4, the canonical functors

$$\mathfrak{Esp}^*_{\mathfrak{r}} \xleftarrow{\mathfrak{F}^!} \mathfrak{Esp}^{\mathfrak{r}}_\infty \xrightarrow{\mathfrak{F}^*} \mathfrak{Esp}^*_{\mathfrak{r}}$$

defined by $\mathfrak{F}^!((X,\mathfrak{E}_X),Y) = (X,\mathfrak{E}_X)$ and $\mathfrak{F}^*((X,\mathfrak{E}_X),Y) = (Y,\mathfrak{E}_Y)$ (cf. 8.8), are 'exact' functors from the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_\infty, \mathfrak{L}^\infty_{\mathfrak{esp}})$ to the left exact category $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{esp}})$. Let $(C_Z, \mathfrak{E}_Z)$ be a right exact category with initial objects and limits of filtered systems. And let G be a functor $(\mathfrak{Esp}^*_{\mathfrak{r}})^{op} \longrightarrow C_Z$. Applying Q8.4.2 to G and each of the functors $\mathfrak{F}^!$ and $\mathfrak{F}^*$, we obtain two universal ∂^*-functors, $G^{\mathfrak{F}^!}_\bullet$ and $G^{\mathfrak{F}^*}_\bullet$ from $(\mathfrak{Esp}^{\mathfrak{r}}_\infty, \mathfrak{L}^\infty_{\mathfrak{esp}})^{op}$ to C_Z whose zero components are respectively the functors $G \circ \mathfrak{F}^{!^{op}}$ and $G \circ \mathfrak{F}^{*^{op}}$; that is

$$G^{\mathfrak{F}^!}_0((X,\mathfrak{E}_X),Y) = G(X,\mathfrak{E}_X) \quad \text{and} \quad G^{\mathfrak{F}^*}_0((X,\mathfrak{E}_X),Y) = G(Y,\mathfrak{E}_Y)$$

for any infinitesimal right exact 'space' $((X,\mathfrak{E}_X),Y)$. The canonical functor morphism $\mathfrak{F}^! \xrightarrow{\rho_{\mathfrak{F}}} \mathfrak{F}^*$ (see 8.8.2) assigns to each infinitesimal right exact 'space' $((X,\mathfrak{E}_X),Y)$ the morphism $(X,\mathfrak{E}_X) \xrightarrow{\mathrm{j}} (Y,\mathfrak{E}_Y)$ whose inverse image functor is the inclusion functor $C_Y \xrightarrow{\mathrm{j}^*} C_X$, induces a morphism

$$G \circ \mathfrak{F}^{*^{op}} \xrightarrow{G\rho_{\mathfrak{F}}} G \circ \mathfrak{F}^{!^{op}}.$$

Thanks to the universality of the ∂^*-functor $G^*_\bullet \circ \mathfrak{F}^!$, the latter morphism determines uniquely a morphism

$$G^{\mathfrak{F}^*}_\bullet \xrightarrow{\rho^{\mathfrak{F}}_\bullet} G^{\mathfrak{F}^!}_\bullet$$

of universal ∂^*-functors. Thanks to the universality of the ∂^*-functors $G^{\mathfrak{F}^!}_\bullet$ and $G^{\mathfrak{F}^*}_\bullet$, there are natural morphisms of ∂^*-functors

$$G_\bullet \circ \mathfrak{F}^{*^{op}} \xrightarrow{\varphi_\bullet} G^{\mathfrak{F}^*}_\bullet \quad \text{and} \quad G_\bullet \circ \mathfrak{F}^{!^{op}} \xrightarrow{\psi_\bullet} G^{\mathfrak{F}^!}_\bullet \tag{1}$$

such that the diagram

$$\begin{array}{ccc} G_\bullet \circ \mathfrak{F}^{*^{op}} & \xrightarrow{G_\bullet \rho_{\mathfrak{F}}} & G_\bullet \circ \mathfrak{F}^{!^{op}} \\ {\scriptstyle \varphi_\bullet}\downarrow & & \downarrow{\scriptstyle \psi_\bullet} \\ G^{\mathfrak{F}^*}_\bullet & \xrightarrow{\rho^{\mathfrak{F}}_\bullet} & G^{\mathfrak{F}^!}_\bullet \end{array} \tag{2}$$

commutes.

8.9.1. Proposition. *Let C_Z be a category with initial objects and limits of filtered systems; and let G be a functor $(\mathfrak{Esp}^*_{\mathfrak{r}}, \mathfrak{L}^*_{\mathfrak{esp}})^{op} \longrightarrow C_Z$. Then the natural morphism*

$$S_- G \circ \mathfrak{F}^{!^{op}} \longrightarrow S_-(G \circ \mathfrak{F}^{!^{op}})$$

is an isomorphism. Here the functor $\mathfrak{F}^!$ *is viewed as a morphism from the left exact category* $(\mathfrak{Esp}^{\mathfrak{r}}_{\infty}, \mathfrak{L}^{\infty}_{\mathfrak{esp}})$ *to the left exact category* $(\mathfrak{Esp}^{*}_{\mathfrak{r}}, \mathfrak{L}^{*}_{\mathfrak{esp}})$.

In particular, the morphism $\mathcal{K}^{\star}_{\bullet} \circ \mathfrak{F}^{!^{op}} \xrightarrow{\psi_{\bullet}} K^{\mathfrak{F}^!}_{\bullet}$ *is an isomorphism of* ∂^**-functors.*

Proof. (a) Fix an object $((X, \mathfrak{E}_X), Y)$ of the category $\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$. The functor $\mathfrak{F}^!$ induces an isomorphism from the category $((X, \mathfrak{E}_X), Y)\backslash\mathfrak{L}^{\infty}_{\mathfrak{esp}}$ of inflations of $((X, \mathfrak{E}_X), Y)$ onto the category $(X, \mathfrak{E}_X)\backslash\mathfrak{L}^{*}_{\mathfrak{esp}}$ of inflations of the the object $(X, \mathfrak{E}_X) = \mathfrak{F}^!((X, \mathfrak{E}_X), Y)$.

In fact, each morphism $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}})$ of $\mathfrak{L}^{*}_{\mathfrak{esp}})$ determines uniquely a morphism $((X, \mathfrak{E}_X), Y) \xrightarrow{\mathfrak{q}} ((\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}), \mathcal{Z}_0)$, where $C_{\mathcal{Z}_0} = \mathfrak{q}^{*^{-1}}(C_Y)$, which belongs to the class $\mathfrak{L}^{\infty}_{\mathfrak{esp}}$. This correspondence extends (uniquely) to a functor

$$(X, \mathfrak{E}_X)\backslash\mathfrak{L}^{*}_{\mathfrak{esp}} \xrightarrow{\Phi_Y} ((X, \mathfrak{E}_X), Y)\backslash\mathfrak{L}^{\infty}_{\mathfrak{esp}}$$

which is inverse to the functor

$$((X, \mathfrak{E}_X), Y)\backslash\mathfrak{L}^{\infty}_{\mathfrak{esp}} \xrightarrow{\mathfrak{F}^!_Y} (X, \mathfrak{E}_X)\backslash\mathfrak{L}^{*}_{\mathfrak{esp}}$$

induced by $\mathfrak{F}^!$.

(b) By definition of the satellite, we have

$$\begin{aligned} S_-G \circ \mathfrak{F}^{!^{op}}(X, \mathfrak{E}_X) &= S_-G((X, \mathfrak{E}_X), Y) \\ &= \lim\big(Ker(G((\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}) \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} Cok((X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}})))\big), \end{aligned}$$

where $(X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}})$ runs through inflations of $(X, \mathfrak{E}_X)$.

On the other hand,

$$\begin{aligned} &S_-(G \circ \mathfrak{F}^{!^{op}})((X, \mathfrak{E}_X), Y) = \\ &\lim\big(Ker(G \circ \mathfrak{F}^{!^{op}}((\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}) \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} Cok(((X, \mathfrak{E}_X), Y) \xrightarrow{\mathfrak{q}} ((\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}), \mathcal{Z}_0))\big) = \\ &\lim\big(Ker(G((\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}) \xrightarrow{\mathfrak{c}_{\mathfrak{q}}} Cok((X, \mathfrak{E}_X) \xrightarrow{\mathfrak{q}} (\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}})))\big), \end{aligned}$$

where $((X, \mathfrak{E}_X), Y) \xrightarrow{\mathfrak{q}} ((\mathcal{Z}, \mathfrak{E}_{\mathcal{Z}}), \mathcal{Z}_0)$ runs through inflations of $((X, \mathfrak{E}_X), Y)$. It follows from (a) above that these two limits are identical. ∎

8.9.2. Constructions related to the functor $\mathfrak{F}^*$. Fix an object of the category $\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$ and an inflation (i.e. a morphism of $\mathfrak{L}^{*}_{\mathfrak{esp}}$) $(Y, \mathfrak{E}_Y) \xrightarrow{\wp} (\mathcal{Z}_0, \mathfrak{E}_{\mathcal{Z}_0})$ with an inverse image functor $C_{\mathcal{Z}_0} \xrightarrow{\wp^*} C_Y$. Let $C_{\mathcal{Z}(\wp)}$ denote the category whose objects are triples $(M, L; \mathfrak{e})$, where $M \in ObC_X$, $L \in ObC_{\mathcal{Z}_0}$, and $\mathfrak{e}$ is a deflation $M \longrightarrow \wp^*(L)$. Morphisms from $(M, L; \mathfrak{e})$ to $(M', L'; \mathfrak{e}')$ are given by pairs of arrows $M \xrightarrow{\xi} M'$, $L \xrightarrow{\gamma} L'$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}} & \wp^*(L) \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\wp^*(\gamma)} \\ M' & \xrightarrow{\widetilde{\mathfrak{e}}} & \wp^*(L') \end{array}$$

commutes. The composition is defined obviously.

There are natural functors (projections) $C_{\mathcal{Z}_0} \xleftarrow{\mathfrak{p}^*} C_{\mathcal{Z}(\wp)} \xrightarrow{\pi^*} C_X$ defined by

$$\pi^*((M, L; \mathfrak{e}) \xrightarrow{(\xi,\gamma)} (M', L'; \mathfrak{e}')) = (M \xrightarrow{\xi} M')$$
$$\mathfrak{p}^*((M, L; \mathfrak{e}) \xrightarrow{(\xi,\gamma)} (M', L'; \mathfrak{e}')) = (L \xrightarrow{\gamma} L')$$

The functor $\mathfrak{p}^*$ has a right adjoint, $C_{\mathcal{Z}_0} \xrightarrow{\mathfrak{p}_*} C_{\mathcal{Z}(\wp)}$, which assigns to each object L of the category $C_{\mathcal{Z}_0}$ the object $(\wp^*(L), L; id_{\wp^*(L)})$ and to each morphism $L \xrightarrow{\gamma} L'$ the morphism

$$(\wp^*(L), L; id_{\wp^*(L)}) \xrightarrow{(\wp^*(\gamma),\gamma)} (\wp^*(L'), L'; id_{\wp^*(L')}).$$

The adjunction morphism $Id_{C_{\mathcal{Z}(\wp)}} \xrightarrow{\eta} \mathfrak{p}_*\mathfrak{p}^*$ assigns to each object $(M, L; \mathfrak{e})$ of the category $C_{\mathcal{Z}(\wp)}$ the morphism $(M, L; \mathfrak{e}) \xrightarrow{(\mathfrak{e}, id_L)} (\wp^*(L), L; id_{\wp^*(L)})$. The other adjunction morphism is the identity. The latter implies that the functor $\mathfrak{p}_*$ is fully faithful, or, equivalently, $\mathfrak{p}^*$ is a localization functor.

It follows from the construction that the diagram

$$\begin{array}{ccc} C_{\mathcal{Z}_0} & \xrightarrow{\wp^*} & C_Y \\ {\scriptstyle \mathfrak{p}_*}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}^*} \\ C_{\mathcal{Z}(\wp)} & \xrightarrow{\pi^*} & C_X \end{array}$$

commutes.

8.9.2.1. Proposition. *The class $\mathfrak{E}_{\mathcal{Z}(\wp)}$ of all arrows $(M, L; \mathfrak{e}) \xrightarrow{(\mathfrak{t},\mathfrak{u})} (M', L'; \mathfrak{e}')$ of the category $C_{\mathcal{Z}(\wp)}$ such that $\mathfrak{t} \in \mathfrak{E}_X$ and $\mathfrak{u} \in \mathfrak{E}_{\mathcal{Z}_0}$ is a structure of a right exact category on $C_{\mathcal{Z}(\wp)}$.*

Proof. Obviously, the class $\mathfrak{E}_{\mathcal{Z}(\wp)}$ is closed under compositions and contains all isomorphisms. It remains to show that it is stable by a base change and defines a subcanonical topology. Let $(M', L'; \mathfrak{e}') \xrightarrow{(\mathfrak{t},\mathfrak{u})} (M, L; \mathfrak{e})$ be a morphism of $\mathfrak{E}_{\mathcal{Z}(\wp)}$ and $(M'', L''; \mathfrak{e}'') \xrightarrow{(f,g)} (M, L; \mathfrak{e})$ an arbitrary morphism. Since $\mathfrak{t}$ and $\mathfrak{u}$ are deflations, there are cartesian squares

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathfrak{t}'} & M'' \\ {\scriptstyle \widetilde{f}}\downarrow & cart & \downarrow{\scriptstyle f} \\ M' & \xrightarrow{\mathfrak{t}} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{L} & \xrightarrow{\mathfrak{u}'} & L'' \\ {\scriptstyle \widetilde{g}}\downarrow & cart & \downarrow{\scriptstyle g} \\ L' & \xrightarrow{\mathfrak{u}} & L \end{array} \qquad (1)$$

whose horizontal arrows are deflations.

Since $\wp^*$ is an 'exact' functor $(C_{\mathcal{Z}_0}, \mathfrak{E}_{\mathcal{Z}_0}) \longrightarrow (C_Y, \mathfrak{E}_Y)$, it maps the right cartesian square (1) to a cartesian square. So we have a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{M}_3 & \xrightarrow{\mathfrak{e}_3} & \mathcal{M}_1 & \xrightarrow{\mathfrak{e}_1} & M'' & & \\
\widetilde{\mathfrak{p}}_1 \downarrow & cart & \mathfrak{p}_1 \downarrow & cart & \downarrow \mathfrak{e}'' & & \\
\mathcal{M}_2 & \xrightarrow{\mathfrak{e}_2} & \wp^*(\widetilde{L}) & \xrightarrow{\wp^*(\mathfrak{u}')} & \wp^*(L'') & \xleftarrow{\mathfrak{e}''} & M'' \\
\widetilde{\mathfrak{p}}_2 \downarrow & cart & \wp^*(\widetilde{g}) \downarrow & cart & \downarrow \wp^*(g) & & \downarrow f \\
M' & \xrightarrow{\mathfrak{e}'} & \wp^*(L') & \xrightarrow{\wp^*(\mathfrak{u})} & \wp^*(L) & \xleftarrow{\mathfrak{e}} & M \\
 & & \mathfrak{e}' \uparrow & & \uparrow \mathfrak{e} & & \\
 & & M' & \xrightarrow{\mathfrak{t}} & M & &
\end{array} \qquad (2)$$

with cartesian squares as indicated. It follows that all horizontal arrows and upper and lower vertical arrows of (2) are deflations. Therefore, the composition $\mathcal{M}_3 \xrightarrow{\mathfrak{p}_1 \circ \mathfrak{e}_3} \wp^*(\widetilde{L})$ is a deflation. On the other hand, it is easy to see that the outer square of (2),

$$\begin{array}{ccc}
\mathcal{M}_3 & \xrightarrow{\mathfrak{e}_1 \mathfrak{e}_3} & \mathcal{M}_1 \\
\widetilde{\mathfrak{p}}_2 \widetilde{\mathfrak{p}}_1 \downarrow & & \downarrow f \\
M' & \xrightarrow{\mathfrak{t}} & M
\end{array}$$

is cartesian. Therefore, since the left square (1) is cartesian. there exists a unique isomorphism $\widetilde{M} \xrightarrow{\sigma} \mathcal{M}_3$ such that $\widetilde{f} = \widetilde{\mathfrak{p}}_1 \widetilde{\mathfrak{p}}_2 \sigma$ and $\mathfrak{t}' = \mathfrak{e}_1 \mathfrak{e}_3 \sigma$. We denote by $\widetilde{\mathfrak{e}}$ the composition $\widetilde{M} \longrightarrow \wp^*(\widetilde{L})$ of the isomorphism σ and the deflation $\mathcal{M}_3 \xrightarrow{\mathfrak{p}_1 \circ \mathfrak{e}_3} \wp^*(\widetilde{L})$. It follows from this argument that $(\widetilde{M}, \widetilde{L}; \widetilde{\mathfrak{e}})$ is an object of the category $C_{\mathcal{Z}(\wp)}$ and

$$\begin{array}{ccc}
(\widetilde{M}, \widetilde{L}; \widetilde{\mathfrak{e}}) & \xrightarrow{(\widetilde{f}, \widetilde{g})} & (M', L'; \mathfrak{e}') \\
(\mathfrak{t}', \mathfrak{u}') \downarrow & & \downarrow (\mathfrak{t}, \mathfrak{u}) \\
(M'', L''; \mathfrak{e}'') & \xrightarrow{(f,g)} & (M, L; \mathfrak{e})
\end{array} \qquad (3)$$

is a cartesian square in $C_{\mathcal{Z}(\wp)}$ whose vertical arrows belong $\mathfrak{E}_{\mathcal{Z}(\wp)}$.

Thus, $\mathfrak{E}_{\mathcal{Z}(\wp)}$ is (the class of covers of) a pretopology. This pretopology is subcanonical; i.e. for any $(\mathfrak{t}, \mathfrak{u}) \in \mathfrak{E}_{\mathcal{Z}(\wp)}$,, the corresponding cartesian square

$$\begin{array}{ccc}
(\widetilde{M}, \widetilde{L}; \widetilde{\mathfrak{e}}) & \xrightarrow{(\widetilde{\mathfrak{t}}, \widetilde{\mathfrak{u}})} & (M', L'; \mathfrak{e}') \\
(\mathfrak{t}', \mathfrak{u}') \downarrow & & \downarrow (\mathfrak{t}, \mathfrak{u}) \\
(M', L'; \mathfrak{e}') & \xrightarrow{(\mathfrak{t}, \mathfrak{u})} & (M, L; \mathfrak{e})
\end{array}$$

is cocartesian, because the squares

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathfrak{t}'} & M' \\ {\scriptstyle\widetilde{\mathfrak{t}}}\downarrow & cart & \downarrow{\scriptstyle\mathfrak{t}} \\ M' & \xrightarrow{\mathfrak{t}} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widetilde{L} & \xrightarrow{\mathfrak{u}'} & L' \\ {\scriptstyle\widetilde{\mathfrak{u}}}\downarrow & cart & \downarrow{\scriptstyle\mathfrak{u}} \\ L' & \xrightarrow{\mathfrak{u}} & L \end{array}$$

are cocartesian. ■

8.9.2.2. Lemma. *Suppose that any pair $N \longleftarrow M \longrightarrow L$ of morphisms of $\mathfrak{E}_X$ can be completed to a commutative square*

$$\begin{array}{ccc} M & \longrightarrow & L \\ \downarrow & & \downarrow \\ N & \longrightarrow & \mathcal{L} \end{array}$$

whose arrows are belong to $\mathfrak{E}_X$. Then the pair $((C_{\mathcal{Z}(\wp)}, \mathfrak{E}_{\mathcal{Z}(\wp)}), C_{\mathcal{Z}_0})$, where $C_{\mathcal{Z}_0}$ is identified with its image in $C_{\mathcal{Z}(\wp)}$, is an object of the category $\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$ of right infinitesimal 'spaces'.

Proof. Let $(M, L; \mathfrak{e})$ be any object of $C_{\mathcal{Z}(\wp)}$, and let

$$M = M_n \xrightarrow{\mathfrak{t}_n} M_{n-1} \xrightarrow{\mathfrak{t}_{n-1}} \ldots \xrightarrow{\mathfrak{t}_1} M_0$$

be a C_Y-cofiltration of the object M. By hypothesis, there exists a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_n} & M_{n-1} \\ {\scriptstyle\mathfrak{e}}\downarrow & & \downarrow{\scriptstyle\mathfrak{e}'} \\ \wp^*(L) & \xrightarrow{\beta'} & \mathcal{L}' \end{array} \tag{4}$$

whose all arrows are deflations. Since β' is a deflation $\wp^*(L) \in ObC_Y$, and C_Y is a topologizing subcategory, of $(C_X, \mathfrak{E}_X)$, $\mathcal{L}' \in ObC_Y$. Since $C_{\mathcal{Z}_0} \xrightarrow{\wp^*} C_Y$ is a localization functor, the object $\mathcal{L}'$ is isomorphic to an object $\wp^*(L'')$ for some $L'' \in ObC_{\mathcal{Z}_0}$. The morphism $(Y, \mathfrak{E}_Y) \xrightarrow{\wp} (\mathcal{Z}_0, \mathfrak{E}_{\mathcal{Z}_0})$ belongs to the class $\mathfrak{L}^*_{\mathfrak{esp}}$. Therefore, there exists an isomorphism $\wp^*(L'') \xrightarrow{\sim} \wp^*(L_{n-1})$ and a deflations $L \xrightarrow{\gamma_n} L_{n-1}$ such that the diagram

$$\begin{array}{ccc} \wp^*(L) & \xrightarrow{\wp^*(\gamma_n)} & \wp^*(L_{n-1}) \\ {\scriptstyle\beta'}\downarrow & & \uparrow{\scriptstyle\wr} \\ \mathcal{L}' & \xrightarrow{\sim} & \wp^*(L'') \end{array}$$

commutes. Combining this with the diagram (4), we obtain the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_n} & M_{n-1} \\ {\scriptstyle\mathfrak{e}}\downarrow & & \downarrow{\scriptstyle\mathfrak{e}_{n-1}} \\ \wp^*(L) & \xrightarrow{\wp^*(\gamma_n)} & \wp^*(L_{n-1}) \end{array}$$

whose all arrows are deflations. Continuing this process, we obtain a commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\mathfrak{t}_n} & M_{n-1} & \xrightarrow{\mathfrak{t}_{n-1}} & \dots & \xrightarrow{\mathfrak{t}_1} & M_0 \\ {\scriptstyle\mathfrak{e}}\downarrow & & \downarrow{\scriptstyle\mathfrak{e}_{n-1}} & & & & \downarrow{\scriptstyle\mathfrak{e}_0} \\ \wp^*(L) & \xrightarrow{\wp^*(\gamma_n)} & \wp^*(L_{n-1}) & \xrightarrow{\wp^*(\gamma_n)} & \dots & \xrightarrow{\wp^*(\gamma_1)} & \wp^*(L_0) \end{array}$$

which encodes a $C_{\mathcal{Z}_0}$-cofiltration

$$(M,L;\mathfrak{e}) \xrightarrow{(\mathfrak{t}_n,\gamma_n)} (M_{n-1},L_{n-1};\mathfrak{e}_{n-1}) \xrightarrow{(\mathfrak{t}_{n-1},\gamma_{n-1})} \dots \xrightarrow{(\mathfrak{t}_1,\gamma_1)} (M_0,L_0;\mathfrak{e}_0)$$

of the object $(M,L;\mathfrak{e})$ in the right exact category $(C_{\mathcal{Z}(\wp)},\mathfrak{E}_{\mathcal{Z}(\wp)})$. ∎

8.9.3. Proposition. *Let $((X,\mathfrak{E}_X),Y)$ be an object of $\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$. Suppose that the right exact category $(C_X,\mathfrak{E}_X)$ satisfies the following conditions:*

(a) Any pair of arrows $L \xleftarrow{\mathfrak{e}} M \xrightarrow{f} M_1$ of C_X, where $\mathfrak{e}$ is a deflation, can be completed to a commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & M_1 \\ {\scriptstyle\mathfrak{e}}\downarrow & & \downarrow{\scriptstyle\mathfrak{e}'} \\ L & \xrightarrow{\widetilde{f}} & L_1 \end{array} \tag{3}$$

where $\mathfrak{e}'$ is a deflation too.

(b) If both morphisms $\mathfrak{e}$ and f in the condition (a) are deflations, then the morphisms $\mathfrak{e}'$ and $\widetilde{f}$ in the diagram (3) can be chosen to be deflations.

Then the natural morphism

$$S_-G(Y,\mathfrak{E}_Y) = S_-G\circ\mathfrak{F}^{*^{op}}((X,\mathfrak{E}_X),Y) \longrightarrow S_-(G\circ\mathfrak{F}^{*^{op}})((X,\mathfrak{E}_X),Y) \tag{4}$$

is an isomorphism for any functor G from $(\mathfrak{Esp}^{\mathfrak{r}}_{\infty},\mathfrak{L}^{\infty}_{\mathfrak{esp}})^{op}$ to any category C_Z with initial objects and limits of filtered systems.

Here the functor $\mathfrak{F}^*$ is viewed as a morphism from the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_{\infty},\mathfrak{L}^{\infty}_{\mathfrak{esp}})$ to the left exact category $(\mathfrak{Esp}^{*}_{\mathfrak{r}},\mathfrak{L}^{*}_{\mathfrak{esp}})$.

Proof. (i) Let $C_{\mathbb{X}}$ denote the category whose objects are triples $(\mathcal{M},\mathcal{L};\mathfrak{e})$, where $\mathcal{M}\in ObC_X$, $\mathcal{L}\in ObC_Y$, and $\mathfrak{e}$ is a deflation $\mathcal{M}\longrightarrow\mathcal{L}$. Morphisms $(\mathcal{M},\mathcal{L};\mathfrak{e})\longrightarrow(\mathcal{M}',\mathcal{L}';\mathfrak{e}')$ are pairs of arrows $\mathcal{M}\xrightarrow{f}\mathcal{M}'$, $\mathcal{L}\xrightarrow{g}\mathcal{L}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \\ f\downarrow & & \downarrow g \\ \mathcal{M}' & \xrightarrow{\mathfrak{e}'} & \mathcal{L}' \end{array}$$

commutes. The composition is defined naturally.

(ii) The class $\mathfrak{E}_{\mathbb{X}}$ of all morphisms of $C_{\mathbb{X}}$ $(\mathcal{M},\mathcal{L};\mathfrak{e}) \xrightarrow{(\mathfrak{t},\mathfrak{u})} (\mathcal{M}',\mathcal{L}';\mathfrak{e}')$ such that both $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{M}'$ and $\mathcal{L} \xrightarrow{\mathfrak{u}} \mathcal{L}'$ are deflations, is a right exact structure on the category $C_{\mathbb{X}}$.

This fact is a special case of 8.9.2.1.

(iii) The functor $C_{\mathbb{X}} \xrightarrow{\rho^*} C_Y$ which maps each object $(M,L;\mathfrak{e})$ of $C_{\mathbb{X}}$ to L and each morphism (f,g) to g is a continuous localization with the canonical right adjoint which maps each object L of C_Y to the object $(L,L;id_L)$.

(iv) Fix an inflation (– a morphism of $\mathfrak{L}^*_{\mathfrak{esp}}$) $(Y,\mathfrak{E}_Y) \xrightarrow{\wp} (\mathcal{Z}_0,\mathfrak{E}_{\mathcal{Z}_0})$ with an inverse image functor $C_{\mathcal{Z}_0} \xrightarrow{\wp^*} C_Y$. Let $C_{\mathcal{Z}(\wp)} \xrightarrow{\mathfrak{q}^*} C_{\mathbb{X}}$ be the functor which assigns to each object $(M.L;\mathfrak{e})$ of $C_{\mathcal{Z}(\wp)}$ the object $(M,\wp^*(L);\mathfrak{e})$ of the category $C_{\mathbb{X}}$ and to each morphism $(\mathcal{M},\mathcal{L};\mathfrak{e}) \xrightarrow{(f,g)} (\mathcal{M}',\mathcal{L}';\mathfrak{e}')$ the morphism $(\mathcal{M},\wp^*(L);\mathfrak{e}) \xrightarrow{(f,\wp^*(g))} (\mathcal{M}',\wp^*(L');\mathfrak{e}')$.

It is easy to see that the square

$$\begin{array}{ccc} C_{\mathcal{Z}_\wp} & \xrightarrow{\mathfrak{q}^*} & C_{\mathbb{X}} \\ {\scriptstyle \mathfrak{p}^*}\downarrow & & \downarrow{\scriptstyle \rho^*} \\ C_{\mathcal{Z}_0} & \xrightarrow{\wp^*} & C_Y \end{array}$$

is cartesian. In particular, since $\wp^*$ and ρ^* are 'exact' localizations, the functors $\mathfrak{p}^*$ and $\mathfrak{q}^*$ are 'exact' localizations too.

(v) The functor $C_{\mathbb{X}} \xrightarrow{\pi^*} C_X$, $(M,L;\mathfrak{e}) \mapsto M$, is an 'exact' localization functor $(C_{\mathbb{X}},\mathfrak{E}_{\mathbb{X}}) \longrightarrow (C_X,\mathfrak{E}_X)$. In other words, the unique functor $\Sigma^{-1}_{\mathfrak{q}^*} C_{\mathbb{X}} \xrightarrow{\widetilde{\pi}^*} C_X$ is an equivalence of categories.

(v') It follows from the definition of right exact structure on $C_{\mathbb{X}}$ and (the argument of) 8.9.2.1 that the functor π^* is 'exact'.

(v") It is easy to verify that the conditions (a) and (b) imply that $\Sigma_{\mathfrak{q}^*}$ is a left multiplicative system.

(v"') For each $M \in ObC_X$, we choose an object $(M,L_M;\mathfrak{t}_M)$ of the category $C_{\mathbb{X}}$ and set $\widetilde{\pi}_*(M) = (M,L_M;\mathfrak{t}_M)$, where $(M,L_M;\mathfrak{t}_M)$ is regarded as an object of the quotient category $\Sigma^{-1}_{\mathfrak{q}^*} C_{\mathbb{X}}$. Thanks to the condition (b), for any morphism $M \xrightarrow{f} N$, there exists a commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \mathfrak{t}_M}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}} \\ L_M & \xrightarrow{\widetilde{f}} & \widetilde{\mathcal{N}} \end{array}$$

with $\mathfrak{e} \in \mathfrak{E}_X$. By the condition (b), there exists a commutative square

$$\begin{array}{ccc} N & \xrightarrow{\mathfrak{t}_N} & L_N \\ {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathfrak{e}}} \\ \widetilde{\mathcal{N}} & \xrightarrow{\mathfrak{t}'} & \mathcal{L} \end{array}$$

whose all arrows are deflations. Thus, we have morphisms

$$(M, L_M; \mathfrak{t}_M) \xrightarrow{(f, \mathfrak{t}' \circ \widetilde{f})} (N, \mathcal{L}; \mathfrak{t}'\mathfrak{e}) \xleftarrow{(id_N, \widetilde{\mathfrak{e}})} (N, L_N; \mathfrak{t}_N)$$

Since the left arrow here belongs to $\Sigma_{\mathfrak{q}^*}$, this pair of morphisms determines a morphism $(M, L_M; \mathfrak{t}_M) = \widetilde{\pi}_*(M) \xrightarrow{\widetilde{\pi}_*(f)} \widetilde{\pi}_*(N) = (N, L_N; \mathfrak{t}_N)$ of $\Sigma^{-1}_{\mathfrak{q}^*} C_{\mathbb{X}}$. The map $\widetilde{\pi}_*$ is a functor $C_X \longrightarrow \Sigma^{-1}_{\mathfrak{q}^*} C_{\mathbb{X}}$. It follows from the construction that $\widetilde{\pi}^* \widetilde{\pi}_* = Id_{C_X}$. On the other hand, it follows from the condition (b) that, for each $(M, L; \mathfrak{e}) \in Ob C_{\mathbb{X}}$, there exists an isomorphism $(M, L; \mathfrak{e}) \longrightarrow \widetilde{\pi}_* \widetilde{\pi}^* (M, L; \mathfrak{e}) = (M, L_M; \mathfrak{t}_M)$. Altogether shows that $\widetilde{\pi}_*$ is a quasi-inverse to the functor $\widetilde{\pi}^*$.

(vi) The constructions of (iv) and (v) assigns, in a functorial way, to each morphism $(Y, \mathfrak{E}_Y) \xrightarrow{\wp} (\mathcal{Z}_0, \mathfrak{E}_{\mathcal{Z}_0})$ of $\mathfrak{L}^*_{\mathfrak{esp}}$) with an inverse image functor $C_{\mathcal{Z}_0} \xrightarrow{\wp^*} C_Y$ two 'exact' localizations, $C_{\mathcal{Z}(\wp)} \xrightarrow{\mathfrak{q}^*} C_{\mathbb{X}}$ and $C_{\mathbb{X}} \xrightarrow{\pi^*} C_X$. Their composition, $\pi^* \mathfrak{q}^*$, is an inverse image functor of an inflation $((X, \mathfrak{E}_X), Y) \xrightarrow{\mathfrak{q}_\wp} ((\mathcal{Z}_\wp, \mathfrak{E}_{\mathcal{Z}_\wp}), \mathcal{Z}_0)$ of infinitesimal right exact 'spaces' which 'lifts' the deflation $(Y, \mathfrak{E}_Y) \xrightarrow{\wp} (\mathcal{Z}_0, \mathfrak{E}_{\mathcal{Z}_0})$ of right exact 'spaces'.

(vii) By the argument of 3.3.2 and the observation 8.8.3.1, we have

$$\begin{aligned} S_-(G \circ \mathfrak{F}^{*^{op}})((X, \mathfrak{E}_X), Y) &= \lim Ker\big(G \circ \mathfrak{F}^{*^{op}}(((X', \mathfrak{E}_{X'}), Y') \xrightarrow{\mathfrak{c}_\mathfrak{q}} ((X'', \mathfrak{E}_{X''}), Y'')\big) \\ &= \lim Ker\big(G((Y', \mathfrak{E}_{Y'}) \xrightarrow{\mathfrak{c}_\wp} (Y'', \mathfrak{E}_{Y''})\big), \end{aligned} \tag{5}$$

where $((X, \mathfrak{E}_X), Y) \xrightarrow{\mathfrak{q}} ((X', \mathfrak{E}_{X'}), Y')$ runs through the category $((X, \mathfrak{E}_X), Y) \backslash \mathfrak{L}^\infty_{\mathfrak{esp}}$ of inflations of $((X, \mathfrak{E}_X), Y)$ and $(Y, \mathfrak{E}_Y) \xrightarrow{\wp} (Y', \mathfrak{E}_{Y'})$ is the inflation of $(Y, \mathfrak{E}_Y)$ determined by the inflation $\mathfrak{q}$. Since the inflation $((X, \mathfrak{E}_X), Y) \xrightarrow{\mathfrak{q}_\wp} ((\mathcal{Z}_\wp, \mathfrak{E}_{\mathcal{Z}_\wp}), \mathcal{Z}_0)$ (see (vi) above) induces the same inflation $(Y, \mathfrak{E}_Y) \xrightarrow{\wp} (Y', \mathfrak{E}_{Y'})$, the arbitrary inflations of $((X, \mathfrak{E}_X), Y)$ can be replaced by the inflations of the form $\mathfrak{q}_\wp$, where $\wp$ runs through the category $(Y, \mathfrak{E}_Y) \backslash \mathfrak{L}^{\mathfrak{r}}_{\mathfrak{esp}}$ of inflations of the right exact category $(Y, \mathfrak{E}_Y)$. But, the limit of $Ker\big(G((Y', \mathfrak{E}_{Y'}) \xrightarrow{\mathfrak{c}_\wp} (Y'', \mathfrak{E}_{Y''})\big)$, where $\wp$ runs through the category $(Y, \mathfrak{E}_Y) \backslash \mathfrak{L}^{\mathfrak{r}}_{\mathfrak{esp}}$ is isomorphic to $S_- G(Y, \mathfrak{E}_Y)$. ∎

8.9.4. The subcategory of 'spaces' $\mathfrak{Esp}^{\rtimes}$. We denote this way the full subcategory of the category $\mathfrak{Esp}^*_{\mathfrak{r}}$ whose objects are right exact 'spaces' $(X, \mathfrak{E}_X)$ such that C_X has final objects and all morphisms to final objects are deflations. Thus, we have a left exact category $(\mathfrak{Esp}^{\rtimes}, \mathfrak{L}^{\rtimes}_{\mathfrak{esp}})$, where $\mathfrak{L}^{\rtimes}_{\mathfrak{esp}}$ is the restriction to the subcategory $\mathfrak{Esp}^{\rtimes}$ of the left exact structure $\mathfrak{L}^{\mathfrak{r}}_{\mathfrak{esp}}$ on $\mathfrak{Esp}^*_{\mathfrak{r}}$.

We denote by $\mathfrak{Esp}^\infty_{\rtimes}$ the full subcategory of the category $\mathfrak{Esp}^\infty_{\mathfrak{r}}$ of right exact infinitesimal 'spaces' generated by all $((X, \mathfrak{E}_X), Y)$ such that $(X, \mathfrak{E}_X)$ is an object of $\mathfrak{Esp}^{\rtimes}$. We denote by $\mathfrak{L}^\infty_{\rtimes}$ the restriction of the left exact structure $\mathfrak{L}^\infty_{\mathfrak{esp}}$ to the subcategory $\mathfrak{Esp}^\infty_{\rtimes}$.

The functors $\mathfrak{F}_*$, $\mathfrak{F}^*$, and $\mathfrak{F}^!$ (cf. 8.8) induce the functors

$$\mathfrak{Esp}^{\rtimes} \xrightarrow{\widetilde{\mathfrak{F}}_*} \mathfrak{Esp}^{\rtimes}_\infty, \quad \mathfrak{Esp}^{\rtimes}_\infty \xrightarrow{\widetilde{\mathfrak{F}}^!} \mathfrak{Esp}^{\rtimes}, \quad \text{and} \quad \mathfrak{Esp}^{\rtimes}_\infty \xrightarrow{\widetilde{\mathfrak{F}}^*} \mathfrak{Esp}^{\rtimes}.$$

8.9.5. Proposition. *Let C_Z be a category with initial objects and limits of filtered systems; and let G be a functor $(\mathfrak{Esp}^{\rtimes}, \mathfrak{L}^{\rtimes}_{\mathfrak{esp}})^{op} \longrightarrow C_Z$. Then the natural morphisms*

$$S_-G \circ \widetilde{\mathfrak{F}}^{!^{op}} \longrightarrow S_-(G \circ \widetilde{\mathfrak{F}}^{!^{op}}) \quad \textit{and} \quad S_-G \circ \widetilde{\mathfrak{F}}^{*^{op}} \longrightarrow S_-(G \circ \widetilde{\mathfrak{F}}^{*^{op}})$$

are isomorphisms. Here the functors $\widetilde{\mathfrak{F}}^!$ and $\widetilde{\mathfrak{F}}^*$ are viewed as morphisms from the left exact category $(\mathfrak{Esp}^{\infty}_{\rtimes}, \mathfrak{L}^{\infty}_{\rtimes})$ to the left exact category $(\mathfrak{Esp}^{\rtimes}, \mathfrak{L}^{\rtimes}_{\mathfrak{esp}})$.

Proof. Fix an object $((X, \mathfrak{E}_X), Y)$ of the category $\mathfrak{Esp}^{\rtimes}_{\infty}$.

(a) By the argument (a) of 8.9.1, the functor $\widetilde{\mathfrak{F}}^!$ induces an isomorphism from the category $((X, \mathfrak{E}_X), Y)\backslash\mathfrak{L}^{\infty}_{\rtimes}$ of inflations of $((X, \mathfrak{E}_X), Y)$ onto the category $(X, \mathfrak{E}_X)\backslash\mathfrak{L}^{\rtimes}_{\mathfrak{esp}}$ of inflations of the object $(X, \mathfrak{E}_X) = \widetilde{\mathfrak{F}}^!((X, \mathfrak{E}_X), Y)$, which implies a canonical isomorphism

$$S_-G(X, \mathfrak{E}_X) = S_-G \circ \widetilde{\mathfrak{F}}^{!^{op}}((X, \mathfrak{E}_X), Y) \longrightarrow S_-(G \circ \widetilde{\mathfrak{F}}^{!^{op}})((X, \mathfrak{E}_X), Y).$$

(b) If $((X, \mathfrak{E}_X), Y)$ is an object of $\mathfrak{Esp}^{\infty}_{\rtimes}$, than any pair of arrows $L \xleftarrow{\mathfrak{e}} M \xrightarrow{f} N$ of C_X is a part of the commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle\mathfrak{e}}\downarrow & & \downarrow{\scriptstyle\mathfrak{t}_N} \\ L & \xrightarrow{\mathfrak{t}_L} & x_\bullet \end{array}$$

where $x_\bullet$ is a final object of C_X, hence, by hypothesis, the arrows $\mathfrak{t}_L$ and $\mathfrak{t}_N$ are deflations. Therefore, the conditions of 8.9.3 hold, which implies that the canonical morphism

$$S_-G(Y, \mathfrak{E}_Y) = S_-G \circ \mathfrak{F}^{*^{op}}((X, \mathfrak{E}_X), Y) \longrightarrow S_-(G \circ \mathfrak{F}^{*^{op}})((X, \mathfrak{E}_X), Y)$$

is an isomorphism for any functor G from $(\mathfrak{Esp}^{\mathfrak{r}}_{\infty}, \mathfrak{L}^{\infty}_{\mathfrak{esp}})^{op}$ to any category C_Z with initial objects and limits of filtered systems. ∎

8.9.6. Corollary. *Let $(C_Z, \mathfrak{E}_Z)$ be a right exact category with initial objects and limits of filtered systems. Then for any universal ∂^*-functor $G_\bullet$ from $(\mathfrak{Esp}^{\rtimes}, \mathfrak{L}^{\rtimes}_{\mathfrak{esp}})^{op}$ to $(C_Z, \mathfrak{E}_Z)$, the compositions $G_\bullet \circ \widetilde{\mathfrak{F}}^{!^{op}}$ and $G_\bullet \circ \widetilde{\mathfrak{F}}^{*^{op}}$ are universal ∂^*-functors from $(\mathfrak{Esp}^{\infty}_{\rtimes}, \mathfrak{L}^{\infty}_{\rtimes})^{op}$ to $(C_Z, \mathfrak{E}_Z)$. If the ∂^*-functor $G_\bullet$ is 'exact', then the ∂^*-functors $G_\bullet \circ \widetilde{\mathfrak{F}}^{!^{op}}$ and $G_\bullet \circ \widetilde{\mathfrak{F}}^{*^{op}}$ are 'exact'.*

Proof. The assertion follows from 8.9.4, 5.3.2, and the fact that $\widetilde{\mathfrak{F}}^!$ and $\widetilde{\mathfrak{F}}^*$ are 'exact' functors from the left exact category $(\mathfrak{Esp}^{\infty}_{\rtimes}, \mathfrak{L}^{\infty}_{\rtimes})$ to the left exact category $(\mathfrak{Esp}^{\rtimes}, \mathfrak{L}^{\rtimes}_{\mathfrak{esp}})$. Details are left to the reader. ∎

8.9.7. Corollary. *Let $G_\bullet$ be a universal ∂^*-functor from $(\mathfrak{Esp}^{\rtimes}, \mathfrak{L}^{\rtimes}_{\mathfrak{esp}})^{op}$ to a category C_Y. Suppose that the functor G maps a natural morphism $\widetilde{\mathfrak{F}}^{*^{op}} \xrightarrow{\rho^{op}_{\mathfrak{F}}} \widetilde{\mathfrak{F}}^{!^{op}}$ to an isomorphism. Then $G_\bullet \circ \widetilde{\mathfrak{F}}^{*^{op}} \xrightarrow{G_\bullet \rho^{op}_{\mathfrak{F}}} G_\bullet \circ \widetilde{\mathfrak{F}}^{!^{op}}$ is an isomorphism of ∂^*-functors. In other words, for any*

object $((X,\mathfrak{E}_X),Y)$ *of the category* $\mathfrak{Esp}^{\rtimes}_{\infty}$*, there is a natural isomorphism* $G_\bullet(Y,\mathfrak{E}_Y)\xrightarrow{\sim} G_\bullet(X,\mathfrak{E}_X)$.

Proof. By 8.9.7, the ∂^*-functors $G_\bullet\circ\widetilde{\mathfrak{F}}^{*^{op}}$ and $G_\bullet\circ\widetilde{\mathfrak{F}}^{!^{op}}$ are universal. Therefore a morphism from $G_\bullet\circ\widetilde{\mathfrak{F}}^{*^{op}}$ to $G_\bullet\circ\widetilde{\mathfrak{F}}^{!^{op}}$ is an isomorphism iff its zero component is an isomorphism, whence the assertion. ∎

8.9.8. Corollary. *Let* $\mathcal{K}^{\rtimes}_\bullet=(\mathcal{K}^{\rtimes}_i,\mathfrak{d}^{\rtimes}_i\mid i\geq 0)$ *be the* K*-functor on the left exact category* $(\mathfrak{Esp}^{\rtimes},\mathfrak{L}^{\times}_{\mathfrak{esp}})$ *of right exact 'spaces'; that is* $\mathcal{K}^{\rtimes}_\bullet$ *is a universal* ∂^**-functor from* $(\mathfrak{Esp}^{\rtimes},\mathfrak{L}^{\rtimes}_{\mathfrak{esp}})^{op}$ *whose zero component is* $(X,\mathfrak{E}_X)\longmapsto K_0(X,\mathfrak{E}_X)$*. Then* $\mathcal{K}^{\rtimes}_\bullet\circ\widetilde{\mathfrak{F}}^{!^{op}}$ *and* $\mathcal{K}^{\rtimes}_\bullet\circ\widetilde{\mathfrak{F}}^{*^{op}}$ *are 'exact', universal* ∂^**-functors from* $(\mathfrak{Esp}^{\infty}_{\rtimes},\mathfrak{L}^{\infty}_{\rtimes})^{op}$ *to* $\mathbb{Z}-mod$.

Proof. This is a special case of 8.9.6. ∎

8.10. The k-linear setting. Instead of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}},\mathfrak{L}^{\mathfrak{r}}_{\mathfrak{esp}})$ of right exact 'spaces', we consider the left exact category $(\mathcal{E}\mathfrak{sp}^{\mathfrak{r}}_k,\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k)$ of right exact k-linear 'spaces' (cf. Q8.6). Here $\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k$ is the left exact structure induced by $\mathfrak{L}^{\mathfrak{r}}_{\mathfrak{esp}}$; i.e. $\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k=\mathfrak{F}_{\mathfrak{r}}^{-1}(\mathfrak{L}^{\mathfrak{r}}_{\mathfrak{esp}})$, where $\mathfrak{F}_{\mathfrak{r}}$ is the natural forgetful functor $\mathfrak{Esp}^{\mathfrak{r}}_k\longrightarrow\mathfrak{Esp}_{\mathfrak{r}}$.

8.10.1. The left exact category of right exact infinitesimal k-'spaces'. A *right exact infinitesimal k-'space'* is a pair $((X,\mathfrak{E}_X),Y)$, where $(C_X,\mathfrak{E}_X)$ is a right exact k-linear category and C_Y a topologizing subcategory of $(C_X,\mathfrak{E}_X)$ such that $C_X=(C_Y)_{(\infty)}$. A morphism from $((X,\mathfrak{E}_X),Y)$ to $((\mathfrak{X},\mathfrak{E}_{\mathfrak{X}}),\mathfrak{Y})$ is a morphism $(X,\mathfrak{E}_X)\xrightarrow{f}(\mathfrak{X},\mathfrak{E}_{\mathfrak{X}})$ right exact k-'spaces' which maps Y to $\mathfrak{Y}$; i.e. f^* is a k-linear 'exact' functor from $(C_{\mathfrak{X}},\mathfrak{E}_{\mathfrak{X}})$ to $(C_X,\mathfrak{E}_X)$ such that $f^*(C_{\mathfrak{Y}})\subseteq C_Y$. This defines a category denoted by $\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}$. The left exact structure $\mathfrak{L}^{\infty}_{\mathfrak{esp}}$ on $\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$ induces (via the forgetful functor $\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}\longrightarrow\mathfrak{Esp}^{\mathfrak{r}}_{\infty}$, a left exact structure, $\mathfrak{L}^{\infty}_{\mathfrak{r},k}$, on $\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}$.

We denote by $\mathfrak{F}_{k*}$ the embedding $\mathfrak{Esp}^{\mathfrak{r}}_k\longrightarrow\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}$ which assigns to each object $(X,\mathfrak{E}_X)$ the object $((X,\mathfrak{E}_X),X)$ of the category $\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}$. This functor is fully faithful and has a left adjoint

$$\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}\xrightarrow{\mathfrak{F}^*_k}\mathfrak{Esp}^{\mathfrak{r}}_k,\quad((X,\mathfrak{E}_X),Y)\longmapsto(Y,\mathfrak{E}_Y),$$

and a right adjoint

$$\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}\xrightarrow{\mathfrak{F}^!_k}\mathfrak{Esp}^{\mathfrak{r}}_k,\quad((X,\mathfrak{E}_X),Y)\longmapsto(X,\mathfrak{E}_X).$$

All three functors, $\mathfrak{F}_{k*}$, $\mathfrak{F}^*_k$, and $\mathfrak{F}^!_k$, are 'exact'.

8.10.2. Proposition. *Let* C_Z *be a* k*-linear category with limits of filtered systems; and let* G *be a functor* $(\mathfrak{Esp}^{\mathfrak{r}}_k,\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k)^{op}\longrightarrow C_Z$*. Then the natural morphisms*

$$S_-G\circ\mathfrak{F}^{!^{op}}_k\longrightarrow S_-(G\circ\mathfrak{F}^{!^{op}}_k)\quad\textit{and}\quad S_-G\circ\mathfrak{F}^{*^{op}}_k\longrightarrow S_-(G\circ\mathfrak{F}^{*^{op}}_k)$$

are isomorphisms. Here the functors $\mathfrak{F}^!$ *and* $\mathfrak{F}^*$ *are viewed as morphisms from the left exact category* $(\mathfrak{Esp}^{\infty}_{\mathfrak{r},k},\mathfrak{L}^{\infty}_{\mathfrak{r},k})$ *to the left exact category* $(\mathfrak{Esp}^{\mathfrak{r}}_k,\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k)$.

Proof. Fix an object $((X,\mathfrak{E}_X),Y)$ of the category $\mathfrak{Esp}^{\infty}_{\mathfrak{r},k}$.

(a) By the (k-linear version of the) argument (a) of 9.9.1, the functor $\mathfrak{F}^!$ induces an isomorphism from the category $((X,\mathfrak{E}_X),Y)\backslash\mathfrak{L}^\infty_{\mathfrak{esp}}$ of inflations of $((X,\mathfrak{E}_X),Y)$ onto the category $(X,\mathfrak{E}_X)\backslash\mathfrak{L}^*_{\mathfrak{esp}}$ of inflations of the the object $(X,\mathfrak{E}_X)=\mathfrak{F}^!((X,\mathfrak{E}_X),Y)$, which implies a canonical isomorphism

$$S_-G(X,\mathfrak{E}_X)=S_-G\circ\mathfrak{F}_k^{!^{op}}((X,\mathfrak{E}_X),Y)\longrightarrow S_-(G\circ\mathfrak{F}_k^{!^{op}})((X,\mathfrak{E}_X),Y).$$

(b) Similarly, the k-linear version of the arguments of 8.9.5 and 8.9.3 shows that the canonical morphism $S_-G\circ\mathfrak{F}_k^{*^{op}}\longrightarrow S_-(G\circ\mathfrak{F}_k^{*^{op}})$ is an isomorphism. ∎

8.10.3. Corollary. *Let $(C_Z,\mathfrak{E}_Z)$ be a k-linear right exact category with limits of filtered systems. Then for any universal ∂^*-functor $G_\bullet$ from $(\mathfrak{Esp}^{\mathfrak{r}}_k,\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k)^{op}$ to $(C_Z,\mathfrak{E}_Z)$, the compositions $G_\bullet\circ\mathfrak{F}_k^{!^{op}}$ and $G_\bullet\circ\mathfrak{F}_k^{*^{op}}$ are universal ∂^*-functors from $(\mathfrak{Esp}^\infty_{\mathfrak{r},k},\mathfrak{L}^\infty_{\mathfrak{r},k})^{op}$ to $(C_Z,\mathfrak{E}_Z)$. If the ∂^*-functor $G_\bullet$ is 'exact', then the ∂^*-functors $G_\bullet\circ\mathfrak{F}_k^{!^{op}}$ and $G_\bullet\circ\mathfrak{F}_k^{*^{op}}$ are 'exact'.*

Proof. See the argument of 8.9.6. ∎

8.10.4. Corollary. *Let $\mathcal{K}^{\mathfrak{r}}_\bullet=(\mathcal{K}^{\mathfrak{r}}_i,\mathfrak{d}^{\mathfrak{r}}_i\mid i\geq 0)$ be the K-functor on the left exact category $(\mathfrak{Esp}^{\mathfrak{r}}_k,\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k)$ of right exact k-linear 'spaces'; that is $\mathcal{K}^{\mathfrak{r}}_\bullet$ is a universal ∂^*-functor from $(\mathfrak{Esp}^{\mathfrak{r}}_k,\widetilde{\mathfrak{L}}^{\mathfrak{r}}_k)^{op}$ whose zero component is $(X,\mathfrak{E}_X)\longmapsto K_0(X,\mathfrak{E}_X)$. Then $\mathcal{K}^{\mathfrak{r}}_\bullet\circ\mathfrak{F}_k^{!^{op}}$ and $\mathcal{K}^{\mathfrak{r}}_\bullet\circ\mathfrak{F}_k^{*^{op}}$ are 'exact', universal ∂^*-functors from $(\mathfrak{Esp}^\infty_{\mathfrak{r},k},\mathfrak{L}^\infty_{\mathfrak{r},k})^{op}$ to $\mathbb{Z}-mod$.*

Proof. This is a special case of 8.10.3. ∎

8.11. An application to K-functors: devissage.

8.11.1. Proposition. (Devissage for K_0.) *Let $((X,\mathfrak{E}_X),Y)$ be an infinitesimal 'space' such that $(X,\mathfrak{E}_X)$ has the following property (which appeared in 8.4.1):*

(‡) *for any pair of deflations $M_1\xleftarrow{\mathfrak{t}_1}M\xrightarrow{\mathfrak{t}_2}M_2$, there is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{t}_1} & M_1 \\ {\scriptstyle\mathfrak{t}_2}\downarrow & & \downarrow{\scriptstyle\mathfrak{p}_2} \\ M_2 & \xrightarrow{\mathfrak{p}_1} & M_3 \end{array}$$

of deflations such that the unique morphism $M\longrightarrow M_1\times_{M_3}M_2$ is a deflation.

Then the natural morphism

$$K_0(Y,\mathfrak{E}_Y)\longrightarrow K_0(X,\mathfrak{E}_X) \tag{1}$$

is an isomorphism.

Proof. Let M be an object of C_X. Since $C_X=(C_Y)_{(\infty)}$, there exists a C_Y-cofiltration

$$M=M_n\xrightarrow{\mathfrak{e}_n}\ldots\xrightarrow{\mathfrak{e}_2}M_1\xrightarrow{\mathfrak{e}_1}M_0. \tag{2}$$

That is the arrows of (2) are deflations such that the object M_0 and objects $Ker(\mathfrak{e}_i)$ belong to the subcategory C_Y for every $1 \leq i \leq n$. Since the subcategory C_Y is (semi)topological, any refinement of a C_Y-cofiltration is a C_Y-cofiltration.

(a1) The map

$$[M] \longmapsto [M_0]_{C_Y} + \sum_{1 \leq i \leq n} [Ker(\mathfrak{e}_i)]_{C_Y} \tag{3}$$

applied to a refinement of the cofiltration (2) gives the same result. Here $[N]_{C_Y}$ denotes the image of the object N in $K_0(Y)$.

In fact, for any sequence of deflations $\mathcal{M}_m \xrightarrow{\mathfrak{t}_m} \ldots \xrightarrow{\mathfrak{t}_2} \mathcal{M}_1 \xrightarrow{\mathfrak{t}_1} \mathcal{M}_0$, we have a commutative diagram

$$\begin{array}{ccccccccccc}
\mathfrak{K}_m & \xrightarrow{\widetilde{\mathfrak{t}}_m} & \mathfrak{K}_{m-1} & \xrightarrow{\widetilde{\mathfrak{t}}_{m-1}} & \ldots & \xrightarrow{\widetilde{\mathfrak{t}}_3} & \mathfrak{K}_2 & \xrightarrow{\widetilde{\mathfrak{t}}_2} & \mathfrak{K}_1 & \xrightarrow{\widetilde{\mathfrak{t}}_1} & x \\
\downarrow & cart & \downarrow & cart & \ldots & cart & \downarrow & cart & \downarrow & cart & \downarrow \\
\mathcal{M}_m & \xrightarrow{\mathfrak{t}_m} & \mathcal{M}_{m-1} & \xrightarrow{\mathfrak{t}_{m-1}} & \ldots & \xrightarrow{\mathfrak{t}_3} & \mathcal{M}_2 & \xrightarrow{\mathfrak{t}_2} & \mathcal{M}_1 & \xrightarrow{\mathfrak{t}_1} & \mathcal{M}_0
\end{array} \tag{4}$$

formed by cartesian squares. Here x is an initial object of the category C_X. Since the 'composition' of cartesian squares is a cartesian square, it follows that $\mathfrak{K}_1 = Ker(\mathfrak{t}_1)$, $\mathfrak{K}_2 = Ker(\mathfrak{t}_1\mathfrak{t}_2)$, $\ldots$, $\mathfrak{K}_m = Ker(\mathfrak{t}_1\mathfrak{t}_2 \ldots \mathfrak{t}_m)$. Since each square

$$\begin{array}{ccc}
\mathfrak{K}_\ell & \xrightarrow{\widetilde{\mathfrak{t}}_\ell} & \mathfrak{K}_{\ell-1} \\
\downarrow & cart & \downarrow \\
\mathcal{M}_\ell & \xrightarrow{\mathfrak{t}_\ell} & \mathcal{M}_{\ell-1}
\end{array}$$

of the diagram (4) is cartesian, all morphisms $\widetilde{\mathfrak{t}}_\ell$ are deflations and $Ker(\widetilde{\mathfrak{t}}_\ell) \simeq Ker(\mathfrak{t}_\ell)$ for all $1 \leq \ell \leq m$. Therefore,

$$[Ker(\mathfrak{t}_1\mathfrak{t}_2 \ldots \mathfrak{t}_m)] = [\mathfrak{K}_m] = \sum_{1 \leq i \leq n} [Ker(\widetilde{\mathfrak{t}}_i)] = \sum_{1 \leq i \leq n} [Ker(\mathfrak{t}_i)]. \tag{5}$$

This shows that the right hand side of (3) remains the same when each of the deflations $\mathfrak{e}_i$ is futher decomposed into a sequence of deflations.

(a2) By 8.4.1, any two finite cofiltrations of an object of C_X have equivalent refinements. Together with (a1) above, this implies that the map (3) does not depend on the choice of C_Y-cofiltrations of objects. Thus, (3) defines a map, $\widetilde{\psi}$, from the set $|C_X|$ of isomorphism classes of objects of the category C_X to the group $K_0^*(Y, \mathfrak{E}_Y)$.

(a3) For any conflation $M' \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} M''$ in $(C_X, \mathfrak{E}_X)$, we have

$$\widetilde{\psi}([M]) = \widetilde{\psi}([M']) + \widetilde{\psi}([M'']).$$

Indeed, let $M \longrightarrow \ldots \longrightarrow M_0$ be some C_Y-cofiltration of M. By 8.4.1, this cofiltration and the cofiltration $M \xrightarrow{\mathfrak{e}} M''$ have equivalent refinements which are, forcibly, C_Y-cofiltrations. Consider the obtained this way refinement

$$M = M_n \xrightarrow{\mathfrak{e}_n} \ldots \xrightarrow{\mathfrak{e}_m} M_{m-1} = M'' \xrightarrow{\mathfrak{e}_m} \ldots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$$

and the associated commutative diagram

$$\begin{array}{ccccccccccc}
\mathfrak{K}_n & \xrightarrow{\widetilde{\mathfrak{e}}_n} & \mathfrak{K}_{n-1} & \xrightarrow{\widetilde{\mathfrak{e}}_{n-1}} & \dots & \xrightarrow{\widetilde{\mathfrak{e}}_m} & x & & & & \\
\downarrow & cart & \downarrow & cart & \dots & cart & \downarrow & & & & \\
M & \xrightarrow{\mathfrak{e}_n} & \mathcal{M}_{n-1} & \xrightarrow{\mathfrak{e}_{n-1}} & \dots & \xrightarrow{\mathfrak{e}_m} & M'' & \xrightarrow{\mathfrak{e}_{m-1}} & \dots & \xrightarrow{\mathfrak{e}_1} & \mathcal{M}_0
\end{array} \tag{6}$$

built of cartesian squares. Here x is an initial object of the category C_X. Since $M \xrightarrow{\mathfrak{e}} M''$ equals to the composition $\mathfrak{e}_m \circ \ldots \circ \mathfrak{e}_n$, it follows from the argument of (a1) that $\mathfrak{K}_n \simeq M' = Ker(\mathfrak{e})$ and the upper row of (6) is a C_Y-filtration of the object M'. The latter implies that

$$\widetilde{\psi}([M']) = \sum_{m \le i \le n} [Ker(\widetilde{\mathfrak{e}}_i)]_{C_Y} = \sum_{m \le i \le n} [Ker(\mathfrak{e}_i)]_{C_Y}$$

(see (5) above). From the lower row, we obtain

$$\widetilde{\psi}([M'']) = [M_0]_{C_Y} + \sum_{m < i \le n} [Ker(\mathfrak{e}_i)]_{C_Y} \quad \text{and}$$
$$\widetilde{\psi}([M]) = [M_0]_{C_Y} + \sum_{1 \le i \le n} [Ker(\mathfrak{e}_i)]_{C_Y}.$$

Therefore, $\widetilde{\psi}([M]) = \widetilde{\psi}([M']) + \widetilde{\psi}([M''])$.

(a4) The map $|C_X| \xrightarrow{\widetilde{\psi}} K_0(Y, \mathfrak{E}_Y)$ extends uniquely to a $\mathbb{Z}$-module morphism

$$\mathbb{Z}|C_X| \xrightarrow{\mathbb{Z}\widetilde{\psi}} K_0(Y, \mathfrak{E}_Y). \tag{7}$$

It follows from (a3) that the morphism (7) factors through a (uniquely determined) $\mathbb{Z}$-module morphism

$$K_0(X, \mathfrak{E}_X) \xrightarrow{\psi_0} K_0(Y, \mathfrak{E}_Y).$$

The claim is that the morphism ψ_0 is invertible and its inverse is

$$K_0(Y, \mathfrak{E}_Y) \xrightarrow{K_0(\mathrm{j})} K_0(X, \mathfrak{E}_X).$$

It is immediate that $\psi_0 \circ K_0(\mathrm{j}) = id_{K_0(Y, \mathfrak{E}_Y)}$.

The equality $K_0(\mathrm{j}) \circ \psi_0 = id_{K_0(X, \mathfrak{E}_X)}$ is also easy to see.: if M is an object of C_X endowed with a C_Y-cofiltration $M = M_n \xrightarrow{\mathfrak{e}_n} \ldots \xrightarrow{\mathfrak{e}_2} M_1 \xrightarrow{\mathfrak{e}_1} M_0$, then

$$K_0(\mathrm{j}) \circ \psi_0([M]) = K_0(\mathrm{j})([M_0]_{C_Y} + \sum_{1 \le i \le n} [Ker(\mathfrak{e}_i)]_{C_Y}) = [M_0] + \sum_{1 \le i \le n} [Ker(\mathfrak{e}_i)] = [M].$$

This proves the assertion. ∎

8.11.2. The ∂^*-functor $K^{\mathfrak{sq}}_\bullet$. Let $\mathfrak{L}^{\mathfrak{es}}_{\rtimes}$ denote the left exact structure on the category $\mathfrak{Esp}^{\rtimes}$ of $\mathfrak{Esp}_{\mathfrak{r}}$ (cf. 8.9.4) induced by the (defined in 6.8.3.3) left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$ on the category $\mathfrak{Esp}_{\mathfrak{r}}$ of right exact 'spaces'. Let $K^{\mathfrak{sq}}_i(X, \mathfrak{E}_X)$ denote the i-th satellite of the functor K_0 with respect to the left exact structure $\mathfrak{L}^{\mathfrak{es}}_{\rtimes}$.

8.11.3. Proposition. *Let $((X, \mathfrak{E}_X), Y)$ be an infinitesimal 'space' such that the right exact 'space' $(X, \mathfrak{E}_X)$ has the property (‡) of 9.11.1, the category C_X has final objects, and all morphisms to final objects are deflations. Then the natural morphism*

$$K^{\mathfrak{sq}}_i(Y, \mathfrak{E}_Y) \longrightarrow K^{\mathfrak{sq}}_i(X, \mathfrak{E}_X) \tag{8}$$

is an isomorphism for all $i \geq 0$.

Proof. (a) Consider the full subcategory $\mathfrak{Esp}^{\infty}_{\mathfrak{r},\ddagger}$ of the category $\mathfrak{Esp}^{\infty}_{\mathfrak{r}}$ of infinitesimal 'spaces' whose objects are infinitesimal 'spaces' $((X, \mathfrak{E}_X), Y)$ such that $(X, \mathfrak{E}_X)$ satisfies the property (‡). The claim is that for any functor G from $(\mathfrak{Esp}^{\infty}_{\mathfrak{r}}, \mathfrak{L}^{\mathfrak{es}^{\infty}}_{\mathfrak{sq}})$ to a category $C_{\mathcal{Z}}$ with filtered limits, the satellite of the composition of G with the inclusion functor $\mathfrak{Esp}^{\infty}_{\mathfrak{r},\ddagger} \longrightarrow \mathfrak{Esp}^{\infty}_{\mathfrak{r}}$ is naturally isomorphic to the composition of the satellite of G with the inclusion functor.

In fact, it is easy to see that if the right exact 'space' $(X, \mathfrak{E}_X)$ has the property (‡), then this property holds for the right exact 'space' $\mathfrak{Pa}(X, \mathfrak{E}_X)$ of paths of $(X, \mathfrak{E}_X)$. By 6.11.1, $\mathfrak{Pa}(X, \mathfrak{E}_X)$ is an injective object of the left exact category $(\mathfrak{Esp}_{\mathfrak{r}}, \mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}})$ and the canonical morphism $(X, \mathfrak{E}_X) \longrightarrow \mathfrak{Pa}(X, \mathfrak{E}_X)$ is an inflation.

(b) One can check that 8.9.5 (hence its corollary 8.9.6) remains true, when the left exact structure $\mathfrak{L}^{\rtimes}_{\mathfrak{esp}}$ is replaced by the left exact structure on $\mathfrak{Esp}^{\rtimes}$ induced by $\mathfrak{L}^{\mathfrak{es}}_{\mathfrak{sq}}$.

(c) The assertion follows now from (a) above, 8.11.1, and 8.9.6. ∎

Complementary facts.

C1. Complements on kernels and cokernels.

C1.1. Kernels of morphisms of 'spaces'. The category $|Cat|^o$ of 'spaces' has an initial object x represented by the category with one object and one (identical) morphism. By [KR, 2.2], the category $|Cat|^o$ has small limits (and colimits). In particular, any morphism of $|Cat|^o$ has a kernel. The kernel of a morphism $X \stackrel{f}{\longrightarrow} Y$ of $|Cat|^o$ can be explicitly described as follows.

Let $C_Y \stackrel{f^*}{\longrightarrow} C_X$ be an inverse image functor of f. For any two objects L, M of the category C_X, we denote by $\mathfrak{I}_f(L,M)$ the set of all arrows $L \longrightarrow M$ which factor through an object of the subcategory $f^*(C_Y)$. The class $\mathfrak{I}_f$ of arrows of C_X obtained this way is a two-sided ideal; i.e. it is closed under compositions on both sides with arbitrary arrows of C_X. We denote by C_{X_f} the quotient of the category C_X by the ideal $\mathfrak{I}_f$; that is $ObC_{X_f} = ObC_X$, $C_{X_f}(L,M) = C_X(L,M)/\mathfrak{I}_f(L,M)$ for all objects L, M, and the compostion is induced by the composition in C_X. Each object M of the image of the subcategory $f^*(C_Y)$ in C_{X_f} has the property that $C_{X_f}(L,M)$ and $C_{X_f}(M,L)$ consist of at most one arrow. This allows to define a category $C_{K(f)}$ by replacing the image of $f^*(C_Y)$ by one object z and one morphism, id_z. (i.e. $ObC_{K(f)} = ObC_X/f^*(C_Y)$). If objects L and M are not equal to z, then we set $C_{K(f)}(L,M) = C_{X_f}(L,M)$. The set $C_{K(f)}(L,z)$ (resp. $C_{K(f)}(z,M)$) consists of one element iff there exists a morphism from L to an object of $f^*(C_Y)$ (resp. from an object of $f^*(C_Y)$ to M); otherwise, it is empty.

We denote by $\mathfrak{k}(f)^*$ the natural projection $C_X \longrightarrow C_{K(f)}$. Thus, we have a commutative square of functors

$$\begin{array}{ccc} C_{K(f)} & \stackrel{\mathfrak{c}(f)^*}{\longleftarrow} & C_X \\ \pi_z^* \uparrow & & \uparrow f^* \\ C_x & \longleftarrow & C_Y \end{array}$$

where π_z^* maps the unique object of C_x to z. This square corresponds to a cartesian square

$$\begin{array}{ccc} K(f) & \stackrel{\mathfrak{c}(f)}{\longrightarrow} & X \\ \pi_z \downarrow & & \downarrow f \\ x & \longrightarrow & Y \end{array}$$

of morphisms of 'spaces'; i.e. the morphism $K(f) \stackrel{\mathfrak{c}(f)}{\longrightarrow} X$ is the kernel of $X \stackrel{f}{\longrightarrow} Y$.

Similarly to $Sets$, the category $|Cat|^o$ has a unique final object represented by the empty category. Since there are no functors from non-empy categories to the empty category, the cokernel of any morphism of $|Cat|^o$ is the unique morphism to the final object.

C1.2. Kernels and cokernels of morphisms of relative objects. Fix an object V of a category C_X and consider the category C_X/V. This category has a final object, (V, id_V), so we can discuss cokernels of its morphisms. Notice that the forgetful functor $C_X/V \longrightarrow C_X$ is exact, in particular, it preserves push-forwards. Therefore, the kernel

of a morphism $(M,g) \xrightarrow{f} (N,h)$ exists iff a push-forward $N \coprod\limits_{M} V = N \coprod\limits_{f,g} V$ exists and is equal to $(N \coprod\limits_{M} V, h')$, where $N \coprod\limits_{M} V \xrightarrow{h'} V$ is determined by $N \xrightarrow{h} V$.

C1.2.1. Kernels. If the category C_X has an initial object x, then $(x, x \to V)$ is an initial object of the category C_X/V. The forgetful functor $C_X/V \longrightarrow C_X$ preserves pull-backs; in particular, it preserves kernels of morphisms. So that the kernel of a morphism $(M,g) \xrightarrow{f} (N,h)$ exists iff the kernel $Ker(f) \xrightarrow{\mathfrak{k}(f)} M$ of $M \xrightarrow{f} N$ exists; and it is equal to $(Ker(f), g \circ \mathfrak{k}(f)) \xrightarrow{\mathfrak{k}(f)} (M,g)$.

C1.3. Application: cokernels of morphisms of relative 'spaces'. Fix a 'space' $\mathcal{S}$ and consider the category $|Cat|^o/\mathcal{S}$ of 'spaces' over $\mathcal{S}$. According to C1.2, the cokernel of a morphism $(X,g) \xrightarrow{f} (Y,h)$ of 'spaces' over $\mathcal{S}$ is the pair $(Cok(f), \widetilde{h})$, where $C_{Cok(f)}$ is the pull-back (in the pseudo-categorical sense) of the pair of inverse image functors $C_{\mathcal{S}} \xrightarrow{g^*} C_X \xleftarrow{h^*} C_Y$. That is objects of the category $C_{Cok(f)}$ are triples $(M,N;\phi)$, where $M \in ObC_{\mathcal{S}}$, $N \in ObC_Y$ and ϕ an isomorphism $g^*(M) \xrightarrow{\sim} f^*(N)$. Morphisms from $(M,N;\phi)$ to $(M',N';\phi')$ are given by a pair of arrows $M \xrightarrow{\mathfrak{u}} M'$, $N \xrightarrow{\mathfrak{v}} N'$ such that the square

$$\begin{array}{ccc} g^*(M) & \xrightarrow{g^*(\mathfrak{u})} & g^*(M') \\ \phi \big\downarrow \wr & & \wr \big\downarrow \phi' \\ f^*(N) & \xrightarrow{f^*(\mathfrak{v})} & f^*(N') \end{array}$$

commutes. The functor $C_{\mathcal{S}} \xrightarrow{\widetilde{h}^*} C_{Cok(f)}$ which assigns to every object L of the category $C_{\mathcal{S}}$ the object $(g^*(L), h^*(L); \psi(L))$, where ψ is an isomorphism $g^* \xrightarrow{\sim} f^*h^*$, is an inverse image functor of the morphism $\widetilde{h}$.

C1.4. Categories with initial objects and associated pointed categories. Let C_X be a category with an initial object, x. Then the category $C_{X_x} = C_X/x$ is a pointed category with a zero object (x, id_x).

C1.4.1. Example: augmented monads. Let $C_{\mathfrak{X}}$ be the category $\mathfrak{Mon}(X)$ of monads on the category C_X. The category $C_{\mathfrak{X}}$ has a canonical initial object x which is the identical monad (Id_{C_X}, id). The category $C_{\mathfrak{X}_x}$ coincides with the category $\mathfrak{Mon}^+(X)$ of *augmented* monads. Its objects are pairs $(\mathcal{F}, \epsilon)$, where $\mathcal{F} = (F, \mu)$ is a monad on C_X and ϵ is a monad morphism $\mathcal{F} \longrightarrow (Id_{C_X}, id)$ called an *augmentation morphism*. One can see that a functor morphism $F \xrightarrow{\epsilon} Id_{C_X}$ is an augmentation morphism iff $(M, \epsilon(M))$ is an $\mathcal{F}$-module morphism for every $M \in ObC_X$. In other words, there is a bijective correspondence between augmentation morphisms and sections $C_X \longrightarrow \mathcal{F} - mod$ of the forgetful functor $\mathcal{F} - mod \xrightarrow{f_*} C_X$.

C1.5. Pointed category of 'spaces'. Consider first a simpler case – the category Cat^{op}. It has an initial object, x, which is represented by the category with one object and

one (identical) morphism. The associated pointed category Cat^{op}/x is equivalent to the category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a 'space' and $\mathcal{O}_X$ an object of the category C_X representing X. Morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ are pairs (f^*, ϕ), where f^* is a functor $C_Y \longrightarrow C_X$ and ϕ is an isomorphism $f^*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$. The composition of $(X, \mathcal{O}_X) \xrightarrow{(f^*,\phi)} (Y, \mathcal{O}_Y) \xrightarrow{(g^*,\psi)} (Z, \mathcal{O}_Z)$ is given by $(g^*, \psi) \circ (f^*, \phi) = (f^* \circ g^*, \phi \circ f^*(\psi))$.

The pointed category $|Cat|^o/x$ associated with the category of 'spaces' $|Cat|^o$ admits a similar realization after fixing a pseudo-functor

$$|Cat|^o \longrightarrow Cat^{op}, \quad X \longmapsto C_X,\ f \longmapsto f^*;\ (gf)^* \xrightarrow{\mathfrak{c}_{f,g}} f^*g^*,$$

– a section of the natural projection $Cat^{op} \longrightarrow |Cat|^o$. Namely, it is equivalent to a category $|Cat|^o_x$ whose objects are (as above) pairs $(X, \mathcal{O}_X)$, where $\mathcal{O}_X \in ObC_X$, morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ are pairs (f, ϕ), where f is a morphism of 'spaces' $X \longrightarrow Y$ and ϕ is an isomorphism $f^*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$. The composition of $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y) \xrightarrow{(g,\psi)} (Z, \mathcal{O}_Z)$ is the morphism $(g \circ f, \phi \circ f^*(\psi) \circ \mathfrak{c}_{f,g})$.

C1.5.1. Cokernels of morphisims. One can deduce from the description of cokernels in C1.3 in terms of the realization of the category $|Cat|^o_x$ given above, that the cokernel of a morphism $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y)$ is isomorphic to $(Y, \mathcal{O}_Y) \xrightarrow{(\mathfrak{c}(f),\psi)} (\mathfrak{C}(f), \mathcal{O}_{\mathfrak{C}(f)})$, where $C_{\mathfrak{C}(f)}$ is a subcategory of C_Y whose objects are $M \in ObC_Y$ such that $f^*(M) \simeq \mathcal{O}_Y$ and morphisms are all arrows between these objects which f^* transforms into isomorphisms. The 'structure' object $\mathcal{O}_{\mathfrak{C}(f)}$ coincides with $\mathcal{O}_Y$; the inverse image functor of $\mathfrak{c}(f)$ is the inclusion functor $C_{\mathfrak{C}(f)} \longrightarrow C_Y$; and the isomorphism ψ is identical.

C1.6. The category of k-'spaces'. We call 'spaces' represented by k-linear additive categories *k-spaces*. We denote by $|Cat_k|^o$ the category whose objects are k-'spaces' and morphisms $X \longrightarrow Y$ are isomorphism classes of k-linear functors $C_Y \longrightarrow C_X$. The category $|Cat_k|^o$ is pointed: its zero object is represented by the zero category. It is easy to see that every morphism $X \xrightarrow{f} Y$ has a canonical cokernel $Y \xrightarrow{\mathfrak{c}} Cok(f)$, where $C_{Cok(f)}$ is the subcategory $Ker(f^*)$ of C_Y (– the full subcategory generated by all objects L such that $f^*(L) = 0$) and $\mathfrak{c}^*$ is the inclusion functor $Ker(f^*) \longrightarrow C_Y$.

The kernel $Ker(f) \xrightarrow{\mathfrak{k}(f)} X$ of f admits a simple description which is a linear version of the one in C1.1. Namely, $C_{Ker(f)}$ is the quotient of the category C_X by the ideal $\mathfrak{I}_f$ formed by all morphisms of C_X which factor through objects of $f^*(C_Y)$. The inverse image of $\mathfrak{k}(f)^*$ is the canonical projection $C_X \longrightarrow C_X/\mathfrak{I}_f$.

C1.6.1. k-'Spaces' over $\mathbf{Sp}(k)$. Consider now the full subcategory $|Cat_k|^o_{\mathbf{Sp}(k)}$ of the category of k-'spaces' over the affine scheme $\mathbf{Sp}(k)$ whose objects are pairs (X, f) where $X \xrightarrow{f} \mathbf{Sp}(k)$ is continuous (i.e. f^* has a right adjoint, f_*). This category admits a realization in the style of C1.5. Namely, it is equvalent to the category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a k-'space' and $\mathcal{O}_X$ is an object of the category C_X such that there exist infinite coproducts of copies of $\mathcal{O}_X$ and cokernels of morphisms between these coproducts. Morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ are pairs (f, ϕ), where f^* is a

k-linear functor $C_Y \longrightarrow C_X$ and ϕ an isomorphism $f^*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$. The composition is defined as in C1.5 (see [KR, 4.5]). By C1.2, kernels of morphisms (as well as other limits) are inherited from $|Cat_k|^o$. That is the kernel of a morphism $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y)$ is the morphism $(Ker(f), \mathcal{O}_{Ker(f)}) \xrightarrow{(\mathfrak{k}(f), id)} (X, \mathcal{O}_X)$, where $C_{Ker(f)} = C_X/\mathfrak{I}_f$, $\mathfrak{k}(f)^*$ is the canonical projection $C_X \longrightarrow C_X/\mathfrak{I}_f$, and $\mathcal{O}_{Ker(f)}$ is the image of $\mathcal{O}_X$.

The cokernel $(Y, \mathcal{O}_Y) \xrightarrow{(\mathfrak{c}(f), \psi)} (\mathfrak{C}_f, \mathcal{O}_{\mathfrak{C}_f})$ of (f, ϕ) is described following C1.3. Objects of the category $C_{\mathfrak{C}_f}$ are triples $(M, N; \phi)$, where $M \in ObC_Y$, $N \in Obk - mod$, and α is an isomorphism $f^*(M) \longrightarrow \gamma^*(N)$. Here γ^* is a functor $k - mod \longrightarrow C_X$ which maps k to $\mathcal{O}_X$ and preserves colimits (which determines γ^* uniquely up to isomorphism). Morphisms are defined as in C1.3. The structure object $\mathcal{O}_{\mathfrak{C}_f}$ is $(\mathcal{O}_Y, k, \phi)$. The inverse image functor $\mathfrak{c}(f)^*$ of $\mathfrak{c}(f)$ is the projection $(M, N; \alpha) \longmapsto M$.

C1.7. The (bi)categories $Cat_\star$ and $Cat_{\mathfrak{pt}}$. Let $Cat_\star$ denote the category whose objects are pairs (C_X, x), where C_X is a category and x its initial object; morphisms $(C_X, x) \longrightarrow (C_Y, y)$ are pairs (F, ϕ), where F is a functor $C_X \longrightarrow C_Y$ and ϕ a morphism $F(x) \longrightarrow y$. The composition of two morphisms, $C_X, x) \xrightarrow{(F,\phi)} (C_Y, y) \xrightarrow{(G,\gamma)} (C_Z, z)$, is given by $(G, \gamma) \circ (F, \phi) = (G \circ F, \gamma \circ G(\phi))$.

Every morphism $C_X, x) \xrightarrow{(F,\phi)} (C_Y, y)$ defines a functor $C_{X_x} \xrightarrow{F_\phi} C_{Y_y}$ between the corresponding pointed categories, and the map $(F, \phi) \longmapsto F_\phi$ respects compositions and maps identical morphisms to identical functors; i.e. the correspondence

$$(C_X, x) \longmapsto C_{X_x},\ (F, \phi) \longmapsto F_\phi$$

is a functor, $\mathfrak{J}_*$, from the category $Cat_\star$ onto the full subcategory $Cat_{\mathfrak{pt}}$ of Cat whose objects are pointed categories. The functor $\mathfrak{J}_*$ is a right adjoint to the functor $Cat_{\mathfrak{pt}} \xrightarrow{\mathfrak{J}^*} Cat_\star$ which assigns to each pointed category C_X an object (C_X, x) of the category $Cat_\star$ and to every functor $C_X \xrightarrow{F} C_Y$ between pointed categories the morphism $C_X, x) \xrightarrow{(F,\phi)} (C_Y, y)$ in which the arrow $F(x) \xrightarrow{\phi} y$ is uniquely defined. The adjunction arrow $Id_{Cat_{\mathfrak{pt}}} \xrightarrow{\eta} \mathfrak{J}_*\mathfrak{J}^*$ assigns to each pointed category C_X the natural isomorphism $C_X \xrightarrow{\sim} C_{X_x}$ (where x is the zero object of C_X involved in the definition of $\mathfrak{J}^*$). The other adjunction arrow, $\mathfrak{J}^*\mathfrak{J}_* \xrightarrow{\epsilon} Id_{Cat_\star}$, assigns to each object (C_Y, y) of $Cat_\star$ the forgetful functor $C_{Y_y} \longrightarrow C_Y$. Notice by passing that the image of this forgetful functor is the full subcategory of C_Y generated by all objects having a morphism to an initial object.

C1.8. Induced right exact structures. A pretopology τ on C_X induces a pretopology τ_V on the category C_X/V for any $V \in ObC_X$; hence τ induces a pretopology τ_x on C_{X_x}. In particular, a structure $\mathfrak{E}_X$ of a right exact category on C_X induces a structure $\mathfrak{E}_{X_x}$ of a right exact category on C_{X_x}. If $(C_X, \mathfrak{E}_X)$ has enough projectives, then $(C_{X_x}, \mathfrak{E}_{X_x})$ has enough projectives. Finally, if the class $\mathfrak{E}_X^{spl}$ of split epimorphisms of C_X is stable under base change, then the class $\mathfrak{E}_{X_x}^{spl}$ of split epimorphisms of C_{X_x} has this property.

C1.9. Monads on categories with an initial object and monads on corresponding pointed categories.

C1.9.1. Definition. Fix an object (C_X, x) of the category $Cat_\star$. A *monad on* (C_X, x) as a pair$(\mathcal{F}, \phi)$, where $\mathcal{F} = (F, \mu)$ is a monad on C_X and $F(x) \xrightarrow{\phi} x$ is an $\mathcal{F}$-module structure on the initial object x.

We denote by $\mathfrak{Mon}(C_X, x)$ the category whose objects are monads on (C_X, x); morphisms from $(\mathcal{F}, \phi)$ to $(\mathcal{F}', \phi')$ are monad morphisms $\mathcal{F} \xrightarrow{g} \mathcal{F}'$ such that $\phi = \phi' \circ g(x)$.

C1.9.2. Lemma. *Every monad $(\mathcal{F}, \phi)$ on (C_X, x) defines a monad $\mathcal{F}_\phi = (F_\phi, \mu_\phi)$ on the corresponding pointed category C_{X_x}. The map $(\mathcal{F}, \phi) \longmapsto \mathcal{F}_\phi$ extends to an isomorphism between the category $\mathfrak{Mon}(C_X, x)$ of monads on C_{X_x} and the category $\mathfrak{Mon}(C_{X_x})$ of monads on C_{X_x}.*

Proof is left to the reader. ∎

C1.9.3. A remark on augmented monads. Every augmented monad $(\mathcal{F}, \epsilon)$ on the category C_X (see C1.4.1) defines a monad $(\mathcal{F}, \epsilon(x))$ on (C_X, x), hence a monad on the associated pointed category C_{X_x}. The map $(\mathcal{F}, \epsilon) \longmapsto (\mathcal{F}, \epsilon(x))$ is functorial; so that we have functors

$$\mathfrak{Mon}^+(C_X) \longrightarrow \mathfrak{Mon}(C_X, x) \xrightarrow{\sim} \mathfrak{Mon}(C_{X_x}).$$

On the other hand, it is easy to see that there is a natural isomorphism between the category $\mathfrak{Mon}^+(C_X)$ of augmented monads on C_X and the category $\mathfrak{Mon}^+(C_{X_x})$ of augmented monads on the pointed category C_{X_x}.

C2. Diagram chasing.

C2.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with an initial object x and kernels of morphisms; and let*

$$\begin{array}{ccccccc}
 & & & & & & x \\
 & & & & & & \downarrow \\
 & & Ker(f') & \xrightarrow{\beta_1'} & Ker(f) & \xrightarrow{\alpha_1'} & Ker(f'') \\
 & & \mathfrak{k}' \downarrow & & \mathfrak{k} \downarrow & & \downarrow \mathfrak{k}'' \\
 & & A_1' & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
 & & f' \downarrow & & f \downarrow & & \downarrow f'' \\
x & \longrightarrow & A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2''
\end{array} \qquad (1)$$

be a commutative diagram whose two lower rows and the right column are 'exact'. Then its upper row, $Ker(f') \xrightarrow{\beta_1'} Ker(f) \xrightarrow{\alpha_1'} Ker(f'')$, is 'exact'.

Proof. Let x be an initial object of C_X; and let $Ker(\beta_2) = x$. The diagram (1) gives rise to the comutative diagram

$$\begin{array}{ccccccc}
Ker(f') & \xrightarrow{\mathfrak{e}'_1} & Ker(\widetilde{f}) & \xrightarrow{\mathfrak{j}'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\
\mathfrak{k}'\downarrow & cart & \widetilde{\mathfrak{k}}\downarrow & cart & \mathfrak{k}\downarrow & & \downarrow\mathfrak{k}'' \\
A'_1 & \xrightarrow{\mathfrak{e}_1} & Ker(\alpha_1) & \xrightarrow{\mathfrak{j}_1} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\
f'\downarrow & & \widetilde{f}\downarrow & & f\downarrow & & \downarrow f'' \\
A'_2 & \xrightarrow{id} & Ker(\alpha_2) = A'_2 & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2
\end{array} \qquad (1.1)$$

where $\mathfrak{e}_1$ is a deflation, $\mathfrak{j}_1 \circ \mathfrak{e}_1 = \beta_1$, $\mathfrak{j}'_1 \circ \mathfrak{e}'_1 = \beta'_1$, and $A'_2 \xrightarrow{\beta_2} A_2$ is the kernel of α_2. The claim is that the morphism $Ker(f) \xrightarrow{\mathfrak{e}'_1} Ker(\alpha'_1)$ is a deflation.

By 2.3.4.1 (or 2.3.4.3), the upper left square of (1.1) is cartesian, because the left lower horizontal arrow is identical. Since $A'_1 \xrightarrow{\mathfrak{e}_1} Ker(\alpha_1)$ is a deflation, this implies that $Ker(f') \xrightarrow{\mathfrak{e}'_1} Ker(\widetilde{f})$ is a deflation.

Since $Ker(\beta_2)$ is trivial, it follows from 2.3.4.3 (applied to the middle section of the diagram (1.1)) that the upper middle square of (1.1) is cartesian.

Notice that $Ker(\widetilde{f}) \xrightarrow{\mathfrak{j}'_1} Ker(f)$ is the kernel of $Ker(f) \xrightarrow{\alpha'_1} Ker(f'')$.

In fact, by hypothesis, the kernel of $Ker(f'') \xrightarrow{\mathfrak{k}''} A''_1$ is trivial. Therefore, by 3.3.4.3, the right square of the commutative diagram

$$\begin{array}{ccccc}
Ker(\alpha'_1) & \xrightarrow{\mathfrak{j}'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\
\widetilde{\mathfrak{k}}'\downarrow & cart & \mathfrak{k}\downarrow & & \downarrow\mathfrak{k}'' \\
Ker(\alpha_1) & \xrightarrow{\mathfrak{j}_1} & A_1 & \xrightarrow{\alpha_1} & A''_1
\end{array}$$

is cartesian (whenever $Ker(\alpha'_1)$ exists). Therefore, by the universality of cartesian squares, there is a natural isomorphism $Ker(\widetilde{f}) \xrightarrow{\sim} Ker(\alpha'_1)$. ∎

The following assertion is a non-additive verstion of the 'snake lemma'. Its proof is not reduced to the element-wise diagram chasing, like the argument of the classical 'snake lemma'. Therefore, it requires more elaboration than its abelian prototype.

C2.2. Proposition ('snake lemma'). *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with*

an initial object x; and let

$$
\begin{array}{ccccccc}
 & & & & & & x \\
 & & & & & & \downarrow \\
 & & Ker(f') & \xrightarrow{\beta'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\
 & & \mathfrak{k}'\downarrow & & \mathfrak{k}\downarrow & & \downarrow\mathfrak{k}'' \\
 & & A'_1 & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\
 & & f'\downarrow & & f\downarrow & & \downarrow f'' \\
x & \longrightarrow & A'_2 & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \\
 & & \mathfrak{e}'\downarrow & & \mathfrak{e}\downarrow & & \downarrow\mathfrak{e}'' \\
 & & A'_3 & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A''_3
\end{array}
\tag{2}
$$

be a commutative diagram whose vertical columns and middle rows are 'exact', the arrows $\alpha_1,\ \mathfrak{e}',\ \mathfrak{e},\ \mathfrak{e}''$ are deflations, and the kernel of $Ker(f'') \xrightarrow{\mathfrak{k}''} A''_1$ is trivial.

(a) Suppose that each deflation of $(C_X, \mathfrak{E}_X)$ is isomorphic to its coimage and the unique arrow $x \longrightarrow A''_3$ is a monomorphism. Then there exists a natural morphism $Ker(f'') \xrightarrow{\mathfrak{d}} A'_3$ such that the sequence

$$
\begin{array}{ccccccc}
Ker(f') & \xrightarrow{\beta'_1} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') & & \\
 & & & & \downarrow\mathfrak{d} & & \\
 & & & & A'_3 & \xrightarrow{\beta_3} A_3 \xrightarrow{\alpha_3} & A''_3
\end{array}
\tag{3}
$$

is a complex. Moreover, its subsequences $Ker(f') \xrightarrow{\beta'_1} Ker(f) \xrightarrow{\alpha'_1} Ker(f'')$ and $Ker(f'') \xrightarrow{\mathfrak{d}} A'_3 \xrightarrow{\beta_2} A_3$ are 'exact'.

(b) Suppose, in addition, that

(b1) $\mathfrak{E}_X$ is saturated in the following sense: if $\lambda\circ\mathfrak{s}$ is a deflation and $\mathfrak{s}$ is a deflation, then λ is a deflation;

(b2) the following condition holds:

(#) If $M \xrightarrow{\mathfrak{e}} N$ is a deflation and $M \xrightarrow{p} M$ an idempotent morphism (i.e. $p^2 = p$) which has a kernel and such that the composition $\mathfrak{e}\circ p$ is a trivial morphism, then the composition of the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and $M \xrightarrow{\mathfrak{e}} N$ is a deflation.

Then the entire sequence (3) is 'exact'.

Proof. (i) Since α_1 is a deflation, there exists a cartesian square

$$
\begin{array}{ccc}
\widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
\widetilde{\mathfrak{k}}''\downarrow & cart & \downarrow\mathfrak{k}'' \\
A_1 & \xrightarrow{\alpha_1} & A''_1
\end{array}
$$

where $\widetilde{\alpha}_1$ is a deflation too. It follows from 2.3.4.1 that $\widetilde{A}_1 = Ker(\alpha_2 f) = Ker(f''\alpha_1)$. This is seen from the commutative diagram

$$\begin{array}{ccccc}
\widetilde{A}_1 & \xrightarrow{id} & Ker(f''\alpha_1) & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
{\scriptstyle id}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
{\scriptstyle h}\downarrow & cart & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \\
{\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\
A_3' & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A_3''
\end{array} \tag{4}$$

with cartesian squares as indicated.

(ii) By 2.3.3, we have a commutative diagram

$$\begin{array}{ccccc}
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & \widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
{\scriptstyle \wr}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\alpha_1) & \xrightarrow{\mathfrak{k}(\alpha_1)} & A_1 & \xrightarrow{\alpha_1} & A_1''
\end{array}$$

whose (rows are conflations and the) left vertical arrow is an isomorphism. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc}
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & \widetilde{A}_1 & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
{\scriptstyle \mathfrak{e}_1}\uparrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
A_1' & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\alpha_1} & A_1'' \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
A_2' & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A_2'' \\
{\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\
A_3' & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A_3''
\end{array} \tag{5}$$

Since the second row of the diagram (2) is 'exact', the morphism $\mathfrak{e}_1$ is a deflation.

(iii) Combining the diagram (4) with (the left upper corner of) (5), we obtain a

commutative diagram

$$\begin{array}{ccccccc}
A'_1 & \xrightarrow{\mathfrak{e}_1} & Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(f''\alpha_1) & \xrightarrow{\widetilde{\alpha}_1} & Ker(f'') \\
 & & {\scriptstyle \mathfrak{k}(\widetilde{\alpha}_1)}\downarrow & & {\scriptstyle \widetilde{\mathfrak{k}}''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}''} \\
 & & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\
 & & {\scriptstyle h}\downarrow & cart & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
 & & A'_2 & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \\
 & & {\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\
 & & A'_3 & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A''_3
\end{array} \tag{6}$$

where $\widetilde{\mathfrak{k}}''\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1 = \beta_1$. Therefore, $\beta_2\circ(h\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1) = f\circ(\widetilde{\mathfrak{k}}''\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1) = f\circ\beta_1 = \beta_2\circ f'$. Since the left middle square of (6) is cartesian, this implies that $h\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1 = f'$.

Therefore, $\mathfrak{e}'\circ h\circ\mathfrak{k}(\widetilde{\alpha}_1)\circ\mathfrak{e}_1 = \mathfrak{e}'\circ f'$ is a trivial morphism.

(iv) Notice that, by 2.1.2, the kernel morphism $Ker(\mathfrak{e}'\circ h\circ\mathfrak{k}(\widetilde{\alpha}_1))\longrightarrow Ker(\widetilde{\alpha}_1)$ is a monomorphism, because A'_3 has a morphism to x, hence $x\longrightarrow A'_3$ is a (split) monomorphism. Since $\mathfrak{e}_1$ is a deflation, in particular a strict epimorphism, it follows from 2.3.4.4 that the composition $(\mathfrak{e}'\circ h)\circ\mathfrak{k}(\widetilde{\alpha}_1)$ is trivial. By hypothesis, $\widetilde{\alpha}_1$ (being a deflation) is isomorphic to the coimage morphism, i.e. $Ker(f'')$ is naturally isomorphic to $Coim(\widetilde{\alpha}_1)$. Therefore, the morphism $\mathfrak{e}'\circ h$ factors through $\widetilde{\alpha}_1$, i.e. $\mathfrak{e}'\circ h = \mathfrak{d}\circ\widetilde{\alpha}_1$. Since $\widetilde{\alpha}_1$ is a deflation, in particular an epimorphism, the latter equality determines the morphism $Ker(f'')\xrightarrow{\mathfrak{d}} A'_3$ uniquely.

(v) By C2.1, the sequence $Ker(f')\xrightarrow{\beta'_1} Ker(f)\xrightarrow{\alpha'_1} Ker(f'')$ is 'exact'.

(vi) The composition of $Ker(f)\xrightarrow{\alpha'_1} Ker(f'')$ and $Ker(f'')\xrightarrow{\mathfrak{d}} A'_3$ is trivial.

In fact, the diagram (6) induces a commutative diagram

$$\begin{array}{ccccccc}
A'_1 & & Ker(h) & \xrightarrow{\sim} & Ker(f) & \xrightarrow{\alpha'_1} & Ker(f'') \\
{\scriptstyle \mathfrak{e}_1}\downarrow & & {\scriptstyle \mathfrak{k}(h)}\downarrow & & {\scriptstyle \mathfrak{k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}''} \\
Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\
 & & {\scriptstyle h}\downarrow & cart & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
 & & A'_2 & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \\
 & & {\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\
 & & A'_3 & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A''_3
\end{array} \tag{7}$$

where the isomorphism $Ker(h)\xrightarrow{\sim} Ker(f)$ is due the fact that the left middle square of the diagram (7) is cartesian. We can and will assume that this isomorphism is identical. The morphism $Ker(f)\xrightarrow{\alpha'_1} Ker(f'')$ is the composition of $Ker(f)\xrightarrow{\mathfrak{k}(h)} Ker(\alpha_2 f)$ and

$Ker(\alpha_2 f) \xrightarrow{\widetilde{\alpha}_1} Ker(f'')$. Therefore, $\mathfrak{d} \circ \alpha'_1 = \mathfrak{d} \circ \widetilde{\alpha}_1 \circ \mathfrak{k}(h) = \mathfrak{e}' \circ h \circ \mathfrak{k}(h)$, which shows that the composition $\mathfrak{d} \circ \alpha'_1$ is trivial, because already the composition $h \circ \mathfrak{k}(h)$ is trivial.

(vii) The argument above can be summarized in the commutative diagram

$$\begin{array}{ccccc} A'_1 & & Ker(f) & \xrightarrow{\gamma} & Ker(\mathfrak{d}) \\ {\scriptstyle \mathfrak{e}_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}(h)} & & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{d})} \\ Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}(\widetilde{\alpha}_1)} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\alpha}_1} & K(f'') \\ {\scriptstyle \mathfrak{e}_2}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathfrak{d}} \\ Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & A'_2 & \xrightarrow{\mathfrak{e}'} & A'_3 \end{array} \tag{8}$$

where $Ker(\widetilde{\alpha}_1) \xrightarrow{\mathfrak{e}_2} Ker(\mathfrak{e}')$ is a deflation. Taking into consideration the cartesian square

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\widetilde{\gamma}} & Ker(\mathfrak{d}) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{d})} \\ Ker(\alpha_2 f) & \xrightarrow{\widetilde{\alpha}_1} & K(f'') \end{array} \tag{9}$$

we extend (8) to the commutative diagram

$$\begin{array}{ccccccc} & & Ker(f) & \xrightarrow{id} & Ker(h) & \xrightarrow{\gamma} & Ker(\mathfrak{d}) \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle \mathfrak{k}(\widetilde{h})}\downarrow & & \downarrow{\scriptstyle id} \\ A'_1 & & Ker(\widetilde{h}) & \xrightarrow{\mathfrak{k}(\widetilde{h})} & \mathcal{M} & \xrightarrow{\widetilde{\gamma}} & Ker(\mathfrak{d}) \\ {\scriptstyle \mathfrak{e}_1}\downarrow & & {\scriptstyle \mathfrak{k}(\widetilde{h})}\downarrow & & {\scriptstyle \mu}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{d})} \\ Ker(\widetilde{\alpha}_1) & \xrightarrow{\mathfrak{k}_1} & \mathcal{M} & \xrightarrow{\mu} & Ker(\alpha_2 f) & \xrightarrow{\widetilde{\alpha}_1} & K(f'') \\ {\scriptstyle \mathfrak{e}_2}\downarrow & & {\scriptstyle \widetilde{h}}\downarrow & cart & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathfrak{d}} \\ Ker(\mathfrak{e}') & \xrightarrow{id} & Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & A'_2 & \xrightarrow{\mathfrak{e}'} & A'_3 \end{array} \tag{10}$$

where $\mu \circ \mathfrak{k}(\widetilde{h}) = \mathfrak{k}(h)$, and $\mu \circ \mathfrak{k}_1 = \mathfrak{k}(\widetilde{\alpha}_1)$.

Since the square (9) is cartesian and $\widetilde{\alpha}_1$ is a deflation, its pull-back, $\widetilde{\gamma}$, is a deflation too. Notice that the commutativity of the left lower square and the fact that $\mathfrak{e}_2$ is a strict epimorphism imply that $\widetilde{h}$ is a strict epimorphism.

Consider the cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\mathfrak{e}'_3} & \mathcal{M} \\ {\scriptstyle \mathfrak{p}}\downarrow & & \downarrow{\scriptstyle \widetilde{h}} \\ A'_1 & \xrightarrow{\mathfrak{e}_3} & Ker(\mathfrak{e}') \end{array} \tag{11}$$

where $\mathfrak{e}_3 = \mathfrak{e}_2 \circ \mathfrak{e}_1$. Since $\mathfrak{e}_3$ is a deflation, the arrow $\widetilde{\mathcal{M}} \xrightarrow{\mathfrak{e}'_3} \mathcal{M}$ is a deflation. Since $\widetilde{h} \circ \mathfrak{k}_1 \circ \mathfrak{e}_1 = \mathfrak{e}_3$, the projection $\mathfrak{p}$ has a splitting, $A'_1 \xrightarrow{\mathfrak{s}} \widetilde{\mathcal{M}}$; i.e. $\mathfrak{p} \circ \mathfrak{s} = id$. Set $p = \mathfrak{s} \circ \mathfrak{p}$ and $\varphi = \widetilde{\gamma} \circ \mathfrak{e}'_3$. It follows that $\widetilde{\mathcal{M}} \xrightarrow{p} \widetilde{\mathcal{M}}$ is an idempotent (– a projector), φ is a deflation, the composition $\varphi \circ p = \widetilde{\gamma} \circ (\mathfrak{e}'_3 \circ \mathfrak{s}) \circ \mathfrak{p} = \widetilde{\gamma} \circ (\mathfrak{k}_1 \circ \mathfrak{e}_1) \circ \mathfrak{p}$ is trivial, because $\mathfrak{k}(\mathfrak{d}) \circ \widetilde{\gamma} \circ \mathfrak{k}_1 = \widetilde{\alpha}_1 \circ \mathfrak{k}(\widetilde{\alpha}_1)$ is trivial and $Ker(\mathfrak{d}) \xrightarrow{\mathfrak{k}(\mathfrak{d})} K(f'')$ is a monomorphism. The latter follows from the fact that A'_3 has a morphism to x, hence the unique arrow $x \longrightarrow A'_3$ is a (split) monomorphism.

Since the square (11) is cartesian, it follows from 2.3.4.1 that $Ker(\mathfrak{p})$ is naturally isomorphic to $Ker(\widetilde{h}) = Ker(f)$. And, by 2.3.4.3, $Ker(p)$ is naturally isomorphic to $Ker(\mathfrak{p})$, because $p = \mathfrak{s} \circ \mathfrak{p}$ and $\mathfrak{s}$ is a monomorphism.

Thus, $Ker(p)$ is naturally isomorphic to $Ker(f)$.

(viii) Suppose that the condition (#) of the proposition holds. Then the composition of $Ker(p) \longrightarrow \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}} \xrightarrow{\varphi} Ker(\mathfrak{d})$ is a deflation; hence the composition of the morphisms $Ker(f) \xrightarrow{\mathfrak{k}(\widetilde{h})} \mathcal{M}$ and $\mathcal{M} \xrightarrow{\widetilde{\gamma}} Ker(\mathfrak{d})$ (i.e. the morphism $Ker(f) \xrightarrow{\gamma} Ker(\mathfrak{d})$ in the diagram (10)) is a deflation.

(ix) The claim is that $\mathfrak{d}$ is the composition of the morphism $Ker(\beta_3) \xrightarrow{\mathfrak{k}(\beta_3)} A'_3$ and a deflation $Ker(f'') \xrightarrow{\mathfrak{d}'} Ker(\beta_3)$. Since $\widetilde{\alpha}_1$ is a deflation, it suffices to prove a similar assertion for $\mathfrak{d} \circ \widetilde{\alpha}_1 = \mathfrak{e}' \circ h$.

We have a commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{B} & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 & \xrightarrow{\alpha_1} & A''_1 \\
{\scriptstyle \mathfrak{t}_h}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{t}_f} & & \downarrow{\scriptstyle \mathfrak{t}''} \\
\mathcal{B} & \xrightarrow{\widetilde{\beta}_3} & Ker(\mathfrak{e}) & \xrightarrow{\lambda} & Ker(\mathfrak{e}'') \\
{\scriptstyle \psi}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{e})} & & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{e}'')} \\
\mathcal{B} & \xrightarrow{\psi} & A'_2 & \xrightarrow{\beta_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \\
{\scriptstyle \widetilde{\mathfrak{e}}'}\downarrow & cart & {\scriptstyle \mathfrak{e}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}} & & \downarrow{\scriptstyle \mathfrak{e}''} \\
Ker(\beta_3) & \xrightarrow{\mathfrak{k}(\beta_3)} & A'_3 & \xrightarrow{\beta_3} & A_3 & \xrightarrow{\alpha_3} & A''_3
\end{array} \qquad (12)$$

where $f = \mathfrak{k}(\mathfrak{e}) \circ \mathfrak{t}_f$, $f'' = \mathfrak{k}(\mathfrak{e}'') \circ \mathfrak{t}''$ and the remaining new arrows are determined by the commutativity of the diagram (12) and by being a part of a cartesian square. By hypothesis, the columns of the diagram (2) are 'exact'; in particular, the morphism $\mathfrak{t}_f$ is a deflation. Therefore, the morphism $\mathfrak{B} \xrightarrow{\mathfrak{t}_h} \mathcal{B}$ is a deflation. Being the composition of two cartesian diagrams, the diagram

$$\begin{array}{ccc}
\mathfrak{B} & \xrightarrow{\widetilde{\mathfrak{k}}''} & A_1 \\
{\scriptstyle \psi \circ \mathfrak{t}_h}\downarrow & & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{e}) \circ \mathfrak{t}_f = f} \\
A'_2 & \xrightarrow{\beta_2} & A_2
\end{array}$$

is cartesian, as well as the diagram

$$\begin{array}{ccc} Ker(\alpha_2 f) & \xrightarrow{\widetilde{\mathfrak{k}''}} & A_1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ A'_2 & \xrightarrow{\beta_2} & A_2 \end{array}$$

Therefore, they are isomorphic to each other. So, we can and will assume that $\mathfrak{B} = Ker(\alpha_2 f)$ and $h = \psi \circ \mathfrak{t}_h$. It follows from (the left part of) the diagram (12) that

$$\mathfrak{e}' \circ h = \mathfrak{e}' \circ \psi \circ \mathfrak{t}_h = \mathfrak{k}(\beta_3) \circ (\widetilde{\mathfrak{e}}' \circ \mathfrak{t}_h),$$

that is $\mathfrak{e} \circ h$ is the composition of $Ker(\beta_3) \xrightarrow{\mathfrak{k}(\beta_3)} A'_3$ and the deflation $\widetilde{\mathfrak{e}}' \circ \mathfrak{t}_h$.

(x) The composition $\alpha_3 \circ \beta_3$ is trivial by 2.3.4.4, because the composition $\alpha_3 \circ \beta_3 \circ \mathfrak{e}' = \mathfrak{e}'' \circ \alpha_2 \circ \beta_2$ is trivial, $x \longrightarrow A''_3$ is a monomorphism (by hypothesis), and $\mathfrak{e}'$ is a deflation, hence a strict epimorphism. The claim is that, if $(C_X, \mathfrak{E}_X)$ has the propery (#), then the morphism $A'_3 \xrightarrow{\beta_3} A_3$ is the composition of the kernel morphism $Ker(\alpha_3) \xrightarrow{\mathfrak{k}_3} A''_3$ and a deflation $A'_3 \xrightarrow{\mathfrak{t}_3} Ker(\alpha_3)$.

Since in the upper right square of the diagram (12), the arrows α_1, $\mathfrak{t}''$, and $\mathfrak{t}_f$ are deflations, the forth arrow, $Ker(\mathfrak{e}) \xrightarrow{\lambda} Ker(\mathfrak{e}'')$, is a deflation too (due to the saturatedness condition (b1)). Consider the commutative diagram

$$\begin{array}{ccccccc} & & & & Ker(\mathfrak{e}) & \xrightarrow{\lambda} & Ker(\mathfrak{e}'') \\ & & & & {\scriptstyle \mathfrak{v}}\downarrow & & \downarrow{\scriptstyle id} \\ A'_2 & & \xrightarrow{\mathfrak{t}'_3} & & \mathfrak{D} & \xrightarrow{\mathfrak{p}_2} & Ker(\mathfrak{e}'') \\ {\scriptstyle \mathfrak{t}'_3}\downarrow & & & & {\scriptstyle \beta'_2}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{k}(\mathfrak{e}'')} \\ A'_2 & \xrightarrow{\mathfrak{t}'_3} & \mathfrak{D} & \xrightarrow{\beta'_2} & A_2 & \xrightarrow{\alpha_2} & A''_2 \\ {\scriptstyle \mathfrak{e}'}\downarrow & & {\scriptstyle \mathfrak{u}}\downarrow & cart & {\scriptstyle \mathfrak{e}}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\ A'_3 & \xrightarrow{\mathfrak{t}_3} & Ker(\alpha_3) & \xrightarrow{\mathfrak{k}(\alpha_3)} & A_3 & \xrightarrow{\alpha_3} & A''_3 \end{array} \qquad (13)$$

where $\beta'_2 \circ \mathfrak{v} = \mathfrak{k}(\mathfrak{e})$, $\beta'_2 \circ \mathfrak{t}'_2 = \beta_2$.

The upper left corner of the commutative diagram (13) gives rise to the commutative diagram

$$\begin{array}{ccccc} \widetilde{A}'_2 & \xrightarrow{\widetilde{\mathfrak{t}}'_3} & \widetilde{\mathfrak{D}} & \xrightarrow{\widetilde{\mathfrak{p}}_2} & Ker(\mathfrak{e}) \\ {\scriptstyle \lambda'}\downarrow & cart & {\scriptstyle \widetilde{\lambda}}\downarrow & cart & \downarrow{\scriptstyle \lambda} \\ A'_2 & \xrightarrow{\mathfrak{t}'_3} & \mathfrak{D} & \xrightarrow{\mathfrak{p}_2} & Ker(\mathfrak{e}'') \end{array} \qquad (14)$$

whose both squares are cartesian. Since λ is a deflation, all vertical arrows of (14) are deflations, as well as the arrows $\mathfrak{p}_2$ and $\widetilde{\mathfrak{p}}_2$. The morphism $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{v}} \mathfrak{D}$ determines a

splitting $Ker(\mathfrak{e}) \xrightarrow{\mathfrak{s}_2} \widetilde{\mathfrak{D}}$ of the projection $\widetilde{\mathfrak{p}}_2$. Let p_2 denote the composition $\mathfrak{s}_2 \circ \mathfrak{p}_2$. It follows that p_2 is an idempotent $\widetilde{\mathfrak{D}} \longrightarrow \widetilde{\mathfrak{D}}$ and the composition

$$\mathfrak{k}(\alpha_3) \circ (\mathfrak{u} \circ \widetilde{\lambda}) \circ p_2 = \mathfrak{e} \circ \beta_2' \circ (\widetilde{\lambda} \circ \mathfrak{s}_2) \circ \widetilde{\mathfrak{p}}_2 = \mathfrak{e} \circ \beta_2' \circ \mathfrak{v} \circ \widetilde{\mathfrak{p}}_2 = (\mathfrak{e} \circ \mathfrak{k}(\mathfrak{e})) \circ \widetilde{\mathfrak{p}}_2$$

is trivial. Therefore, $(\mathfrak{u} \circ \widetilde{\lambda}) \circ p_2$ is trivial. The kernel of the idempotent p_2 is isomorphic to the kernel of $\widetilde{\mathfrak{p}}_2$. Since the right square of (14) is cartesian, there is a natural isomorphism $Ker(\mathfrak{p}_2) \simeq Ker(\widetilde{\mathfrak{p}}_2)$. It follows from the right cartesian square of (13) that there is a natural isomorphism $Ker(\mathfrak{p}_2) \simeq Ker(\alpha_2) = A_2'$.

If the right exact category $(C_X, \mathfrak{E}_X)$ has the property (#), then the above implies that the morphism $A_2' \xrightarrow{\mathfrak{u} \circ \mathfrak{t}_3'} Ker(\alpha_3)$ is a deflation. Since $\mathfrak{u} \circ \mathfrak{t}_3' = \mathfrak{t}_3 \circ \mathfrak{e}'$ and $\mathfrak{e}'$ is a deflation, the morphism $A_3' \xrightarrow{\mathfrak{t}_3} Ker(\alpha_3)$ is a deflation. ∎

C2.3. Remarks about conditions of the 'snake lemma'. Fix a right exact category $(C_X, \mathfrak{E}_X)$. The main condition of the 'snake lemma' C2.2, the one which garantees the existence of the *connecting morphism* $\mathfrak{d}$, is that each deflation $M \xrightarrow{\mathfrak{e}} N$ is isomorphic to its coimage morphism $M \xrightarrow{\mathfrak{c}(\mathfrak{e})} Coim(\mathfrak{e}) = M/Ker(\mathfrak{e})$.

If the category C_X is additive, then every strict epimorphism which has a kernel, in particular, every deflation, is isomorphic to its coimage morphism.

The latter property holds in many non-additive categories, for instance in the category Alg_k of unital associative k-algebras (see 2.3.5.3).

Similarly, the property

(#) If $M \xrightarrow{\mathfrak{e}} N$ is a deflation and $M \xrightarrow{p} M$ an idempotent morphism (i.e. $p^2 = p$) which has a kernel and such that the composition $\mathfrak{e} \circ p$ is a trivial morphism, then the composition of the canonical morphism $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and the deflation $M \xrightarrow{\mathfrak{e}} N$ is a deflation.

which ensures 'exactness' of the 'snake' sequence (3) holds in any additive category.

In fact, if the category C_X is additive, then the existence of the kernel of p means precisely that the idempotent $q = id_M - p$ is splittable; i.e. $M \xrightarrow{q} M$ is the composition of $Ker(p) \xrightarrow{\mathfrak{k}(p)} M$ and a (strict) epimorphism $M \xrightarrow{\mathfrak{t}} Ker(p)$ such that $\mathfrak{t} \circ \mathfrak{k}(p) = id$. The condition $\mathfrak{e} \circ p$ is trivial (that is $\mathfrak{e} \circ p = 0$) is equivalent to the equalities $\mathfrak{e} = \mathfrak{e} \circ q = (\mathfrak{e} \circ \mathfrak{k}(p)) \circ \mathfrak{t}$ which imply (under saturatedness condition, cf. C2.2(b1)) that $\mathfrak{e} \circ \mathfrak{k}(p)$ is a deflation.

C2.3.1. Example. The property (#) holds in the category Alg_k. In fact, let $A \xrightarrow{\varphi} B$ be a strict algebra epimorphism, and $A \xrightarrow{p} A$ an idempotent endomorphism such that the composition $\varphi \circ p$ is a trivial morphism; that it equals to the composition of an augmentation morphism $A \xrightarrow{\pi} k$ and the k-algebra structure $k \xrightarrow{i_B} B$. In particular, $A = k \oplus A_+$, where $A_+ = K(\pi)$ is the kernel of the augmentation π in the usual sense.

On the other hand, $Ker(p) = k \oplus K(p)$, and, since $p \circ p = p$ and the ideal $K(p) \stackrel{\text{def}}{=} \{y \in A \mid p(y) = 0\}$ coincides $\{x - p(x) \mid x \in A\}$. Similarly, $Ker(\varphi \circ p) = k \oplus K(\varphi \circ p)$, and it follows that $K(\varphi \circ p) = A_+$.

Every element x of A is uniquely represented as $\lambda \cdot 1_A + x_+$, where 1_A is the unit element of the algebra A and $x_+ \in A_+$. Therefore, $x - p(x) = x_+ - p(x_+)$ and

$$\varphi(\mu \cdot 1_A + (x - p(x))) = \mu \cdot 1_B + \varphi(x_+ - p(x_+)) = \mu \cdot 1_B + \varphi(x_+) = \varphi(\mu \cdot 1_A + x_+).$$

Since $\mu \in k$ and $x_+ \in A_+$ are arbitrary and φ is a strict epimorphism (that is a surjective map), this shows that $\varphi \circ \mathfrak{k}(p)$ is a strict epimorphism.

C3. Localizations of exact categories and (co)quasi-suspended categories. t-Structures.

C3.1. Remarks on localizations. Let $C_X \xrightarrow{u^*} C_Z$ be a functor. Suppose that the category C_Z is *cocomplete*, i.e. it has colimits of arbitrary small diagrams (equivalently, it has infinite coproducts and cokernels of pairs of arrows). By [GZ, II.1], the functor u^* equals to the composition of the Yoneda embedding $C_X \xrightarrow{h_X} \widehat{C}_X$ of the category C_X into the category $\widehat{C}_X$ of presheaves of sets on C_X and a continuous (that is having a right adjoint) functor $\widehat{C}_X \xrightarrow{\widetilde{u}^*} C_Z$. Since every presheaf of sets on a category is a colimit of a canonical diagram of representable presheaves and the functor $\widetilde{u}^*$ preserves colimits, it is determined uniquely up to isomorphism.

In particular, every functor $C_X \xrightarrow{q^*} C_Y$ gives rise to a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{q^*} & C_Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ \widehat{C}_X & \xrightarrow{\widehat{q^*}} & \widehat{C}_Y \end{array} \tag{1}$$

with a continuous functor $\widehat{q}^*$ determined by the commutativity of (1) uniquely up to isomorphism.

C3.1.1. Lemma. *(a) The functor*

$$\widehat{C}_Y \xrightarrow{\widehat{q_*}} \widehat{C}_X, \quad F \longmapsto F \circ q^*, \tag{2}$$

is a canonical right adjoint to $\widehat{q}^*$.

(b) If the functor q^* has a right adjoint q_*, then the diagram

$$\begin{array}{ccc} C_X & \xleftarrow{q_*} & C_Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ \widehat{C}_X & \xleftarrow{\widehat{q_*}} & \widehat{C}_Y \end{array} \tag{1_*}$$

quasi-commutes.

Proof. (a) Recall that the functor $\widehat{q}^*$ is determined uniquely up to isomorphism by the equality $\widehat{q}^*(h_Y(L)) = h_Y(q^*(L))$ for all $L \in ObC_X$. For every $L \in ObC_X$ and $F \in Ob\widehat{C}_Y$, we have

$$\widehat{C}_X(h_X(L), F \circ q^*) \simeq F(q^*(L)) \simeq \widehat{C}_Y(h_Y(q^*(L)), F) \simeq \widehat{C}_Y(\widehat{q}^*(h_Y(L)), F).$$

Since all isomorphisms here are functorial, it follows that the functor (2) is a right adjoint to $\widehat{q}^*$.

(b) For any $L \in ObC_Y$,

$$\widehat{q}_*(h_Y(L)) = h_Y(L) \circ q^* = C_Y(q^*(-), L) \simeq C_X(-, q_*(L)) = h_X(q_*(L)),$$

hence the assertion. ∎

C3.1.1.1. Corollary. *For every functor $C_X \xrightarrow{q^*} C_Y$, the functor $\widehat{q}_*$ has a right adjoint, $\widehat{q}^!$. In particular, $\widehat{q}_*$ is exact.*

Proof. The fact follows from C3.1.1(a). ∎

C3.1.2. Proposition. *If $C_X \xrightarrow{q^*} C_Y$ is a localization, then the continuous functor $\widehat{q}^*$ in (1) is a localization too.*

Proof. The functor $\widehat{C}_X \xrightarrow{\widehat{q^*}} \widehat{C}_Y$ is decomposed into a localization $\widehat{C}_X \xrightarrow{\widehat{q^*_{\mathfrak{f}}}} C_Z$ at $\Sigma_{\widehat{q^*}} = \{s \in Hom\widehat{C}_X \mid \widehat{q}^*(s) \text{ is invertible}\}$ and a conservative functor $C_Z \xrightarrow{\widehat{q^*_c}} \widehat{C}_Y$. Since q^* is a localization and the composition $\widehat{q}^*_{\mathfrak{f}} \circ h_X$ makes invertible all arrows of $\Sigma_{q^*} = \{s \in Hom\widehat{C}_X \mid q^*(s) \text{ is invertible}\}$, there exists a unique functor $C_Y \xrightarrow{\Psi} C_Z$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{q^*} & C_Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \widehat{C}_X & \xrightarrow{\widehat{q}^*_{\mathfrak{f}}} & C_Z \end{array} \qquad (2)$$

commutes. The localization $\widehat{q}^*_{\mathfrak{f}}$ is continuous, i.e. it has a right adjoint which is, forcibly, a fully faithful functor. Therefore, by [GZ, I.1.4], the category C_Z has limits and colimits of arbitrary (small) diagrams. Therefore, the functor $C_Y \xrightarrow{\Psi} C_Z$ is the composition of the Yoneda imbedding $C_Y \xrightarrow{h_Y} \widehat{C}_Y$ and a continuous functor $\widehat{C}_Y \xrightarrow{\Psi'} C_Z$; the latter is defined uniquely up to isomorphism. Thus, we have the equalities

$$\begin{aligned} &\widehat{q}^*_{\mathfrak{f}} \circ h_X = \Psi \circ q^* = \Psi' \circ h_Y \circ q^* = \Psi' \circ \widehat{q}^* \circ h_X = (\Psi' \circ \widehat{q}^*_c) \circ \widehat{q}^*_{\mathfrak{f}} \circ h_X \\ &(\widehat{q}^*_c \circ \Psi') \circ h_Y \circ q^* = \widehat{q}^*_c \circ \Psi \circ q^* = \widehat{q}^*_c \circ \widehat{q}^*_{\mathfrak{f}} \circ h_X = \widehat{q}^* \circ h_X \simeq h_Y \circ q^* \end{aligned} \qquad (3)$$

The equality $\widehat{q}^*_{\mathfrak{f}} \circ h_X = (\Psi' \circ \widehat{q}^*_c) \circ \widehat{q}^*_{\mathfrak{f}} \circ h_X$ implies, thanks to the continuity of the functors $\Psi' \circ \widehat{q}^*_c$ and $\widehat{q}^*_{\mathfrak{f}}$ and the universal properties of the localization $\widehat{q}^*_{\mathfrak{f}}$, that the composition $\Psi' \circ \widehat{q}^*_c$ is isomorphic to the identity functor.

Similarly, thanks to the universal properties of the localization q^*, the isomorphism $(\widehat{q}^*_c \circ \Psi') \circ h_Y \circ q^* \simeq h_Y \circ q^*$ implies that $(\widehat{q}^*_c \circ \Psi') \circ h_Y \simeq h_Y$. Since the functor $\widehat{q}^*_c \circ \Psi'$ is continuous and every presheaf of sets on C_Y is a colimit of a (canonical) diagram of representable presheaves, it follows from the latter isomorphism that the composition $\widehat{q}^*_c \circ \Psi'$ is isomorphic to the identical functor. All together shows that $\widehat{q}^*_c$ and Ψ' are mutually quasi-inverse category equivalences. ∎

C3.1.3. Note. Suppose that C_X and C_Y are k-linear categories and $C_X \xrightarrow{q^*} C_Y$ a k-linear functor. If the category C_Y is cocomplete, then it follows from the assertion [GZ, II.1] mentioned above that there exists a unique up to isomorphism continuous functor $\mathcal{M}_k(X) \xrightarrow{\widehat{q^*}} C_Y$ such that $q^* = \widehat{q}^* \circ h_X$. Here, as above, $\mathcal{M}_k(X)$ is the category of k-presheaves on the category C_X. This establishes an equivalence between the category $\mathcal{H}om_k(C_X, C_Y)$ of k-linear functors $C_X \longrightarrow C_Y$ and the category $\mathcal{H}om^c_k(C_X, C_Y)$ of continuous k-linear functors $\mathcal{M}_k(X) \longrightarrow C_Y$.

If a k-linear functor $C_X \xrightarrow{q^*} C_Y$ is equivalent to a localization functor (i.e. it is the composition of the localization functor at $\Sigma_{q^*} \stackrel{\text{def}}{=} \{s \in HomC_X \mid q^*(s) \text{ is invertible}\}$ and a category equivalence $\Sigma^{-1}_{q^*} C_X \longrightarrow C_Y$), then the argument of C3.1.1 with the categories of presheaves of sets replaced by the categories of k-presheaves shows that the natural extension $\mathcal{M}_k(X) \xrightarrow{\widehat{q^*}} \mathcal{M}_k(Y)$ is equivalent to a continuous localization.

C3.2. Right weakly 'exact' functors and 'exact' localizations. Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact categories. A *right weakly 'exact'* functor $(C_X, \mathcal{E}_X) \longrightarrow (C_Y, \mathcal{E}_Y)$ is a functor $C_X \xrightarrow{\varphi^*} C_Y$ such that for every conflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$, there is a commutative diagram

$$\begin{array}{ccccc} \varphi^*(L) & \xrightarrow{\varphi^*(\mathfrak{j})} & \varphi^*(M) & \xrightarrow{\varphi^*(\mathfrak{e})} & \varphi^*(N) \\ & {\scriptstyle \mathfrak{e}'} \searrow & \nearrow {\scriptstyle \mathfrak{j}'} & & \\ & & L_1 & & \end{array}$$

in which $\mathfrak{e}'$ is a deflation and $L_1 \xrightarrow{\mathfrak{j}'} \varphi^*(M) \xrightarrow{\varphi^*(\mathfrak{e})} \varphi^*(N)$ is a conflation.

Recall that the Gabriel-Quillen embedding $C_X \xrightarrow{\mathfrak{j}^*_X} C_{X_{\mathfrak{E}}}$ is the composition of the Yoneda embedding $C_X \xrightarrow{h_X} \mathcal{M}_k(X)$ and the sheafification functor $\mathcal{M}_k(X) \xrightarrow{\mathfrak{q}^*_X} C_{X_{\mathfrak{E}}}$.

C3.2.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories and $(C_X, \mathcal{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathcal{E}_Y)$ a right 'exact' k-linear functor.*

(a) There is a unique up to isomorphism continuous k-linear functor $C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ {\scriptstyle \mathfrak{j}^*_X}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}^*_Y} \\ C_{X_{\mathfrak{E}}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_{\mathfrak{E}}} \end{array}$$

commutes. Here the vertical arrows are the Gabriel-Quillen embeddings.

(b) If the functor $C_X \xrightarrow{\varphi^*} C_Y$ *is a localization, then the functor* $C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$ *is a localization.*

(c) Suppose that the following condition holds: for every $L \in ObC_X$ *and every deflation* $N \xrightarrow{\mathfrak{e}} \varphi^*(L)$*, there exist a deflation* $M \xrightarrow{\mathfrak{t}} L$ *and a commutative diagram*

$$\begin{array}{ccc} \varphi^*(M) & \xrightarrow{g} & N \\ {\scriptstyle \varphi^*(\mathfrak{t})}\searrow & & \swarrow{\scriptstyle \mathfrak{e}} \\ & \varphi^*(L) & \end{array}$$

Then a right adjoint $C_{Y_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}_*} C_{X_{\mathfrak{E}}}$ *to the functor* $\widetilde{\varphi}^*$ *has a right adjoint,* $\widetilde{\varphi}^!$*. In particular, the functor* $\widetilde{\varphi}_*$ *is exact.*

Proof. (a) Objects of the category $C_{X_{\mathfrak{E}}}$ – k-sheaves on the pretopology $(C_X, \mathfrak{E}_X)$, are naturally identified with right 'exact' k-linear functors from C_X to the abelian category $\mathcal{M}_k(X)^{op}$. Therefore, since the functor $C_X \xrightarrow{\varphi^*} C_Y$ is right 'exact', the composition with it maps $C_{Y_{\mathfrak{E}}}$ to $C_{X_{\mathfrak{E}}}$. By C3.1.1, we can (and will) assume that the functor $\mathcal{M}_k(Y) \xrightarrow{\widehat{\varphi}_*} \mathcal{M}_k(X)$ is given by $F \longmapsto F \circ \varphi^*$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_k(X) & \xleftarrow{\widehat{\varphi}_*} & \mathcal{M}_k(Y) \\ {\scriptstyle \mathfrak{q}_{X*}}\uparrow & & \uparrow{\scriptstyle \mathfrak{q}_{Y*}} \\ C_{X_{\mathfrak{E}}} & \xleftarrow{\widetilde{\varphi}_*} & C_{Y_{\mathfrak{E}}} \end{array}$$

whose vertical arrows are inclusion functors. This diagram yields, by adjunction, a quasi-commutative diagram

$$\begin{array}{ccc} \mathcal{M}_k(X) & \xrightarrow{\widehat{\varphi}^*} & \mathcal{M}_k(Y) \\ {\scriptstyle \mathfrak{q}^*_X}\downarrow & & \downarrow{\scriptstyle \mathfrak{q}^*_Y} \\ C_{X_{\mathfrak{E}}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_{\mathfrak{E}}} \end{array} \qquad (1)$$

where the vertical arrows are sheafification functors. The sheafification functors are exact localizations. An isomorphism $\mathfrak{q}^*_Y\widehat{\varphi}^* \simeq \widetilde{\varphi}^*\mathfrak{q}^*_X$ implies that $\widetilde{\varphi}^* \simeq \mathfrak{q}^*_Y\widehat{\varphi}^*\mathfrak{q}_{X*}$, because the adjunction arrow $Id_{C_{X_{\mathfrak{E}}}} \longrightarrow \mathfrak{q}^*_X\mathfrak{q}_{X*}$ is an isomorphism. Together with the isomorphism $\mathfrak{q}^*_Y\widehat{\varphi}^* \simeq \widetilde{\varphi}^*\mathfrak{q}^*_X$, this implies that the canonical morphism $\mathfrak{q}^*_Y\widehat{\varphi}^* \longrightarrow \mathfrak{q}^*_Y\widehat{\varphi}^*\mathfrak{q}_{X*}\mathfrak{q}^*_X$ is an isomorphism. The claim is that the functor $\widetilde{\varphi}_* \stackrel{\text{def}}{=} \mathfrak{q}^*_X\widehat{\varphi}_*\mathfrak{q}_{Y*}$ is a right adjoint to $\widetilde{\varphi}^*$. In fact, the composition of morphisms

$$Id_{C_{Y_{\mathfrak{E}}}} \xrightarrow{\sim} \mathfrak{q}^*_Y\mathfrak{q}_{Y*} \longrightarrow \mathfrak{q}^*_Y\widehat{\varphi}_*\widehat{\varphi}^*\mathfrak{q}_{Y*} \longrightarrow \mathfrak{q}^*_Y\widehat{\varphi}_*\mathfrak{q}_{X*}\mathfrak{q}^*_X\widehat{\varphi}^*\mathfrak{q}_{Y*} \xrightarrow{\sim} \widetilde{\varphi}_*\widetilde{\varphi}^*$$

and

$$\widetilde{\varphi}^*\widetilde{\varphi}_* \xrightarrow{\sim} \mathfrak{q}^*_X\widehat{\varphi}^*\mathfrak{q}_{Y*}\mathfrak{q}^*_Y\widehat{\varphi}_*\mathfrak{q}_{X*} \xrightarrow{\sim} \mathfrak{q}^*_X\widehat{\varphi}^*\widehat{\varphi}_*\mathfrak{q}_{X*} \longrightarrow \mathfrak{q}^*_X\mathfrak{q}_{X*} \xrightarrow{\sim} Id_{C_{X_{\mathfrak{E}}}}$$

are adjunction arrows.

(b) By C3.1.1 (and C3.1.2), the continuous functor $\mathcal{M}_k(X) \xrightarrow{\widehat{\varphi}^*} \mathcal{M}_k(Y)$ is a localization. Thus, the three arrows of the quasi-commutaive diagram (1) are localizations, hence the forth one, $\widetilde{\varphi}^*$, is a localization.

(c) The condition (c) means that for every $L \in ObC_Y$ and every presheaf F of k-modules on C_X, the value of the associated sheaf $\mathfrak{q}_X(F)$ at $\varphi^*(L)$ can be computed using only deflations (– covers) of the form $\varphi^*(M \xrightarrow{\mathsf{t}} L)$, where $M \xrightarrow{\mathsf{t}} L$ is a deflation. This implies that the diagram

$$\begin{array}{ccc} \mathcal{M}_k(X) & \xleftarrow{\widehat{\varphi}_*} & \mathcal{M}_k(Y) \\ {\scriptstyle \mathfrak{q}^*_X}\downarrow & & \downarrow{\scriptstyle \mathfrak{q}^*_Y} \\ C_{X_{\mathfrak{E}}} & \xleftarrow{\widetilde{\varphi}_*} & C_{Y_{\mathfrak{E}}} \end{array} \qquad (1_*)$$

quasi-commutes. Therefore, by the argument similar to (a) above, the functor $\mathfrak{q}^*_Y\widehat{\varphi}^!\mathfrak{q}_{X*}$ is a right adjoint to $\widetilde{\varphi}_*$. ∎

C3.3. Example. Suppose that C_X is a k-linear category with the smallest exact structure (given by split conflations). Then any k-linear functor (in particular, any right or left 'exact' k-linear functor) $C_X \xrightarrow{\varphi^*} C_Y$ is 'exact'. The category $C_{X_{\mathfrak{E}}}$ coincides with the category $\mathcal{M}_k(X)$ of k-presheaves on C_X, and the functor

$$C_{X_{\mathfrak{E}}} = \mathcal{M}_k(X) \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$$

is isomorphic to the composition of the functor $\mathcal{M}_k(X) \xrightarrow{\widehat{\varphi}^*} \mathcal{M}_k(Y)$ and the sheafification functor $\mathcal{M}_k(Y) \xrightarrow{\mathfrak{q}^*_Y} C_{Y_{\mathfrak{E}}}$. Therefore, a right adjoint $\widetilde{\varphi}_*$ to $\widetilde{\varphi}^*$ is isomorphic to the composition $\widehat{\varphi}_*\mathfrak{q}_{Y*}$, which is not, usually, an exact functor.

C3.3.1. Example. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category. Suppose that C_Y is an additive k-linear category endowed with the smallest exact structure, $\mathcal{E}^{spl}_Y$. Then a functor $C_X \xrightarrow{\varphi^*} C_Y$ is right 'exact' functor from $(C_X, \mathcal{E}_X)$ to $(C_X, \mathcal{E}^{spl}_Y)$ iff it maps every deflation of the exact category $(C_X, \mathcal{E}_X)$ to a split epimorphism (i.e. a coretraction). Notice that the condition (c) of C3.2.1 holds because every deflation in C_Y splits. Therefore, by C3.2.1(c), the functor $\widetilde{\varphi}_*$ has a right adjoint, $\widetilde{\varphi}^!$.

If the exact structure on $(C_X, \mathcal{E}_X)$ is also the smallest one (i.e. $\mathcal{E}_X = \mathcal{E}^{spl}_X$), then $C_{X_{\mathfrak{E}}} = \mathcal{M}_k(X)$ and $C_{Y_{\mathfrak{E}}} = \mathcal{M}_k(Y)$; i.e. in this case $\widetilde{\varphi}^* = \widehat{\varphi}^*$ and, therefore, a right adjoint to the functor $\widetilde{\varphi}_*$ coincides with $\widehat{\varphi}^!$.

C3.4. Remark. Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact categories. If $C_Y \xrightarrow{\varphi^*} C_X$ is an arbitrary functor, one can still define functors

$$C_{Y_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}_*} C_{Y_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^!} C_{X_{\mathfrak{E}}}$$

by the formulas

$$\widetilde{\varphi}^* = \mathfrak{q}^*_X\widehat{\varphi}^*\mathfrak{q}_{Y*},\ \widetilde{\varphi}_* = \mathfrak{q}^*_Y\widehat{\varphi}_*\mathfrak{q}_{X*},\ \widetilde{\varphi}^! = \mathfrak{q}^*_X\widehat{\varphi}^!\mathfrak{q}_{Y*}. \qquad (2)$$

The functors $\widetilde{\varphi}^*$, $\widetilde{\varphi}_*$, and $\widetilde{\varphi}^!$ might be regarded as *derived* functors of respectively $\widehat{\varphi}^*$, $\widehat{\varphi}_*$, and $\widehat{\varphi}^!$ (this viewpoint is discussed in Section D).

C3.5. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories and $(C_X, \mathcal{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathcal{E}_Y)$ a right 'exact' k-linear functor. Suppose that φ^* is a localization functor. Then φ^* is 'exact' iff the class of arrows $\Sigma_{\varphi^*} = \{s \in HomC_X \mid \varphi^*(s)$ is an isomorphism$\}$ satisfies the following condition:*

(#) If the rows of a commutative diagram

$$\begin{array}{ccccc} L & \longrightarrow & M & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ L' & \longrightarrow & M' & \longrightarrow & N' \end{array} \tag{2}$$

are conflations and any two of its vertical arrows belong to Σ_{φ^*}, then the remaining arrow belongs to Σ_{φ^*}.

Proof. (i) Consider first the case when φ^* is the identical functor. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \end{array} \tag{3}$$

be a commutative diagram in C_Y such that $L \longrightarrow M \longrightarrow N$ and $L' \longrightarrow M' \longrightarrow N'$ are conflations. If two of the three vertical arrows are isomorphisms, then the third arrow is an isomorphism as well.

In fact, the Gabriel-Quillen embedding transforms the diagram (2) into a commutative diagram with exact rows. If two of the vertical arrows of such diagram are isomorphisms, then the third one is an isomorphism. The Gabriel-Quillen embedding is a fully faithful functor, in particular, it is conservative. Therefore, all vertical arrows in the original diagram are isomorphisms.

(ii) Suppose that the functor $C_X \xrightarrow{\varphi^*} C_Y$ is 'exact'; i.e. it maps conflations to conflations. In particular, φ^* maps a diagram (2) with two arrows from Σ_{φ^*} to a diagram whose rows are conflations and two vertical arrows are isomorphisms. By (i) above, the third arrow is an isomorphism too; i.e. all vertical arrows of the diagram (2) belong to Σ_{φ^*}.

(iii) Suppose now that $C_X \xrightarrow{\varphi^*} C_Y$ is a localization functor which is right 'exact' and satisfies the condition (#). The claim is that the functor φ^* is 'exact'.

Let $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ be a conflation in C_X. The functor φ^* being right 'exact' means that there is a commutative diagram

$$\begin{array}{ccccc} \varphi^*(L) & \xrightarrow{\varphi^*(\mathfrak{j})} & \varphi^*(M) & \xrightarrow{\varphi^*(\mathfrak{e})} & \varphi^*(N) \\ & {\scriptstyle \mathfrak{e}'} \searrow \quad \nearrow {\scriptstyle \mathfrak{j}'} & & & \\ & \widetilde{L} & & & \end{array} \tag{4}$$

such that $\widetilde{L} \xrightarrow{\mathfrak{j}'} \varphi^*(M) \xrightarrow{\varphi^*(\mathfrak{e})} \varphi^*(N)$ is a conflation in C_Y and $\mathfrak{e}' \in \mathfrak{E}_Y$. Since φ^* is a localization, we can and will assume that $\widetilde{L} = \varphi^*(L')$ for some $L' \in ObC_X$. Let $\mathfrak{j}'$ be the composition of arrows $\varphi^*(L') \xrightarrow{\varphi^*(\mathfrak{j}'')} \varphi^*(M_1)$ and $\varphi^*(M_1) \xrightarrow{{\varphi^*}^{-1}(s)} \varphi^*(M)$ for some $s \in \Sigma_{\varphi^*}$.

Consider the cocartesian square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ M_1 & \xrightarrow{\mathfrak{e}_1} & N_1 \end{array} \tag{4}$$

By hypothesis, $\mathfrak{e}_1$ is a deflation and φ^* maps (4) to a cocartesian square. The square (4) is embedded into a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle s''}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ L_1 & \xrightarrow{\mathfrak{j}_1} & M_1 & \xrightarrow{\mathfrak{e}_1} & N_1 \end{array} \tag{5}$$

whose rows are deflations. Since the vertical arrows $s,\ s'$ in (5) belong to Σ_{φ^*}, the remaining vertical arrow, s'', belongs to Σ_{φ^*}.

The equality $\varphi^*(\mathfrak{e}_1 \circ \mathfrak{j}'') = 0$ means that $\mathfrak{e}_1 \circ \mathfrak{j}'' \circ \mathfrak{t} = 0$ for some $\mathfrak{t} \in \Sigma_{\varphi^*}$. Therefore we have a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle s''}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ L_1 & \xrightarrow{\mathfrak{j}_1} & M_1 & \xrightarrow{\mathfrak{e}_1} & N_1 \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \mathfrak{j}''} & & \\ L'' & \xrightarrow{\mathfrak{t}} & L' & & \end{array} \tag{6}$$

with a uniquely defined $L'' \xrightarrow{g} L_1$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \varphi^*(L) & \xrightarrow{\varphi^*(s'')} & \varphi^*(L_1) \\ {\scriptstyle \mathfrak{e}_2}\searrow & & \nearrow{\scriptstyle \varphi^*(g)} \\ & \varphi^*(L'') & \end{array} \tag{7}$$

where the arrow $\mathfrak{e}_2$ is the composition of the deflation $\varphi^*(L) \xrightarrow{\mathfrak{e}'} \varphi^*(L)$ and the isomorphism $\varphi^*(L') \xrightarrow{\varphi^*(s'')^{-1}} \varphi^*(L'')$. Since $\varphi^*(L) \xrightarrow{\varphi^*(s'')} \varphi^*(L_1)$ is an isomorphism, it follows from the commutativity of (7) that the arrow $\mathfrak{e}_2$ is a retraction; in particular it is a strict monomorphism. On the other hand, $\mathfrak{e}_2$ is a deflation, hence an epimorphism. Therefore, $\mathfrak{e}_2$ is an isomorphism, which implies that the deflation $\mathfrak{e}'$ in the diagram (4) is an isomorphism. Therefore, φ^* maps the deflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ to a deflation. ∎

C3.5.1. Corollary. *Let $(C_{\mathfrak{X}}, \mathcal{E}_X)$ and $(C_{\mathfrak{Y}}, \mathcal{E}_Y)$ be exact categories and $C_X \xrightarrow{\varphi^*} C_Y$ a left 'exact' functor. Suppose that φ^* is a localization functor. Then the functor φ^* is*

'exact' iff the class of arrows $\Sigma_{\varphi^*} = \{s \in HomC_X \mid \varphi^*(s)$ *is an isomorphism*$\}$ *satisfies the condition (#) of C3.5.*

Proof. The assertion is dual to that of C3.5. ∎

C3.6. Proposition. *Let* $(C_{\mathfrak{X}}, \mathcal{E}_X)$ *and* $(C_{\mathfrak{Y}}, \mathcal{E}_Y)$ *be exact categories,* $C_{\mathfrak{X}} \xrightarrow{\phi^*} C_{\mathfrak{Y}}$ *an 'exact' functor, and*

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\phi^*} & C_{\mathfrak{Y}} \\ {\scriptstyle \phi^*_s}\searrow & & \nearrow{\scriptstyle \phi^*_c} \\ & \Sigma^{-1}_{\phi^*} C_{\mathfrak{X}} & \end{array}$$

its canonical decomposition into a localization and a conservative functor. The functors ϕ^*_s *and* ϕ^*_c *are 'exact'.*

Proof. We call a pair of arrows $L \longrightarrow M \longrightarrow N$ in $\Sigma^{-1}_{\phi^*} C_{\mathfrak{X}}$ a *conflation* if it is isomorphic to the image of a conflation of $C_{\mathfrak{X}}$. We leave to the reader verifying that this defines a structure of an exact category on the quotient category $\Sigma^{-1}_{\phi^*} C_{\mathfrak{X}}$. It follows that the functors ϕ^*_s and ϕ^*_c are 'exact'. ∎

C3.7. Proposition. *Let* $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ *be an exact svelte category,* S *a family of arrows of* $C_{\mathfrak{X}}$*; and let* $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -))$ *be the pseudo-functor which assigns to every exact category* $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ *the category of 'exact' functors from* $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ *to* $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ *mapping every arrow of* S *to an isomorphism. The pseudo-functor* $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -)$ *is representable.*

Proof. Let $\mathfrak{F}_S$ be the family of all 'exact' functors which map S to isomorphisms, and let $\bar{S}$ denote the family of all arrows which are transformed into isomorphisms by all functors from $\mathfrak{F}_S$. Since the category $C_{\mathfrak{X}}$ is svelte, there exists a subset Ω of $\mathfrak{F}_S$ such that the family of all arrows of $C_{\mathfrak{X}}$ made invertible by functors of Ω coincides with $\bar{S}$.

The product of any set of exact categories is an exact category. In particular, the product $C_{\mathfrak{X}_\Omega}$ of targets of functors of Ω is an exact category and the canonical functor $C_{\mathfrak{X}} \xrightarrow{F_\Omega} C_{\mathfrak{X}_\Omega}$ is an 'exact' functor. By C3.6, the functor F_Ω factors through an 'exact' localization $C_{\mathfrak{X}} \xrightarrow{F_S} C_{\bar{S}^{-1}\mathfrak{X}}$. The 'exact' functor F_S is the universal arrow representing the pseudo-functor $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -)$. ∎

By the reason of C4.3.4, we need versions of the facts above for exact categories with actions.

C3.7.1. Proposition. *Let* $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ *be an exact* $\mathbb{Z}_+$*-category,* S *a family of arrows of* $C_{\mathfrak{X}}$*; and let* $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -))$ *be the pseudo-functor which assigns to every exact* $\mathbb{Z}_+$*-category* $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ *the category of 'exact'* $\mathbb{Z}_+$*-functors from* $(C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}})$ *to* $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$ *mapping every arrow of* S *to an isomorphism. The pseudo-functor* $\mathcal{E}x_S((C_{\mathfrak{X}}, \mathcal{E}_{\mathfrak{X}}), -)$ *is representable.*

C3.8. Multiplicative systems in quasi-(co)suspended categories. Fix a quasi-cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$. We call a class Σ of arrows of $C_{\mathfrak{X}}$ a *multiplicative system* of the quasi-cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}}$ if it is $\theta_{\mathfrak{X}}$-invariant, closed under composition, contains all isomorphisms, and satisfies the following condition:

(L1) for every pair of triangles

$$\theta_X(L) \xrightarrow{\mathfrak{d}} N \xrightarrow{g} M \xrightarrow{f} L \quad \text{and} \quad \theta_X(L') \xrightarrow{\mathfrak{d}'} N' \xrightarrow{g'} M' \xrightarrow{f'} L'$$

and a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ M' & \xrightarrow{f'} & L' \end{array}$$

where s and t are elements of Σ, there exists a morphism $N \xrightarrow{u} N'$ in Σ such that (u,t,s) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \theta_X(L) & \xrightarrow{\mathfrak{d}} & N & \xrightarrow{g} & M & \xrightarrow{f} & L \\ {\scriptstyle \theta_{\mathfrak{X}}(s)}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ \theta_X(L') & \xrightarrow{\mathfrak{d}'} & N' & \xrightarrow{g'} & M' & \xrightarrow{f'} & L' \end{array}$$

commutes.

We denote by $\mathcal{SM}_-(\mathfrak{X})$ the preorder (with respect to the inclusion) of all multiplicative systems and by $\mathcal{SM}^{\mathfrak{s}}_-(\mathfrak{X})$ the preorder of saturated multiplicative systems of the quasi-cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}}$.

Recall that a multiplicative system Σ in $C_{\mathfrak{X}}$ is *saturated* iff the following condition holds: if α, β, γ are arrows of $C_{\mathfrak{X}}$ such that the compositions $\alpha\beta$ and $\beta\gamma$ belong to Σ, then $\beta \in \Sigma$ (equivalently, all three arrows belong to Σ).

C3.8.1. Proposition. *(a) Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ and $\mathfrak{T}_-C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, Tr_{\mathfrak{Y}})$ be quasi-cosuspended categories and $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}_-C_{\mathfrak{Y}}$ a triangle functor. The family of arrows $\Sigma_F = \{s \in HomC_{\mathfrak{X}} \mid F(s)$ is invertible$\}$ is a saturated multiplicative system in $\mathfrak{T}_-C_{\mathfrak{X}}$.*

(b) Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a quasi-cosuspended category, $(C_Z, \mathcal{E}_Z)$ an exact category and $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{H} (C_Z, \mathcal{E}_Z)$ a homological functor. Then

$$\Sigma_{H,\theta_{\mathfrak{X}}} = \{s \in HomC_{\mathfrak{X}} \mid H\theta^n_{\mathfrak{X}}(s) \text{ is invertible for all } n \geq 0\}$$

is a saturated multiplicative system in $\mathfrak{T}_-C_{\mathfrak{X}}$.

Proof. (a) For any functor F, the family Σ_F is closed under composition and contains all isomorphisms. The $\theta_{\mathfrak{X}}$-invariance of Σ_F and the property (L1) follow from the axioms of quasi-cosuspended categories.

(b) The system $\Sigma_{H,\theta_{\mathfrak{X}}}$ is closed under composition, contains all isomorphisms, and is $\theta_{\mathfrak{X}}$-invariant by construction. It remains to verify the property (L1). Let

$$\theta_X(L) \xrightarrow{\mathfrak{d}} N \xrightarrow{g} M \xrightarrow{f} L \quad \text{and} \quad \theta_X(L') \xrightarrow{\mathfrak{d}'} N' \xrightarrow{g'} M' \xrightarrow{f'} L'$$

be a pair of triangles and

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ M' & \xrightarrow{f'} & L' \end{array}$$

a commutative diagram with s and t elements of $\Sigma_{H,\theta_{\mathfrak{X}}}$. By the property (S3) of quasi-cosuspended categories, there exists a morphism $N \xrightarrow{u} N'$ in Σ such that (u,t,s) is a morphism of triangles, i.e. the diagram

$$\begin{array}{ccccccc} \theta_X(L) & \xrightarrow{\mathfrak{d}} & N & \xrightarrow{g} & M & \xrightarrow{f} & L \\ {\scriptstyle \theta_{\mathfrak{X}}(s)}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ \theta_X(L') & \xrightarrow{\mathfrak{d}'} & N' & \xrightarrow{g'} & M' & \xrightarrow{f'} & L' \end{array} \tag{1}$$

commutes. Let $\mathcal{H}$ denote the composition of the homological functor $C_{\mathfrak{X}} \xrightarrow{H} C_Z$ with the Gabriel-Quillen embedding $C_Z \longrightarrow C_{Z_{\mathfrak{E}}}$, we obtain for every nonnegative integer n a commutative diagram

$$\begin{array}{ccccccc} \mathcal{H}\theta_X^{n+1}(L) & \xrightarrow{\mathcal{H}\theta_X^n(\mathfrak{d})} & \mathcal{H}\theta_X^n(N) & \xrightarrow{\mathcal{H}\theta_X^n(g)} & \mathcal{H}\theta_X^n(M) & \xrightarrow{\mathcal{H}\theta_X^n(f)} & \mathcal{H}\theta_X^n(L) \\ {\scriptstyle \mathcal{H}\theta_{\mathfrak{X}}^{n+1}(s)}\downarrow\wr & & {\scriptstyle \mathcal{H}\theta_X^n(u)}\downarrow & & {\scriptstyle \mathcal{H}\theta_X^n(t)}\downarrow\wr & & \wr\downarrow{\scriptstyle \mathcal{H}\theta_X^n(s)} \\ \mathcal{H}\theta_X^{n+1}(L') & \xrightarrow{\mathcal{H}\theta_X^n(\mathfrak{d}')} & \mathcal{H}\theta_X^n(N') & \xrightarrow{\mathcal{H}\theta_X^n(g')} & \mathcal{H}\theta_X^n(M') & \xrightarrow{\mathcal{H}\theta_X^n(f')} & \mathcal{H}\theta_X^n(L') \end{array} \tag{2}$$

in the abelian category $C_{Z_{\mathfrak{E}}}$ whose rows are exact sequences and three of the for vertical arrows are isomorphisms. Therefore the fourth vertical arrow, $\mathcal{H}\theta_X^n(u)$ is an isomorphism for all $n \geq 0$; i.e. u belongs to $\Sigma_{H,\theta_{\mathfrak{X}}}$. ∎

C3.8.2. Proposition. *(a) Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ and $\mathfrak{T}_- C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, Tr_{\mathfrak{Y}})$ be quasi-cosuspended categories. Every triangle functor $\mathfrak{T}_- C_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}_- C_{\mathfrak{Y}}$ is uniquely represented as the composition of a triangle localization $\mathfrak{T}_- C_{\mathfrak{X}} \xrightarrow{F_{\mathfrak{s}}} \mathfrak{T}_- C_{\mathfrak{X}_{\mathfrak{s}}}$ and a conservative triangle functor $\mathfrak{T}_- C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{F_{\mathfrak{c}}} \mathfrak{T}_- C_{\mathfrak{Y}}$.*

(b) Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a quasi-cosuspended category and $(C_Z, \mathcal{E}_Z)$ an exact category. Every homological functor $\mathfrak{T}_- C_{\mathfrak{X}} \xrightarrow{H} (C_Z, \mathcal{E}_Z)$ is uniquely represented as the composition of a triangle localization $\mathfrak{T}_- C_{\mathfrak{X}} \xrightarrow{H_{\mathfrak{s}}} \mathfrak{T}_- C_{\mathfrak{X}_{\mathfrak{s}}}$ and a conservative homological functor $\mathfrak{T}_- C_{\mathfrak{X}_{\mathfrak{s}}} \xrightarrow{H_{\mathfrak{c}}} (C_Z, \mathcal{E}_Z)$.

Proof. Let Σ denote the multiplicative system Σ_F of C3.8.1(a), or $\Sigma_{H,\theta_{\mathfrak{X}}}$ of C3.8.1(b). Then the quotient category $\Sigma^{-1}C_{\mathfrak{X}}$ is an additive k-linear category having a unique structure $(\widetilde{\theta}, Tr_{\Sigma^{-1}\mathfrak{X}})$ of a quasi-cosuspended category such that the localization functor

$$C_{\mathfrak{X}} \xrightarrow{q_\Sigma^*} \Sigma^{-1}C_{\mathfrak{X}} = C_{\Sigma^{-1}\mathfrak{X}}$$

is a strict triangle functor. Here *strict* means that the quasi-cosuspension functor $\widetilde{\theta} = \theta_{\Sigma^{-1}\mathfrak{X}}$ is uniquely determined by the equality $\widetilde{\theta} \circ q^*_\Sigma = q^*_\Sigma \circ \theta_{\mathfrak{X}}$, and $Tr^-_{\Sigma^{-1}\mathfrak{X}}$ is the class of all sequences $\widetilde{\theta}(L) \longrightarrow N \longrightarrow M \longrightarrow L$ in $C_{\Sigma^{-1}\mathfrak{X}}$ which are isomorphic to the images of triangles of $Tr_{\mathfrak{X}}$ by the localization functor q^*_Σ. Details are left to the reader. ∎

C3.8.3. Proposition. *Let $\mathfrak{T}_{-}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ be a svelte quasi-cosuspended category, S a family of arrows of the category $C_{\mathfrak{X}}$, and $Tr_S(\mathfrak{T}_{-}C_{\mathfrak{X}}, -)$ the pseudo-functor which assigns to every quasi-cosuspended category $\mathfrak{T}_{-}C_{\mathfrak{Y}}$ the category of all triangular functors F from $\mathfrak{T}_{-}C_{\mathfrak{X}}$ to $\mathfrak{T}_{-}C_{\mathfrak{Y}}$ transforming all arrows of S into isomorphisms. The pseudo-functor $Tr_S(\mathfrak{T}_{-}C_{\mathfrak{X}}, -)$ is representable.*

Proof. Let $\mathfrak{F}_S$ be the family of all triangular functors which map S to isomorphisms, and let $\bar{S}$ denote the family of arrows which are transformed into isomorphisms by all functors from $\mathfrak{F}_S$. Since the category $C_{\mathfrak{X}}$ is svelte, there exists a subset Ω of $\mathfrak{F}_S$ such that the family of all arrows of $C_{\mathfrak{X}}$ made invertible by functors of Ω coincides with $\bar{S}$.

The product of any set of quasi-cosuspended categories is a quasi-cosuspended category. In particular, the product $C_{\mathfrak{X}_\Omega}$ of targets of functors of Ω is a quasi-cosuspended category and the canonical functor $C_{\mathfrak{X}} \xrightarrow{F_\Omega} C_{\mathfrak{X}_\Omega}$ is a triangle functor. By C3.8.2, the functor F_Ω factors through a triangle localization $\mathfrak{T}_{-}C_{\mathfrak{X}} \xrightarrow{F_S} \mathfrak{T}_{-}C_{\bar{S}^{-1}\mathfrak{X}}$. The triangle functor F_S is the universal arrow representing the pseudo-functor $Tr_S(\mathfrak{T}_{-}C_{\mathfrak{X}}, -)$. ∎

C3.9. Triangle subcategories. A full subcategory $\mathcal{B}$ of the category $C_{\mathfrak{X}}$ is called a *triangle subcategory* of $\mathfrak{T}_{-}C_{\mathfrak{X}}$ if it is $\theta_{\mathfrak{X}}$-stable and has the following property: any morphism $M \xrightarrow{f} L$ of $\mathcal{B}$ is embedded into a triangle

$$\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$$

such that $N \in Ob\mathcal{B}$.

A full triangle subcategory $\mathcal{B}$ of $\mathfrak{T}_{-}C_{\mathfrak{X}}$ is called a *thick triangle subcategory* if it is closed under extensions, i.e. if $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$ is a triangle with L and N objects of $\mathcal{B}$, then M belongs to $\mathcal{B}$.

C3.9.1. Saturated triangle subcategories. A full triangle subcategory $\mathcal{B}$ of a quasi-cosuspended category $\mathfrak{T}_{-}C_{\mathfrak{X}}$ is called *saturated* if it coincides with its Karoubian envelope in $\mathfrak{T}_{-}C_{\mathfrak{X}}$; i.e. any retract of an object of $\mathcal{B}$ is an object of $\mathcal{B}$.

Evidently, every thick triangle subcategory of $\mathfrak{T}_{-}C_{\mathfrak{X}}$ is saturated.

It is known that the converse is true if $\mathfrak{T}_{-}C_{\mathfrak{X}}$ is a triangulated category: a full triangle subcategory of a triangulated category is thick iff it is saturated.

C3.10. Triangle subcategories and multiplicative systems. Let $\mathfrak{T}_{-}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a quasi-cosuspended k-linear category; and let $\mathcal{B}$ be its triangle subcategory. Let $\Sigma(\mathcal{B})$ denote the family of all arrows $N \xrightarrow{t} M$ of the category $C_{\mathfrak{X}}$ such that there exists a triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{t} M \xrightarrow{f} L$ with $L \in Ob\mathcal{B}$. Set

$$\Sigma_\infty(\mathcal{B}) = \{s \in HomC_{\mathfrak{X}} \mid \theta^n(s) \in \Sigma(\mathcal{B}) \text{ for some } n \geq 0\}.$$

C3.10.1. Proposition. *Let $\mathcal{B}$ be a full triangle subcategory of a quasi-cosuspended category $\mathfrak{T}_{-}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$. Then the class $\Sigma_{\infty}(\mathcal{B})$ is a multiplicative system. It is saturated iff the subcategory $\mathcal{B}$ is saturated.*

Proof. It follows from the definitions of $\Sigma(\mathcal{B})$ and $\Sigma_{\infty}(B)$ that both systems are $\theta_{\mathfrak{X}}$-stable and contain all isomorphisms. ∎

For a full triangle subcategory $\mathcal{B}$ of the quasi-cosuspended category $\mathfrak{T}_{-}C_{\mathfrak{X}}$, we set $C_{\mathfrak{X}}/\mathcal{B} = \Sigma(\mathcal{B})^{-1}C_{\mathfrak{X}}$.

C3.10.2. Proposition. *Let $\mathfrak{T}_{-}C_{\mathfrak{X}}$ and $\mathfrak{T}_{-}C_{\mathfrak{Y}}$ be quasi-cosuspended categories, and let $\mathfrak{T}_{-}C_{\mathfrak{X}} \xrightarrow{F} \mathfrak{T}_{-}C_{\mathfrak{Y}}$ be a triangle functor. Then*

(a) $Ker(F)$ is a thick triangle subcategory of $\mathfrak{T}_{-}C_{\mathfrak{X}}$;

(b) $\theta_{\mathfrak{X}}(\Sigma_F) \subseteq \Sigma(Ker(F)) \subseteq \Sigma_F$. In particular, $\Sigma_F = \Sigma(Ker(F))$ if the quasi-cosuspension $\theta_{\mathfrak{X}}$ is a conservative functor.

Proof. (a) If $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$ is a triangle in $C_{\mathfrak{X}}$ with L and N objects of $Ker(F)$, then the functor F maps it to the triangle $0 \longrightarrow 0 \longrightarrow F(M) \longrightarrow 0$, hence $F(M) = 0$.

(b) Let $N \xrightarrow{t} M$ be a morphism of $\Sigma(F)$; i.e. there exists a triangle

$$\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{t} M \xrightarrow{f} L$$

with $L \in ObKer(F)$. The functor F maps it to the triangle

$$0 \longrightarrow F(N) \xrightarrow{F(t)} F(M) \longrightarrow 0$$

which means, precisely, that $F(t)$ is an isomorphism. This shows that $\Sigma(Ker(F)) \subseteq \Sigma_F$.

Conversely, let $M \xrightarrow{s} L$ be a morphism of Σ_F and $\theta_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{s} L$ a triangle. The functor F maps it to the triangle

$$\ldots \longrightarrow F\theta_{\mathfrak{X}}(M) \xrightarrow{\sim} F\theta_{\mathfrak{X}}(L) \xrightarrow{F(h)} F(N) \xrightarrow{F(g)} F(M) \xrightarrow{\sim} F(L).$$

Therefore, $F\theta^n_{\mathfrak{X}}(N) = 0$ for all $n \geq 0$. This shows that $\theta^n_{\mathfrak{X}}(s) \in \Sigma(Ker(F))$ for $n \geq 1$. ∎

C3.11. Coaisles and t-structures in a quasi-cosuspended category.

C3.11.1. Coaisles in a quasi-cosuspended category. Let $\mathfrak{T}_{-}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a quasi-cosuspended category. Its thick triangle subcategory $\mathcal{U}$ is called a *coaisle* if the inclusion functor $\mathcal{U} \xrightarrow{\mathfrak{j}_*} C_{\mathfrak{X}}$ has a left adjoint, $\mathfrak{j}^*$.

C3.11.2. Proposition [KeV1]. *Let $\mathfrak{T}_{-}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr_{\mathfrak{X}})$ be a triangulated k-linear category (i.e. the quasi-cosuspension $\theta_{\mathfrak{X}}$ is an auto-equivalence). Then a strictly full subcategory $\mathcal{U}$ of $C_{\mathfrak{X}}$ is a coaisle iff it is $\theta_{\mathfrak{X}}$-stable and for each $M \in ObC_{\mathfrak{X}}$, there is a triangle*

$$\theta_{\mathfrak{X}}(M^{\mathcal{U}}) \longrightarrow M_{\perp\mathcal{U}} \longrightarrow M \longrightarrow M^{\mathcal{U}}, \tag{1}$$

where $M^{\mathcal{U}}$ is an object of $\mathcal{U}$ and $M_{\perp\mathcal{U}}$ is an object of ${}^{\perp}\mathcal{U}$. The triangle (1) is unique up to isomorphism.

Proof. Suppose $\mathcal{U}$ is a coaisle in $\mathfrak{T}_{-}C_{\mathfrak{X}}$, i.e. it is $\theta_{\mathfrak{X}}$-stable and the inclusion functor $\mathcal{U} \xrightarrow{\mathfrak{j}_*} C_{\mathfrak{X}}$ has a left adjoint, $\mathfrak{j}^*$. Fix an adjunction morphism $Id_{C_{\mathfrak{X}}} \xrightarrow{\eta} \mathfrak{j}_*\mathfrak{j}^*$. Then we have, for any $M \in ObC_{\mathfrak{X}}$, a triangle

$$\theta_{\mathfrak{X}}\mathfrak{j}_*\mathfrak{j}^*(M) = \theta_{\mathfrak{X}}(M^{\mathcal{U}}) \xrightarrow{\partial(M)} M_{\perp\mathcal{U}} = \mathfrak{K}(M) \xrightarrow{\mathfrak{k}(M)} M \xrightarrow{\eta(M)} \mathfrak{j}_*\mathfrak{j}^*(M) = M^{\mathcal{U}} \qquad (2)$$

Since, by hypothesis, j^* is a triangle functor, its application to the triangle (2) produces a triangle in the quasi-cosuspended category $\mathfrak{T}_{-}\mathcal{U}$. Since $\mathfrak{j}^*\eta$ is an isomorphism, $\mathfrak{j}^*(\mathfrak{K}(M)) = 0$, i.e. $\mathfrak{K}(M) = M_{\perp\mathcal{U}}$ belongs to the kernel of the localization functor $\mathfrak{j}^*$. It is easy to see that $Ker(\mathfrak{j}^*)$ coincides with ${}^{\perp}\mathcal{U}$.

Conversely, suppose that for every $M \in ObC_{\mathfrak{X}}$, there exists a triangle (1) with $M^{\mathcal{U}} \in Ob\mathcal{U}$ and $M_{\perp\mathcal{U}} \in Ob{}^{\perp}\mathcal{U}$. ∎

C3.11.2. Cores of t-structures. The core of a t-structure $\mathcal{U} \xrightarrow{\mathfrak{j}_*} C_{\mathfrak{X}}$ is the subcategory $\mathcal{U} \cap {}^{\perp}\theta_{\mathfrak{X}}(\mathcal{U})$.

C4. Cohomological functors on suspended categories. Universal cohomological and homological functors.

See preliminaries on exact and (co)suspended categories in Appendix K.

Categories (suspended, cosuspended, exact) and functors of this section are k-linear for some fixed commutative unital ring k.

C4.1. k-Presheaves on a k-linear $\mathbb{Z}_+$-category. Fix a k-linear $\mathbb{Z}_+$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$. Let $\mathcal{M}_k(\mathfrak{X}) \xrightarrow{\Theta^*_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X})$ denote the continuous (i.e. having a right adjoint) extension of the functor $C_{\mathfrak{X}} \xrightarrow{\theta_{\mathfrak{X}}} C_{\mathfrak{X}}$. This extension is determined uniquely up to isomorphism by the quasi-commutativity of the diagram

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\theta_{\mathfrak{X}}} & C_{\mathfrak{X}} \\ {\scriptstyle h_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle h_{\mathfrak{X}}} \\ \mathcal{M}_k(\mathfrak{X}) & \xrightarrow{\Theta^*_{\mathfrak{X}}} & \mathcal{M}_k(\mathfrak{X}) \end{array}$$

where $h_{\mathfrak{X}}$ is the Yoneda embedding.

Let Θ_* be a right adjoint to $\Theta^*_{\mathfrak{X}}$. Notice that the projective objects of the category $\mathcal{M}_k(\mathfrak{X})$ are direct summands of coproducts of representable presheaves. Since $\Theta^*_{\mathfrak{X}}$ maps representable presheaves to representable objects and preserves arbitrary coproducts, it maps projective objects of $\mathcal{M}_k(\mathfrak{X})$ to projective objects. Therefore, thanks to the fact that the category $\mathcal{M}_k(\mathfrak{X})$ has enough projectives, the functor Θ_* is exact.

C4.1.1. Note. Whenever it is convenient, we shall identify a k-linear $\mathbb{Z}_+$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ with the equivalent to it full subcategory of the $\mathbb{Z}_+$-category $(\mathcal{M}_k(\mathfrak{X}), \Theta^*_{\mathfrak{X}})$ generated by representable presheaves.

C4.2. Cohomological and homological functors. Let $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^+_{\mathfrak{X}})$ be a suspended category and $(C_Z, \mathcal{E}_Z)$ an exact category. A functor $C_{\mathfrak{X}} \xrightarrow{\Phi} C_Z$ is called a *cohomological* functor on $\mathfrak{T}_+C_{\mathfrak{X}}$ with values in $(C_Z, \mathcal{E}_Z)$ (and we write $\mathfrak{T}_+C_{\mathfrak{X}} \xrightarrow{\Phi} (C_Z, \mathcal{E}_Z)$), if for any triangle $L \longrightarrow M \longrightarrow N \longrightarrow \theta_{\mathfrak{X}}(L)$, the sequence

$$\Phi(L) \longrightarrow \Phi(M) \longrightarrow \Phi(N) \longrightarrow \Phi(\theta_{\mathfrak{X}}(L)) \longrightarrow \Phi(\theta_{\mathfrak{X}}(M)) \longrightarrow \dots \tag{1}$$

is 'exact' and for any morphism $L \xrightarrow{f} M$ of $C_{\mathfrak{X}}$, there exists a kernel of $\Phi(f)$ and the canonical monomorphism $Ker(\Phi(f)) \longrightarrow \Phi(L)$ is an inflation.

Dually, if $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ is a cosuspended category, then a functor $C_{\mathfrak{X}} \xrightarrow{\Psi} C_Z$ is called a *homological* functor $\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{\Psi} (C_Z, \mathcal{E}_Z)$ if the dual functor $C^{op}_{\mathfrak{X}} \xrightarrow{\Psi^{op}} C^{op}_Z$ is cohomological. In other words, for any triangle $\theta_{\mathfrak{X}}(N) \longrightarrow L \longrightarrow M \longrightarrow N$, the sequence

$$\dots \longrightarrow \Psi(\theta_{\mathfrak{X}}(M)) \longrightarrow \Psi(\theta_{\mathfrak{X}}(N)) \longrightarrow \Psi(L) \longrightarrow \Psi(M) \longrightarrow \Psi(N)$$

is 'exact' and for any morphism $L \xrightarrow{f} M$ of $C_{\mathfrak{X}}$, there exists a cokernel of $\Psi(f)$ and the canonical epimorphism $\Psi(M) \longrightarrow Cok(\Psi(f))$ is a deflation in $(C_Z, \mathcal{E}_Z)$.

C4.2.1. Example. Let $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ be a k-linear cosuspended category. Then for every $W \in ObC_{\mathfrak{X}}$, the sequence

$$\dots \longrightarrow C_{\mathfrak{X}}(W, \theta_{\mathfrak{X}}(M)) \longrightarrow C_{\mathfrak{X}}(W, \theta_{\mathfrak{X}}(L)) \longrightarrow C_{\mathfrak{X}}(W, N) \longrightarrow C_{\mathfrak{X}}(W, M) \longrightarrow C_{\mathfrak{X}}(W, L) \tag{2}$$

is exact. This means precisely that the Yoneda embedding

$$C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X}), \quad M \longmapsto C_{\mathfrak{X}}(-, M),$$

is a homological functor.

Let $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^+_{\mathfrak{X}})$ be a suspended category. For every object V of $C_{\mathfrak{X}}$ and every triangle $L \longrightarrow M \longrightarrow N \longrightarrow \theta_{\mathfrak{X}}(L)$, the sequence

$$C_{\mathfrak{X}}(L, V) \longleftarrow C_{\mathfrak{X}}(M, V) \longleftarrow C_{\mathfrak{X}}(N, V) \longleftarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(L), V) \longleftarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(M), V) \longleftarrow \dots \tag{3}$$

is exact. In other words, the functor $h^o_{\mathfrak{X}}$ dual to the Yoneda embedding

$$C^{op}_{\mathfrak{X}} \longrightarrow \mathcal{M}_k(\mathfrak{X}^o), \quad M \longmapsto C_{\mathfrak{X}}(M, -),$$

is a cohomological functor.

C4.3. Universal homological functors.

C4.3.1. The category $C_{X_{\mathfrak{a}}}$. For any k-linear category C_X, let $C_{X_{\mathfrak{a}}}$ denote the full subcategory of the category $\mathcal{M}_k(X)$ of k-presheaves on C_X whose objects are k-presheaves having a left resolution formed by representable presheaves.

C4.3.2. Proposition. *(a) The subcategory $C_{X_{\mathfrak{a}}}$ is closed under extensions; i.e. $C_{X_{\mathfrak{a}}}$ is an exact subcategory of the abelian category $\mathcal{M}_k(X)$. In particular, $C_{X_{\mathfrak{a}}}$ is an additive k-linear category.*

(b) Suppose that the category C_X is Karoubian. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence in $\mathcal{M}_k(X)$. If two of the objects M', M, M'' belong to the subcategory $C_{X_{\mathfrak{a}}}$, then the third object belongs to $C_{X_{\mathfrak{a}}}$.

(b1) More generally, if C_X is Karoubian and

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow 0$$

is an exact sequence in $\mathcal{M}_k(X)$ with at least $n-1$ objects from the subcategory $C_{X_{\mathfrak{a}}}$, then the remaining object belongs to $C_{X_{\mathfrak{a}}}$.

Proof. (a) Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence in $\mathcal{M}_k(X)$. Let $P' \longrightarrow M'$ and $P'' \longrightarrow M''$ be projective resolutions. Then, by [Ba, I.6.7], there exists a differential on the graded object $P = P' \oplus P''$ such that the splitting exact sequence $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$ is an exact sequence of complexes which are resolutions of the exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$. If the complexes P' and P'' are formed by representable presheaves, then P is a complex of representable presheaves, hence M is an object of the subcategory $C_{X_{\mathfrak{a}}}$.

(b) The assertion (b) follows from [Ba, I.6.8] and (b1) is a special case of [Ba, I.6.9]. ∎

C4.3.3. Lemma. *If $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{T}\mathfrak{r}^-_{\mathfrak{X}})$ is a cosuspended category, then objects of $C_{\mathfrak{X}_{\mathfrak{a}}}$ are all objects M of $\mathcal{M}_k(X)$ such that there exists an exact sequence*

$$M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0,$$

where M_0 and M_1 are representable presheaves.

Proof. In fact, let $M_1 \xrightarrow{f} M_0 \xrightarrow{e} M \longrightarrow 0$ be such an exact sequence. Since M_0 and M_1 are representable, there exists a triangle $\Theta^*_{\mathfrak{X}}(M_0) \xrightarrow{\mathfrak{d}} M_2 \xrightarrow{g} M_1 \xrightarrow{f} M_0$ which gives rise to a resolution

$$\ldots \longrightarrow \Theta^*_{\mathfrak{X}}(M_1) \xrightarrow{\Theta^*_{\mathfrak{X}}(f)} \Theta^*_{\mathfrak{X}}(M_0) \xrightarrow{\mathfrak{d}} M_2 \xrightarrow{g} M_1 \xrightarrow{f} M_0 \xrightarrow{e} M$$

of the object M. ∎

C4.3.4. Proposition. *Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{T}\mathfrak{r}^-_{\mathfrak{X}})$ be a cosuspended category. Then the corestriction $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ of the Yoneda embedding $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} \mathcal{M}_k(X)$ to the subcategory $C_{\mathfrak{X}_{\mathfrak{a}}}$ is a universal homological functor in the following sense: for any exact category $(C_Z, \mathcal{E}_Z)$ and a homological functor $\mathfrak{T}_- C_{\mathfrak{X}} \xrightarrow{\mathcal{H}} (C_Z, \mathcal{E}_Z)$, there exists a unique up to isomorphism 'exact' functor $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}) \xrightarrow{\mathcal{H}_{\mathfrak{a}}} (C_Z, \mathcal{E}_Z)$ such that $\mathcal{H} \simeq \mathcal{H}_{\mathfrak{a}} \circ \mathfrak{H}_{\mathfrak{X}}$.*

The category $C_{\mathfrak{X}_{\mathfrak{a}}}$ has a unique up to isomorphism $\mathbb{Z}_+$-category structure $C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\theta_{\mathfrak{X}_{\mathfrak{a}}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ such that the functor $\mathfrak{H}_{\mathfrak{X}}$ is a $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) \longrightarrow (C_{\mathfrak{X}_{\mathfrak{a}}}, \theta_{\mathfrak{X}_{\mathfrak{a}}})$.

Proof. (a) Fix an exact category $(C_Z, \mathcal{E}_Z)$ with the class of deflations $\mathfrak{E}_Z$. Let $\mathfrak{q}_Z^{\mathfrak{I}}$ denote the Gabriel-Quillen embedding $C_Z \longrightarrow C_{Z_{\mathfrak{E}}}$. Since $C_{Z_{\mathfrak{E}}}$ is a Grothendieck category, in particular it is cocomplete (i.e. closed under colimits), any functor $C_{\mathfrak{X}} \xrightarrow{\mathcal{H}} C_Z$ gives a rise to a quasi-commutative diagram

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\mathcal{H}} & C_Z \\ {\scriptstyle h_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{q}_Z^{\mathfrak{I}}} \\ \mathcal{M}_k(\mathfrak{X}) & \xrightarrow{\mathcal{H}^*} & C_{Z_{\mathfrak{E}}} \end{array} \tag{1}$$

in which the functor $\mathcal{H}^*$ has a right adjoint, $\mathcal{H}_*$. Since the functor $\mathcal{H}^*$ preserves colimits of small diagrams (thanks to the existence of a right adjoint) and every object of the category $\mathcal{M}_k(\mathfrak{X})$ is a colimit of a small diagram of representable presheaves, $\mathcal{H}^*$ is determined uniquely up to isomorphism by the quasi-commutativity of the diagram (1).

If $C_{\mathfrak{X}} \xrightarrow{\mathcal{H}} C_Z$ is a homological functor $\mathfrak{T}_- C_{\mathfrak{X}} \longrightarrow (C_Z, \mathcal{E}_Z)$, then the composition of $\mathcal{H}$ and $C_Z \xrightarrow{\mathfrak{q}_Z^{\mathfrak{I}}} C_{Z_{\mathfrak{E}}}$ is a homological functor, because the functor $\mathfrak{q}_Z^{\mathfrak{I}}$ is 'exact' and homological functors are stable under the composition with 'exact' functors.

(b) The diagram (1) induces the quasi-commutative diagram

$$\begin{array}{ccc} C_{\mathfrak{X}} & \xrightarrow{\mathcal{H}} & C_Z \\ {\scriptstyle \mathfrak{H}_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{q}_Z^{\mathfrak{I}}} \\ C_{\mathfrak{X}_{\mathfrak{a}}} & \xrightarrow{\mathcal{H}^*_{\mathfrak{a}}} & C_{Z_{\mathfrak{E}}} \end{array} \tag{2}$$

The claim is that the functor $\mathcal{H}^*_{\mathfrak{a}}$ (– the restriction of the functor $\mathcal{H}^*$ to $C_{\mathfrak{X}_{\mathfrak{a}}}$) is 'exact'.

In fact, let $M' \longrightarrow M \longrightarrow M''$ be a conflation in $C_{\mathfrak{X}_a}$. Since the functor $\mathcal{H}^*$ is right exact, the sequence $\mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M) \longrightarrow \mathcal{H}^*(M'') \longrightarrow 0$ is exact. It remains to show that $\mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M)$ is a monomorphism.

Let $P_1' \xrightarrow{f'} P_0' \xrightarrow{e'} M' \longrightarrow 0$ and $P_1'' \xrightarrow{f''} P_0'' \xrightarrow{e''} M'' \longrightarrow 0$ be exact sequences in $C_{\mathfrak{X}_{\mathfrak{a}}}$ such that the objects P_i', P_i'', $i = 0, 1$, are representable. The morphisms $P_1' \xrightarrow{f'} P_0'$ and $P_1'' \xrightarrow{f''} P_0''$ can be inserted into triangles resp. $\Theta^*_{\mathfrak{X}}(P_0') \xrightarrow{\mathfrak{d}'} P_2' \xrightarrow{g'} P_1' \xrightarrow{f'} P_0'$ and $\Theta^*_{\mathfrak{X}}(P_0'') \xrightarrow{\mathfrak{d}''} P_2'' \xrightarrow{g''} P_1'' \xrightarrow{f''} P_0''$ which give rise to the complexes

$$\mathcal{P}' = \big(\ldots \longrightarrow \Theta^*_{\mathfrak{X}}(P_1') \xrightarrow{\Theta^*_{\mathfrak{X}}(f')} \Theta^*_{\mathfrak{X}}(P_0') \xrightarrow{\mathfrak{d}'} P_2' \xrightarrow{g'} P_1' \xrightarrow{f'} P_0' \big)$$

and

$$\mathcal{P}'' = \big(\ldots \longrightarrow \Theta^*_{\mathfrak{X}}(P_1'') \xrightarrow{\Theta^*_{\mathfrak{X}}(f'')} \Theta^*_{\mathfrak{X}}(P_0'') \xrightarrow{\mathfrak{d}''} P_2'' \xrightarrow{g''} P_1'' \xrightarrow{f''} P_0'' \big)$$

By (the argument of) C4.3.2(a), there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}' & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{P}'' & \longrightarrow & 0 \\ & & {\scriptstyle e'}\downarrow & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e''} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array} \tag{3}$$

in which $\mathcal{P}' \xrightarrow{e'} M'$, $\mathcal{P}'' \xrightarrow{e''} M''$, and $\mathcal{P} \xrightarrow{e} M$ are projective resolutions and

$$0 \longrightarrow \mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}'' \longrightarrow 0$$

is an exact sequence of projective complexes. Since $\mathcal{H}^* \circ h_{\mathfrak{X}}$ is a cohomological functor, the complexes $\mathcal{H}^*(\mathcal{P}')$ and $\mathcal{H}^*(\mathcal{P}'')$ are exact. Together with the exactness of the sequence

$$0 \longrightarrow \mathcal{H}^*(\mathcal{P}') \longrightarrow \mathcal{H}^*(\mathcal{P}) \longrightarrow \mathcal{H}^*(\mathcal{P}'') \longrightarrow 0$$

this implies the exactness of the complex $\mathcal{H}^*(\mathcal{P})$. Now it follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}^*(\mathcal{P}') & \longrightarrow & \mathcal{H}^*(\mathcal{P}) & \longrightarrow & \mathcal{H}^*(\mathcal{P}'') & \longrightarrow & 0 \\ & & {\scriptstyle \mathcal{H}^*(e')}\downarrow & & {\scriptstyle \mathcal{H}^*(e)}\downarrow & & \downarrow{\scriptstyle \mathcal{H}^*(e'')} & & \\ 0 & \longrightarrow & \mathcal{H}^*(M') & \longrightarrow & \mathcal{H}^*(M) & \longrightarrow & \mathcal{H}^*(M'') & \longrightarrow & 0 \end{array}$$

that $\mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M)$ is a monomorphism; hence the sequence

$$0 \longrightarrow \mathcal{H}^*(M') \longrightarrow \mathcal{H}^*(M) \longrightarrow \mathcal{H}^*(M'') \longrightarrow 0$$

is exact.

(c) There is a unique up to isomorphism functor $C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\mathcal{H}_{\mathfrak{a}}} C_Z$ such that $\mathcal{H}^*_{\mathfrak{a}} \simeq \mathfrak{q}^{\mathfrak{I}}_Z \circ \mathcal{H}_{\mathfrak{a}}$. The functor $\mathcal{H}_{\mathfrak{a}}$ is an 'exact' functor $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}) \longrightarrow (C_Z, \mathcal{E}_Z)$.

Let M be an object of $C_{\mathfrak{X}_{\mathfrak{a}}}$, and let $P_1 \xrightarrow{f} P_0 \xrightarrow{e} M \longrightarrow 0$ be an exact sequence with representable objects P_0 and P_1. Since $\mathcal{H}$ is a homological functor, there exists a cokernel of the morphism $\mathcal{H}(f)$. We set $\mathcal{H}_{\mathfrak{a}}(M) = Cok(\mathcal{H}(f))$. Since the functor $\mathcal{H}^*$ is right exact, it maps $P_1 \xrightarrow{f} P_0 \xrightarrow{e} M \longrightarrow 0$ to an exact sequence. Therefore, because the Gabriel-Quillen embedding $(C_Z, \mathcal{E}_Z) \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_Z} (C_{Z_{\mathfrak{E}}}, \mathcal{E}_{Z_{\mathfrak{E}}})$ is an 'exact' functor, we have an isomorphism $\mathfrak{q}^{\mathfrak{I}}_Z(\mathcal{H}_{\mathfrak{a}}(M)) \simeq \mathcal{H}^*(M)$. Since the functor $\mathfrak{q}^{\mathfrak{I}}_Z$ is fully faithful, it follows that the object $\mathcal{H}_{\mathfrak{a}}(M)$ is defined uniquely up to isomorphism. By a standard argument, once the objects $\mathcal{H}_{\mathfrak{a}}(M)$ and $\mathcal{H}_{\mathfrak{a}}(N)$ are fixed, any morphism $M \xrightarrow{g} N$ determines uniquely a morphism $\mathcal{H}_{\mathfrak{a}}(M) \longrightarrow \mathcal{H}_{\mathfrak{a}}(N)$.

The 'exactness' of $\mathcal{H}_{\mathfrak{a}}$ follows from the isomorphism $\mathcal{H}^*_{\mathfrak{a}} \simeq \mathfrak{q}^{\mathfrak{I}}_Z \circ \mathcal{H}_{\mathfrak{a}}$, because the functor $\mathcal{H}^*_{\mathfrak{a}}$ is 'exact' (by (b) above) and the functor $\mathfrak{q}^{\mathfrak{I}}_Z$ reflects 'exactness': if $L' \longrightarrow L \longrightarrow L''$ is a sequence in C_Z such that the sequence $0 \longrightarrow \mathfrak{q}^{\mathfrak{I}}_Z(L') \longrightarrow \mathfrak{q}^{\mathfrak{I}}_Z(L) \longrightarrow \mathfrak{q}^{\mathfrak{I}}_Z(L'') \longrightarrow 0$ is exact, then $L' \longrightarrow L \longrightarrow L''$ is a conflation.

(d) The isomorphism $\mathcal{H}^*_{\mathfrak{a}} \simeq \mathfrak{q}^{\mathfrak{I}}_Z \circ \mathcal{H}_{\mathfrak{a}}$ implies that $\mathfrak{q}^{\mathfrak{I}}_Z \circ (\mathcal{H}_{\mathfrak{a}} \circ \mathfrak{H}_{\mathfrak{X}}) \simeq \mathcal{H}^*_{\mathfrak{a}} \circ \mathfrak{H}_{\mathfrak{X}} \simeq \mathfrak{q}^{\mathfrak{I}}_Z \circ \mathcal{H}$. Since the functor $\mathfrak{q}^{\mathfrak{I}}_Z$ is fully faithful, it follows that $\mathcal{H} \simeq \mathcal{H}_{\mathfrak{a}} \circ \mathfrak{H}_{\mathfrak{X}}$. It follows from the definition

of the exact category $C_{\mathfrak{X}_\mathfrak{a}}$ and the exactness of the functor $\mathcal{H}_\mathfrak{a}$ that it is determined by the isomorphism $\mathcal{H} \simeq \mathcal{H}_\mathfrak{a} \circ \mathfrak{H}_\mathfrak{X}$ uniquely up to isomorphism.

(e) The extension $\mathcal{M}_k(\mathfrak{X}) \xrightarrow{\Theta^*_\mathfrak{X}} \mathcal{M}_k(\mathfrak{X})$ of the functor $C_\mathfrak{X} \xrightarrow{\theta^*_\mathfrak{X}} C_\mathfrak{X}$ maps representable presheaves to representable presheaves and has a right adjoint functor. In particular, $\Theta^*_\mathfrak{X}$ is a right exact functor, and it maps an exact sequence $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ in $\mathcal{M}_k(\mathfrak{X})$ with representable presheaves P_1 and P_0 to an exact sequence of the same type. By C4.3.3, this implies that the subcategory $C_{\mathfrak{X}_\mathfrak{a}}$ is $\Theta^*_\mathfrak{X}$-stable. Therefore, $\Theta^*_\mathfrak{X}$ induces a functor $C_{\mathfrak{X}_\mathfrak{a}} \xrightarrow{\theta_{\mathfrak{X}_\mathfrak{a}}} C_{\mathfrak{X}_\mathfrak{a}}$ such that the diagram

$$\begin{array}{ccc} C_\mathfrak{X} & \xrightarrow{\theta_\mathfrak{X}} & C_\mathfrak{X} \\ {\scriptstyle \mathfrak{H}_\mathfrak{X}}\downarrow & & \downarrow{\scriptstyle \mathfrak{H}_\mathfrak{X}} \\ C_{\mathfrak{X}_\mathfrak{a}} & \xrightarrow{\theta_{\mathfrak{X}_\mathfrak{a}}} & C_{\mathfrak{X}_\mathfrak{a}} \end{array}$$

quasi-commutes, i.e. $\mathfrak{H}_\mathfrak{X}$ is a $\mathbb{Z}_+$-functor $(C_\mathfrak{X}, \theta_\mathfrak{X}) \longrightarrow (C_{\mathfrak{X}_\mathfrak{a}}, \theta_{\mathfrak{X}_\mathfrak{a}})$. ∎

C4.3.5. Remarks. (a) The universal property described in C4.3.4 determines the exact category $(C_{\mathfrak{X}_\mathfrak{a}}, \mathcal{E}_{\mathfrak{X}_\mathfrak{a}})$ and the functor $C_\mathfrak{X} \xrightarrow{\mathfrak{H}_\mathfrak{X}} C_{\mathfrak{X}_\mathfrak{a}}$ uniquely up to equivalence.

(b) It follows from the definition of the category $C_{\mathfrak{X}_\mathfrak{a}}$ that its projective objects are retracts of representable presheaves. In particular, if the category $C_\mathfrak{X}$ is Karoubian, then every projective object of the exact category $C_{\mathfrak{X}_\mathfrak{a}}$ is isomorphic to an object of the form $\mathfrak{H}_X(M)$ for some $M \in Ob C_\mathfrak{X}$. In other words, the canonical embedding $C_\mathfrak{X} \xrightarrow{\mathfrak{H}_\mathfrak{X}} C_{\mathfrak{X}_\mathfrak{a}}$ induces an equivalence between $C_\mathfrak{X}$ and the full subcategory of the category $C_{\mathfrak{X}_\mathfrak{a}}$ generated by all projective objects of $C_{\mathfrak{X}_\mathfrak{a}}$.

The following proposition is a *cosuspended* version of Theorem 2.2.1 in [Ve2].

C4.3.6. Proposition. *The map which assigns to each cosuspended category* $\mathfrak{T}_- C_\mathfrak{X} = (C_\mathfrak{X}, \theta_\mathfrak{X}, \widetilde{\mathfrak{T}\mathfrak{r}^-_\mathfrak{X}})$ *the exact category* $C_{\mathfrak{X}_\mathfrak{a}}$ *is functorial in the following sense: to every triangle functor* $\Phi = (\Phi, \phi)$ *from a cosuspended category* $\mathfrak{T}_- C_\mathfrak{X}$ *to a cosuspended category* $\mathfrak{T}_- C_\mathfrak{Y}$*, there corresponds an 'exact'* $\mathbb{Z}_+$*-functor* $(C_{\mathfrak{X}_\mathfrak{a}}, \mathcal{E}_{\mathfrak{X}_\mathfrak{a}}) \xrightarrow{\Phi_\mathfrak{a}} (C_{\mathfrak{Y}_\mathfrak{a}}, \mathcal{E}_{\mathfrak{Y}_\mathfrak{a}})$ *which maps projectives to projectives. The functor* $\Phi_\mathfrak{a}$ *is determined uniquely up to isomorphism by the quasi-commutativity of the diagram*

$$\begin{array}{ccc} C_\mathfrak{X} & \xrightarrow{\Phi} & C_\mathfrak{Y} \\ {\scriptstyle \mathfrak{H}_\mathfrak{X}}\downarrow & & \downarrow{\scriptstyle \mathfrak{H}_\mathfrak{Y}} \\ C_{\mathfrak{X}_\mathfrak{a}} & \xrightarrow{\Phi_\mathfrak{a}} & C_{\mathfrak{Y}_\mathfrak{a}} \end{array} \qquad (1)$$

Proof. (a) Since $\widetilde{\Phi} = (\Phi, \phi)$ is a triangle functor and $\mathfrak{H}_\mathfrak{Y}$ is a homological functor, the composition, $\mathfrak{H}_\mathfrak{Y} \circ \Phi$ is a homological functor. By the universal property of the homological functor $C_\mathfrak{X} \xrightarrow{\mathfrak{H}_\mathfrak{X}} C_{\mathfrak{X}_\mathfrak{a}}$ (see C4.3.4), there exists a unique (up to isomorphism) exact functor $C_{\mathfrak{X}_\mathfrak{a}} \xrightarrow{\widetilde{\Phi}_\mathfrak{a}} C_{\mathfrak{Y}_\mathfrak{a}}$ such that the diagram (1) quasi-commutes. The quasi-commutativity of

the diagram (1) implies that $\widetilde{\Phi}_{\mathfrak{a}}$ maps representable presheaves to representable presheaves. Since projective objects of the categories $C_{\mathfrak{X}_{\mathfrak{a}}}$ and $C_{\mathcal{Y}_{\mathfrak{a}}}$ are all possible retracts (direct summands) of representable presheaves, it follows that $\widetilde{\Phi}_{\mathfrak{a}}$ maps projectives to projectives.

The isomorphism $\Phi \circ \theta_{\mathfrak{X}} \xrightarrow{\phi} \theta_{\mathcal{Y}} \circ \Phi$ induces an isomorphism $\Phi_{\mathfrak{a}} \circ \Theta^{\mathfrak{a}}_{\mathfrak{X}} \xrightarrow{\phi_{\mathfrak{a}}} \Theta^{\mathfrak{a}}_{\mathcal{Y}} \circ \Phi_{\mathfrak{a}}$, where $\Theta^{\mathfrak{a}}_{\mathfrak{X}}$ is the endofunctor $C_{\mathfrak{X}_{\mathfrak{a}}} \longrightarrow C_{\mathfrak{X}_{\mathfrak{a}}}$ induced by $\Theta^{*}_{\mathfrak{X}}$. So that the pair $(\Phi_{\mathfrak{a}}, \phi_{\mathfrak{a}})$ is a $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}_{\mathfrak{a}}}, \Theta^{\mathfrak{a}}_{\mathfrak{X}}) \longrightarrow (C_{\mathfrak{X}_{\mathfrak{a}}}, \Theta^{\mathfrak{a}}_{\mathfrak{X}})$ and the diagram (1) is a diagram of $\mathbb{Z}_+$-functors. ∎

Let $\mathfrak{T}_{-}C_{\mathfrak{X}}$ be a cosuspended category and $(C_Z, \mathcal{E}_Z)$ an exact category. We denote by $\mathfrak{Ex}((C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}), (C_Z, \mathcal{E}_Z))$ the category whose objects are 'exact' functors from $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}})$ to $(C_Z, \mathcal{E}_Z)$ and morphisms are morphisms of functors. Let $\mathcal{H}om(C_{\mathfrak{X}}, C_Z)$ denote the category whose objects are functors from $C_{\mathfrak{X}}$ to C_Z and morphisms are morphisms of functors.

C4.3.7. Proposition. *The composition with the functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}}$ defines a fully faithful functor*

$$\mathfrak{Ex}((C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}), (C_Z, \mathcal{E}_Z)) \longrightarrow \mathcal{H}om(C_{\mathfrak{X}}, C_Z)$$

which induces an equivalence of the category $\mathfrak{Ex}((C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}}), (C_Z, \mathcal{E}_Z))$ with the full subcategory of $\mathcal{H}om(C_{\mathfrak{X}}, C_Z)$ generated by homological functors.

Proof. The assertion is a corollary of (actually, it is equivalent to) C4.3.4. ∎

C4.3.8. Triangle functors. Let $\mathfrak{T}_{-}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^{-}_{\mathfrak{X}})$ and $\mathfrak{T}_{-}C_{\mathcal{Y}} = (C_{\mathcal{Y}}, \theta_{\mathcal{Y}}, \mathfrak{Tr}^{-}_{\mathcal{Y}})$ be cosuspended categories, and let $\widetilde{\Phi} = (\Phi, \phi)$ be a triangle functor $\mathfrak{T}_{-}C_{\mathfrak{X}} \longrightarrow \mathfrak{T}_{-}C_{\mathcal{Y}}$. Then we have a quasi-commutative diagram of $\mathbb{Z}_+$-categories and $\mathbb{Z}_+$-functors

$$\begin{array}{ccccc} (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) & \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} & (C_{\mathfrak{X}_{\mathfrak{a}}}, \theta_{\mathfrak{X}_{\mathfrak{a}}}) & \xrightarrow{\mathfrak{Q}_{\mathfrak{X}_{\mathfrak{a}}}} & (C_{\mathfrak{X}^{\mathfrak{E}}_{\mathfrak{a}}}, \Theta_{\mathfrak{X}^{\mathfrak{E}}_{\mathfrak{a}}}) \\ \Phi \downarrow & & \downarrow \Phi_{\mathfrak{a}} & & \downarrow \Phi^{*}_{\mathfrak{E}} \\ (C_{\mathcal{Y}}, \theta_{\mathcal{Y}}) & \xrightarrow{\mathfrak{H}_{\mathcal{Y}}} & (C_{\mathcal{Y}_{\mathfrak{a}}}, \theta_{\mathcal{Y}_{\mathfrak{a}}}) & \xrightarrow{\mathfrak{Q}_{\mathcal{Y}_{\mathfrak{a}}}} & (C_{\mathcal{Y}^{\mathfrak{E}}_{\mathfrak{a}}}, \Theta_{\mathcal{Y}^{\mathfrak{E}}_{\mathfrak{a}}}) \end{array} \qquad (1)$$

in which $\mathfrak{Q}_{\mathfrak{X}_{\mathfrak{a}}}$ and $\mathfrak{Q}_{\mathcal{Y}_{\mathfrak{a}}}$ are Gabriel-Quillen embeddings, the functor $\Phi_{\mathfrak{a}}$ is exact, and the functor $\Phi^{*}_{\mathfrak{E}}$ has a right adjoint, $\Phi_{\mathfrak{E}*}$, which is an exact functor.

C4.4. The category $C_{\mathfrak{X}_{\mathfrak{m}}}$ and abelianization of triangulated categories. Fix a k-linear cosuspended category $\mathfrak{T}_{-}C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^{-}_{\mathfrak{X}})$. We denote by $C_{\mathfrak{X}_{\mathfrak{m}}}$ the strictly full subcategory of the category $\mathcal{M}_k(\mathfrak{X})$ of k-presheaves on C_X whose objects are (isomorphic to) images of morphisms between representable presheaves. In other words, an object of $\mathcal{M}_k(\mathfrak{X})$ belongs to $C_{\mathfrak{X}_{\mathfrak{m}}}$ iff it is a subobject and a quotient object of some representable presheaves. An immediate consequence of this description is that the category $C_{\mathfrak{X}_{\mathfrak{m}}}$ is Karoubian. It is easy to show that the subcategory $C_{\mathfrak{X}_{\mathfrak{m}}}$ is closed under finite coproducts in $\mathcal{M}_k(\mathfrak{X})$; i.e. $C_{\mathfrak{X}_{\mathfrak{m}}}$ is an additive subcategory of $\mathcal{M}_k(\mathfrak{X})$.

Notice that $C_{\mathfrak{X}_{\mathfrak{m}}}$ is a subcategory of $C_{\mathfrak{X}_{\mathfrak{a}}}$. In fact, by the definition of the subcategory $C_{\mathfrak{X}_{\mathfrak{m}}}$, for every its object M, there exist an epimorphism $M_0 \xrightarrow{\mathfrak{e}} M$ and a monomorphism $M \xrightarrow{\mathfrak{j}} L_0$, where M_0 and L_0 are representable presheaves. There is a triangle

$$\Theta^{*}_{\mathfrak{X}}(L_0) \xrightarrow{\mathfrak{d}} M_1 \xrightarrow{g} M_0 \xrightarrow{\mathfrak{j}\circ\mathfrak{e}} L_0.$$

Since this triangle is an exact sequence, we have an exact sequence

$$M_1 \xrightarrow{g} M_0 \xrightarrow{\mathfrak{e}} M \longrightarrow 0$$

with M_0, M_1 representable presheaves. By C4.3.3, M is an object of $C_{\mathfrak{X}_\mathfrak{a}}$.

It follows that an object of $C_{\mathfrak{X}_\mathfrak{a}}$ belongs to the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$ iff it is a subobject of a representable presheaf.

C4.4.1. Proposition. *(a) The subcategory $C_{\mathfrak{X}_\mathfrak{m}}$ is $\Theta^*_{\mathfrak{X}}$-stable.*

(b) For every morphism α of $C_{\mathfrak{X}_\mathfrak{m}}$, the kernel and cokernel of $\Theta^*_{\mathfrak{X}}(\alpha)$ belong to the subcategory $C_{\mathfrak{X}_\mathfrak{m}}$.

Proof. (i) Let $K \xrightarrow{\alpha} K'$ be morphism of $C_{\mathfrak{X}_\mathfrak{m}}$; i.e. there exist $M \xrightarrow{f} L$ and $M' \xrightarrow{f'} L'$ such that $K = Im(f)$, $K' = Im(f')$, and presheaves M and M' are representable. Let $\Theta^*_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M \xrightarrow{f} L$ and $\Theta^*_{\mathfrak{X}}(L') \xrightarrow{h'} N' \xrightarrow{g'} M' \xrightarrow{f'} L'$ be triangles. Then there is a commutative diagram

$$\begin{array}{ccccccccccc}
\Theta^*_{\mathfrak{X}}(M) & \xrightarrow{\Theta^*_{\mathfrak{X}}(f)} & \Theta^*_{\mathfrak{X}}(L) & \xrightarrow{h} & N & \xrightarrow{g} & M & \xrightarrow{\mathfrak{e}} & K & \xrightarrow{\mathfrak{j}} & L \\
{\scriptstyle \Theta^*_{\mathfrak{X}}(\xi_1)}\downarrow & & {\scriptstyle \widetilde{\xi}_0}\downarrow & & \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \alpha} & & \\
\Theta^*_{\mathfrak{X}}(M') & \xrightarrow{\Theta^*_{\mathfrak{X}}(f')} & \Theta^*_{\mathfrak{X}}(L') & \xrightarrow{h'} & N' & \xrightarrow{g'} & M' & \xrightarrow{\mathfrak{e}'} & K' & \xrightarrow{\mathfrak{j}'} & L'
\end{array} \tag{7}$$

constructed as follows. The arrow $M \xrightarrow{\xi_1} M'$ is due to the fact that M is a projective object of $\mathcal{M}_k(\mathfrak{X})$ and $M' \xrightarrow{\mathfrak{e}'} K'$ is an epimorphism. Similarly, the morphism $N \xrightarrow{\xi_2} N'$ exists because the sequence $N' \xrightarrow{g'} M' \xrightarrow{\mathfrak{e}'} K'$ is exact and the object N is projective. By the property (SP2), the sequences

$$\Theta^*_{\mathfrak{X}}(M) \xrightarrow{-\Theta^*_{\mathfrak{X}}(f)} \Theta^*_{\mathfrak{X}}(L) \xrightarrow{h} N \xrightarrow{g} M$$

and

$$\Theta^*_{\mathfrak{X}}(M') \xrightarrow{-\Theta^*_{\mathfrak{X}}(f')} \Theta^*_{\mathfrak{X}}(L') \xrightarrow{h'} N' \xrightarrow{g'} M'$$

are triangles. By (SP3), there exists a morphism $\Theta^*_{\mathfrak{X}}(L) \xrightarrow{\widetilde{\xi}_0} \Theta^*_{\mathfrak{X}}(L')$ such that the diagram

$$\begin{array}{ccccccc}
\Theta^*_{\mathfrak{X}}(M) & \xrightarrow{-\Theta^*_{\mathfrak{X}}(f)} & \Theta^*_{\mathfrak{X}}(L) & \xrightarrow{h} & N & \xrightarrow{g} & M \\
{\scriptstyle \Theta^*_{\mathfrak{X}}(\xi_1)}\downarrow & & {\scriptstyle \widetilde{\xi}_0}\downarrow & & \downarrow{\scriptstyle \xi_2} & & \downarrow{\scriptstyle \xi_1} \\
\Theta^*_{\mathfrak{X}}(M') & \xrightarrow{-\Theta^*_{\mathfrak{X}}(f')} & \Theta^*_{\mathfrak{X}}(L') & \xrightarrow{h'} & N' & \xrightarrow{g'} & M'
\end{array}$$

commutes. Therefore, the diagram (7) commutes.

(ii) Since the functor $\Theta^*_{\mathfrak{X}}$ is right exact, the arrows $\Theta^*_{\mathfrak{X}}(\mathfrak{e})$ and $\Theta^*_{\mathfrak{X}}(\mathfrak{e}')$ in the commutative diagram

$$\begin{array}{ccccccccc}
\Theta^*_{\mathfrak{X}}(N) & \xrightarrow{\Theta^*_{\mathfrak{X}}(g)} & \Theta^*_{\mathfrak{X}}(M) & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{e})} & \Theta^*_{\mathfrak{X}}(K) & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{j})} & \Theta^*_{\mathfrak{X}}(L) & \xrightarrow{h} & N \\
{\scriptstyle \Theta^*_{\mathfrak{X}}(\xi_2)}\downarrow & & \downarrow{\scriptstyle \Theta^*_{\mathfrak{X}}(\xi_1)} & & {\scriptstyle \Theta^*_{\mathfrak{X}}(\alpha)}\downarrow & & {\scriptstyle \widetilde{\xi}_0}\downarrow & & \downarrow{\scriptstyle \xi_2} \\
\Theta^*_{\mathfrak{X}}(N') & \xrightarrow{\Theta^*_{\mathfrak{X}}(g')} & \Theta^*_{\mathfrak{X}}(M') & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{e}')} & \Theta^*_{\mathfrak{X}}(K') & \xrightarrow{\Theta^*_{\mathfrak{X}}(\mathfrak{j}')} & \Theta^*_{\mathfrak{X}}(L') & \xrightarrow{h'} & N'
\end{array} \tag{8}$$

are epimorphisms. It follows from the exactness of the rows in (7) that the arrows $\Theta^*_{\mathfrak{X}}(\mathfrak{j})$ and $\Theta^*_{\mathfrak{X}}(\mathfrak{j}')$ are monomorphisms.

An argument similar to that of [Ve2, 3.2.5] applied to the commutative diagram (7) shows that the kernel and cokernel of the morphism $\Theta^*_{\mathfrak{X}}(\alpha)$ belong to the subcategory $C_{\mathfrak{X}_{\mathfrak{m}}}$. Since α is an arbitrary morphism of $C_{\mathfrak{X}_{\mathfrak{m}}}$, it follows, in particular, that the subcategory $C_{\mathfrak{X}_{\mathfrak{m}}}$ is $\Theta^*_{\mathfrak{X}}$-stable; i.e. it has a natural structure of a $\mathbb{Z}_+$-category and the Yoneda embedding induces a $\mathbb{Z}_+$-functor $(C_{\mathfrak{X}}, \theta) \longrightarrow (C_{\mathfrak{X}_{\mathfrak{m}}}, \Theta^*_{\mathfrak{X}})$. ∎

C4.4.2. Note. Since the Yoneda functor $C_{\mathfrak{X}} \xrightarrow{h_{\mathfrak{X}}} \mathcal{M}_k(\mathfrak{X})$ takes values in $C_{\mathfrak{X}_{\mathfrak{m}}}$, the $\mathbb{Z}_+$-category $(C_{\mathfrak{X}_{\mathfrak{m}}}, \Theta_{\mathfrak{X}_{\mathfrak{m}}})$ has enough projectives. It follows that the 'translation' functor $C_{\mathfrak{X}_{\mathfrak{m}}} \xrightarrow{\Theta_{\mathfrak{X}_{\mathfrak{m}}}} C_{\mathfrak{X}_{\mathfrak{m}}}$ induced by $\Theta^*_{\mathfrak{X}}$ maps projectives to projectives.

C4.4.3. Proposition. *Suppose that the cosuspension functor $C_{\mathfrak{X}} \xrightarrow{\theta_{\mathfrak{X}}} C_{\mathfrak{X}}$ is a category equivalence, i.e. $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ is a triangulated category. Then $C_{\mathfrak{X}_{\mathfrak{m}}}$ is an abelian category which coincides with $C_{\mathfrak{X}_{\mathfrak{a}}}$.*

Proof. If the suspension functor $C_{\mathfrak{X}} \xrightarrow{\theta_{\mathfrak{X}}} C_{\mathfrak{X}}$ is a category equivalence, then its extension $\Theta^*_{\mathfrak{X}}$ is a category equivalence. In this case, it follows from C4.4.1(ii) that the subcategory $C_{\mathfrak{X}_{\mathfrak{m}}}$ contains kernels and cokernels of all its morphisms, hence $C_{\mathfrak{X}_{\mathfrak{m}}}$ is an abelian subcategory of $\mathcal{M}_k(\mathfrak{X})$. Since every object of the category $C_{\mathfrak{X}_{\mathfrak{a}}}$ is the cokernel of a morphism between representable objects, it follows that $C_{\mathfrak{X}_{\mathfrak{a}}} \subseteq C_{\mathfrak{X}_{\mathfrak{m}}}$. Therefore $C_{\mathfrak{X}_{\mathfrak{a}}} = C_{\mathfrak{X}_{\mathfrak{m}}}$. ∎

C4.4.4. Note. Proposition C4.4.3 together with 3.2.4 and 5.2.6 recover, in particular, the 'abelianization' theory for triangulated categories [Ve2, II.3].

C4.5. Triangulation and abelianization of cosuspended categories.

C4.5.1. Inverting endofunctors. A $\mathbb{Z}$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ is called *strict* if the endofunctor $\theta_{\mathfrak{X}}$ is an auto-morphism of the category $C_{\mathfrak{X}}$.

There is a standard construction which assigns to each $\mathbb{Z}_+$-category $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}})$ a strict $\mathbb{Z}$-category $(C_{\mathfrak{X}_{\mathfrak{s}}}, \theta_{\mathfrak{X}_{\mathfrak{s}}})$. Objects of the category $C_{\mathfrak{X}_{\mathfrak{s}}}$ are pairs (n, M), where $n \in \mathbb{Z}$ and $M \in Ob C_{\mathfrak{X}}$. Morphisms are defined by

$$C_{\mathfrak{X}_{\mathfrak{s}}}((s, M), (t, N)) \stackrel{\text{def}}{=} \operatorname{colim}_{n \geq s,t} C_{\mathfrak{X}}(\theta^{n-s}_{\mathfrak{X}}(M), \theta^{n-t}_{\mathfrak{X}}(N)). \tag{1}$$

The composition is determined by the compositions

$$C_{\mathfrak{X}}(\theta^{n-r}_{\mathfrak{X}}(L), \theta^{n-s}_{\mathfrak{X}}(M)) \times C_{\mathfrak{X}}(\theta^{n-s}_{\mathfrak{X}}(M), \theta^{n-t}_{\mathfrak{X}}(N)) \longrightarrow C_{\mathfrak{X}}(\theta^{n-r}_{\mathfrak{X}}(L), \theta^{n-t}_{\mathfrak{X}}(N)).$$

The functor θ_{X_s} is defined on objects by $\theta_{X_s}(s,M) = (s-1,M)$. It follows from (1) above that there is a natural isomorphism

$$C_{\mathfrak{X}_s}((s,M),(t,N)) \xrightarrow{\sim} C_{\mathfrak{X}_s}(\theta_{X_s}(s,M),\theta_{X_s}(t,N)) = C_{\mathfrak{X}_s}((s-1,M),(t-1,N)),$$

which is the action of θ_{X_s} on morphisms.

There is a functor $C_{\mathfrak{X}} \xrightarrow{\Phi_{\mathfrak{X}}} C_{\mathfrak{X}_s}$ which maps an object M of $C_{\mathfrak{X}}$ to the object $(0,M)$ and a morphism $M \longrightarrow N$ to its image in

$$C_{\mathfrak{X}_s}((0,M),(0,N)) \stackrel{\text{def}}{=} \operatorname{co}\lim_{n\geq 1} C_{\mathfrak{X}}(\theta^n_{\mathfrak{X}}(M),\theta^n_{\mathfrak{X}}(N)).$$

The morphism

$$\theta_{\mathfrak{X}_s} \circ \Phi_{\mathfrak{X}}(M) = (-1,M) \xrightarrow{\varphi_{\mathfrak{X}}(M)} \Phi_{\mathfrak{X}} \circ \theta_{\mathfrak{X}}(M) = (0,\theta_{\mathfrak{X}}(M))$$

is the image of the identical morphism $\theta_{\mathfrak{X}}(M) \longrightarrow \theta_{\mathfrak{X}}(M)$.

Let $\mathbb{Z}_+ - Cat_k$ denote the category of svelte k-linear $\mathbb{Z}_+$-categories, and let $\mathbb{Z} - \mathfrak{Cat}_k$ denote its full subcategory generated by k-linear strict $\mathbb{Z}$-categories.

C4.5.1.1. Proposition. *The map which assigns to a $\mathbb{Z}_+$-category $(C_{\mathfrak{X}},\theta_{\mathfrak{X}})$ the strict $\mathbb{Z}$-category $(C_{\mathfrak{X}_s},\theta_{\mathfrak{X}_s})$ extends to a functor $\mathbb{Z}_+ - Cat_k \xrightarrow{\mathfrak{J}^*} \mathbb{Z} - \mathfrak{Cat}_k$ which is a left adjoint to the inclusion functor $\mathbb{Z} - \mathfrak{Cat}_k \xrightarrow{\mathfrak{J}_*} \mathbb{Z}_+ - Cat_k$.*

Proof. The morphisms $(C_{\mathfrak{X}},\theta_{\mathfrak{X}}) \xrightarrow{(\Phi_{\mathfrak{X}},\varphi_{\mathfrak{X}})} (C_{\mathfrak{X}_s},\theta_{\mathfrak{X}_s})$ defined above form an adjunction morphism from identical functor on $\mathbb{Z} - Cat_k$ to the composition $\mathfrak{J}_*\mathfrak{J}^*$. The second adjunction morphism is a natural isomorphism. ∎

C4.5.2. Cosuspended categories and strict triangulated categories. The construction of C4.5.1 extends to a functor from the category of cosuspended categories to the category of *strict triangulated* categories. Recall that a triangulated category $\mathfrak{T}C_{\mathfrak{X}} = (C_{\mathfrak{X}},\theta_{\mathfrak{X}},Tr^-_{\mathfrak{X}})$ is *strict* if $\theta_{\mathfrak{X}}$ is an auto-morphism of the category $C_{\mathfrak{X}}$.

C4.5.2.1. Proposition [KeV]. *To any cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}},\theta_{\mathfrak{X}},Tr^-_{\mathfrak{X}})$, there corresponds a strict triangulated category $\mathfrak{T}_-C_{\mathfrak{X}_s}$ and a triangle functor*

$$\mathfrak{T}_-C_{\mathfrak{X}} \xrightarrow{(\Phi_{\mathfrak{X}},\varphi_{\mathfrak{X}})} \mathfrak{T}_-C_{\mathfrak{X}_s}$$

such that for every triangulated category $\mathfrak{T}C_{\mathfrak{Y}}$, the functor

$$\widetilde{\mathfrak{T}\mathfrak{r}^-_k}(\mathfrak{T}_-C_{\mathfrak{X}_s},\mathfrak{T}C_{\mathfrak{Y}}) \longrightarrow \widetilde{\mathfrak{T}\mathfrak{r}^-_k}(\mathfrak{T}_-C_{\mathfrak{X}},\mathfrak{T}C_{\mathfrak{Y}}) \qquad (1)$$

of composition with $(\Phi_{\mathfrak{X}},\varphi_{\mathfrak{X}})$ is an equivalence of categories.

(a) If $\mathfrak{T}C_{\mathfrak{Y}}$ is a strict triangulated category, then (1) is an isomorphism of categories.

(b) If $\mathfrak{T}_-C_{\mathfrak{X}}$ is a triangulated category, then $(\Phi_{\mathfrak{X}},\varphi_{\mathfrak{X}})$ is a triangle equivalence.

Proof. By C4.5.1, objects of the category $\mathfrak{T}_{-}C_{\mathfrak{X}_{\mathfrak{s}}}$ are pairs (n, M), where $n \in \mathbb{Z}$ and $M \in ObC_{\mathfrak{X}}$. The triangles are sequences

$$\theta_{X_{\mathfrak{s}}}(r, L) = (r-1, L) \longrightarrow (t, N) \longrightarrow (s, M) \longrightarrow (r, L)$$

associated to sequences

$$\theta_{\mathfrak{X}}\theta^{n-r}_{\mathfrak{X}}(L) \xrightarrow{w} \theta^{n-t}_{\mathfrak{X}}(N) \xrightarrow{v} \theta^{n-s}_{\mathfrak{X}}(M) \xrightarrow{u} \theta^{n-r}_{\mathfrak{X}}(L)$$

such that $((-1)^n w, v, u)$ is a triangle. ∎

Let $\mathfrak{T}_{-}Cat_k$ (resp. $\mathfrak{Tr}\mathfrak{Cat}_k$) denote the category whose objects are svelte cosuspended (resp. svelte triangulated strict) k-linear categories and morphisms are triangle functors.

C4.5.3. Proposition. *The map which assigns to each cosuspended category the corresponding strict triangulated category extends to a functor* $\mathfrak{T}_{-}Cat_k \xrightarrow{{}^{\mathfrak{t}}\mathfrak{J}^*} \mathfrak{Tr}\mathfrak{Cat}_k$ *which is a left adjoint to the inclusion functor.*

Proof. See C4.5.1.1. ∎

C4.5.4. Proposition. *Let* $\mathfrak{T}_{-}C_{\mathfrak{X}}$ *be a cosuspended* k*-linear category. The functor* $\mathbb{Z}_{+} - Cat_k \xrightarrow{\mathfrak{J}^*} \mathbb{Z} - \mathfrak{Cat}_k$ *maps the natural embedding* $C_{\mathfrak{X}_{\mathfrak{m}}} \longrightarrow C_{\mathfrak{X}_{\mathfrak{a}}}$ *of* $\mathbb{Z}_{+}$*-categories to an equivalence between abelian strict* $\mathbb{Z}$*-categories.*

Proof. It follows from the construction of the functor $\mathfrak{J}^*$ that it is compatible with the 'triangularization' functor $\mathfrak{T}_{-}Cat_k \xrightarrow{{}^{\mathfrak{t}}\mathfrak{J}^*} \mathfrak{Tr}\mathfrak{Cat}_k$ of C4.5.3. The constructions of the categories $C_{\mathfrak{X}_{\mathfrak{m}}}$ and $C_{\mathfrak{X}_{\mathfrak{a}}}$ are also compatible with the functors triangularization functor and the functor $\mathfrak{J}^*$. By C4.4.3, the categories $C_{\mathfrak{X}_{\mathfrak{m}}}$ and $C_{\mathfrak{X}_{\mathfrak{a}}}$ coincide if $\mathfrak{T}_{-}C_{\mathfrak{X}}$ is a triangulated category, hence the assertion. ∎

C4.6. Complements.

C4.6.0. Exact categories and exact categories with enough projectives. Let $(C_X, \mathfrak{E}_X)$ be an exact category and $C_{X_{\mathfrak{P}}}$ its full subcategory generated by all objects M of C_X such that there exists a deflation $P \longrightarrow M$, where P is a projective object of $(C_X, \mathfrak{E}_X)$. It follows from (the argument of) C4.3.2 that the subcategory $C_{X_{\mathfrak{P}}}$ is *fully exact* (i.e. it is closed under extensions). In particular, it is an exact subcategory of $(C_X, \mathfrak{E}_X)$. By construction, this exact subcategory, $(C_{X_{\mathfrak{P}}}, \mathfrak{E}_{X_{\mathfrak{P}}})$, has enough projectives.

Let $\mathfrak{Cat}_{\mathfrak{ex}}$ denote the bicategory of exact categories (whose 1-morphisms are 'exact' functors) and $\mathfrak{Cat}^{\mathfrak{P}}_{\mathfrak{ex}}$ its full subcategory generated by exact categories with enough projectives. The map which assigns to every exact category $(C_X, \mathfrak{E}_X)$ its fully exact subcategory $(C_{X_{\mathfrak{P}}}, \mathfrak{E}_{X_{\mathfrak{P}}})$ extends to a 2-functor from $\mathfrak{Cat}_{\mathfrak{ex}}$ to $\mathfrak{Cat}^{\mathfrak{P}}_{\mathfrak{ex}}$ which is left adjoint to the inclusion functor $\mathfrak{Cat}^{\mathfrak{P}}_{\mathfrak{ex}} \longrightarrow \mathfrak{Cat}_{\mathfrak{ex}}$ (in the 2-categorical sense).

C4.6.1. Costable categories in terms of complexes. Let $(C_X, \mathfrak{E}_X)$ be an exact category. Consider the full subcategory $C_{\mathcal{P}_0 X}$ of the homotopy category $\mathcal{H}(C_X)$ whose objects are acyclic complexes $\mathcal{P} = (\ldots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0)$ such that objects $P_i,\ i \geq 0$, are projective.

The category $C_{\mathcal{P}_0 X}$ has a natural $\mathbb{Z}_+$-action given by the 'translation' functor θ_- which assigns to every object $\mathcal{P} = (\ldots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0)$ the object $\theta_-(\mathcal{P}) = (\ldots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} Cok(d_0) \longrightarrow 0)$.

C4.6.1.1. Lemma. *Let $(C_X, \mathcal{E}_X)$ be an exact category with enough projectives. Then the costable category $C_{\mathfrak{S}_- X}$ of C_X is $\mathbb{Z}_+$-equivalent to the category $C_{\mathcal{P}_0 X}$.*

Proof. The equivalence is given by the functor $C_{\mathcal{P}_0 X} \longrightarrow C_{\mathfrak{S}_- X}$ which assigns to every object $\mathcal{P} = (\ldots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0)$ of $C_{\mathcal{P}_0 X}$ the (image in $C_{\mathfrak{S}_- X}$ of the) cokernel of $P_1 \xrightarrow{d_0} P_0$. The quasi-inverse functor assigns to each object M of $C_{\mathfrak{S}_- X}$ (the image in $C_{\mathcal{P}_0 X}$ of) its projective resolution $(\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0)$.

It follows from the definitions that both functors are compatible with the $\mathbb{Z}_+$-actions on the respective categories. ∎

C4.6.2. Homological dimension.

C4.6.2.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be an exact category with enough projectives, $\mathfrak{T}_- C_{\mathfrak{S}_- X} = (C_{\mathfrak{S}_- X}, \theta, \mathfrak{Tr}_{\mathfrak{S}_- X})$ its costable cosuspended category, and $C_X \xrightarrow{\mathfrak{P}_X} C_{\mathfrak{S}_- X}$ the canonical projection.*

(a) The following condition on an object M of C_X are equivalent:

(a1) $hd(M) \leq n$;

(a2) $\theta^n(\mathfrak{P}_X(M)) = 0$.

(b) An object M of C_X is projective iff its image in the costable category is zero.

Proof. Consider first the case $n = 0$. Then the condition (a1) means that the object M is projective. The condition (a2) reads: the image of M in the costable category is zero. The implication $(a) \Rightarrow (b)$ follows from the definition of the costable category.

On the other hand, the image of M in the costable category is zero iff the image of the identical morphism id_M is zero. The latter means that id_M factors through a projective object, i.e. M is a retract of a projective object, hence it is projective.

Suppose now that $n \geq 1$. Let $(\ldots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow M)$ be a projective resolution of the object M. By the definition of the cosuspension θ, there is an isomorphism $\theta(\mathfrak{P}_X(M)) \simeq \mathfrak{P}_X(im(d_0))$. Therefore, $\theta^n(\mathfrak{P}_X(M)) \simeq \mathfrak{P}_X(im(d_{n-1})$. The homological dimension of M is less or equal to n iff $im(d_{n-1})$ is a projective object, or, equivalently, $\mathfrak{P}_X(im(d_{n-1}) = 0$. ∎

C4.6.2.1.1. Corollary. *Let $(C_X, \mathcal{E}_X)$ be an exact category with enough projectives. The following conditions are equivalent:*

(a) $hd(C_X, \mathcal{E}_X) \leq n$;

(b) $\theta^n = 0$.

In particular, $hd(C_X, \mathcal{E}_X) = 0$ iff the costable category of $(C_X, \mathcal{E}_X)$ is trivial.

C4.6.2.2. Homological dimension of objects of a cosuspended category. Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ be a cosuspended category. We say that an object M of $C_{\mathfrak{X}}$ has homological dimension n if $\theta^n(M) = 0$ and $\theta^{n-1}(M) \neq 0$. In particular, an object of $C_{\mathfrak{X}}$ is of homological dimension zero iff it is zero.

C4.6.2.3. Proposition. *Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}^-_{\mathfrak{X}})$ be a cosuspended category.*

(a) The full subcategory $C_{\mathfrak{X}_{h\omega}}$ of the category $C_{\mathfrak{X}}$ generated by the objects of finite homological dimension is a thick cosuspended subcategory of $\mathfrak{T}_- C_{\mathfrak{X}}$.

(b) The subcategory $C_{\mathfrak{X}_{h\omega}}$ is contained in the kernel of the canonical "triangularization" functor $\mathfrak{T}_- C_{\mathfrak{X}} \stackrel{(\Phi_{\mathfrak{X}}, \varphi_{\mathfrak{X}})}{\longrightarrow} \mathfrak{T}_- C_{\mathfrak{X}(\mathbb{Z})}$ (see C4.5.2.1.)

Proof. (a) Recall that a full cosuspended subcategory $\mathcal{B}$ of $\mathfrak{T}_- C_{\mathfrak{X}}$ is called a *thick cosuspended subcategory* if it is closed under extensions, i.e. if $\theta_{\mathfrak{X}}(L) \stackrel{h}{\longrightarrow} N \stackrel{g}{\longrightarrow} M \stackrel{f}{\longrightarrow} L$ is a triangle and L and N are objects of $\mathcal{B}$, then M is an object of $\mathcal{B}$ too.

By K8.4(b), for every triangle $\theta_{\mathfrak{X}}(L) \stackrel{w}{\longrightarrow} N \stackrel{v}{\longrightarrow} M \stackrel{u}{\longrightarrow} L$, the sequence of representable functors

$$\ldots \longrightarrow C_{\mathfrak{X}}(-, \theta_{\mathfrak{X}}(L)) \stackrel{C_{\mathfrak{X}}(-,w)}{\longrightarrow} C_{\mathfrak{X}}(-, N) \stackrel{C_{\mathfrak{X}}(-,v)}{\longrightarrow} C_{\mathfrak{X}}(-, M) \stackrel{C_{\mathfrak{X}}(-,u)}{\longrightarrow} C_{\mathfrak{X}}(-, L)$$

is exact. In particular, there is an exact sequence of representable functors

$$\ldots \longrightarrow C_{\mathfrak{X}}(-, \theta^n_{\mathfrak{X}}(N)) \longrightarrow C_{\mathfrak{X}}(-, \theta^n_{\mathfrak{X}}(M)) \longrightarrow C_{\mathfrak{X}}(-, \theta^n_{\mathfrak{X}}(L)) \longrightarrow \ldots \qquad (1)$$

for every positive integer n. If the objects L and N have finite homological dimension, i.e. $\theta^n_{\mathfrak{X}}(L))$ and $\theta^n_{\mathfrak{X}}(N))$ are zero objects for some n, then it follows from the exactness of the sequence (1) that $\theta^n_{\mathfrak{X}}(M)) = 0$.

(b) Triangulated categories are precisely cosuspended categories whose cosuspension functor is an auto-equivalence. Therefore, every nonzero object of a triangulated category has an infinite homological dimension. ∎

C4.6.2.4. Homological dimension of a cosuspended category. Homological dimension of the cosuspended category $\mathfrak{T}_- C_{\mathfrak{X}}$ is, by definition, the supremum of homological dimensions of its objects. In particular, $hd(C_{\mathfrak{X}}) \leq n$ for some finite n iff $\theta^n_{\mathfrak{X}} = 0$.

C4.6.3. The stable and costable categories of an arbitrary exact category. Let $(C_X, \mathcal{E}_X)$ be an exact category with the class of deflations (resp. inflations) $\mathfrak{E}_X$ (resp. $\mathfrak{M}_X$). Let $C_X \stackrel{\mathfrak{q}^{\mathfrak{I}}_X}{\longrightarrow} C_{X_{\mathfrak{E}}}$ be the Gabriel-Quillen embedding. Since $C_{X_{\mathfrak{E}}}$ is a Grothendieck category, it has enough injectives. In particular, $C_{X_{\mathfrak{E}}}$ has the stable suspended category $(C_{\mathfrak{S}_+ X_{\mathfrak{E}}}, \Theta_{X_{\mathfrak{E}}}, \mathfrak{Tr}_{\mathfrak{S}_+ X_{\mathfrak{E}}})$ with infinite coproducts and products.

The composition of the Gabriel-Quillen embedding and the projection $C_{X_{\mathfrak{E}}} \longrightarrow C_{\mathfrak{S}_+ X_{\mathfrak{E}}}$ gives a functor $C_X \longrightarrow C_{\mathfrak{S}_+ X_{\mathfrak{E}}}$. We call the *stable category* of the exact category C_X the triple $(C_{\mathfrak{T}_+ X}, \Theta_X, \mathfrak{Tr}_{\mathfrak{T}_+ X})$, where $C_{\mathfrak{T}_+ X}$ is the smallest $\Theta_{X_{\mathfrak{E}}}$-invariant full subcategory of $C_{\mathfrak{S}_+ X_{\mathfrak{E}}}$ containing the image of C_X, Θ_X is the endofunctor of $C_{\mathfrak{T}_+ X}$ induced by $\Theta_{X_{\mathfrak{E}}}$, and $\mathfrak{Tr}_{\mathfrak{T}_+ X}$ is the class of all triangles from $\mathfrak{Tr}_{\mathfrak{S}_+ X_{\mathfrak{E}}}$ which belong to the subcategory $C_{\mathfrak{T}_+ X}$.

One can see that $(C_{\mathfrak{T}_+ X}, \Theta_X, \mathfrak{Tr}_{\mathfrak{T}_+ X})$ is a full suspended subcategory of the suspended category $(C_{\mathfrak{T}_+ X_{\mathfrak{E}}}, \Theta_{X_{\mathfrak{E}}})$. If the exact category C_X has enough injectives, then the suspended category $(C_{\mathfrak{T}_+ X}, \Theta_X)$ is equivalent to the stable category of C_X defined earlier.

The costable category $(C_{\mathfrak{T}_- X}, \theta_X, \mathfrak{Tr}_{\mathfrak{T}_- X})$ of the exact category C_X is defined dually.

C4.6.4. Canonical resolutions.

C4.6.4.1. The resolution of a cosuspended category. Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{T}\mathfrak{r}_{\mathfrak{X}})$ be a cosuspended category. The universal homological functor is the full embedding the $\mathbb{Z}_+$-categories $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}) \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} (C_{\mathfrak{X}_{\mathfrak{a}}}, \Theta_{\mathfrak{X}_{\mathfrak{a}}})$ which realizes $C_{\mathfrak{X}}$ as a subcategory of the full sucategory of $C_{\mathfrak{X}_{\mathfrak{a}}}$ generated by projective objects of $(C_{\mathfrak{X}_{\mathfrak{a}}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a}}})$. Since the exact category $C_{\mathfrak{X}_{\mathfrak{a}}}$ has enough projectives, its costable category $C_{\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}}}$ is (the underlying category of) a cosuspended category with the cosuspension functor θ_2. Since the functor $\Theta_{\mathfrak{X}_{\mathfrak{a}}}$ maps projectives to projectives, it induces an endofunctor θ_1 on the costable category $C_{\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}}}$. It follows from the exactness of the functor $\Theta_{\mathfrak{X}_{\mathfrak{a}}}$ that $\theta_1 \circ \theta_2 \simeq \theta_2 \circ \theta_1$; i.e. $C_{\mathfrak{T}\mathfrak{X}_{\mathfrak{a}}}$ is a cosuspended $\mathbb{Z}_+$-category. In particular, it is a $\mathbb{Z}_+ \times \mathbb{Z}_+$-category. The canonical universal homological functor embeds the cosuspended $\mathbb{Z}_+$-category $C_{\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}}}$ into an exact $\mathbb{Z}_+ \times \mathbb{Z}_+$-category $C_{(\mathfrak{S}_-\mathfrak{X}_{\mathfrak{a}})_{\mathfrak{a}}}$, etc.. As a result of this procedure, we obtain a sequence of categories and functors

$$\begin{aligned} &C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a}}}} C_{\mathfrak{X}_1} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_1}} C_{\mathfrak{X}_{\mathfrak{a},1}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},1}}} \dots \\ &\dots \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n-1}}} C_{\mathfrak{X}_n} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_n}} C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{n+1}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_{n+1}}} \dots \end{aligned} \tag{1}$$

where $\mathfrak{X}_{\mathfrak{a},n} = (\mathfrak{X}_n)_{\mathfrak{a}}$, $\mathfrak{X}_{n+1} = \mathfrak{S}_-\mathfrak{X}_{\mathfrak{a},n}$ for $n \geq 0$ and $\mathfrak{X}_0 = \mathfrak{X}$.

It follows that $\mathfrak{X}_n$ is represented by a cosuspended $\mathbb{Z}_+^n$-category (hence a $\mathbb{Z}_+^{n+1}$-category), $\mathfrak{X}_{\mathfrak{a},n}$ is represented by an exact $\mathbb{Z}_+^{n+1}$-category; and the universal homological functor $\mathfrak{H}_{\mathfrak{X}_n}$ and the canonical projections $\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n}}$ are $\mathbb{Z}_+^n$-functors. All exact categories $(C_{\mathfrak{X}_{\mathfrak{a},n}}, \mathcal{E}_{\mathfrak{X}_{\mathfrak{a},n}})$ have enough projectives.

For every exact category $(C_X, \mathcal{E}_X)$ with enough projectives, let Φ_X denote the composition of the projection $C_X \xrightarrow{\mathfrak{P}_X} C_{\mathfrak{S}_-X}$ to the costable category and the universal homological functor $C_{\mathfrak{S}_-X} \xrightarrow{\mathfrak{H}_{\mathfrak{S}_-X}} C_{\mathfrak{S}_-X_{\mathfrak{a}}}$.

Set $\Phi_n = \mathfrak{H}_{\mathfrak{X}_n} \circ \mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n-1}}$. Then we have a sequence of functors

$$\begin{aligned} &C_{\mathfrak{X}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}}} C_{\mathfrak{X}_{\mathfrak{a}}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a}}}} C_{\mathfrak{X}_{\mathfrak{a},1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},1}}} C_{\mathfrak{X}_{\mathfrak{a},2}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},2}}} \dots \\ &\dots \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n-2}}} C_{\mathfrak{X}_{\mathfrak{a},n-1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n-1}}} C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{\mathfrak{a},n+1}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n+1}}} \dots \end{aligned} \tag{2}$$

in which the composition of any two consecutive arrows equals to zero. The kernel of the functor $C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{\mathfrak{a},n+1}}$ coincides with the full subcategory of the category $C_{\mathfrak{X}_{\mathfrak{a},n}}$ generated by all its projective objects. It coincides with the Karoubian envelope in $C_{\mathfrak{X}_{\mathfrak{a},n}}$ of the image of the functor $\Phi_{\mathfrak{X}_{\mathfrak{a},n-1}}$.

C4.6.4.2. The resolution of an exact category with enough projectives. Let $(C_X, \mathcal{E}_X)$ be an exact category with enough projectives. Let $C_{\mathcal{P}X}$ denote the full subcategory of the category C_X generated by all projectives of $(C_X, \mathcal{E}_X)$. Then we have a sequence

$$\begin{aligned} &C_{\mathcal{P}X} \xrightarrow{\mathfrak{K}_X} C_X \xrightarrow{\mathfrak{P}_X} C_{\mathfrak{X}_0} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_0}} C_{\mathfrak{X}_{\mathfrak{a},0}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},0}}} C_{\mathfrak{X}_1} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_1}} C_{\mathfrak{X}_{\mathfrak{a},1}} \dots \\ &\dots \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n-1}}} C_{\mathfrak{X}_n} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_n}} C_{\mathfrak{X}_{\mathfrak{a},n}} \xrightarrow{\mathfrak{P}_{\mathfrak{X}_{\mathfrak{a},n}}} C_{\mathfrak{X}_{n+1}} \xrightarrow{\mathfrak{H}_{\mathfrak{X}_{n+1}}} \dots \end{aligned} \tag{3}$$

where $\mathfrak{X}_0 = \mathfrak{S}_- X$, i.e. $C_{\mathfrak{X}_0}$ is the costable category of the exact category $(C_X, \mathcal{E}_X)$, and the rest is defined as in (1) above. Again, one can ignore the intermediate cosuspended categories and obtain a complex of exact categories

$$\begin{array}{ccccccccc} C_{\mathcal{P}X} & \xrightarrow{\mathfrak{K}_X} & C_X & \xrightarrow{\Phi_X} & C_{\mathfrak{X}_{\mathfrak{a},0}} & \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},0}}} & C_{\mathfrak{X}_{\mathfrak{a},1}} & \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},1}}} & \dots \\ \dots & \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n-1}}} & C_{\mathfrak{X}_{\mathfrak{a},n}} & \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n}}} & C_{\mathfrak{X}_{\mathfrak{a},n+1}} & \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n+1}}} & C_{\mathfrak{X}_{\mathfrak{a},n+2}} & \xrightarrow{\Phi_{\mathfrak{X}_{\mathfrak{a},n+2}}} & \dots \end{array} \tag{4}$$

C4.6.4.3. Note. If $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, \mathfrak{Tr}_{\mathfrak{X}})$ is a triangulated category (i.e. $\theta_{\mathfrak{X}}$ is an auto-equivalence), then all the cosuspended $\mathbb{Z}^n_+$-categories $\mathfrak{T}_- C_{\mathfrak{X}_n} = (C_{\mathfrak{X}_n}, \theta_{\mathfrak{X}_n}, \mathfrak{Tr}_{\mathfrak{X}_n})$ constructed above are triangulated $\mathbb{Z}^n$-categories and all exact $\mathbb{Z}^n_+$-categories $C_{\mathfrak{X}_{\mathfrak{a},n}}$ are abelian $\mathbb{Z}^n$-categories.

C5. The weak costable category of a right exact category.

C5.1. Definition. Let $(C_X, \mathfrak{E}_X)$ be a right exact category such that the category C_X has an initial object, x. We denote by $\mathfrak{Pr}(X, \mathfrak{E}_X)$ the full subcategory of C_X whose objects are projectives. Let $\widetilde{\mathcal{S}}_X$ denote the class of all arrows t_1 in the commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & P & \xrightarrow{\mathfrak{e}} & M \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle id_M} \\ Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & V & \xrightarrow{\mathfrak{e}} & M \end{array}$$

where $\mathfrak{e}$, $\mathfrak{e}'$ are deflations, t_0 (hence t_1) are split epimorphisms, and P (hence V) is an object of $\mathfrak{Pr}(X, \mathfrak{E}_X)$. Let $\mathcal{S}_X$ be the smallest saturated system containing $\widetilde{\mathcal{S}}_X$ and all deflations $P \longrightarrow P'$ with P and P' in $\mathfrak{Pr}(X, \mathfrak{E}_X)$. We call the quotient category $\mathcal{S}_X^{-1} C_X$ the *weak costable* category of the right exact category $(C_X, \mathfrak{E}_X)$ and denote it by $C_{\mathcal{S}_- X}$.

C5.1.1. Proposition. *Let $(C_X, \mathfrak{E}_X)$ be a right exact category with initial objects and enough projectives. For any object N of the costable category, let $\theta^w_X(N)$ denote the image in $C_{\mathcal{S}_- X}$ of $Ker(\mathfrak{e})$, where $P \xrightarrow{\mathfrak{e}} N$ is a deflation with P projective (we identify objects of $C_{\mathcal{S}_- X}$ with objects of C_X). The object $\theta^w_X(N)$ is determined uniquely up to isomorphism. The map $N \longmapsto \theta^w_X(N)$ extends to a functor $C_{\mathcal{S}_- X} \longrightarrow C_{\mathcal{S}_- X}$.*

Proof. Let $P' \xrightarrow{\mathfrak{e}'} N \xleftarrow{\mathfrak{e}''} P''$ be deflations with P' and P'' projective objects. Since $(C_X, \mathfrak{E}_X)$ has enough projectives, there exists (by the argument C5.3.1(a)) a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{t'_0} & P' \\ {\scriptstyle t''_0}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}'} \\ P'' & \xrightarrow{\mathfrak{e}''} & N \end{array}$$

whose arrows are deflations and the object P is projective. Therefore, we have a commutative diagram

$$\begin{array}{ccccc} Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}'} & P' & \xrightarrow{\mathfrak{e}''} & N \\ {\scriptstyle t'_1}\uparrow & & \uparrow{\scriptstyle t'_0} & & \uparrow{\scriptstyle id_N} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}} & P & \xrightarrow{\mathfrak{e}''} & N \\ {\scriptstyle t''_1}\downarrow & & \downarrow{\scriptstyle t''_0} & & \downarrow{\scriptstyle id_N} \\ Ker(\mathfrak{e}'') & \xrightarrow{\mathfrak{k}''} & P'' & \xrightarrow{\mathfrak{e}''} & N \end{array}$$

Since t'_0 and t''_0 are deflations to projective objects, they are split epimorphisms. Therefore, t'_1 and t''_1 are split epimorphisms, i.e. they belong to $\widetilde{\mathcal{S}}_X$ (cf. 2.5).

Consider a diagram $N \xrightarrow{f} L \xleftarrow{\mathfrak{e}'} M$, where $\mathfrak{e}'$ is a deflation. Then we have a commutative diagram

$$\begin{array}{ccccc} Ker(\sigma) & \xrightarrow{\mathfrak{k}(\sigma)} & P & \xrightarrow{\sigma} & N \\ {\scriptstyle \mathfrak{t}_1}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}_0} & & \downarrow{\scriptstyle id_N} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}(\mathfrak{e})} & \mathfrak{N} & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ Ker(\mathfrak{e}') & \xrightarrow{\mathfrak{k}(\mathfrak{e}')} & M & \xrightarrow{\mathfrak{e}'} & L \end{array} \qquad (1)$$

in which the right lower square is cartesian, the morphism f_1 is uniquely determined by the choice of f_0 (hence both f_0 and f_1 are determined by f uniquely up to isomorphism), $\mathfrak{t}_0$ is a deflation, and $\mathfrak{t}_1$ is (a deflation) uniquely determined by $\mathfrak{t}_0$. Applying the localization $C_X \xrightarrow{\mathfrak{q}^*_{\mathcal{S}_X}} C_{\mathcal{S}_X}$, we obtain morphisms

$$\theta^w_X(N) \xrightarrow{\sim} \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\sigma)) \xrightarrow{\mathfrak{q}^*_{\mathcal{S}_X}(\mathfrak{t}_1)} \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\mathfrak{e})) \xrightarrow{\mathfrak{q}^*_{\mathcal{S}_X}(f_1)} \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\mathfrak{e}')). \qquad (2)$$

The only choice in this construction is that of the deflation $P \xrightarrow{\mathfrak{t}_0} \mathfrak{N}$. If $P' \xrightarrow{\mathfrak{s}_0} \mathfrak{N}$ is another choice, then there exists a commutative square

$$\begin{array}{ccc} P'' & \xrightarrow{\mathfrak{s}''_0} & P \\ {\scriptstyle \mathfrak{t}''_0}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}_0} \\ P' & \xrightarrow{\mathfrak{s}_0} & \mathfrak{N} \end{array}$$

whose arrows are deflations and the object P'' is a projective. Therefore, $\mathfrak{t}''_0$ and $\mathfrak{s}''_0$ are

split deflations, and we have a commutative diagram

$$\begin{array}{ccccc} Ker(\sigma') & \xrightarrow{\mathfrak{k}(\sigma')} & P' & \xrightarrow{\sigma'} & N \\ {\scriptstyle \mathfrak{s}_1''}\uparrow & & \uparrow{\scriptstyle \mathfrak{s}_0''} & & \uparrow{\scriptstyle id_N} \\ Ker(\sigma'') & \xrightarrow{\mathfrak{k}(\sigma'')} & P'' & \xrightarrow{\sigma''} & N \\ {\scriptstyle \mathfrak{t}_1''}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}_0''} & & \downarrow{\scriptstyle id_N} \\ Ker(\sigma) & \xrightarrow{\mathfrak{k}(\sigma)} & P & \xrightarrow{\sigma} & N \end{array} \qquad (3)$$

whose vertical arrows belong to $\mathcal{S}_X$, i.e. their images in the costable category are isomorphisms. This implies that the composition $\theta_X^w(N) \longrightarrow \mathfrak{q}^*_{\mathcal{S}_X}(Ker(\mathfrak{e}'))$ of morphisms of (2) does not depend on the choice of the deflation $P \xrightarrow{\mathfrak{t}_0} \mathfrak{N}$. Taking M projective, we obtain a morphism $\theta_X^w(N) \xrightarrow{\theta_X(f)} \theta_X^w(L)$ which is uniquely defined once the choice of objects $\theta_X^w(N)$ and $\theta_X^w(L)$ is fixed. ∎

C5.2. The weak cosuspension functor. Let $(C_X, \mathfrak{E}_X)$ be a right exact category with enough projectives and initial objects. Let $C_{\mathcal{S}\_X}$ its cosuspended category. The functor $C_{\mathcal{S}\_X} \xrightarrow{\theta_X^w} C_{\mathcal{S}\_X}$ defined in C5.1.1 is called the *weak cosuspension* functor.

Notice that the weak costable category $C_{\mathcal{S}\_X}$ of $(C_X, \mathfrak{E}_X)$ has initial objects. If the category C_X is pointed, then $C_{\mathcal{S}\_X}$ is pointed and the image in $C_{\mathcal{S}\_X}$ of each projective object of $(C_X, \mathfrak{E}_X)$ is a zero object.

C5.2.1. Note. It follows from C6.7.1 that if the category C_X is additive, then the weak costable category $C_{\mathcal{S}\_X}$ with the weak cosuspension functor θ_X^w is equivalent to the costable category $C_{\mathfrak{T}\_X}$ with the cosuspension functor θ_X.

C5.3. Right exact categories of modules over monads and their weak costable categories. Suppose that C_X is a category with initial objects and such that the class $\mathfrak{E}_X^{spl}$ of split epimorphisms of C_X is stable under base change; so that $(C_X, \mathfrak{E}_X^{spl})$ is a right exact category. Let $\mathcal{F} = (F, \mu)$ be a monad on C_X. Set $C_{\mathfrak{X}} = \mathcal{F} - mod$. Let $C_{\mathfrak{X}} \xrightarrow{f_*} C_X$ be the forgetful functor, f^* its canonical left adjoint, and ε the standard adjunction morphism $f^*f_* \longrightarrow Id_{C_{\mathfrak{X}}}$. We denote by $\mathfrak{E}_{\mathfrak{X}}$ the right exact structure on $C_{\mathfrak{X}}$ induced by $\mathfrak{E}_X^{spl}$ via the forgetful functor f_*. By 5.5, $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$ is a right exact category with enough projectives: all modules of the form $(F(L), \mu(L))$, $L \in ObC_X$, are projective objects of $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, and every module $\mathcal{M} = (M, \xi)$ has a canonical deflation $f^*f_*(\mathcal{M}) \xrightarrow{\varepsilon(\mathcal{M})} \mathcal{M}$.

We denote by $\Omega_{\mathcal{F}}$ the kernel of the adjunction morphism $f^*f_* \xrightarrow{\varepsilon} Id_{C_{\mathfrak{X}}}$ and call it the functor of *Kähler differentials*.

C5.3.1. Standard triangles. Let $\mathcal{M} = (M, \xi_{\mathcal{M}})$ and $\mathcal{L} = (L, \xi_{\mathfrak{L}})$ be $\mathcal{F}$-modules and $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$ a deflation (i.e. the epimorphism $M \xrightarrow{f_*(\mathfrak{e})} L$ splits). Then we have a

commutative diagram

$$\begin{array}{ccccc} \Omega_{\mathcal{F}}(\mathcal{L}) & \xrightarrow{\mathfrak{k}_{\mathcal{F}}(\mathcal{L})} & f^*f_*(\mathcal{L}) & \xrightarrow{\varepsilon(\mathcal{L})} & \mathcal{L} \\ \partial\downarrow & & \downarrow \mathfrak{t}_0 & & \downarrow id_{\mathcal{L}} \\ Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{k}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{L} \end{array} \tag{4}$$

which contains (and defines) the *standard triangle*

$$\Omega_{\mathcal{F}}(\mathcal{L}) \xrightarrow{\partial} Ker(\mathfrak{e}) \xrightarrow{\mathfrak{k}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L} \tag{5}$$

corresponding to the deflation $\mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{L}$.

The image of (5) in the weak costable category $C_{\mathcal{S}\_X}$ is a standard triangle of $C_{\mathcal{S}\_X}$.

C5.4. Example: right exact categories of unital algebras. Let C_X be the category Alg_k of associative unital k-algebras. The category C_X has an initial object – the k-algebra k, and the associated pointed category C_{X_k} is the category of augmented k-algebras.

C5.4.1. The functor of Kähler differentials. Kähler differentials appear when we have a pair of adjoint functors $C_X \xrightarrow{f_*} C_Y \xrightarrow{f^*} C_X$. Presently, the role of the category C_Y is played by the category of k-modules. The forgetful functor $Alg_k \xrightarrow{f_*} k-mod$ has a canonical left adjoint f^* which assigns to every k-module M the tensor algebra $T_k(M) = \bigoplus_{n\geq 0} M^{\otimes n}$. Therefore, the class of all split k-module epimorphisms induces via f_* a structure $\mathfrak{E}_X$ of a right exact category on $C_X = Alg_k$. In this case, the tensor algebra $f^*(M) = T_k(M)$ is a projective object of $(C_X, \mathfrak{E}_X)$ for every k-module M; and for every k-algebra A, the adjunction morphism

$$f^*f_*(A) = T_k(f_*(A)) \xrightarrow{\varepsilon(A)} A,$$

(determined by the k-algebra structure and the multiplication $f_*(A) \otimes_k f_*(A) \longrightarrow f_*(A)$ in A) is a canonical projective deflation. By definition, the functor Ω_k of *Kähler differentials* assigns to each k-algebra A the kernel of the adjunction morphism $\varepsilon(A)$, which coincides with the augmented k-algebra $k \oplus \Omega_k^+(A)$, where $\Omega_k^+(A)$ is the kernel $K(\varepsilon(A))$ of the algebra morphism $\varepsilon(A)$ in the usual sense (i.e. in the category of non-unital algebras).

C5.4.2. The functor of non-additive Kähler differentials. The category Alg_k has small products and kernels of pairs of arrows $A \rightrightarrows B$, hence it has limits of arbitrary small diagrams. As any functor having a left adjoint, the forgetful functor $Alg_k \xrightarrow{f_*} k-mod$ preserves limits. In particular, f_* preserves pull-backs and, therefore, kernel pairs of algebra morphisms. Therefore, each k-algebra morphism $A \xrightarrow{\varphi} B$ has a canonical kernel pair $A \times_B A \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$. Using the fact that $A \times_B A$ is computed as the pull-back of k-modules, we can represent $A \times_B A$ as the k-module $f_*(A) \oplus K(f_*(\varphi))$ with the multiplication induced by the isomorphism

$$f_*(A) \oplus Ker(f_*(\varphi)) \xrightarrow{\sim} f_*(A) \times_{f_*(B)} f_*(A), \quad x \oplus y \longmapsto (x, x+y).$$

That is the multiplication is given by the formula $(a \oplus b)(a' \oplus b') = aa' \oplus (ab' + ba' + bb')$. We denote this algebra by $A\#K(\varphi)$.

Applying this to the adjunction arrow $f^*f_* \xrightarrow{\varepsilon} Id_{C_X}$, we obtain a canonical isomorphism between the functor $\widetilde{\Omega}_k$ of non-additive Kähler differentials and $f^*f_*\#\Omega_k^+$, where $\Omega_k^+(A)$ is the kernel of the algebra morphism $T_k(f_*(A)) \xrightarrow{\varepsilon(A)} A$ in the category of non-unital k-algebras (cf. C5.4.1). Thus, for every k-algebra A, we have a commutative diagram similar to the one in the additive case:

$$\begin{array}{ccccccc}
k \oplus \Omega_k^+(A) & \xrightarrow{\sim} & \Omega_k(A) & \overset{\mathfrak{k}(\varepsilon)}{\underset{0_k}{\rightrightarrows}} & T_k(f_*(A)) & \xrightarrow{\varepsilon} & A \\
\widetilde{\mathfrak{j}}_k \downarrow & & \mathfrak{j}_k \downarrow & & \downarrow id & & \downarrow id \\
T_k(f_*(A))\#\Omega_k^+(A) & \xrightarrow{\sim} & \widetilde{\Omega}_k(A) & \overset{\lambda_1}{\underset{\lambda_2}{\rightrightarrows}} & T_k(f_*(A)) & \xrightarrow{\varepsilon} & A
\end{array} \qquad (6)$$

Here $0_k = 0_k(A)$ is the 'zero' morphism – the composition of the augmentation morphism $\Omega_k(A) \longrightarrow k$ and the k-algebra structure $k \longrightarrow T_k(f_*(A))$.

The morphism $\widetilde{\mathfrak{j}}_k$ (hence $\mathfrak{j}_k$) becomes an isomorphism in the costable category.

C5.4.3. Another canonical right exact structure. Let $\mathfrak{E}_X^{\mathfrak{s}}$ denote the class of all strict epimorphisms of k-algebras. The class $\mathfrak{E}_X^{\mathfrak{s}}$ is stable under base change, i.e. $(C_X, \mathfrak{E}_X)$ is a right exact category. For every projective k-module V, the tensor algebra $T_k(V)$ is a projective object of $(C_X, \mathfrak{E}_X^{\mathfrak{s}})$, because the forgetful functor $Alg_k \xrightarrow{f_*} k-mod$ is exact (hence it maps strict epimorphisms to epimorphisms of k-modules). By 5.3.1, its left adjoint f^* maps projectives of $k-mod$ to projectives of $(C_X, \mathfrak{E}_X^{\mathfrak{s}})$. That is for every projective k-module V the tensor algebra $T_k(V)$ of V is a projective object of $(C_X, \mathfrak{E}_X^{\mathfrak{s}})$. Since the adjunction arrow $f^*f_* \xrightarrow{\varepsilon} Id_{C_X}$ is a strict epimorphism and $k-mod$ has enough projectives, the right exact category $(C_X, \mathfrak{E}_X^{\mathfrak{s}})$ has enough projectives: for any k-algebra A, there exists a strict k-algebra epimorphism $T_k(V) \xrightarrow{\mathfrak{e}} A$ for some projective k-module V. By 2.2.1, the kernel $Ker(\mathfrak{e})$ coincides with the augmented k-algebra $k \oplus K(\mathfrak{e})$, where $K(\mathfrak{e})$ is the kernel of $\mathfrak{e}$ in the usual sense – a two-sided ideal in $T_k(V)$.

C5.4.4. Remarks. (a) The forgetful functor $Alg_k \xrightarrow{f_*} k-mod$ is conservative and preserves cokernels of pairs of arrows. Therefore, by Beck's Theorem, there is a canonical equivalence (in this case, an isomorphism) between the category Alg_k and the category $\mathcal{F}-mod$ of modules over the monad $\mathcal{F} = (f_*f^*, \mu) = (T_k(-), \mu)$ associated with the pair of adjoint functors f_*, f^* and the adjunction morphism $f^*f_* \xrightarrow{\varepsilon} Id_{Alg_k}$.

(b) Consider the category $Aff_k = Alg_k^{op}$ of affine (noncommutative) k-schemes. Right exact structures on Alg_k define left exact structures on Aff_k and vice versa. Inflations in Aff_k corresponding strict epimorphisms of algebras are precisely closed immersions of (noncommutative) affine schemes.

(c) The example C5.4 is generalized to algebras in an additive monoidal category.

C5.5. The left exact category of comodules over a comonad and its weak stable category. Fix a comonad $\mathcal{G} = (G, \delta)$ on a category C_X with final objects and split monomorphisms stable under cobase change; i.e. $(C_X, \mathfrak{I}_X^{spl})$ is a left exact category.

C5.5.1. The suspension functor. Let $\mathcal{G}_+$ denote the functor $C_{\mathfrak{Y}} \longrightarrow C_{\mathfrak{Y}}$ which assigns to every $\mathcal{G}$-comodule $\mathcal{M} = (M, \nu)$ the cokernel of the adjunction morphism

$$\mathcal{M} \xrightarrow{\nu} g_* g^*(\mathcal{M}) = (G(M), \delta(M))$$

(see 5.5.2(2)). The functor $\mathcal{G}_+$ is a canonical *suspension* functor on the category $C_{\mathfrak{Y}} = \mathcal{G} - comod = (\mathfrak{Y}\backslash\mathcal{G}) - comod$ which induces a suspension functor on the stable category $\mathcal{S}_+ C_{\mathfrak{Y}}$ of the exact category $(C_{\mathfrak{Y}}, \mathcal{E}_{\mathfrak{Y}})$.

C5.5.2. Lemma. *A morphism $\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$ of $C_{\mathfrak{Y}}$ becomes a trivial morphism in the stable category $\mathcal{T}_+ C_{\mathfrak{Y}}$ iff it factors through an adjunction arrow (3); i.e. there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{M}' \\ {\scriptstyle\nu}\searrow & & \nearrow{\scriptstyle h} \\ & g_* g^*(\mathcal{M}) & \end{array}$$

for some morphism $g_* g^*(\mathcal{M}) = (G(M), \delta(M)) \xrightarrow{h} \mathcal{M}'$.

Proof. By definition of the stable category, the image of an arrow $\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$ of $C_{\mathfrak{Y}}$ in the stable category $\mathcal{T}_+ C_{\mathfrak{Y}}$ is trivial iff it factors through an $\mathcal{E}_{\mathfrak{Y}}$-injective object $\mathcal{N}$. By 5.5.3, the adjunction arrow $\mathcal{N} \longrightarrow g_* g^*(\mathcal{N})$ splits. Therefore, the arrow $\mathcal{M} \xrightarrow{\phi} \mathcal{M}'$ becomes trivial in the stable category iff it factors through a morphism $\mathcal{M} \longrightarrow g_*(N)$ for an object N of C_X. Every such arrow factors through the adjunction morphism $\mathcal{M} \longrightarrow g_* g^*(\mathcal{M})$; hence the assertion. ∎

C5.5.3. Standard triangles. For any conflation $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N}$ in $C_{\mathfrak{Y}} = \mathcal{G} - comod$, the standard triangle

$$\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M} \xrightarrow{\mathfrak{e}} \mathcal{N} \xrightarrow{\mathfrak{d}} \mathcal{G}_+(\mathcal{L})$$

is defined via a commutative diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\mathfrak{j}} & \mathcal{M} & \xrightarrow{\mathfrak{e}} & \mathcal{N} \\ {\scriptstyle id_{\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\mathfrak{d}} \\ \mathcal{L} & \xrightarrow{\eta_\gamma(\mathcal{L})} & \mathcal{G}(\mathcal{L}) & \xrightarrow{\lambda_\gamma(\mathcal{L})} & \mathcal{G}_+(\mathcal{L}) \end{array} \tag{1}$$

where $\mathcal{G} = g_* g^*$ and $G \xrightarrow{\lambda_g} \mathcal{G}_+$ is the canonical deflation. The morphism γ in (1) exists by the $\mathcal{E}_{\mathfrak{Y}}$-injectivity of $G(\mathcal{L})$. The morphism $\mathcal{N} \xrightarrow{\mathfrak{d}} \mathcal{G}_+(\mathcal{L})$ is uniquely determined by the choice of γ (because $\mathfrak{e}$ is an epimorphism). The image of $\mathfrak{d}$ in the stable category $\mathcal{T}_+ C_{\mathfrak{Y}}$ does not depend on the choice of γ.

C5.6. Frobenious morphisms of 'spaces' and Frobenious monads. Let $Y \xrightarrow{f} X$ be a continuous morphism of 'spaces' with an inverse image functor f^* and a direct image

functor f_*. We say that f is a *Frobenious* morphism if there exists an auto-equivalence Ψ of the category C_X such that the composition $f^* \circ \Psi$ is a right adjoint to f_*.

It is clear that every isomorphism is a Frobenious morphism and the composition of Frobenious morphisms is a Frobenious morphism.

It follows that every Frobenious morphism $Y \xrightarrow{f} X$ with a conservative direct image functor is affine. Therefore, the category C_Y can be identified with the category $\mathcal{F}-mod$ of modules over the monad $\mathcal{F} = (F, \mu)$ on a category C_X associated with the pair of adjoint functors f^*, f_*. Conversely, we call a monad $\mathcal{F}$ on the category C_X a *Frobenious* monad if the forgetful functor $\mathcal{F}-mod \xrightarrow{f_*} C_X$ is a direct image functor of a Frobenious morphism; i.e. there exists an equivalence $C_X \xrightarrow{\Psi} C_X$ such that the functor

$$C_X \xrightarrow{f^* \circ \Psi} \mathcal{F}-mod, \quad V \longmapsto (F(\Psi(V)), \mu(\Psi(V)),$$

is a right adjoint to the forgetful functor f_*. In particular, the monad $\mathcal{F}$ is continuous.

C5.6.1. Proposition. *Let $\mathcal{F}$ be a Frobenious monad on a category C_X such that $(C_X, \mathfrak{E}_X^{spl})$ is a right exact category. Then the right exact category $(C_{\mathfrak{X}}, \mathfrak{E}_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ is the categor $\mathcal{F}-mod$ of $\mathcal{F}$-modules and $\mathfrak{E}_{\mathfrak{X}}$ is a right exact structure induced by $\mathfrak{E}_X^{spl}$, is a Frobenious category.*

Proof. Let f_* denote the forgetful functor $\mathcal{F}-mod \longrightarrow C_X$ and f^* its canonical left adjoint. Let Ψ be a functor $C_X \longrightarrow C_X$ such that the composition $f^! = f^* \circ \Psi$ is a right adjoint to f_*. Then every injective object of the category $\mathcal{F}-mod$ is a retract of an object of the form $f^*(\Psi(V))$ for some $V \in ObC_X$. On the other hand, every projective object of $\mathcal{F}-mod$ is a retract of an object of the form $f^*(L)$ for some $L \in ObC_X$. Therefore, every injective $\mathcal{F}$-module is projective. If the functor Ψ is an auto-equivalence, then $f^* \simeq f^! \circ \Psi^*$, where Ψ^* is a quasi-inverse to Ψ. That the functor $f^! \circ \Psi$ is a left adjoint to f_*. By duality, it follows from the argument above that every projective object of $\mathcal{F}-mod$ is injective. ∎

C5.7. The costable category associated with an augmented monad. Let $\mathcal{F} = (F, \mu)$ be an augmented monad on a k-linear additive category C_X; i.e. $F = Id_{C_X} \oplus F_+$. The category $\mathcal{F}-mod$ of $\mathcal{F}$-modules is isomorphic to the category $\mathcal{F}_+-mod_1$ of $\mathcal{F}_+$-actions. Recall that the objects of $\mathcal{F}_+-mod_1$ are pairs (M, ξ), where $M \in ObC_X$ and ξ is a morphism $F_+(M) \longrightarrow M$ satisfying associativity condition with respect to multiplication $F_+^2 \xrightarrow{\mu_+} F_+$, i.e. $\xi \circ \mu_+(M) = \xi \circ F_+(\xi)$. Morphisms are defined naturally.

Notice that the monad $\mathcal{F}$ is continuous (i.e. the functor F has a right adjoint) iff the functor F_+ has a right adjoint.

It follows that $\Omega_{\mathcal{F}} \longrightarrow f^* f_*$ factors through the subfunctor $\mathcal{F}_+$ of $f^* f_*$ corresponding to the subsemimonad (F_+, μ^+) of $\mathcal{F}$. The full subcategory $\mathcal{T}_{\mathcal{F}_+}$ of $\mathcal{F}-mod$ generated by all $\mathcal{F}$-modules $\mathcal{M}$ such that $\Omega_{\mathcal{F}}(\mathcal{M}) \longrightarrow \mathcal{F}_+(\mathcal{M})$ is an isomorphism (i.e. the action of F_+ on $\mathcal{M}$ is zero) is isomorphic to the category C_X.

C5.7.1. Infinitesimal neighborhoods. Let $\mathcal{T}_{\mathcal{F}_+}^{(n)}$ denote the *n-th infinitesimal neighborhood* of $\mathcal{T}_{\mathcal{F}_+}$, $n \geq 1$. It is the full subcategory of $\mathcal{F}-mod$ generated by modules

$\mathcal{M} = (M, \xi)$ such that the n-th iteration $F_+^n(M) \xrightarrow{\xi_n^+} M$ of the action of F_+ on M is zero. In particular, $\mathcal{T}_{\mathcal{F}_+}^{(1)} = \mathcal{T}_{\mathcal{F}_+}$.

Since ξ_n^+ is an $\mathcal{F}$-module morphism for any $n \geq 1$, an $\mathcal{F}$-module $\mathcal{M} = (M, \xi)$ is an object of $\mathcal{T}_{\mathcal{F}_+}^{(n)}$ iff $F_+^{\cdot n} \hookrightarrow \Omega_{\mathcal{F}}$, where $F_+^{\cdot n}$ is the image of the iterated multiplication $F_+^n \xrightarrow{\mu_n^+} F_+$. One can see that $F_+^{\cdot n}$ is a two-sided ideal in the monad $\mathcal{F}$. If the quotient functor $F/F_+^{\cdot n}$ is well defined (which is the case if cokernels of morphisms exist in C_X), then there is a unique monad structure μ_n on the quotient $F/F_+^{\cdot n}$ such that the quotient morphism $F \longrightarrow F/F_+^{\cdot n}$ is a monad morphism from $\mathcal{F}$ to $\mathcal{F}/F_+^{\cdot n} = (F/F_+^{\cdot n}, \mu_n)$ and the category $\mathcal{T}_{\mathcal{F}_+}^{(n)}$ is equivalent to the category $\mathcal{F}/F_+^{\cdot n}$-modules. Clearly, $\mathcal{F}/F_+^{\cdot n}$ is an augmented monad: $F/F_+^{\cdot n} \simeq Id_{C_X} \oplus F_+/F_+^{\cdot n}$.

It follows from the preceeding discussion that $F_+^{\cdot (n-1)}/F_+^{\cdot n} \hookrightarrow \Omega_{\mathcal{F}/F_+^{\cdot n}} \hookrightarrow F_+/F_+^{\cdot n}$. In particular, $\Omega_{\mathcal{F}/F_+^{\cdot 2}} = \mathcal{F}_+/F_+^{\cdot 2}$.

C5.7.2. Free actions. Let C_X be a k-linear category with the exact structure $\mathcal{E}^{spl}$; and let $\mathfrak{L}$ be a k-linear endofunctor on C_X. Consider the category $\mathfrak{L} - \mathfrak{act}$ whose objects are pairs (M, ξ), where $M \in ObC_X$ and ξ is a morphism $\mathfrak{L}(M) \longrightarrow M$. Morphisms between actions are defined in a standard way. We endow $\mathfrak{L} - \mathfrak{act}$ with the exact structure induced by the forgetful functor $\mathfrak{L} - \mathfrak{act} \xrightarrow{f_*} C_X$. If C_X has countable coproducts and the functor $\mathfrak{L}$ preserves countable coproducts, then the category $\mathfrak{L} - \mathfrak{act}$ is isomorphic to $\mathbb{T}(\mathfrak{L}) - mod$, where $\mathbb{T}(\mathfrak{L}) = (T(\mathfrak{L}), \mu)$ is a free monad generated by the endofunctor $\mathfrak{L}$; i.e. $T(\mathfrak{L}) = \oplus_{n \geq 0} \mathfrak{L}^n$ and μ is the multiplication defined by the identical morphisms $\mathfrak{L}^n \circ \mathfrak{L}^m \longrightarrow \mathfrak{L}^{n+m}$, $n, m \geq 0$.

The category C_X is isomorphic to the full subcategory $\mathcal{T}_{\mathfrak{L}}$ of $\mathfrak{L} - \mathfrak{act}$ generated by zero actions. The n-th infinitesimal neighborhood of $\mathcal{T}_{\mathfrak{L}}$ is the full subcategory $\mathcal{T}_{\mathfrak{L}}^{(n)}$ of $\mathfrak{L} - \mathfrak{act}$ generated by all actions (M, ξ) such that the n-th iteration $\mathfrak{L}^n(M) \xrightarrow{\xi_n} M$ of the action ξ is zero. The category $\mathcal{T}_{\mathfrak{L}}^{(n+1)}$ is equivalent to the category $\mathbb{T}_{\mathfrak{L},n} - mod$ of modules over the monad $\mathbb{T}_{\mathfrak{L},n} = (T_{\mathfrak{L},n}, \mu_n)$, where $T_{\mathfrak{L},n} = \bigoplus_{0 \leq m \leq n} \mathfrak{L}^m$ and the multiplication defined by morphisms $\mathfrak{L}^k \circ \mathfrak{L}^m \longrightarrow \mathfrak{L}^{k+m}$, $0 \leq k, m \leq n$, which are identical if $k + m < n$ and zero otherwise.

It follows from C5.7.1 that $\mathfrak{L}^n \hookrightarrow \Omega_{\mathbb{T}_{\mathfrak{L},n}} \hookrightarrow T_{\mathfrak{L},n}^+ \stackrel{\text{def}}{=} \bigoplus_{1 \leq m \leq n} \mathfrak{L}^m$.

In particular, $\Omega_{\mathbb{T}_{\mathfrak{L},2}} = \mathfrak{L}$. Here $\mathfrak{L}$ denotes the functor $\mathfrak{L} - \mathfrak{act} \longrightarrow \mathfrak{L} - \mathfrak{act}$ which assigns to an object (M, ξ) the object $(\mathfrak{L}(M), \mathfrak{L}(\xi))$ and acts on morphisms accordingly.

C5.7.2.1. Projectives and injectives of an infinitesimal neighborhood. Projective objects of the category $\mathcal{T}_{\mathfrak{L}}^{(n+1)} = \mathbb{T}_{\mathfrak{L},n} - mod$ are retracts of relatively free objects. The latter are $\mathbb{T}_{\mathfrak{L},n}$-modules of the form $\mathbb{T}_{\mathfrak{L},n}(V)$, $V \in ObC_X$.

Suppose that $\mathfrak{L}$ has a right adjoint functor, $\mathfrak{L}_*$. Then the functor $T_{\mathfrak{L},n} = \bigoplus_{0 \leq m \leq n} \mathfrak{L}^m$ has a right adjoint equal to $T_{\mathfrak{L},n}^! = \bigoplus_{0 \leq m \leq n} \mathfrak{L}_*^m$; that is $\mathbb{T}_{\mathfrak{L},n}$ is a continuous monad.

Therefore, by G1.4, the injective objects of $\mathbb{T}_{\mathfrak{L},n} - mod$ are retracts of $\mathbb{T}_{\mathfrak{L},n}$-modules of the form $\mathbb{T}^!_{\mathfrak{L},n}(V) = (T^!_{\mathfrak{L},n}(V), \gamma_n(V)),\ V \in ObC_X$.

C5.7.2.2. Proposition. *Suppose that $\mathfrak{L}$ is an autoequivalence of the category C_X. Then $\mathcal{T}^{(n+1)}_{\mathfrak{L}} = \mathbb{T}_{\mathfrak{L},n} - mod$ is a Frobenius category.*

Proof. It suffices to show that $\mathbb{T}_{\mathfrak{L},n}$ is a Frobenious monad. An adjunction arrow $\mathfrak{L} \circ \mathfrak{L}_* \longrightarrow Id_{C_X}$ induces a canonical morphism from $\mathbb{T}_{\mathfrak{L},n}(\mathfrak{L}^n_*(V))$ to the injective object $\mathbb{T}^!_{\mathfrak{L},n}(V)$. If $\mathfrak{L}$ is an autoequivalence, then this canonical morphism is an isomorphism. ∎

C5.7.3. Example. Let C_X be the product of $\mathbb{Z}$ copies of a k-linear category C_Y; i.e. objects of C_X are sequences $M = (M_i |\ i \in \mathbb{Z})$ of objects of C_Y. Let $\mathfrak{L}$ be the translation functor: $\mathfrak{L}(M)_i = M_{i-1}$. Objects of the category $\mathfrak{L} - \mathfrak{act}$ of $\mathfrak{L}$-actions are arbitrary sequences of arrows $(\ldots \stackrel{d_{n+1}}{\longrightarrow} M_{n+1} \stackrel{d_n}{\longrightarrow} M_n \stackrel{d_{n-1}}{\longrightarrow} \ldots)$. Objects of the subcategory $\mathcal{T}^{(n)}_{\mathfrak{L}}$ are sequences such that the composition of any n consecutive arrows is zero. In particular, $\mathcal{T}^{(2)}_{\mathfrak{L}}$ coincides with the category of complexes on C_Y and its subcategory $\mathcal{T}_{\mathfrak{L}} = \mathcal{T}^{(1)}_{\mathfrak{L}}$ is the category of complexes with zero differential. By C5.7.2.2, $\mathcal{T}^{(n)}_{\mathfrak{L}}$ is a Frobenious category for every n. Therefore, its costable category is triangulated. Notice that the costable category of $\mathcal{T}^{(2)}_{\mathfrak{L}}$ coincides with the homotopy category of unbounded complexes.

Appendix K. Exact categories and their (co)stable categories.

K1. Exact categories. We follow here the approach of B. Keller [Ke1]. For the convenience of applications, we consider mostly k-linear categories and k-linear functors, where k is a commutative associative unital ring. For a k-linear category C_X, we denote by $\mathcal{M}_k(X)$ the category of k-linear functors $C_X^{op} \longrightarrow k-mod$ and will call it the category of k-*presheaves* on C_X.

K1.1. Definition. Let C_X be a k-linear category and $\mathcal{E}_X$ a class of pairs of morphisms $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ of C_X such that the sequence $0 \longrightarrow L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N \longrightarrow 0$ is exact (i.e. $\mathfrak{j}$ is a kernel of $\mathfrak{e}$ and $\mathfrak{e}$ a cokernel of $\mathfrak{j}$). The elements of $\mathcal{E}_X$ are called *conflations*. The morphism $\mathfrak{e}$ (resp. $\mathfrak{j}$) of a conflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ is called a *deflation* (resp. *inflation*). The pair $(C_X, \mathcal{E}_X)$ is called an *exact* category if $\mathcal{E}_X$ is closed under isomorphisms and the following conditions hold.

(Ex0) id_0 is a deflation.

(Ex1) The composition of two deflations is a deflation.

(Ex2) For every diagram $M' \xrightarrow{f} M \xleftarrow{\mathfrak{e}} L$, where $\mathfrak{e}$ is a deflation, there is a cartesian square

$$\begin{array}{ccc} L' & \xrightarrow{\mathfrak{e}'} & M' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ L & \xrightarrow{\mathfrak{e}} & M \end{array}$$

where $\mathfrak{e}'$ is a deflation.

(Ex2^{op}) For every diagram $M' \xleftarrow{f} M \xrightarrow{\mathfrak{j}} L$, where $\mathfrak{j}$ is an inflation, there is a cocartesian square

$$\begin{array}{ccc} L' & \xleftarrow{\mathfrak{j}'} & M' \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ L & \xleftarrow{\mathfrak{j}} & M \end{array}$$

where $\mathfrak{j}'$ is an inflation.

For an exact category $(C_X, \mathcal{E}_X)$, we denote by $\mathfrak{E}_X$ the class of all deflations and by $\mathfrak{M}_X$ the class of all inflations of $(C_X, \mathcal{E}_X)$.

K1.2. Remarks.

K1.2.1. Applying (Ex2) to (Ex0), we obtain that $id_{_M}$ is a deflation for every $M \in ObC_X$. Thus, axioms (Ex0), (Ex1), (Ex2) mean simply that the class $\mathfrak{E}_X$ of deflations forms a right multiplicative system, or, what is the same, a pretopology on C_X. The invariance of $\mathcal{E}_X$ under isomorphisms implies that all isomorphisms of C_X are deflations.

The fact that all arrows of $\mathfrak{E}_X$ are strict epimorphisms means precisely that the pretopology $\mathfrak{E}_X$ on C_X is *subcanonical*, i.e. every representable presheaf of sets on C_X is a sheaf on $(C_X, \mathfrak{E}_X)$. Thus, one can start from a class $\mathfrak{E}_X$ of arrows of C_X which forms a subcanonical pretopology (equivalently, it is a right multiplicative system formed by strict

epimorphisms) and define $\mathfrak{M}_X$ as kernels of arrows of $\mathfrak{E}_X$. The only remaining requirement is the axiom (Ex^{op}) – the invariance of the class $\mathfrak{M}_X$ of inflations under a cobase change.

This shows, in particular, that the first three axioms make sense in any category and the last axiom, (Ex2^{op}), makes sense in any pointed category.

The fact that all identical morphisms are deflations implies that arrows $0 \longrightarrow M$ are inflations for all objects M of C_X. Applying the axiom (Ex2^{op}) to arbitrary pair of inflations $L \longleftarrow 0 \longrightarrow M$, we obtain the existence of coproducts of any two objects; i.e. the category C_X is additive.

K1.2.2. Quillen's original definition of an exact category contains some additional axioms. B. Keller showed that they follow from the axioms (Ex0) – (Ex2) and (Ex2^{op}) (cf. [Ke1, Appendix A]). Moreover, he observes (in [Ke1, A.2]) that the axiom (Ex2) follows from (Ex2^{op}) and a weaker version of (Ex2):

(Ex2') For every diagram $M' \xrightarrow{f} M \xleftarrow{\mathfrak{e}} L$, where $\mathfrak{e}$ is a deflation, there is a commutative square

$$\begin{array}{ccc} L' & \xrightarrow{\mathfrak{e}'} & M' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ L & \xrightarrow{\mathfrak{e}} & M \end{array}$$

where $\mathfrak{e}'$ is a deflation.

Quillen's description of exact categories is self-dual which implies self-duality of Keller's axioms: if $(C_X, \mathcal{E}_X)$ is an exact category, then $(C_X^{op}, \mathcal{E}_X^{op})$ is an exact category too.

K1.2.3. In the axioms (Ex2) and (Ex2^{op}), the conditions "there exists a cartesian (resp. cocartesian) square" can be replaced by "for any cartesian (resp. cocartesian) square". This implies that for any family $\{\mathcal{E}_i \mid i \in J\}$ of exact category structures on an additive category C_X, the intersection $\mathcal{E}_J = \bigcap_{i \in J} \mathcal{E}_i$ is a structure of an exact category.

K2. Examples of exact categories.

K2.1. The smallest exact structure. For any additive k-linear category C_X, let $\mathcal{E}_X^{spl}$ denote the class of all split sequences $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$. Then the pair $(C_X, \mathcal{E}_X^{spl})$ is an exact category. Notice that $\mathcal{E}_X^{spl}$ is the smallest exact structure on C_X.

K2.2. The category of complexes. Let $\mathbf{C}(\mathcal{A})$ be the category of complexes of an additive k-linear category $\mathcal{A}$. Conflations are diagrams $L^\bullet \xrightarrow{\mathfrak{j}^\bullet} M^\bullet \xrightarrow{\mathfrak{e}^\bullet} N^\bullet$ such that the diagram $L^n \xrightarrow{\mathfrak{j}^n} M^n \xrightarrow{\mathfrak{e}^n} N^n$ is split for every $n \in \mathbb{Z}$.

K2.3. Quasi-abelian categories. A *quasi-abelian* k-linear category is an additive k-linear category C_X with kernels and cokernels and such that every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism. It follows from definitions that the pair $(C_X, \mathcal{E}_s)$, where $\mathcal{E}_s$ is the class of all short exact sequences in C_X, is an exact category.

Every abelian k-linear category is quasi-abelian.

K2.4. Filtered objects. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category. Objects of the *filtered category* $\mathfrak{F}(C_X, \mathcal{E}_X)$ are sequences of inflations

$$\mathcal{M} = (\ldots \longrightarrow M_n \xrightarrow{\mathfrak{j}_n} M_{n+1} \longrightarrow \ldots)$$

such that $M_n = 0$ for $n \ll 0$ and $\mathfrak{j}_m$ are isomorphisms for $m \gg 0$. Morphisms of filtered objects are defined in a natural way (componentwise). Conflations are sequences of two morphisms whose components belong to $\mathcal{E}_X$.

If $(C_X, \mathcal{E}_X)$ is a quasi-abelian category, then the filtered category $\mathfrak{F}(C_X, \mathcal{E}_X)$ is quasi-abelian too.

K2.5. The category of Banach spaces. Let C_X be the category of complex Banach spaces. A sequence of morphisms $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ is a conflation if it is an exact sequence of complex vector spaces. Thus defined exact category of Banach spaces is quasi-abelian. In fact, a morphism of Banach spaces is a strict epimorphism iff it is an epimorphism of vector spaces.

K2.6. Categories of functors. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category and C_Z a category. The category $\mathcal{H}om(C_Z, C_X)$ of functors from C_Z to C_X is an exact category: a sequence $F' \xrightarrow{\mathfrak{j}} F \xrightarrow{\mathfrak{e}} F''$ of functor morphisms is a conflation if

$$F'(M) \xrightarrow{\mathfrak{j}(M)} F(M) \xrightarrow{\mathfrak{e}(M)} F''(M)$$

is a conflation for every object M of C_Z.

K3. 'Exact' functors. Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories. A k-linear functor $C_X \xrightarrow{F} C_Y$ is called *'exact'* if it maps conflations to conflations. We denote by $ExCat_k$ the category whose objects are exact k-linear categories and morphisms 'exact' k-linear functors.

K3.1. Example. Let C_X and C_Y be additive k-linear categories. Every k-linear functor $C_X \xrightarrow{F} C_Y$ is an 'exact' functor $(C_X, \mathcal{E}_X^{spl}) \xrightarrow{F} (C_Y, \mathcal{E}_Y^{spl})$ (see K2.1). The map which assigns to an additive k-linear category C_X the exact category $(C_X, \mathcal{E}_X^{spl})$ and to a k-linear functor the corresponding 'exact' functor is a full embedding of the category Add_k of additive k-linear categories and k-linear functors to the category $ExCat_k$ of exact k-linear categories and 'exact' k-linear functors. This embedding is a left adjoint to the forgetful functor $ExCat_k \longrightarrow Add_k$.

K3.2. Example: 'exact' functors from a quasi-abelian category. Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories. If $(C_X, \mathcal{E}_X)$ is quasi-abelian, then a k-linear functor $C_X \xrightarrow{F} C_Y$ is an 'exact' functor from $(C_X, \mathcal{E}_X)$ to $(C_Y, \mathcal{E}_Y)$ iff it preserves finite limits and colimits. In other words, 'exact' functors in this case are precisely exact functors.

K4. Right and left 'exact' functors. Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories. A k-linear functor $C_X \xrightarrow{F} C_Y$ is called *right 'exact'* if it maps deflations to

deflations and for any deflation $M \xrightarrow{\mathfrak{e}} N$ of $(C_X, \mathcal{E}_X)$, the functor F maps the canonical diagram $M \times_N M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M \xrightarrow{\mathfrak{e}} N$ to an exact diagram.

Left 'exact' functors $(C_X, \mathcal{E}_X) \longrightarrow (C_Y, \mathcal{E}_Y)$ are defined dually.

K4.1. Remark. Taking as C_Y the abelian category $k - mod^{op}$ with the canonical exact structure, one can see that right 'exact' functors from $(C_X, \mathcal{E}_X)$ to $(C_Y, \mathcal{E}_Y)$ are, precisely, sheaves of k-modules on the pretopology $(C_X, \mathfrak{E}_X)$. They also can be viewed as left 'exact' functors from $(C_X, \mathcal{E}_X)^{op} \overset{\text{def}}{=} (C_X^{op}, \mathcal{E}_X^{op})$ to $k - mod$.

K4.2. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories and $C_X \xrightarrow{F} C_Y$ a k-linear functor which maps deflations to deflations and for every deflation $M \xrightarrow{\mathfrak{e}} N$ of $(C_X, \mathcal{E}_X)$ the canonical morphism*

$$F(M \times_N M) \longrightarrow F(M) \times_{F(N)} F(M)$$

is a deflation of $(C_Y, \mathcal{E}_Y)$. Then F is right 'exact'.

Proof. The condition that the pretopology $(C_Y, \mathfrak{E}_Y)$ is subcanonical means precisely that for every deflation $K \longrightarrow L$, the diagram $K \times_L K \rightrightarrows K \longrightarrow L$ is exact. Since the functor F maps deflations to deflations, the diagram

$$F(M) \times_{F(N)} F(M) \rightrightarrows F(M) \longrightarrow F(N)$$

is exact for every deflation $M \longrightarrow N$ of $(C_X, \mathcal{E}_X)$. Since by hypothesis, the canonical morphism

$$F(M \times_N M) \longrightarrow F(M) \times_{F(N)} F(M)$$

is exact, this implies the exactness of the diagram

$$F(M \times_N M) \rightrightarrows F(M) \longrightarrow F(N).$$

That is F is a right 'exact' functor. ∎

K4.3. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories and $(C_X, \mathcal{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathcal{E}_Y)$ a k-linear functor. The following conditions are equivalent:*

(a) φ^* is right 'exact';

(b) for any conflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$, the sequence

$$\varphi^*(L) \xrightarrow{\varphi^*(\mathfrak{j})} \varphi^*(M) \xrightarrow{\varphi^*(\mathfrak{e})} \varphi^*(N) \longrightarrow 0 \qquad (1)$$

is 'exact'.

(c) the functor φ^* maps deflations to deflations and $\widetilde{\varphi}_* : \mathfrak{F} \longmapsto \mathfrak{F} \circ \varphi^*$ maps sheaves of k-modules to sheaves of k-modules.

Proof. $(a) \Leftrightarrow (b)$. For any conflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$, we have a commutative diagram

$$\begin{array}{ccccc} Ker(p_2) & \overset{\sim}{\underset{0}{\rightrightarrows}} & L & & \\ {\scriptstyle \mathfrak{k}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}} & & \\ M \times_N M & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & M & \xrightarrow{\mathfrak{e}} & N \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccccc} Ker(p_2) & \xrightarrow{\sim} & L & & \\ {\scriptstyle \mathfrak{k}_2}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}} & & \\ M \times_N M & \xrightarrow{p_1-p_2} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \tag{2}$$

Since $\mathfrak{e} \circ p_1 = \mathfrak{e} \circ p_2$, the morphism $p_1 - p_2$ is the composition $\mathfrak{j} \circ \gamma$ for a uniquely defined morphism γ. It follows from the diagram (2) and the monomorphness of $\mathfrak{j}$ that γ is a split epimorphism. In particular, $\gamma \in \mathfrak{E}_X$. Since γ is a split epimorphism, $\varphi^*(\gamma)$ has this property. Therefore the sequence (1) is exact iff the sequence

$$\varphi^*(M \times_N M) \xrightarrow{\varphi^*(\mathfrak{j}\gamma)} \varphi^*(M) \xrightarrow{\varphi^*(\mathfrak{e})} \varphi^*(N) \longrightarrow 0 \tag{3}$$

is exact. The exactness of the sequence (3) is equivalent to the exactness of the diagram

$$\varphi^*(M \times_N M) \underset{\varphi^*(p_2)}{\overset{\varphi^*(p_1)}{\rightrightarrows}} \varphi^*(M) \xrightarrow{\varphi^*(\mathfrak{e})} \varphi^*(N).$$

$(b) \Rightarrow (c)$. (i) The equivalence of (a) and (b) applied to presheaves of k-modules (i.e. C_Y is the category $k - mod^{op}$ with the canonical exact structure), gives a description of sheaves of k-modules on the pretopology $(C_X, \mathfrak{E}_X)$: a presheaf $\mathfrak{F}$ of k-modules is a sheaf iff for any conflation $L \longrightarrow M \longrightarrow N$, the sequence $0 \longrightarrow \mathfrak{F}(N) \longrightarrow \mathfrak{F}(M) \longrightarrow \mathfrak{F}(L)$ is exact.

(ii) It follows that the sequence $0 \longrightarrow \mathfrak{F}(N) \longrightarrow \mathfrak{F}(M) \longrightarrow \mathfrak{F}(\widetilde{L})$ is exact for any 'exact' sequence $\widetilde{L} \longrightarrow M \longrightarrow N \longrightarrow 0$ and any sheaf of k-modules $\mathfrak{F}$. In fact, the sequence $\widetilde{L} \longrightarrow M \longrightarrow N \longrightarrow 0$ being exact means that the morphism $\widetilde{L} \longrightarrow M$ is the composition of a deflation $\widetilde{L} \longrightarrow L$ and an inflation $L \longrightarrow M$ such that $L \longrightarrow M \longrightarrow N$ is a conflation. By (i) above, the latter implies that the sequence $0 \longrightarrow \mathfrak{F}(N) \longrightarrow \mathfrak{F}(M) \longrightarrow \mathfrak{F}(L)$ is exact and the arrow $\mathfrak{F}(L) \longrightarrow \mathfrak{F}(\widetilde{L})$ is a monomorphism for any sheaf $\mathfrak{F}$ of k-modules. Therefore the sequence $0 \longrightarrow \mathfrak{F}(N) \longrightarrow \mathfrak{F}(M) \longrightarrow \mathfrak{F}(\widetilde{L})$ is exact.

(iii) Let now $\mathfrak{F}$ be a sheaf of k-modules on $(C_Y, \mathfrak{E}_Y)$. If $(C_X, \mathcal{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathcal{E}_Y)$ is a right 'exact' functor and $L \longrightarrow M \longrightarrow N$ is a conflation in $(C_X, \mathcal{E}_X)$, then the sequence $\varphi^*(L) \longrightarrow \varphi^*(M) \longrightarrow \varphi^*(N) \longrightarrow 0$ is 'exact', hence the sequence

$$0 \longrightarrow \mathfrak{F}\varphi^*(N) \longrightarrow \mathfrak{F}\varphi^*(M) \longrightarrow \mathfrak{F}\varphi^*(L) \tag{4}$$

is exact. By (ii), this means that $\mathfrak{F}\varphi^*$ is a sheaf on $(C_X, \mathfrak{E}_X)$.

$(c) \Rightarrow (b)$. It follows from (ii) above that the condition (c) means precisely that the diagram (4) is exact for any conflation $L \longrightarrow M \longrightarrow N$ and any sheaf of k-modules $\mathfrak{F}$. In particular, since every representable presheaf is a sheaf, (4) is exact for every representable presheaf, or, equivalently, the diagram

$$\varphi^*(L) \longrightarrow \varphi^*(M) \longrightarrow \varphi^*(N) \longrightarrow 0$$

is exact. ∎

K4.4. Corollary. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories.*
(a) A functor $C_X \xrightarrow{\varphi^*} C_Y$ is 'exact' iff it is both left and right 'exact'.
(b) A presheaf F of k-modules on $(C_X, \mathcal{E}_X)$ is a sheaf iff the sequence

$$0 \longrightarrow F(N) \longrightarrow F(M) \longrightarrow F(L)$$

is exact for any conflation $L \longrightarrow M \longrightarrow N$ in $(C_X, \mathcal{E}_X)$.

Proof. (a) By K4.3, the functor φ^* is both right and left 'exact' iff for any conflation $L \longrightarrow M \longrightarrow N$ in $(C_X, \mathcal{E}_X)$, the sequences

$$\varphi^*(L) \longrightarrow \varphi^*(M) \longrightarrow \varphi^*(N) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \varphi^*(L) \longrightarrow \varphi^*(M) \longrightarrow \varphi^*(N)$$

are exact, i.e. $\varphi^*(L) \longrightarrow \varphi^*(M) \longrightarrow \varphi^*(N)$ is a conflation.

(b) The assertion is proved in the argument of K4.3. It is, also, a formal consequence of K4.1 and K4.3. ∎

K4.5. Remark: right 'exact' functors between pretopologies. The assertion K4.3(c) suggests the following

K4.5.1. Definition. Let $(C_X, \mathfrak{T}_X)$ and $(C_Y, \mathfrak{T}_Y)$ be pretopologies. We call a functor $C_X \xrightarrow{F} C_Y$ a *right 'exact'* functor from $(C_X, \mathfrak{T}_X)$ to $(C_Y, \mathfrak{T}_Y)$ if it maps elements of covers to elements of covers and induces a functor between the categories of sheaves.

K5. Fully exact subcategories of exact categories. Gabriel-Quillen embedding. Let $(C_X, \mathcal{E}_X)$ be an exact category. A full subcategory $\mathcal{B}$ of C_X is called a *fully exact* subcategory of $(C_X, \mathcal{E}_X)$ if it is closed under extensions; i.e. if objects L and N in a conflation $L \longrightarrow M \longrightarrow N$ belong to $\mathcal{B}$, then M is an object of $\mathcal{B}$ too.

K5.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be an exact category and $\mathcal{B}$ a fully exact subcategory of C_X. Then $\mathcal{E}_X$ induces on $\mathcal{B}$ a structure of an exact category.*

Proof. The condition '$\mathcal{B}$ is closed under extensions' means that for any conflation $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ such that L and N are objects of $\mathcal{B}$, the object M is isomorphic to an object of $\mathcal{B}$. Let $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ be a conflation of $\mathcal{E}_{\mathcal{B}}$ (i.e. a conflation of $\mathcal{E}_X$ which

is a diagram in $\mathcal{B}$), and let $N' \xrightarrow{f} N$ be an arbitrary morphism of $\mathcal{B}$. Then we have a commutative diagram

$$\begin{array}{ccccc} L' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & N' \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ L & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & N \end{array} \qquad (1)$$

whose rows are conflations and the right square is cartesian. It is not difficult to see that the left vertical arrow g in (1) is an isomorphism. Since the subcategory $\mathcal{B}$ is closed under extensions, this implies that M' is isomorphic to an object of $\mathcal{B}$. Hence, the class of deflations $\mathfrak{E}_{\mathcal{B}}$ is invariant under base change (axiom (Ex2)). Thus, $\mathcal{E}_{\mathcal{B}}$ has properties (Ex0), (Ex1), (Ex2). The remaining property, (Ex2^{op}), follows from the fact that everything here is selfdual. ∎

A category C_X is called *svelte* if it is equivalent to a small category.

K5.2. Theorem (Gabriel-Quillen embedding**).** *Let $(C_X, \mathcal{E}_X)$ be a svelte exact k-linear category. Then there exist a Grothendieck k-linear category C_Y and an 'exact' fully faithful k-linear functor $(C_X, \mathcal{E}_X) \longrightarrow (C_Y, \mathcal{E}_Y)$ which induces an equivalence between $(C_X, \mathcal{E}_X)$ and a full subcategory of C_Y closed under extensions.*

Proof. Notice that the category $\mathcal{M}_k(X)$ of k-presheaves on C_X is a Grothendieck category (it has a generator because the category C_X is svelte and the category $\mathcal{M}_k(X)$ has infinite coproducts). Let $C_{X_{\mathfrak{e}}}$ denote the category of sheaves of k-modules on the presite $(C_X, \mathfrak{E}_X)$. By K4.1, $C_{X_{\mathfrak{e}}}$ is a full subcategory of $\mathcal{M}_k(X)$ whose objects are left 'exact' functors $(C_X, \mathcal{E}_X)^{op} \longrightarrow k-mod$ (or right 'exact' functors from $(C_X, \mathcal{E}_X)$ to $k-mod^{op}$; see K4.4(b)). The inclusion functor $C_{X_{\mathfrak{e}}} \longrightarrow \mathcal{M}_k(X)$ has a left adjoint – the sheafification functor $\mathcal{M}_k(X) \longrightarrow C_{X_{\mathfrak{e}}}$, which is exact; i.e. $C_{X_{\mathfrak{e}}}$ is (equivalent to) a quotient category of the Grothendieck category $\mathcal{M}_k(X)$ by a Serre subcategory, $\mathcal{S}_{\mathfrak{E}}$. Therefore, $C_{X_{\mathfrak{e}}}$ is a Grothendieck category itself. Since the pretopology $\mathfrak{E}_X$ on C_X is subcanonical (see K1.2.1), the Yoneda embedding induces an equivalence between the exact category $(C_X, \mathcal{E}_X)$ and a full subcategory of $C_{X_{\mathfrak{e}}}$. It remains to show that this full subcategory is closed under extensions and the embedding $C_X \xrightarrow{\mathfrak{j}_X^*} C_{X_{\mathfrak{e}}}$ is an 'exact' functor which reflects conflations.

(i) The Yoneda embedding $C_X \longrightarrow \mathcal{M}_k(X)$ is a left exact functor, and the sheafification functor $\mathcal{M}_k(X) \longrightarrow C_{X_{\mathfrak{e}}}$ is exact. Therefore, their composition $C_X \xrightarrow{\mathfrak{j}_X^*} C_{X_{\mathfrak{e}}}$ is a left exact functor; in particular, it is left 'exact'. The claim is that the functor $\mathfrak{j}_X^*$ is right 'exact'; i.e. $\mathfrak{j}_X^*$ maps every deflation to an epimorphism of the category $C_{X_{\mathfrak{e}}}$.

In fact, let $M \longrightarrow N$ be a deflation and $M \times_N M \rightrightarrows M \longrightarrow N$ the associated exact diagram. The Yoneda embedding maps this diagram to the diagram

$$\widehat{M} \times_{\widehat{N}} \widehat{M} \rightrightarrows \widehat{M} \longrightarrow \widehat{N}, \qquad (2)$$

where $\widehat{M} = C_X(-, M)$. For any presheaf $\mathfrak{F}$ of k-modules, the functor $\mathcal{M}_k(X)(-, \mathfrak{F})$ maps the diagram (2) to the diagram isomorphic to

$$\mathfrak{F}(N) \longrightarrow \mathfrak{F}(M) \rightrightarrows \mathfrak{F}(M \times_N M).$$

which is exact if $\mathfrak{F}$ is a sheaf on $(C_X, \mathfrak{E}_X)$. This shows that for every sheaf $\mathfrak{F}$, the functor $C_{X_\mathfrak{e}}(-, \mathfrak{F})$ maps the diagram (2) to an exact diagram. Therefore the diagram (2) viewed as a diagram in the category of sheaves, is exact.

(ii) Let $N \in ObC_X$ and $\mathfrak{F} \xrightarrow{\gamma} \widehat{N}$ a morphism of sheaves on $(C_X, \mathfrak{E}_X)$. Regarding γ as a presheaf morphism, we represent it as the composition of the presheaf epimorphism $\mathfrak{F} \longrightarrow Im(\gamma)$ and the embedding $Im(\gamma) \hookrightarrow \widehat{N}$. It follows from the exactness of the sheafification functor that γ is a sheaf epimorphism iff the sheafification functor maps the embedding $Im(\gamma) \hookrightarrow \widehat{N}$ to an isomorphism; i.e. $Im(\gamma) \hookrightarrow \widehat{N}$ is a refinement of N in the topology associated with the pretopology $\mathfrak{E}_X$. The latter means that there exists a deflation $M' \xrightarrow{\mathfrak{e}'} N$ such that the image of $\widehat{\mathfrak{e}'}$ is contained in $Im(\gamma)$, i.e. $\widehat{M'} \xrightarrow{\widehat{\mathfrak{e}'}} \widehat{N}$ is the composition of a morphism $\widehat{M'} \xrightarrow{v'} Im(\gamma)$ and the embedding $Im(\gamma) \hookrightarrow \widehat{N}$. Since representable functors are projective objects in $\mathcal{M}_k(X)$, the morphism v' factors through the presheaf epimorphism $\mathfrak{F} \longrightarrow Im(\gamma)$. Thus, we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{M'} & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle id} \\ \mathfrak{F} & \xrightarrow{\gamma} & \widehat{N} \end{array} \tag{3}$$

Suppose that the kernel of γ is representable by an object L, and let $L' \xrightarrow{\mathfrak{j}'} M'$ be the kernel of $M' \xrightarrow{\mathfrak{e}'} N$. Then the diagram (3) extends to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L'} & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M'} & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle \widehat{u}}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\lambda} & F & \xrightarrow{\gamma} & \widehat{N} & \longrightarrow & 0 \end{array} \tag{4}$$

(iii) Applying (Ex2^{op}) to the innflation $L' \xrightarrow{\mathfrak{j}'} M'$ and the arrow $L \xleftarrow{u} L'$, we obtain a commutative diagram

$$\begin{array}{ccccc} L' & \xrightarrow{\mathfrak{j}'} & M' & \xrightarrow{\mathfrak{e}'} & N \\ {\scriptstyle u}\downarrow & cocart & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle id} \\ L & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}''} & N \end{array} \tag{5}$$

whose left square is cocartesian and both rows are conflations (cf. C5.3.3).

(iv) The Yoneda functor assigns to the diagram (5) the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L'} & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M'} & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle \widehat{u}}\downarrow & & \downarrow{\scriptstyle \widehat{u'}} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \end{array} \tag{$\widehat{5}$}$$

whose rows are (by (i) and (iii) above) exact sequences in the category of sheaves. The latter implies that the left square of the diagram $(\widehat{5})$ is cocartesian.

In fact, let

$$\begin{array}{ccc} \widehat{L}' & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M}' \\ {\scriptstyle \widehat{u}}\downarrow & & \downarrow{\scriptstyle \nu} \\ \widehat{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & \mathcal{G} \end{array}$$

be a cocartesian square. Applying the argument of (iii) above, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L}' & \xrightarrow{\widehat{\mathfrak{j}'}} & \widehat{M}' & \xrightarrow{\widehat{\mathfrak{e}'}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle \widehat{u}}\downarrow & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & \mathcal{G} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \end{array}$$

with exact rows. Therefore, the canonical morphism $\mathcal{G} \xrightarrow{g} \widehat{M}$ gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & \mathcal{G} & \xrightarrow{\widetilde{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \end{array} \tag{6}$$

of sheaves on $(C_X, \mathfrak{E}_X)$ with exact rows, which shows that $\mathcal{G} \xrightarrow{g} \widehat{M}$ is an isomorphism.

(v) The commutative diagrams (4) and $(\widehat{5})$ give rise to the commutative diagram of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{L} & \xrightarrow{\widehat{\mathfrak{j}}} & \widehat{M} & \xrightarrow{\widehat{\mathfrak{e}}} & \widehat{N} & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \widehat{L} & \xrightarrow{\lambda} & F & \xrightarrow{\gamma} & \widehat{N} & \longrightarrow & 0 \end{array} \tag{7}$$

with exact rows, which implies that $\widehat{M} \xrightarrow{\mathfrak{t}} F$ is an isomorphism.

(vi) By (iii) above, $L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N$ is a conflation. Therefore, the isomorphism (7) shows also that the functor $\mathfrak{j}_X^*$ reflects conflations: if $0 \longrightarrow \widehat{\mathcal{L}} \xrightarrow{\widetilde{i}} \widehat{\mathcal{M}} \xrightarrow{\widehat{e}} \widehat{\mathcal{N}} \longrightarrow 0$ is an exact sequence of sheaves on $(C_X, \mathfrak{E}_X)$, then $\mathcal{L} \xrightarrow{i} \mathcal{M} \xrightarrow{e} \mathcal{N}$ is a conflation. ∎

K5.2.1. Note. The canonical embedding $C_X \xrightarrow{\mathfrak{j}_X^*} C_{X_{\mathfrak{E}}}$ of (the argument of) K5.2 is called the *Gabriel-Quillen embedding*.

K5.3. Sheafification functor and effaceable presheaves. Recall a standard construction of a sheafification functor $\mathcal{M}_k(X) \xrightarrow{\mathfrak{t}_X^*} C_{X_{\mathfrak{E}}}$.

Let $\mathcal{H}_X$ denote the functor $\mathcal{M}_k(X) \longrightarrow \mathcal{M}_k(X)$ which assigns to every presheaf F of k-modules the presheaf $\mathcal{H}_X(F)$ defined by

$$\mathcal{H}_X(F)(N) = \mathrm{colim}(Ker(F(M) \rightrightarrows F(M \times_N M))) \tag{8}$$

where colimit is taken by the diagram $\mathfrak{E}_X(N)$ of deflations $M \longrightarrow N$. The morphisms $F(N) \longrightarrow Ker(F(M) \rightrightarrows F(M \times_N M)$ determine a morphism $F(N) \xrightarrow{\tau_F(N)} \mathcal{H}_X(F)(N)$ for every $N \in ObC_X$ which is functorial in N; i.e. it defines a functor morphism $F \xrightarrow{\tau_F} \mathcal{H}_X(F)$. The function $F \longmapsto \tau_F$ is a functor morphism $Id_{\mathcal{M}_k(X)} \xrightarrow{\tau} \mathcal{H}_X(F)$.

A presheaf F on $(C_X, \mathfrak{E}_X)$ is a *monopresheaf* if for every deflation $M \xrightarrow{\mathfrak{e}} N$, the morphism $F(N) \xrightarrow{F(\mathfrak{e})} F(M)$ is a monomorphism. There are the following facts:

(a) A presheaf of k-modules F is a monopresheaf (resp. a sheaf) iff the canonical morphism $F \xrightarrow{\tau_F} \mathcal{H}_X(F)$ is a monomorphism (resp. an isomorphism).

(b) The functor $\mathcal{H}_X$ maps every presheaf of k-modules F to a monopresheaf and every monopresheaf to a sheaf.

It follows that the functor $\mathcal{H}^2_X$ maps presheaves to sheaves and its corestriction to the subcategory $C_{X_\mathfrak{E}}$ of sheaves of k-modules is isomorphic to the sheafification functor $\mathfrak{t}^*_X$. Or, what is the same, $\mathcal{H}^2_X \simeq \mathfrak{t}_{X*}\mathfrak{t}^*_X$.

Another consequence of (a) and (b) is that the kernel of $\mathfrak{t}^*_X$ coincides with the kernel of the functor $\mathcal{H}_X$. It follows from the formula (8) that a presheaf F belongs to the kernel of $\mathcal{H}_X$ iff it is *effaceable*. The latter means that for every pair (N, ξ), where $N \in ObC_X$ and ξ is an element of $F(N)$, there exists a deflation $M \xrightarrow{\mathfrak{e}} N$ such that $F(\mathfrak{e})(\xi) = 0$. Equivalently, for any object N of C_X and any morphism $\widehat{N} \xrightarrow{\widetilde{\xi}} F$, there exists a deflation $M \xrightarrow{\mathfrak{e}} N$ such that the composition of $\widehat{M} \xrightarrow{\widehat{\mathfrak{e}}} \widehat{N}$ and $\widetilde{\xi}$ equals to zero.

Thus, objects of the kernel $\mathcal{S}_{\mathfrak{E}_X}$ of the sheafification functor $\mathcal{M}_k(X) \xrightarrow{\mathfrak{t}^*_X} C_{X_\mathfrak{E}}$ are precisely effaceable presheaves. Since the functor $\mathfrak{j}^*_X$ is a flat localization, $\mathcal{S}_{\mathfrak{E}_X}$ is a Serre subcategory of the category $\mathcal{M}_k(X)$, and the category of sheaves $C_{X_\mathfrak{E}}$ is equivalent to the quotient category $\mathcal{M}_k(X)/\mathcal{S}_{\mathfrak{E}_X}$.

K5.4. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be svelte exact k-linear categories and $(C_X, \mathcal{E}_X) \xrightarrow{\varphi^*} (C_Y, \mathcal{E}_Y)$ a right 'exact' k-linear functor. Then there exists a functor $C_{X_\mathfrak{E}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_\mathfrak{E}}$ such that the diagram*

$$\begin{array}{ccc} C_X & \xrightarrow{\varphi^*} & C_Y \\ {\scriptstyle \mathfrak{j}^*_X}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}^*_Y} \\ C_{X_\mathfrak{E}} & \xrightarrow{\widetilde{\varphi}^*} & C_{Y_\mathfrak{E}} \end{array}$$

quasi commutes, i.e. $\widetilde{\varphi}^*\mathfrak{j}^*_X \simeq \mathfrak{j}^*_Y\varphi^*$. Here $C_X \xrightarrow{\mathfrak{j}^*_X} C_{X_\mathfrak{E}}$ and $C_Y \xrightarrow{\mathfrak{j}^*_Y} C_{Y_\mathfrak{E}}$ are Gabriel-Quillen embeddings. The functor $\widetilde{\varphi}^*$ is defined uniquely up to isomorphism and has a right adjoint, $\widetilde{\varphi}_*$.

Proof. (i) If the functor φ^* is right 'exact', then the functor

$$\mathcal{M}_k(Y) \longrightarrow \mathcal{M}_k(X), \quad \mathfrak{F} \longmapsto \mathfrak{F} \circ \varphi^*,$$

maps sheaves on the pretopology $(C_Y, \mathfrak{E}_Y)$ to sheaves on $(C_X, \mathfrak{E}_X)$; in particular, it induces a functor $C_{Y_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}_*} C_{X_{\mathfrak{E}}}$.

In fact, for any arrow $M \longrightarrow N$ of $\mathfrak{E}_X$, consider the decomposition

$$\varphi^*(M \coprod_N M) \longrightarrow \varphi^*(M) \coprod_{\varphi^*(N)} \varphi^*(M) \rightrightarrows \varphi^*(M) \longrightarrow \varphi^*(N) \tag{1}$$

of the diagram

$$\varphi^*(M \coprod_N M) \rightrightarrows \varphi^*(M) \longrightarrow \varphi^*(N) \tag{2}$$

Since the right and the left arrows of the diagram (1) belong to $\mathfrak{E}_Y$, for any sheaf $\mathfrak{F}$ on $(C_Y, \mathfrak{E}_Y)$ the diagram

$$\mathfrak{F}\varphi^*(N) \longrightarrow \mathfrak{F}\varphi^*(M) \rightrightarrows \mathfrak{F}\big(\varphi^*(M) \coprod_{\varphi^*(N)} \varphi^*(M)\big)$$

is exact and the morphism

$$\mathfrak{F}\big(\varphi^*(M) \coprod_{\varphi^*(N)} \varphi^*(M)\big) \longrightarrow \mathfrak{F}\varphi^*(M \coprod_N M)$$

is a monomorphism. Therefore, the diagram

$$\mathfrak{F}\varphi^*(N) \longrightarrow \mathfrak{F}\varphi^*(M) \rightrightarrows \mathfrak{F}\varphi^*(M \coprod_N M)$$

is exact. This shows that $\mathfrak{F}\varphi^*$ is a sheaf on the pretopology $(C_X, \mathfrak{E}_X)$.

(ii) The functor $\widetilde{\varphi}_*$ has a left adjoint, $\widetilde{\varphi}^*$. It follows that $\widetilde{\varphi}^*\mathfrak{j}_X^* \simeq \mathfrak{j}_Y^*\varphi^*$. ∎

K5.4.1. Note. Even if the functor φ^* is 'exact', the functor $C_{X_{\mathfrak{E}}} \xrightarrow{\widetilde{\varphi}^*} C_{Y_{\mathfrak{E}}}$ need not to be (left) exact. For instance, let $(C_X, \mathcal{E}_X)$ (resp. $(C_Y, \mathcal{E}_Y)$) be the exact category of projective A-modules (resp. B-modules) of finite type, and φ^* the functor $M \longmapsto B \otimes_A M$ corresponding to an algebra morphism $A \longrightarrow B$. Then the category $C_{X_{\mathfrak{E}}}$ is naturally identified with $A - mod$ and the functor $\widetilde{\varphi}^*$ with $A - mod \xrightarrow{B\otimes_A} B - mod$. Therefore, the functor $\widetilde{\varphi}^*$ is exact iff the algebra morphism $A \longrightarrow B$ turns B into a flat right A-module.

K5.5. The Gabriel-Quillen embedding and the smallest abelianization of an exact category. Fix an exact k-linear category $(C_X, \mathcal{E}_X)$. Consider the category $\mathfrak{A}_{X,\mathcal{E}}$ whose objects are fully faithful exact k-linear functors $(C_X, \mathcal{E}_X) \xrightarrow{\mathfrak{j}^*} (C_Y, \mathcal{E}_Y)$ such that C_Y is an abelian k-linear category with the canonical exact structure, $\mathfrak{j}^*$ reflects exact sequences, and C_X is closed under extensions in C_Y. A morphism between two such embeddings, $(C_X, \mathcal{E}_X) \xrightarrow{\mathfrak{j}^*} (C_Y, \mathcal{E}_Y)$ and $(C_X, \mathcal{E}_X) \xrightarrow{\mathfrak{k}^*} (C_Z, \mathcal{E}_Z)$, is a pair (g^*, α), where g^* is a functor $C_Y \longrightarrow C_Z$ and α a functor isomorphism $g^*\mathfrak{j}^* \xrightarrow{\sim} \mathfrak{k}^*$. The composition is

defined naturally. Let $\mathfrak{A}^{\mathfrak{c}}_{X,\mathcal{E}}$ denote the subcategory of $\mathfrak{A}_{X,\mathcal{E}}$ whose objects are $(C_X, \mathcal{E}_X) \xrightarrow{\mathrm{j}^*} (C_Y, \mathcal{E}_Y)$ such that the category C_Y has small coproducts and morphisms are pairs (g^*, α) such that the functor g^* has a right adjoint.

K5.5.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category.*

(a) The Gabriel-Quillen embedding $(C_X, \mathcal{E}_X) \xrightarrow{\mathrm{j}^*_X} (C_{X_{\mathfrak{E}}}, \mathcal{E}_{X_{\mathfrak{E}}})$, is a final object of the category $\mathfrak{A}^{\mathfrak{c}}_{X,\mathcal{E}}$. Here $\mathcal{E}_{X_{\mathfrak{E}}}$ is the canonical exact structure of the abelian category $C_{X_{\mathfrak{E}}}$,

(b) The category $\mathfrak{A}_{X,\mathcal{E}}$ has an initial object.

Proof. (a) The assertion follows from K5.3.

(b) Let $C_{X_{\mathfrak{a}}(\mathcal{E})}$ be the smallest full abelian subcategory of the category $C_{X_{\mathfrak{E}}}$ containing (the image of) C_X. Its objects are kernels and cokernels (taken in $C_{X_{\mathfrak{E}}}$) of (pairs of) arrows in C_X. The embedding $C_X \xrightarrow{\mathrm{j}^*_{\mathfrak{a}}} C_{X_{\mathfrak{a}}(\mathcal{E})}$ is an initial object of the category $\mathfrak{A}_{X,\mathcal{E}}$. ∎

K5.5.2. Example. If $\mathcal{E} = \mathcal{E}^{spl}_X = \{\text{split exact sequences}\}$, then $C_{X_{\mathfrak{E}}}$ coincides with the category $\mathcal{M}_k(X)$ of presheaves of k-modules on C_X, hence $C_{X_{\mathfrak{a}}(\mathcal{E})}$ is the smallest abelian subcategory of $\mathcal{M}_k(X)$ containing the image of C_X.

K6. The Karoubian envelope.

K6.1. Proposition (Karoubi). *Let C_X be an additive k-linear category.*

(a) There exists a Karoubian additive k-linear category C_{X_K} and a fully faithful k-linear functor $C_X \xrightarrow{\mathfrak{K}^*_X} C_{X_K}$ such that any k-linear functor from C_X to any Karoubian k-linear category factors uniquely up to a natural isomorphism through $C_X \xrightarrow{\mathfrak{K}^*_X} C_{X_K}$.

(b) Every object of C_{X_K} is a direct summand of an object in $\mathfrak{K}^*_X(C_X)$.

Proof. Objects of the category C_{X_K} are pairs (M, p), where M is an object of the category C_X and $M \xrightarrow{p} M$ is an idempotent endomorphism, i.e. $p^2 = p$. Morphisms $(M, p) \longrightarrow (M', p')$ are morphisms $M \xrightarrow{f} M'$ such that $fp = f = p'f$. The composition of $(M, p) \xrightarrow{f} (M', p')$ and $(M', p') \xrightarrow{g} (M'', p'')$ is $(M, p) \xrightarrow{gf} (M'', p'')$. It follows from this definition that $(M, p) \xrightarrow{p} (M, p)$ is the identical morphism.

The category C_{X_K} is additive with $(M, p) \oplus (M', p') = (M \oplus M', p \oplus p')$, and an object (M, p) is a direct summand of $\mathfrak{K}^*_X(M) = (M, id_M)$, because

$$(M, p) \oplus (M, id_M - p) = (M \oplus M, p \oplus (id_M - p)) \xrightarrow{\simeq} (M, id_M).$$

Here the isomorphism corresponds to the identical morphisms $M \xrightarrow{id_M} M \xleftarrow{id_M} M$.

The functor $C_X \xrightarrow{\mathfrak{K}^*_X} C_{X_K}$ assigns to every object M of C_X the object (M, id_M). It is fully faithful. A k-linear functor $C_X \xrightarrow{F} C_Z$ to a Karoubian category C_Z gives rise to a functor $C_{X_K} \xrightarrow{F_K} C_Z$ which assigns to every object (M, p) of the category C_{X_K} the kernel of $F(id_M - p)$. It follows that $F_K \circ \mathfrak{K}^*_X \simeq F$. In particular, for every additive functor

$C_X \xrightarrow{F} C_Y$, there exists a natural functor $C_{X_K} \xrightarrow{F_K} C_{X_K}$ such that the diagram

$$\begin{array}{ccc} C_X & \xrightarrow{F} & C_Y \\ {\scriptstyle \mathfrak{K}_X^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{K}_Y^*} \\ C_{X_K} & \xrightarrow{F_K} & C_{X_K} \end{array}$$

quasi-commutes. The map $F \longmapsto F_K$ defines a (pseudo) functor from the category of additive categories to the category of Karoubian categories which is a left adjoint to the inclusion functor. This implies, in particular, the universal property of the correspondence $C_X \longmapsto C_{X_K}$. ∎

The category C_{X_K} in K6.1 is called the *Karoubian envelope* of the category C_X.

K6.2. Proposition. *Let $(C_X, \mathcal{E}_X)$ be an exact category. The Karoubian envelope C_{X_K} has a structure of an exact category, $\mathcal{E}_K$, whose conflations are direct summands of conflations of $\mathcal{E}$.*

Proof. Consider the Gabriel-Quillen embedding $C_X \xrightarrow{\mathrm{j}_X^*} C_{X_\mathfrak{E}}$. The category $C_{X_\mathfrak{E}}$ is abelian, hence Karoubian. It follows from K6.1 that the functor j_X^* factors through $C_X \xrightarrow{\mathfrak{K}_X^*} C_{X_K}$, i.e. there exists a canonical morphism $C_{X_K} \longrightarrow C_{X_\mathfrak{E}}$ which induces an equivalence between the category C_{X_K} and the full subcategory of $C_{X_\mathfrak{E}}$ whose objects are all direct summands of objects of $\mathrm{j}_X^*(C_X)$ (see the argument of K6.1). Since the subcategory $\mathrm{j}_X^*(C_X)$ is closed under extensions in $C_{X_\mathfrak{E}}$, the image of C_{X_K} in $C_{X_\mathfrak{E}}$ has the same property. In fact, let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $C_{X_\mathfrak{E}}$ such that $L \oplus L'$ and $N \oplus N'$ are isomorphic to objects of $\mathrm{j}_X^*(C_X)$ for some objects L' and N' of $C_{X_\mathfrak{E}}$. Since the subcategory $\mathrm{j}_X^*(C_X)$ is closed under extensions in $C_{X_\mathfrak{E}}$ and the sequence $0 \longrightarrow L \oplus L' \longrightarrow M \oplus L' \oplus N' \longrightarrow N \oplus N' \longrightarrow 0$ is exact, the object $M \oplus L' \oplus N'$ is isomorphic to an object of $\mathrm{j}_X^*(C_X)$. This shows that M is a direct summand of an object of $\mathrm{j}_X^*(C_X)$ and that any exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ in $C_{X_\mathfrak{E}}$ whose objects belong to the image of C_{X_K} is a direct summand of an image of an image of a sequence in $\mathcal{E}$. The assertion follows now from K5.1. ∎

K7. Injective and projective objects of an exact category. An object M of a k-linear exact category $(C_X, \mathcal{E}_X)$ is *projective* if $C_X(M, -)$ is an 'exact' functor from $(C_X, \mathcal{E}_X)$ to $k - mod$. Injective objects are defined dually – they correspond to projective objects of the dual exact category $(C_X^{op}, \mathcal{E}_X^{op})$.

Let $\mathfrak{P}(X, \mathcal{E}_X)$ denote the full subcategory of the category C_X generated by projective objects of $(C_X, \mathcal{E}_X)$. It follows that any deflation $N \longrightarrow P$ such that P is a projective object splits. In particular, the subcategory $\mathfrak{P}(X, \mathcal{E}_X)$ is closed under extensions in $(C_X, \mathcal{E}_X)$; i.e. $\mathfrak{P}(X, \mathcal{E}_X)$ is a *fully exact* subcategory of $(C_X, \mathcal{E}_X)$. The exact structure induced on $\mathfrak{P}(X, \mathcal{E}_X)$ is the smallest one: it consists of split conflations.

Similarly, the full subcategory $\mathfrak{I}(X, \mathcal{E}_X)$ of C_X generated by injective objects is a fully exact subcategory of $(C_X, \mathcal{E}_X)$.

K7.1. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories and $C_X \xrightarrow{f_*} C_Y$ a k-linear functor having a right adjoint. If f_* is an 'exact' functor from $(C_X, \mathcal{E}_X)$ to $(C_Y, \mathcal{E}_Y)$, then its right adjoint maps injective objects to injective objects.*

Dually, if f_* has a left adjoint functor, then the latter maps $\mathcal{E}_Y$-projective objects to $\mathcal{E}_X$-projective objects.

Proof. Let $C_Y \xrightarrow{f^!} C_X$ be a right adjoint to the functor f_*. For any conflation $E \in \mathcal{E}_X$ and any $\mathcal{E}_Y$-injective objects M, the sequence $C_Y(f_*(E), M)$ is exact, because, by hypothesis, f_* is an 'exact' functor. Therefore, the sequence $C_X(E, f^!(M))$ is exact, i.e. $f^!(M)$ is an $\mathcal{E}_X$-injective object. ∎

K7.1.1. Remark. Notice by passing that a right (or left) adjoint to a k-linear functor is a k-linear functor. In fact, let $C_Y \xrightarrow{f^!} C_X$ be a right adjoint to a k-linear functor f_*. Then we have a commutative diagram of bifunctors

$$\begin{array}{ccc} C_Y(-,-) & \xrightarrow{f^!} & C_X(f^!(-), f^!(-)) \\ {\scriptstyle \cdot\varepsilon}\downarrow & & \downarrow{\scriptstyle f_*} \\ C_Y(f_*f^!(-),-) & \xleftarrow{\varepsilon\cdot} & C_Y(f_*f^!(-), f_*f^!(-)) \end{array}$$

whose right vertical arrow is a k-module morphism because the functor f_* is k-linear. Its left vertical (resp. the lower horizontal) arrow is a k-module morphism because it is the composition from the left (resp. from the right) with the adjunction morphism $f_*f^! \xrightarrow{\varepsilon} Id_{C_Y}$. The composition of the right vertical and lower horizontal arrow is an adjunction isomorphism. Since it is a k-module isomorphism, its inverse is a k-module isomorphism too. Therefore, the upper horizontal arrow is a k-module morphism, which proves that $f^!$ is a k-linear functor.

K7.2. Exact categories with enough projectives or/and injectives. An exact category $(C_X, \mathcal{E}_X)$ has *enough* projectives (resp. enough injectives) if for every object M of C_X, there exists a deflation $P \longrightarrow M$ (resp. an inflation $M \longrightarrow P$), where P is a projective (resp. injective) object of $(C_X, \mathcal{E}_X)$.

K7.2.1. Proposition. *Let $(C_X, \mathcal{E}_X) \xrightarrow{f_*} (C_Y, \mathcal{E}_Y)$ be an 'exact' functor which reflects inflations and has a right adjoint, $C_Y \xrightarrow{f^!} C_X$ (resp. a left adjoint). Suppose that $\mathcal{E}_Y$ consists of split sequences. Then $(C_X, \mathcal{E}_X)$ has enough injective (resp. projective) objects.*

Proof. If $\mathcal{E}_Y$ consists of split sequences, then every object of C_Y is injective (and projective). Therefore, by K7.1, every object $f^!(M)$, $M \in ObC_Y$, is $\mathcal{E}_X$-injective. For every $M \in ObC_Y$, the adjunciton arrow $M \xrightarrow{\eta(M)} f^!f_*(M)$ belongs, by hypothesis, to the class $\mathfrak{M}_X$ of inflations of $(C_X, \mathcal{E}_X)$, because $f_*(\eta(M))$ is a split monomorphism. ∎

K8. Suspended and cosuspended categories. Suspended categories were introduced in [KeV]. In a sequel, we shall mostly use their dual version – *cosuspended* categories. They are defined as follows.

K8.1. Definitions. A *cosuspended k-linear category* is a triple $(C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ is an additive k-linear category, $\theta_{\mathfrak{X}}$ a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}}$, and $Tr^-_{\mathfrak{X}}$ is a class of sequences of the form

$$\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L \qquad (1)$$

called *triangles* and satisfying the following axioms:

(SP0) Every sequence of the form (1) isomorphic to a triangle is a triangle.

(SP1) For every $M \in ObC_{\mathfrak{X}}$, the sequence $0 \longrightarrow M \xrightarrow{id_M} M \longrightarrow 0$ is a triangle.

(SP2) If $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ is a triangle, then

$$\theta_{\mathfrak{X}}(M) \xrightarrow{-\theta_{\mathfrak{X}}(u)} \theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M$$

is a triangle.

(SP3) Given triangles $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ and $\theta_{\mathfrak{X}}(L') \xrightarrow{w'} N' \xrightarrow{v'} M' \xrightarrow{u'} L'$ and morphisms $L \xrightarrow{\alpha} L'$ and $M \xrightarrow{\beta} M'$ such that the square

$$\begin{array}{ccc} L & \xleftarrow{u} & M \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ L' & \xleftarrow{u'} & M' \end{array}$$

commutes, there exists a morphism $N \xrightarrow{\gamma} N'$ such that the diagram

$$\begin{array}{ccccccc} L & \xleftarrow{u} & M & \xleftarrow{v} & N & \xleftarrow{w} & \theta_{\mathfrak{X}}(L) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\theta_{\mathfrak{X}}(\alpha)} \\ L' & \xleftarrow{u'} & M' & \xleftarrow{v'} & N' & \xleftarrow{w'} & \theta_{\mathfrak{X}}(L') \end{array}$$

commutes.

(SP4) For every pair of morphisms $M \xrightarrow{u} L$ and $M' \xrightarrow{x} M$, there exists a commutative diagram

$$\begin{array}{ccccccc} L & \xleftarrow{u} & M & \xleftarrow{v} & N & \xleftarrow{w} & \theta_{\mathfrak{X}}(L) \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle x} & & \uparrow{\scriptstyle y} & & \uparrow{\scriptstyle id} \\ L & \xleftarrow{u'} & M' & \xleftarrow{v'} & N' & \xleftarrow{w'} & \theta_{\mathfrak{X}}(L) \\ & & {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle \theta_{\mathfrak{X}}(u)} \\ & & \widetilde{M} & \xleftarrow{id} & \widetilde{M} & \xleftarrow{r} & \theta_{\mathfrak{X}}(M) \\ & & {\scriptstyle r}\uparrow & & \uparrow & & \\ & & \theta_{\mathfrak{X}}(M) & \xleftarrow{\theta_{\mathfrak{X}}(v)} & \theta_{\mathfrak{X}}(N) & & \end{array}$$

whose two upper rows and two central columns are triangles.

K8.2. Suspended categories. A *suspended k-linear category* is defined dually; i.e. it is a triple $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^+_{\mathfrak{X}})$, where $C_{\mathfrak{X}}$ is an additive k-linear category, $\theta_{\mathfrak{X}}$ a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{X}}$, and $Tr^+_{\mathfrak{X}}$ is a class of sequences of the form

$$L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w} \theta_{\mathfrak{X}}(L) \tag{2}$$

such that the dual data is a cosuspended category.

K8.3. Triangulated categories and (co)suspended categories. A suspended category $\mathfrak{T}_+C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^+_{\mathfrak{X}})$ (resp. a cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$) is a triangulated category iff the translation functor $\theta_{\mathfrak{X}}$ is an auto-equivalence.

K8.4. Properties of cosuspended and suspended categories. The following properties of a cosuspended category $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$ follow directly from the axioms:

(a) Every morphism $M \xrightarrow{u} L$ of $C_{\mathfrak{X}}$ can be embedded into a triangle

$$\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L.$$

(b) For every triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, the sequence of representable functors

$$\ldots \longrightarrow C_{\mathfrak{X}}(-, \theta_{\mathfrak{X}}(L)) \xrightarrow{C_{\mathfrak{X}}(-,w)} C_{\mathfrak{X}}(-, N) \xrightarrow{C_{\mathfrak{X}}(-,v)} C_{\mathfrak{X}}(-, M) \xrightarrow{C_{\mathfrak{X}}(-,u)} C_{\mathfrak{X}}(-, L) \qquad (3)$$

is exact. In particular, the compositions $u \circ v$, $v \circ w$, $w \circ \theta_{\mathfrak{X}}(u)$ are zero morphisms.

(c) If the rows of the commutative diagram

$$\begin{array}{ccccccc} L & \xleftarrow{u} & M & \xleftarrow{v} & N & \xleftarrow{w} & \theta_{\mathfrak{X}}(L) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\theta_{\mathfrak{X}}(\alpha)} \\ L' & \xleftarrow{u'} & M' & \xleftarrow{v'} & N' & \xleftarrow{w'} & \theta_{\mathfrak{X}}(L') \end{array}$$

are triangles and the two left vertical arrows, α and β, are isomorphisms, then γ is an isomorphism too (see the axiom K8.1 (SP3)).

(d) Direct sum of triangles is a triangle.

(e) If $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, is a triangle, then the sequence

$$\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \longrightarrow 0$$

is split exact iff $u = 0$.

(f) For an arbitrary choice of triangles starting with u, x and xu in the diagram K8.1 (SP4), there are morphisms y and t such that the second central column is a triangle and the diagram commutes.

If $\mathfrak{T}_-C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$ is a triangulated category, i.e. the translation functor $\theta_{\mathfrak{X}}$ is an auto-equivalence, then, in addition, we have the following properties:

(g) A diagram $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, is a triangle if (by (SP2), iff)

$$\theta_{\mathfrak{X}}(M) \xrightarrow{-\theta_{\mathfrak{X}}(w)} \theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M$$

is a triangle.

(h) Given triangles $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ and $\theta_{\mathfrak{X}}(L') \xrightarrow{w'} N' \xrightarrow{v'} M' \xrightarrow{u'} L'$ and morphisms $M \xrightarrow{\beta} M'$ and $N \xrightarrow{\gamma} N'$ such that the square

$$\begin{array}{ccc} N & \xrightarrow{v} & M \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ N' & \xrightarrow{v'} & M' \end{array}$$

commutes, there exists a morphism $L \xrightarrow{\alpha} L'$ such that the diagram

$$\begin{array}{ccccccc} L & \xleftarrow{u} & M & \xleftarrow{v} & N & \xleftarrow{w} & \theta_{\mathfrak{X}}(L) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\theta_{\mathfrak{X}}(\alpha)} \\ L' & \xleftarrow{u'} & M' & \xleftarrow{v'} & N' & \xleftarrow{w'} & \theta_{\mathfrak{X}}(L') \end{array}$$

commutes.

(i) For every triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$, the sequence of corepresentable functors

$$\ldots \longleftarrow C_{\mathfrak{X}}(\theta_{\mathfrak{X}}(L), -) \xleftarrow{C_{\mathfrak{X}}(w,-)} C_{\mathfrak{X}}(N, -) \xleftarrow{C_{\mathfrak{X}}(v,-)} C_{\mathfrak{X}}(M, -) \xleftarrow{C_{\mathfrak{X}}(u,-)} C_{\mathfrak{X}}(L, -) \qquad (3^o)$$

is exact.

K8.5. Triangle functors. Let $\mathfrak{T}_- C_{\mathfrak{X}} = (C_{\mathfrak{X}}, \theta_{\mathfrak{X}}, Tr^-_{\mathfrak{X}})$ and $\mathfrak{T}_- C_{\mathfrak{Y}} = (C_{\mathfrak{Y}}, \theta_{\mathfrak{Y}}, Tr^-_{\mathfrak{Y}})$ be cosuspended k-linear categories. A *triangle* k-linear functor from $\mathfrak{T}_- C_{\mathfrak{X}}$ to $\mathfrak{T}_- C_{\mathfrak{Y}}$ is a pair (F, ϕ), where F is a k-linear functor $C_{\mathfrak{X}} \longrightarrow C_{\mathfrak{Y}}$ and ϕ is a functor morphism $\theta_{\mathfrak{Y}} \circ F \longrightarrow F \circ \theta_{\mathfrak{X}}$ such that for every triangle $\theta_{\mathfrak{X}}(L) \xrightarrow{w} N \xrightarrow{v} M \xrightarrow{u} L$ of $\mathfrak{T}_- C_{\mathfrak{X}}$, the sequence

$$\theta_{\mathfrak{Y}}(F(L)) \xrightarrow{F(w)\phi(L)} F(N) \xrightarrow{F(v)} F(M) \xrightarrow{F(u)} F(L)$$

is a triangle of $\mathfrak{T}_- C_{\mathfrak{Y}}$. It follows from this condition and the property K8.4(b) (applied to the case $M = 0$) that ϕ is invertible. The composition of triangle functors is defined naturally: $(G, \psi) \circ (F, \phi) = (G \circ F, G\phi \circ \psi F)$.

If (F, ϕ) and (F', ϕ') are triangle functors from $\mathfrak{T}_- C_{\mathfrak{X}}$ to $\mathfrak{T}_- C_{\mathfrak{Y}}$. A morphism from (F, ϕ) to (F', ϕ') is given by a functor morphism $F \xrightarrow{\lambda} F'$ such that the diagram

$$\begin{array}{ccc} \theta_{\mathfrak{Y}} \circ F & \xrightarrow{\phi} & F \circ \theta_{\mathfrak{X}} \\ {\scriptstyle\theta_{\mathfrak{Y}}\lambda}\downarrow & & \downarrow{\scriptstyle\lambda\theta_{\mathfrak{X}}} \\ \theta_{\mathfrak{Y}} \circ F' & \xrightarrow{\phi'} & F' \circ \theta_{\mathfrak{X}} \end{array}$$

commutes. The composition is the compsition of the functor morphisms.

Altogether gives the definition of a large bicategory $\widetilde{\mathfrak{Tr}}_k^-$ formed by cosuspended k-linear categories, triangle k-linear functors as 1-morphisms and morphisms between them as 2-morphisms. Restricting to svelte cosuspended categories, we obtain the bicategory $\mathfrak{Tr}_k^-$.

We denote by $\widetilde{\mathfrak{Tr}}_k$ (resp. by $\mathfrak{Tr}_k$) the full subbicategory of $\widetilde{\mathfrak{Tr}}_k^-$ whose objects are triangulated (resp. svelte triangulated) categories.

Finally, dualizing (i.e. inverting all arrows in the constructions above), we obtain the large bicategory $\widetilde{\mathfrak{Tr}}_k^+$ of suspended categories and triangular functors and its subbicategory $\mathfrak{Tr}_k^+$ whose objects are svelte suspended categories. Thus, we have a diagram of natural full embeddings

$$\begin{array}{ccccc} \widetilde{\mathfrak{Tr}}_k^+ & \longleftarrow & \widetilde{\mathfrak{Tr}}_k & \longrightarrow & \widetilde{\mathfrak{Tr}}_k^- \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{Tr}_k^+ & \longleftarrow & \mathfrak{Tr}_k & \longrightarrow & \mathfrak{Tr}_k^- \end{array}$$

K8.6. Triangle equivalences. A triangle k-linear functor $\mathfrak{T}_- C_{\mathfrak{X}} \xrightarrow{(F,\phi)} \mathfrak{T}_- C_{\mathfrak{Y}}$ is called a *triangle equivalence* if there exists a triangle functor $\mathfrak{T}_- C_{\mathfrak{Y}} \xrightarrow{(G.\psi)} \mathfrak{T}_- C_{\mathfrak{X}}$ such that the compositions $(F,\phi)\circ(G,\psi)$ and $(G,\psi)\circ(F,\phi)$ are isomorphic to respective identical triangle functors.

It follows from K7.1.1 that the quasi-inverse triangle functor (G,ψ) is k-linear.

K8.6.1. Lemma [Ke1]. *A triangle k-linear functor (F,ϕ) is a triangle equivalence iff F is an equivalence of the underlying categories.*

K9. Stable and costable categories of an exact category. Let C_X be a k-linear category and $\mathcal{B}$ its full subcategory. The class $\mathcal{J}_{\mathcal{B}}$ of all arrows of C_X which factor through some objects of $\mathcal{B}$ is an ideal in $HomC_X$. We denote by $\mathcal{B}\backslash C_X$, or by $C_{\mathcal{B}\backslash X}$ the category having same objects as C_X; its morphisms are classes of morphisms of C_X modulo the ideal $\mathcal{J}_{\mathcal{B}}$, that is two morphisms with the same source and target are equivalent if their difference belongs to the ideal $\mathcal{J}_{\mathcal{B}}$.

We are particularly interested in this construction when $(C_X,\mathcal{E}_X)$ is an exact k-linear category and $\mathcal{B}$ is the fully exact subcategory of C_X generated by $\mathcal{E}_X$-projective or $\mathcal{E}_X$-injective objects of $(C_X,\mathcal{E}_X)$. In the first case, we denote the category $\mathcal{B}\backslash C_X$ by $C_{\mathfrak{S}_- X}$ and will call it the *costable* category of $(C_X,\mathcal{E}_X)$. In the second case, the notation is $C_{\mathfrak{S}_+ X}$ and the name of this category is the *stable* category of $(C_X,\mathcal{E}_X)$.

K9.1. Example. Let C_X be an additive k-linear category endowed with the smallest exact structure $\mathcal{E}_X^{spl}$ (cf. K2.1). Then the correponding costable category is trivial: all its objects are isomorphic to zero.

K9.2. Exact categories with enough projectives and their costable categories. Let $(C_X,\mathcal{E}_X)$ be an exact k-linear category with *enough projectives*; i.e. for each object M of C_X, there exists a deflation $P \longrightarrow M$, where P is a projective object. Then the costable category $C_{\mathfrak{S}_- X}$ of $(C_X,\mathcal{E}_X)$ has a natural structure of a cosuspended k-linear category defined as follows. The endofunctor $\theta_{\mathfrak{S}_- X}$ assigns to an object M the (image

in $C_{\mathfrak{S}\_X}$ of) the kernel of a deflation $P \longrightarrow M$, where P is a projective object. For any morphism $L \xrightarrow{f} M$, the morphism $\theta_{\mathfrak{S}\_X}(f)$ is the image of the morphism h in the commutative diagram

$$\begin{array}{ccccc} \theta_{\mathfrak{S}\_X}(L) & \xrightarrow{\mathfrak{j}} & P_L & \xrightarrow{\mathfrak{e}} & L \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \theta_{\mathfrak{S}\_X}(M) & \xrightarrow{\mathfrak{j}'} & P_M & \xrightarrow{\mathfrak{e}'} & M \end{array}$$

A standard argument shows that objects $\theta_{\mathfrak{S}\_X}(L)$ are determined uniquely up to isomorphism and the morphism $\theta_{\mathfrak{S}\_X}(f)$ is uniquely determined by the choice of the objects $\theta_{\mathfrak{S}\_X}(L)$ and $\theta_{\mathfrak{S}\_X}(M)$.

With each conflation $N \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} L$ of $(C_X, \mathcal{E}_X)$, it is associated a sequence

$$\theta_{\mathfrak{S}\_X}(L) \xrightarrow{\partial} N \xrightarrow{\bar{\mathfrak{j}}} M \xrightarrow{\bar{\mathfrak{e}}} L$$

called a *standard triangle* and determined by a commutative diagram

$$\begin{array}{ccccc} \theta_{\mathfrak{S}\_X}(L) & \xrightarrow{\widetilde{\mathfrak{j}}} & P_L & \xrightarrow{\widetilde{\mathfrak{e}}} & L \\ {\scriptstyle \widetilde{\partial}}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle id_L} \\ N & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & L \end{array}$$

The morphism g here exists thanks to the projectivity of P_L. The *connecting morphism* $\theta_{\mathfrak{S}\_X}(L) \xrightarrow{\partial} N$ is, by definition, the image of $\widetilde{\partial}$.

Triangles are defined as sequences of the form $\theta_{\mathfrak{S}\_X}(L') \xrightarrow{\partial'} N' \xrightarrow{\mathfrak{j}'} M' \xrightarrow{\mathfrak{e}'} L'$ which are isomorphic to a standard triangle.

K9.2.1. Proposition ([KeV]). *For any exact k-linear category $(C_X, \mathcal{E}_X)$ with enough projectives, the triple $\mathfrak{T}\_C_{\mathfrak{S}\_X} = (C_{\mathfrak{S}\_X}, \theta_{\mathfrak{S}\_X}, \mathfrak{Tr}_{\mathfrak{S}\_X})$ constructed above is a cosuspended k-linear category.*

If $(C_X, \mathcal{E}_X)$ is an exact category with enough injectives, then the dual construction provides a structure of a suspended category on the stable category $C_{\mathfrak{S}_+X}$ of $(C_X, \mathcal{E}_X)$.

K9.2.2. The case of Frobenius categories. Recall that an exact category $(C_X, \mathcal{E}_X)$ is called a Frobenius category, if it has enough injectives and projectives and projectives coincide with injectives.

K9.2.1. Proposition. *If $(C_X, \mathcal{E}_X)$ is a Frobenius category, then its costable cosuspended category $\mathfrak{T}\_C_{\mathfrak{S}\_X}$ and (therefore) the stable suspended category $\mathfrak{T}_+C_{\mathfrak{S}_+X}$ are triangulated, and are triangular equivalent one to another.*

Proof. It is easy to check that if $(C_X, \mathcal{E}_X)$ is a Frobenius category, then the translation functor $\theta_{\mathfrak{S}\_X}$ is an auto-equivalence of the category $C_{\mathfrak{S}\_X}$. The rest follows from this fact. Details are left to the reader. ∎

K9.3. Proposition. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories with enough projectives. Every 'exact' k-linear functor $(C_X, \mathcal{E}_X) \xrightarrow{f^*} (C_Y, \mathcal{E}_Y)$ which maps projectives to projectives induces a triangle k-linear functor $\mathfrak{T}_- C_{\mathfrak{S}_- X} \xrightarrow{\mathfrak{T}_- f^*} \mathfrak{T}_- C_{\mathfrak{S}_- Y}$ between the corresponding costable cosuspended categories.*

Proof. The argument is left to the reader. ∎

K9.3.1. Corollary. *Let $(C_X, \mathcal{E}_X)$ and $(C_Y, \mathcal{E}_Y)$ be exact k-linear categories with enough projectives and*

$$(C_X, \mathcal{E}_X) \xrightarrow{f^*} (C_Y, \mathcal{E}_Y) \xrightarrow{f_*} (C_X, \mathcal{E}_X)$$

a pair of 'exact' functors such that f^* is k-linear and a left adjoint of f_*. Then the functor f^* induces a triangle k-linear functor $\mathfrak{T}_- C_{\mathfrak{S}_- X} \xrightarrow{\mathfrak{T}_- f^*} \mathfrak{T}_- C_{\mathfrak{S}_- Y}$ between the corresponding costable cosuspended categories.

Proof. By K7.1, the functor f^* maps projective objects of $(C_X, \mathcal{E}_X)$ to projective objects of $(C_Y, \mathcal{E}_Y)$. The assertion follows now from K9.3. ∎

K10. Complements.

K10.1. Admissible morphisms. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category with the class of inflations $\mathfrak{M}_X$ and the class of deflations $\mathfrak{E}_X$. We call arrows of $\mathfrak{M}_X \circ \mathfrak{E}_X$ *admissible*. In general, the class of admissible morphisms is not closed under composition.

K10.1.1. Lemma. *Suppose that for any pair of arrows $L \xrightarrow{\mathfrak{j}} M \xleftarrow{\mathfrak{j}'} \widetilde{L}$ of $\mathfrak{M}_X$, there exists a cartesian square*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathfrak{j}''} & \widetilde{L} \\ {\scriptstyle \widetilde{\mathfrak{j}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}} \\ L & \xrightarrow{\mathfrak{j}'} & M \end{array} \tag{1}$$

Then the class of admissible arrows is closed under composition.

Proof. (i) Notice that if (1) is a cartesian square with $\mathfrak{j} \in \mathfrak{M}_X \ni \mathfrak{j}'$, then the remaining two arrows, $\mathfrak{j}''$ and $\widetilde{\mathfrak{j}}$, belong to $\mathfrak{M}_X$ too. In fact, the arrows $\mathfrak{j}''$ and $\widetilde{\mathfrak{j}}$ are (strict) monomorphisms in any category. The Gabriel-Quillen embedding, preserves cartesian squares, maps arrows of $\mathfrak{M}_X$ to monomorphisms, and reflects monomorphisms to arrows of $\mathfrak{M}_X$.

(ii) It suffices to show that $\mathfrak{E}_X \circ \mathfrak{M}_X \subseteq \mathfrak{M}_X \circ \mathfrak{E}_X$. Let $L \xrightarrow{\mathfrak{j}} M$ be a morphism of $\mathfrak{M}_X$ and $M \xrightarrow{\mathfrak{e}} N$ a morphism of $\mathfrak{E}_X$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ker(\widetilde{\mathfrak{e}}) & \xrightarrow{\widetilde{\mathfrak{j}}} & L & \xrightarrow{\widetilde{\mathfrak{e}}} & M' & \longrightarrow & 0 \\ & & {\scriptstyle \mathfrak{j}''}\downarrow & & {\scriptstyle \mathfrak{j}}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}'} & & \\ 0 & \longrightarrow & Ker(\mathfrak{e}) & \xrightarrow{\mathfrak{j}_{\mathfrak{e}}} & M & \xrightarrow{\mathfrak{e}} & N & \longrightarrow & 0 \end{array} \tag{2}$$

with exact rows. Its left square is cartesian and formed by arrows of $\mathfrak{M}_X$. The morphism $L \xrightarrow{\widetilde{\mathfrak{e}}} M'$ is a cokernel of $\widetilde{\mathfrak{j}}$; in particular, belongs to $\mathfrak{E}_X$. The existence of the right vertical arrow in (2), $M' \xrightarrow{\mathfrak{j}'} N$, follows from the exactness of the rows. Applying the Gabriel-Quillen embedding, $\mathfrak{j}^*_X$. to the diagram (2), we reduce to the case of an abelian category with the canonical exact structure. One can see that $\mathfrak{j}^*_X(\mathfrak{j}')$ is a monomorphism. Therefore, $\mathfrak{j}'$ is an arrow of $\mathfrak{M}_X$. Thus, we obtain the equality $\mathfrak{e} \circ \mathfrak{j} = \mathfrak{j}' \circ \widetilde{\mathfrak{e}}$, where $\mathfrak{j}' \in \mathfrak{M}_X$ and $\widetilde{\mathfrak{e}} \in \mathfrak{E}_X$. ∎

K10.1.2. Remarks. (a) If the condition of K10.1.1 holds, then the dual condition holds for deflations. In fact, let $N' \xleftarrow{\mathfrak{e}'} M \xrightarrow{\mathfrak{e}} N$ be a pair of arrows of $\mathfrak{E}_X$. So that we have exact sequences $0 \longrightarrow L' \xrightarrow{\mathfrak{j}'} M \xrightarrow{\mathfrak{e}'} N' \longrightarrow 0$ and $0 \longrightarrow L \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} N \longrightarrow 0$. By hypothesis (and the part (i) of the argument above), there is a cartesian square

$$\begin{array}{ccc} \widetilde{L} & \xrightarrow{\widetilde{\mathfrak{j}}} & L' \\ {\scriptstyle \mathfrak{j}''}\downarrow & & \downarrow{\scriptstyle \mathfrak{j}'} \\ L & \xrightarrow{\mathfrak{j}} & M \end{array}$$

with all arrows from $\mathfrak{M}_X$. Since $\mathfrak{j} \circ \mathfrak{j}'' \in \mathfrak{M}_X$, there is an exact sequence

$$0 \longrightarrow \widetilde{L} \xrightarrow{\mathfrak{j}\circ\mathfrak{j}''} M \xrightarrow{\mathfrak{e}_1} \widetilde{N} \longrightarrow 0.$$

By the universal properties of cokernels, there exists a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{e}} & N \\ {\scriptstyle \mathfrak{e}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}''} \\ N' & \xrightarrow{\widetilde{\mathfrak{e}}} & \widetilde{N} \end{array} \tag{3}$$

with arrows $\mathfrak{e}''$ and $\widetilde{\mathfrak{e}}$ uniquely determined by the equalities $\mathfrak{e}'' \circ \mathfrak{e} = \mathfrak{e}_1 = \widetilde{\mathfrak{e}} \circ \mathfrak{e}'$. Since $\mathfrak{e}_1 \in \mathfrak{E}_X$, it follows, by a property of exact categories, that $\mathfrak{e}''$ and $\widetilde{\mathfrak{e}}$ are arrows of $\mathfrak{E}_X$. It is easy to see that the square (3) is cocartesian.

(b) The assumption of K10.1.1 holds for exact categories associated with quasi-abelian categories (discussed shortly in K10.2 below), because in quasi-abelian categories all fibred products and coproducts exist, $\mathfrak{M}_X$ is the class of all strict monomorphisms, and a pull-back of a strict monomorphism is a strict monomorphism.

K10.1.3. Proposition. *Suppose the condition of K10.1.1 holds. Then the class of all admissible morphisms of the exact category $(C_X, \mathcal{E}_X)$ forms the largest abelian exact subcategory, $C_{X_{\mathfrak{a}}(\mathcal{E})}$, of $(C_X, \mathcal{E}_X)$.*

Proof. Let $M \underset{h}{\overset{g}{\rightrightarrows}} N$ be a pair of morphisms of C_X. Their sum is the composition of the arrows

$$M \xrightarrow{\Delta_M} M \oplus M \xrightarrow{g\oplus h} N \oplus N \xrightarrow{+_N} N, \tag{4}$$

where Δ_M is the diagonal morphism and $+_N$ is the *codiagonal* morphism. Since the composition of Δ_M and any of projections $M \oplus M \longrightarrow M$ is the identical morphism, $\Delta_M \in \mathfrak{M}$. Dually, $+_N$ belongs to $\mathfrak{E}_X$. If both g and h are admissible arrows, then $g \oplus h$ is admissible. Therefore, in this case, $g + h$ is the composition of admissible morphisms. Under the condition of K10.1.1, the composition of admissible morphisms is an admissible morphism. The subcategory $C_{X_{\mathfrak{a}}(\mathcal{E})}$ has same objects as C_X. Therefore, since the category C_X is additive, $C_{X_{\mathfrak{a}}(\mathcal{E})}$ is additive too. It is quasi-abelian, because every admissible morphism has a kernel and a cokernel. An admissible arrow is a monomorphism iff it belongs to $\mathfrak{M}_X$. Since all arrows of $\mathfrak{M}_X$ are strict monomorphisms, an inflation is an epimorphism iff it is an isomorphism. Altogether means that $C_{X_{\mathfrak{a}}(\mathcal{E})}$ is an abelian subcategory. The exact structure $\mathcal{E}_X$ induces the canonical exact structure on the subcategory $C_{X_{\mathfrak{a}}(\mathcal{E})}$. It follows that any other abelian exact subcategory of $(C_X, \mathcal{E}_X)$ is formed by admissible arrows, i.e. it is contained in $C_{X_{\mathfrak{a}}(\mathcal{E})}$. ∎

K10.1.4. Example: the category of torsion-free objects. Let $(C_X, \mathcal{E}_X)$ be an exact k-linear category. Let $\mathcal{T}$ be a full subcategory of C_X such that if $M' \longrightarrow M$ is an inflation and $M \in Ob\mathcal{T}$, then M' is an object of $\mathcal{T}$ too. In particular, the subcategory $\mathcal{T}$ is strictly full. Let $C_{X_{\mathcal{T}}}$ denote the full subcategory of C_X generated by all *$\mathcal{T}$-torsion free* objects; i.e. objects N such that the only inflation $L \longrightarrow N$ with $L \in Ob\mathcal{T}$ is zero.

K10.1.4.1. Lemma. *Suppose that for any pair $L' \longrightarrow L \longleftarrow L''$ of inflations of $(C_X, \mathcal{E}_X)$, there exists a pull-back $L' \times_L L''$. Then the subcategory $C_{X_{\mathcal{T}}}$ of $\mathcal{T}$-torsion free objects is closed under extensions. In particular, $C_{X_{\mathcal{T}}}$ is an exact subcategory of $(C_X, \mathcal{E}_X)$.*

Proof. Let $M' \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} M''$ be a conflation with $M' \in ObC_{X_{\mathcal{T}}}$. Let $L \longrightarrow M$ be an inflation with $L \in Ob\mathcal{T}$. Then we have a commutative diagram

$$\begin{array}{ccccc} L' & \xrightarrow{\mathfrak{j}'} & L & \xrightarrow{\mathfrak{e}'} & L'' \\ \downarrow & & \downarrow & & \downarrow \\ M' & \xrightarrow{\mathfrak{j}} & M & \xrightarrow{\mathfrak{e}} & M'' \end{array} \tag{1}$$

whose left square is cartesian and the both rows are conflations.

In fact, by K10.1.2(a), all arrows of the left square are inflations. The arrow $\mathfrak{e}'$ is the cokernel of $\mathfrak{j}'$. It follows from the argument of K10.1.3 (or direct application of the Gabriel-Quillen embedding and the corresponding fact for abelian categories) that the remaining (right) vertical arrow is an inflation too. Since $L' \in Ob\mathcal{T}$ and M' is $\mathcal{T}$-torsion free, it follows that $L' = 0$ therefore $\mathfrak{e}'$ is an isomorphism. Therefore, if M'' is also $\mathcal{T}$-torsion free, then $L'' = 0$ which implies that $L = 0$. This shows that if the ends of a conflation are $\mathcal{T}$-torsion free, same holds for the middle. ∎

K10.2. Quasi-abelian categories. A *quasi-abelian category* is an additive category C_X with kernels and cokernels and such that every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism.

It follows from definitions that the pair $(C_X, \mathcal{E}_s)$, where $\mathcal{E}_s$ is the class of all short exact sequences in C_X, is an exact category.

Every abelian category is quasi-abelian.

K10.2.1. Proposition. *Let C_X be a quasi-abelian category. There exist two canonical fully faithful functors $C_{\mathfrak{L}X} \hookleftarrow C_X \hookrightarrow C_{\mathfrak{R}X}$ of C_X into abelian categories which preserve and reflect exactness. Moreover, the category C_X is stable under extensions in these embeddings. The category C_X is closed under taking subobjects in $C_{\mathfrak{L}X}$ and every object of $C_{\mathfrak{L}X}$ is a quotient of an object of C_X. Dually, C_X is closed under taking quotients in $C_{\mathfrak{R}X}$ and every object of $C_{\mathfrak{R}X}$ is a subobject of an object of C_X.*

Proof. See [Sch, 1.2.35, 1.2.31]. ∎

K10.2.2. Quasi-abelian categories and torsion pairs. Let C_X be a quasi-abelian category, and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in C_X. That is $\mathcal{T}$ and $\mathcal{F}$ are full subcategories of C_X such that $\mathcal{F} \subseteq \mathcal{T}^{\perp}$ and $C_X = \mathcal{T} \bullet \mathcal{F}$. The latter means that every object M of C_X fits into an exact sequence

$$0 \longrightarrow M' \xrightarrow{\mathfrak{j}} M \xrightarrow{\mathfrak{e}} M'' \longrightarrow 0 \tag{1}$$

with $M' \in \mathcal{T}$ and $M'' \in Ob\mathcal{F}$. Notice that the exact sequence (1) is unique up to isomorphism. In fact, if $N \xrightarrow{f} M$ is a morphism and $N \in Ob\mathcal{T}$, then $\mathfrak{e} \circ f = 0$, hence f factors uniquely through the monomorphism $M' \xrightarrow{\mathfrak{j}} M$.

This implies, in particular, that $\mathcal{T}$ is closed under taking quotients (in C_X) and, dually, $\mathcal{F}$ is closed under taking strict subobjects.

The assignments $M \longmapsto M'$ and $M \longmapsto M''$ in (1) extend to functors $C_X \xrightarrow{\mathfrak{j}_{\mathcal{T}*}} \mathcal{T}$ and $C_X \xrightarrow{\mathfrak{j}_{\mathcal{T}!}} \mathcal{F}$ which are resp. a right and a left adjoint to the inclusion functors $\mathcal{T} \xrightarrow{\mathfrak{j}^*_{\mathcal{T}}} C_X$ and $\mathcal{F} \xrightarrow{\mathfrak{j}^*_{\mathcal{F}}} C_X$. By [GZ, 1], the categories $\mathcal{T}$ and $\mathcal{F}$ have all types of limits and colimits which exist in the category C_X given by the formulas

$$\lim \mathfrak{D} = \mathfrak{j}_{\mathcal{T}*}(\lim(\mathfrak{j}^*_{\mathcal{T}} \circ \mathfrak{D})) \quad \text{and} \quad colim\mathfrak{D} = \mathfrak{j}_{\mathcal{T}*}(colim(\mathfrak{j}^*_{\mathcal{T}} \circ \mathfrak{D})) \tag{2}$$

for any small diagram $\mathcal{D} \xrightarrow{\mathfrak{D}} \mathcal{T}$. In particular, $\mathcal{T}$ has kernels and cokernels given by $Coker_{\mathcal{T}} = Coker_{C_X}$ and $Ker_{\mathcal{T}} = \mathfrak{j}_{\mathcal{T}*}(Ker_{\mathcal{T}})$. Similarly for $\mathcal{F}$.

A torsion pair $(\mathcal{T}, \mathcal{F})$ in C_X is called *tilting* if every object of C_X is a subobject of an object of $\mathcal{T}$. Dually, $(\mathcal{T}, \mathcal{F})$ is called a *cotilting* torsion pair if every object of C_X is a quotient of an object of $\mathcal{F}$.

K10.2.2.1. Proposition. *Let C_X be an additive category. The following conditions are equivalent.*

(a) C_X is quasi-abelian.

(b) There exists a tilting torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category C_Y such that $\mathcal{T}$ is equivalent to C_X.

(c) There exists a cotilting torsion pair $(\mathcal{T}', \mathcal{F}')$ in an abelian category C_W such that $\mathcal{F}'$ is equivalent to C_X.

Proof. It follows from K10.2.1 that $C_Y = C_{\mathfrak{R}X}$ and $C_Y = C_{\mathfrak{L}X}$. See details of the proof in [BOVdB, B.3]. ∎

References

[Ba] H. Bass, Algebraic K-theory, Benjamin 1968

[BOVdB] A. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, math.AG/0204218 v2

[CE] H. Cartan and S. Eilenberg, Homological algebra, Princeton University Press, 1956

[Gab] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France, 90 (1962), 323-449

[GZ] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Springer Verlag, Berlin-Heidelberg-New York, 1967

[Gr] A. Grothendieck, Sur quelques points d'algèbre homologique. Tohuku Math. J., Ser. II, 9, 120–221 (1957)

[GrD] A. Grothendieck, J.A. Dieudonné, Eléments de Géométrie Algébrique, Springer Verlag, New York - Heidelberg - Berlin, 1971

[Ke1] B. Keller, Derived categories and their uses, preprint, 1994.

[Ke2] B. Keller, Chain complexes and stable categories, Manus. Math. 67 (1990), 379–417.

[KeV] B. Keller, D. Vossieck, Sous les catégories dérivées, C. R. Acad. Sci. Paris 305 (1987), 225–228.

[KeV1] B. Keller, D. Vossieck, Aisles in derived categories, Bull. Soc. Math. Belg. Sér. A 40 (1988), no. 2, 239–253.

[KR] M. Kontsevich, A.L. Rosenberg, Noncommutative spaces and flat descent, preprint MPIM, 2004(36), 108 pp.

[ML] S. Mac-Lane, Categories for the working mathematicians, Springer - Verlag; New York - Heidelberg - Berlin (1971)

[MLM] S. Mac-Lane, L. Moerdijk, Sheaves in Geometry and Logic, Springer - Verlag; New York - Heidelberg - Berlin (1992)

[Q] D. Quillen, Higher algebraic K-theory, LNM 341, 85–147 (1973)

[R] A.L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras, Kluwer Academic Publishers, Mathematics and Its Applications, v.330 (1995), 328 pages.

[R1] A.L. Rosenberg, Noncommutative spaces and schemes, preprint MPIM, 1999

[R2] A.L. Rosenberg, Trieste Lectures, to appear in Lecture Notes of the School on Algebraic K-Theory and Applications, ICTP, Italy, 92 pp.

[R3] A.L. Rosenberg, Topics in noncommutative algebraic geometry, homological algebra and K-theory, preprint MPIM, 2008(57), 105 pp

[R4] A.L. Rosenberg, Geometry of right exact 'spaces', preprint MPIM, 2008(85), 66 pp.

[Ve1] J.–L. Verdier, Catégories dérivées, Séminaire de Géométrie Algébrique 4 1/2, Cohomologie Étale, LNM 569, pp. 262–311, Springer, 1977

[Ve2] J.–L. Verdier, Des catégories dérivées des catégories abéliennes, Astérisque, v.239, 1996