On non toric fibrations on lagrangian tori of toric Fano varieties

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Abstract

It is presented a construction of a fibration on lagrangian tori of toric Fano varieties, based on considerations of linear subsystems of divisors of different degrees, which are invariant under the Hamiltonian action of certain distinguished functions – symbols. It is shown that known examples of fibrations (the Clifford fibration, the example of D. Auroux) are particular cases of the construction. As an application one constructs non toric lagrangian fibrations of two dimensional quadric and the projective space.

Introduction

The lagrangian geometry of algebraic varieties is novadays an important and interesting problem. An abstract interest to the question which submanifolds are realized as lagrangian ones with respect to an appropriate Kahler form of the Hodge type is completed by the specified interest to the same problem in connection with recently proposed new approaches to Mirror Symmetry and Geometric Quantization. A programme, proposed by M. Kontsevich and called Homological Mirror Symmetry suggests certain duality of the derived categories of sheaves and of lagrangian submanifolds on algebraic varieties – partners (see [1], [2]). On the other hand, a lagrangian approach to Geometric Quantization, proposed by A. Tyurin and developed in [3], [4], requires the solution of the same problem, being applied to any specified algebraic variety.

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Our interest to the lagrangian approach in Geometric Quantization imposes the restriction to the case of orientable submanifolds only, therefore we will speak in what follows about lagrangian tori and their classes modulo Hamiltonian istopy only. This subject is presented today by a number of observations and examples. The basic example is given for the case of toric Fano varieties; for $\mathbb{CP}^2$ it is the classical Clifford fibration with the degeneration of three lines, whose essence is explained in terms of toric geometry. Further, in paper [5] one presents first non toric example for $\mathbb{CP}^2$ with degeneration on a reducible cubic consists of a conic and a line. Beside of this one refers to a paper of Chekanov and Schlenck about lagrangian tori on $\mathbb{CP}^2$, which is forthcoming but is not published yet.

Intensive studies of the Auroux construction in terms of Hamiltonian systems lead the authors to another observation which makes it possible to construct many new examples of lagrangian fibrations of the projective plane and which can be extended to the higher dimensions and further to the case of appropriate Fano varieties. The observation itself is rather simple: let one has on $\mathbb{CP}^2$ a Morse real function f and a pencil of curves degree k, such that the Hamiltonian vector field X_f preserves each curve from the pencil. Then the chocie of a Morse function on the projective line $\mathbb{CP}^1$, parameterizing the pencil, induces a fibration on $\mathbb{CP}^2$, whose generic fiber is lagrangian torus. This fibration *a priori* contains singular tori which corresponds to the isolated critical points of f which do not belong to the base set of the pencil. The extension to higher dimensions follows the same scheme: let one has on $\mathbb{CP}^n$ certain $n-1$ Morse functions $\{f_1, ..., f_{n-1}\}$ in involution and a one dimensional linear subsystem (pencil) in the complete linear system $|kH|$ of divisors of degree k, such that the Hamiltonian action of each f_i preserves each element of the linear subsystem. Then the choice of a Morse function h on $\mathbb{CP}^1$, parameterizing the pencil, induces a fibration on $\mathbb{CP}^n$ without hypersurfaces which corresponds to critical points of function h, with lagrangian torus as generic fiber. If, moreover, this function h has only two critical points and the union of these hypersurfaces contains all $n-2$ - planes which form the degeneration set of the polyvector field $X_{f_1} \wedge ... \wedge X_{f_{n-1}}$, then the complement to these hypersurfaces in $\mathbb{CP}^n$ is completely fibered on lagrangian tori.

The simplicity of our observation is compensated by difficulties in the problem of searching desired Morse functions of the projective space, but it is simplified if one restricts by real functions of special type — the Berezin symbols. The symbols are the real functions whose Hamiltonian vector fields preserve not just the symplectic structure but whole the Kahler structure. The symbols were introduced by F. Berezin in the quantization problem of Kahler manifolds as analogies of quantum observables. In the case of $\mathbb{CP}^n$ each symbol is given by a self adjoint operator on $\mathbb{C}^{n+1}$, and a set of n commuting functionally independent symbols is precisely corresponds to the choice of a real subtorus in the complex one acting on $\mathbb{CP}^n$. In the framework of toric geometry moment maps are symbols, and at each point below one could replace "symbol" by "moment map", but in view of further applications for non toric manifolds we keep the first notion. Restricting ourself by the considerations of symbols the real Morse functions we simultaneously simplify the searching problem for linear subsystems which consist of invariant divisors, and the observation itself in the framework of toric geometry looks as follows: one chooses real $n-1$ - and one dimensional subtori in the complex n -torus and glues these nontoricaly.

The examples, presented in sections 2 and 3 below, possess a mutual property — we are looking for fibrations on lagrangian tori of a Fano variety with a canceled element from the anticanonical system. This property is dictated by an interesting conjucture, proposed by D. Auroux in [6]. Namely, one expects that if a divisor from the anticanonical system, so a cubic curve, is removed from the projective plane then the resting part can be fibered on lagrangian tori which are special. Our examples follows the idea of this conjecture. And although our method of constructing lagrangian submanifolds doesn't lead to a construction of a fibration on lagrangian tori of the projective plane without a smooth cubic curve (or, more generally, a Fano variety without a smooth element from the anticanonical system), it gives new non toric examples of lagrangian submanifolds in toric varieties.

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1 General construction

Consider $\mathbb{CP}^n$, endowed with the standard Fubini – Study metric which induces the standard Kahler form ω. The space $C^\infty(\mathbb{CP}^n, \mathbb{R})$ contains a subspace of symbols $C_q(\mathbb{CP}^n, \mathbb{R})$ defined as follows (see [7], [4]): a smooth real function f, whose Hamiltonian vector field preserves the riemannian metric

$$Lie_{X_f} g = 0.$$

It's not hard to see, [7], that in the case of $\mathbb{CP}^n$ any such function f is generated by a self adjoint operator F on $\mathbb{C}^{n+1}$. Under this correspondence the eigenvectors of F after the projectivization become the critical points of f, and the eigenvalues are the critical values. It shows that f is a Morse function if and only if all the eigenvalues of F are different. Moreover, symbols f_1, f_2 commute if and only if the corresponding operators F_1, F_2 do (see [7]). We will call a symbol *degenerated* if it has multiple critical values. Below we will speak about *linearly independent* symbols.

In terms of symbols the standard Clifford fibration on lagrangian tori of $\mathbb{CP}^n$ can be described as follows. Choose in $C_q(\mathbb{CP}^n, \mathbb{R})$ a maximal commutative subalgebra, generated by symbols $f_1, ..., f_n$, and consider the mutual level sets

$$T_{(c_1,...,c_n)} = \cap_{i=1}^n \{f_i = c_i\}.$$

Sets of non critical values $c_1, ..., c_n$ form a convex polytop in $\mathbb{R}^n$, whose facets carry degenerations of the Clifford fibration — tori of smaller dimensions.

Consider now the complete linear system $|kH|$ of divisors of degree k on $\mathbb{CP}^n$. Any $f \in C_q(\mathbb{CP}^n, \mathbb{R})$ induces a *symbol* f^k on the projective space $|kH|$. The critical values of this symbol f^k are computed in a simple manner: these are all possible sums of length k of critical values of our given symbol f. We are interested in the case when symbol f^k is degenerated. Then in $|kH|$ one has a linear subsystem $\mathbb{CP}^d$, consists of invariant with respect to the Hamiltonian action of f^k points. These subsystems are of our interest.

Let's formulate our main observation:

Theorem 1. *Let a set of commuting non degenerated symbols* $f_1, ..., f_{n-1}$ *preserves the elements of a one dimensional linear subsystem* $\mathbb{CP}^1_B$ *with the base set* B *in the complete linear system of divisors of certain degree* k *on* $\mathbb{CP}^n$. *Then any Morse function* h *on* $\mathbb{CP}^1_B$ *induces an isotropical fibration on* $\mathbb{CP}^n$ *with compact fibers whose generic one is a smooth lagrangian torus.*

Proof. Let $\Delta_n \subset \mathbb{CP}^n$ be the "degeneration simplex" of the set $\{f_1, ..., f_{n-1}\}$, defined as

$$\Delta_n = \{X_{f_1} \wedge ... \wedge X_{f_{n-1}} = 0\}$$

which consists of $\frac{1}{2}(n+1)n$ projective $n-2$ - subspaces, spanned on the set of $n+1$ mutual critical points of f_i. The the invariance of the pencil elements implies that the base set B of the pencil is contained in Δ_n. Moreover, if the pencil includes singular elements (so reducible divisors or divisors with singularities) then all the singularities as well must be contained by Δ_n. Indeed, let an element $D \in \mathbb{CP}^1 \subset |kH|$ is reducible so

$$D = D_1 \cup D_2, D_1 \cap D_2 = N \neq \emptyset.$$

Then each function f_i preserves N, since its preserves both D_1 and D_2, therefore the restrictions $f_1|_N, ..., f_{n-1}|_N$ induces a linear independent set of functions in involution on a symplectic $n-2$ - dimensional manifold, and it implies that N must be contained by Δ_n.

Consider the map

$$\phi_L : \mathbb{CP}^n - B \to \mathbb{CP}^1,$$

defined by our pencil. Choose on $\mathbb{CP}^1$ a Morse function

$$h : \mathbb{CP}^1 \to \mathbb{R}$$

with critical points $p_{max} = p_1, ...p_s = p_{min}$. Then $\mathbb{CP}^n - B$ carries a set of functions

$$f_1, ..., f_{n-1}, f_n = \phi_L^* h,$$

which commute. Indeed, the map ϕ_L is complex and consequently symplectic, thus the fibers of the map are endowed with a symplectic connection ∇, and the Hamiltonian vector field of the lifted function $\phi_L^* h$ coincides with the lifting by the connection ∇ of the Hamiltonian vector filed X_h. It shows that $X_{\psi_L^* h}$ is symplectically orthogonal to fibers of ϕ_L, but by the condition of the Theorem X_{f_i} is tangent to fibers at each point of $\mathbb{CP}^n - B$, which implies that

$$\omega(X_{f_i}, X_{\phi_L^* h}) = \{f_i, \psi_L^* h\} = 0$$

for any $i = 1, ..., n-1$ on $\mathbb{CP}^n - B$.

Further, denote as D_i the fibers of the pencil which correspond to the critical points of h:

$$D_i = \psi_L^{-1}(p_i).$$

Consider the "action" map

$$F = (f_1, ..., f_n) : \mathbb{CP}^n - B \to \mathbb{R}^n. \qquad (*)$$

Its image $\operatorname{Im} F \subset \mathbb{R}^n$ lies in the direct product

$$P_{n-1} \times I, \quad P_{n-1} \subset \mathbb{R}^{n-1}, I = [\min h; \max h] \subset \mathbb{R},$$

where P_{n-1} is a convex polytop, defined as the image of the map

$$(f_1, ..., f_{n-1}) : \mathbb{CP}^n \to \mathbb{R}^{n-1}. \qquad (**)$$

Let us study the fibers of the map. For a set of values $(c_1, ..., c_n)$, which are not included by the set

$$\Delta_{n+1} = \operatorname{Im} F \cap ((\partial P_{n-1} \times I) \cup (\cup_{i=1}^{s} P_{n-1} \times [h(p_i)])),$$

the mutual level set

$$T_{(c_1,...,c_n)} = \{f_i = c_i\}$$

is a smooth lagrangian torus. Indeed, degenerations of the polyvector field

$$X_{f_1} \wedge ...X_{f_n}$$

on $\mathbb{CP}^n - B$ form exactly the set $\Delta_h \cup D_1 \cup ... \cup D_s$, whose image is Δ_{n+1}. Moreover, for a set of values $(c_1, ...c_{n-1})$, which is not included by ∂P_{n-1}, the mutual level set

$$S_{(c_1,...,c_{n-1})} = \{f_1 = c_1, ..., f_{n-1} = c_{n-1}\}$$

is a smooth compact manifold of dimension $n+1$, which does not intersect the base set B. Thus the function f_n is correctly defined as a smooth function on $S_{(c_1,...,c_{n-1})}$, whose non critical level sets are compact and smooth. Therefore they are lagrangian tori.

Since Δ_{n+1} has codimension 1 in $\operatorname{Im} F$ the case is generic. In studies of singular fibers the main question is how to extend the action map (*) to the base set B. Let $P_B \subset P_{n-1} \subset \mathbb{R}^{n-1}$ be the image of the base set under the map (**), consists of a set of certain $n-2$- dimensional faces of P_{n-1}. Then for a set of values $(c_1, ..., c_{n-1})$, belonging to the inner part of P_B, the mutual level set $S_{(c_1,...,c_{n-1})}$ is a smooth isotropical torus of dimension $n = 2$. Indeed, over an inner point of P_B the rank of the vector system $< X_{f_1}, ..., X_{f_{n-1}} >$ equals to $n-2$, which coincides with the dimension of the mutual level set, which gives the statement. Our isotropical fibration is not smooth near B, but however it is correctly defined. Our saddle critical points of our function h one has singular compact lagrangian tori with self intersections, and over focal critical points one has isotropical tori of smaller dimensions. This completes the proof of Theorem 1.

It's not hard to see that the statement of Theorem 1 remains to be true for any toric variety if we replace term "symbol" by more appropriate in toric geometry term "moment map". As it is shown in subsequent sections the construction gives non toric lagrangian fibrations of toric Fano varieties.

The most general form of our observation can be given due to a fundamental result of S. Donaldson concerning the Lefschetz pencils on symplectic manifolds with integer symplectic forms, see [8], as follows.

Theorem 2. *Let (X, ω) be a simply connected symplectic manifold of real dimension $2n$ with integer symplectic form, $[\omega] \in H^2(X, \mathbb{Z})$. Let sections $s_1, s_2 \in \Gamma(X, L^k)$ induce a Lefschetz pencil as in [8]. Then any set of first integrals $f_1, ..., f_{n-1}$, preserving the sections s_1, s_2 in combination with the prequantization connection ∇, exploited in [8], such that*

$$\nabla_{X_{f_i}} s_j = 0, i = 1, ..., n-1, j = 1, 2,$$

and any Morse function on the projective line $\mathbb{CP}^1$, parameterizing the pencil, induce an isotropical fibration of X, whose generic fiber is a smooth lagrangian torus.

The proof follows precisely the scheme of the proof of Theorem 1. The key points are the following. It remains true that the base set of the Lefschetz pencil

$$B = (s_1)_0 \cap (s_2)_0$$

again is contained in the "degeneration simplex"

$$\Delta_n = \{X_{f_1} \wedge ... \wedge X_{f_{n-1}} = 0\}.$$

The map (**) is correctly defined and sends Δ_n to the boundary of the domain $P_{n-1} \subset \mathbb{R}^{n-1}$, which is the image of whole X. For a generic set of values $(c_1, ..., c_{n-1}$ from the inner part of P_{n-1} the mutual level set $S_{(c_1,...,c_{n-1})}$ is a smooth compact manifold. Indeed, all the level sets $\{f_i = c_i\}$ are smooth (since the critical values go to ∂P_{n-1}) and intersect each other transversally (since the Hamiltonian vector fields X_{f_i} are linearly independent at each point of the intersection). Thus the arguments from the proof of Theorem 1 work at this case as well.

2 Examples in dimension 2.

In his batchelor diploma [9] the first author observed an interesting fact: if a non degenerated symbol f on the projective plane $\mathbb{CP}^2$ preserves an irreducible smooth conic $Q \subset \mathbb{CP}^2$, then it preserves a pencil of conics which includes the given one. Due to this one can reproduce the Auroux construction of non toric lagrangian fibration of $\mathbb{CP}^2$ with degeneration at the union of a conic and a line (see [5]) as follows.

Consider on $\mathbb{CP}^2$ a symbol f with critical values $(1, -1, 0)$. Fix the eigenbasis of the corresponding self adjoint operator in $\mathbb{C}^3$ and consider the corresponding homogeneous coordinates in $\mathbb{CP}^2$. Then the pencil of conic $\alpha_0 z_0 z_1 = \alpha_1 z_2^2$ is an invariant linear subsystem in the complete linear system $|2H|$ on $\mathbb{CP}^2$. The coefficients α_0, α_1 are homogeneous coordinates on the line $\mathbb{CP}^1$ which parameterized the pencil. Choose the following Morse function on $\mathbb{CP}^1$:

$$h = \frac{|\alpha_0|^2}{|\alpha_0|^2 + |a\alpha_1 - \alpha_0|^2}, \tag{1}$$

where a is a fixed real positive number (compare with [5]). Two its critical points p_{max}, p_{min} with critical values 1 and 0 correspond to non degenerated conic $Q = \{z_2^2 = a z_0 z_1\}$ and double line $\{z_2^2 = 0\}$. Furthermore, the base set of the pencil $\mathbb{CP}^1$ doesn't coincide with the set of critical values of f, thus one has a singular torus $T_{(0,1/2)}$ which is modeled by shrinking a loop to point on a smooth torus.

Therefore this non toric example of D. Auroux is presented by our construction from section 1.

This example has a simple extension to higher degrees. As it was shown by the first author in [10], a symbol can not preserve a smooth irreducible curve of degree greater or equal to 3 in $\mathbb{CP}^2$. But it is still possible to consider a pencil of cuspidal curves of the form

$$\alpha_0 z_0^k + \alpha_1 z_1^{k-1} z_2 = 0, \tag{$*$}$$

which are invariant with respect to non degenerated symbol with the critical values $(k-1, k, 0)$. The base set of the symbol is the sum of points $[0:1:0]$ and $[0:0:1]$ with multiplicities. The choice of a Morse function on the line $\mathbb{CP}^1$, parameterizing the pencil, makes it possible to construct different fibrations on tori of $\mathbb{CP}^2$. If one takes the Morse function h of the same form, as it was in the Auroux example, then one gets a lagrangian fibration of $\mathbb{CP}^2$ without a reducible curve of degree $k+1$ consists of a line and an irreducible cuspidal curve of degree k. There is a singular fiber again, which corresponds to the values

$$f = k-1, h = 1/2.$$

This singular torus divides the set of smooth fibers on two types — the Clifford one and the Chekanov one, as it is in the Auroux example from [5].

Let's consider now the case of non degenerated quadric Q in $\mathbb{CP}^3$.

A very simple example of non toric lagrangian fibration of Q was presented in [11] and is based on the birational map $Q \to \mathbb{CP}^2$ and the Auroux example. Below this example is generalized by the application of the construction from section 1.

Consider on $\mathbb{CP}^3$ with fixed homogeneous coordinates $[z_0 : ... : z_3]$ certain degenerated symbol f with critical values $(1, -1, 0, 0)$ and a non degenerated quadric given by the equation

$$Q = \{z_0 z_1 + z_2 z_3 = 0\}.$$

It's easy to see that our quadric Q is invariant with respect to the Hamiltonian action of f. Therefore the restriction

$$f_Q = f|_Q$$

is a symbol on Q, which is already non degenerated (with isolated critical points). In the complete linear system of divisors of bi - degree $(1, 1)$ on Q one has a pencil of hyperplane sections $\{q_{\alpha_0,\alpha_1} \subset Q\}$, given by intersections

$$q_{\alpha_0,\alpha_1} = Q \cap \pi_{\alpha_0,\alpha_1},$$

where plane π_{α_0,α_1} is defined by the equation $\alpha_0 z_2 + \alpha_1 z_3 = 0$. Each conic q_{α_0,α_1} is invariant with respect to f_Q since each plane π_{α_0,α_1} is invariant with respect to symbol f. Thus the choice of a Morse function h on $\mathbb{CP}^1$, parameterizing the pencil $\{q_{\alpha_0,\alpha_1}\}$, induces a lagrangian fibration of our quadric Q.

If h has the form (2), then it gives the standard toric fibration with degeneration on four lines. If one takes h of the form (1) (as in the Auroux example), it gives non toric fibration from [11] with degeneration on a pair of lines and a conic which has one isolated singular torus. At the end, if one takes the function of the form

$$h = \frac{|\alpha_0 + \alpha_1|^2}{|\alpha_0|^2 + |\alpha_1|^2}, \tag{3}$$

then it gives us a non toric fibration on lagrangian tori of the complement $Q - (q_{1,1} \cup q_{1,-1})$ with two isolated singular fibers.

3 Examples for $\mathbb{CP}^3$.

Using the construction of section 1 one can get non toric lagrangian fibrations of the projective space $\mathbb{CP}^3$. Below we place two examples. The first one presents a lagrangian fibration of the projective space without two smooth quadrics, and this fibration doesn't have singular fibers. The second example is degenerated, it is given by removing from $\mathbb{CP}^3$ a plane and a non degenerated cubic. Singular tori in this case form a big set, being non isolated. However this example is useful since it contains deformations of the standard Clifford fibration of the projective plane.

Let's fix homogeneous coordinates $[z_0 : ... : z_3]$ and consider non degenerated symbol f with critical values λ_i at points $z_i \neq 0, z_{\hat{i}} = 0$. In the complete linear system of quadrics consider the pencil $\alpha_0 z_0 z_1 + \alpha_1 z_2 z_3 = 0$. Symbol f preserves the elements of the pencil if and only if the equality holds

$$\lambda_0 + \lambda_1 = \lambda_2 + \lambda_3.$$

It's easy to see that there are two linearly independent non degenerated symbols which satisfy this condition. For example,

$$f_1 \mapsto (1, -1, 2, -2), f_2 \mapsto (2, -2, -1, 1).$$

Now if on the line $\mathbb{CP}^1$, parameterizing the pencil, one chooses function h of the form (2), then one gets the standard Clifford fibration on $\mathbb{CP}^3$ with degeneration on four planes. Consider another function h of the form (3). Then its critical points p_{max}, p_{min} correspond to non degenerated quadrics

$$Q_+ = \{z_0 z_1 + z_2 z_3 = 0\}, Q_- = \{z_0 z_1 - z_2 z_3 = 0\}.$$

Note, that the degeneration simplex of the symbols f_1, f_2 is contained by the union

$$\Delta_{n-1} = \Delta_1 \subset Q_+ \cup Q_-.$$

Indeed, the degeneration simplex consists of six lines, two of which form the support of the base set of the pencil and the remaining four are divided into two pairs each of them lies on $Q_\pm$. It follows that f_1, f_2 h (3) define a smooth fibrations on tori of the complement to these two smooth quadrics in $\mathbb{CP}^3$.

Next example: consider the pencil of cubic surfaces

$$\mathbb{CP}^1 = \{\alpha_0 z_0^3 + \alpha_1 z_1 z_2 z_3 = 0\}$$

in the same coordinates. Consider a pair of symbols: f_1 with critical values $(0,1,2,-3)$ and a degenerated symbol f_2 with critical values $(0,1,0,-1)$. It's not hard to see that both the symbols preserve the elements of the pencil therefore the choice of a Morse function on $\mathbb{CP}^1$ induces a fibration $\mathbb{CP}^3$ without two surfaces $h^{-1}(p_{max}), h^{-1}(p_{min})$. Function h of the form (2) gives the standard Clifford fibration, while function h of the form (1) induces non toric lagrangian fibration on $\mathbb{CP}^3$ without plane $\pi_0 = \{z_0 = 0\}$ and cubic $C = \{z_0^3 + z_1 z_2 z_3 = 0\}$ (let's equal the real parameter a in (1) to one for brevity). Now the degeneration simplex D_{n-1} is not contained by the union $\pi_0 \cup C$, therefore the induced lagrangian fibration $\mathbb{CP}^3 - (\pi_0 \cup C)$ has singular non isolated fibers. For their description it is convenient to make an additional step. Consider a pencil of projective planes π_{β_0,β_1} passing through line $l = \{z_0 z_2 = 0\}$, such that

$$\pi_0 = \pi_{0,1} \in \mathbb{CP}^1.$$

It's easy to see that each plane from the pencil is invariant with respect to the degenerated symbol f_2, therefore the restriction

$$f_2|_{\pi_{\beta_0,\beta_1}} = f_{\beta_0,\beta_1}$$

is a symbol on the plane π_{β_0,β_1}. The intersection

$$C \cap \pi_{\beta_0,\beta_1} = l \cup Q_{\beta_0,\beta_1}$$

is the union of the line l and a smooth conic $Q_{\beta_0,\beta_1} \subset \pi_{\beta_0,\beta_1}$ except the boundary cases when $\beta_i = 0$. If $\beta_0 = 0$ the conic Q_{β_0,β_1} degenerates to a pair of lines, when $\beta_1 = 0$ the conic degenerates to the double line $2l$.

It's not hard to see that the data $f_{\beta_0,\beta_1}, Q_{\beta_0,\beta_1}, l$ induce a lagrangian fibration of the Auroux type of the plane π_{β_0,β_1} if $\beta_i \neq 0$. Indeed, the cubic C and the plane π_{β_0,β_1} both are invariant with respect to the symbol f_2, and it implies that their intersection is invariant with respect to f_2 restricted to the plane π_{β_0,β_1}. This fibration is given by the intersections of the corresponding lagrangian tori of the fibration of $\mathbb{CP}^3 - (\pi_0 \cup C)$ with the plane π_{β_0,β_1}. Under these circumstance 3 - dimensional tori can be reconstructed from these 2-dimensional tori: the $U(1)$ - action associated to the moment map f_1 induces cyclic motion of points of a non degenerated 2 - torus, such that the union of the orbits is a 3 - torus. Singular fibers appear if it is taken a singular 2 - torus with singularity at the intersection point of the line $z_1 z_3 = 0$ and the plane π_{β_0,β_1}. In this setup the plane $\pi_{0,1}$ is fibered by the Clifford type fibration, and $\pi_{1,0}$ carries a degenerated fibration on *lagrangian cilinders* of a projective plane without a single projective line. Therefore the presented example illustrates the deformations of the Auroux fibration to the standard Clifford fibration and to the cilindrical one. And the last cilindrical fibration can be exploited in the framework of the Auroux conjecture as well. The point is that Auroux would like to use the fact stated in the conjecture for the following situation: two Fano varieties X_1, X_2 can be glued along anticanonical divisors and then deform to a smooth Calabi – Yau manifold Y. And doing this one could glue as well special lagrangian fibrations on each X_i with degenerations on the divisors getting a special lagrangian fibration of Y. However lagrangian tori on Y can be constructed by gluing of pairs of lagrangian cilinders, and our last example can be exploited in this way.

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