# Max-Planck-Institut für Mathematik
# Bonn

## Arbeitstagung 2009

## 9. Arbeitstagung der zweiten Serie

Max-Planck-Institut für Mathematik
Preprint Series 2009 (40)

# Arbeitstagung 2009

## 9. Arbeitstagung der zweiten Serie

Max-Planck-Institut
für Mathematik
Vivatsgasse 7
53111 Bonn

Germany

Mathematisches Institut
der Universität Bonn
Endenicher Allee 60
53115 Bonn

Germany

MPI 09-40

## Contents

| Name | First Name | Institution |
|---|---|---|
| A'Campo | Norbert | Universität Basel |
| Ageev | Oleg | MPIM |
| Ahmadian | Razieh | University of Teheran |
| Alexandrov | Alexander | Intitute for Central Sciences RAS |
| Amerik | Ekaterina | MPIM/Université Paris-Sud |
| Angel | Andees | HCM Bonn |
| Asgari | Mahdi | Oklahoma State University |
| Auroux | Denis | MIT |
| Baditiou | Gabriel | MPIM/Institute of Mathematics Bucharest |
| Ballmann | Werner | MPIM/Universität Bonn |
| Belolipetsky | Misha | Durham University |
| Beraldo | Dario | University of California, Berkeley |
| Boissy | Corentin | MPIM/Université Aix-Marseille 3 |
| Cacic | Branimir | MPIM |
| Carrilo | Paulo | MPIM |
| Ceyhan | Özgür | MPIM |
| Choi | Ka | University of California, Berkeley |
| Dabrowski | Andrzej | Universisty of Szczecin |
| Dais | Dimitrios | University of Crete |
| Dalawat | Chandan | Harish-Chandra Institute |
| Dessai | Anand | Université de Fribourg |
| Dobrwoski | Peter | Universität Köln |
| Dobrynskiy | Vladimir | |
| Drozd | Yuri | MPIM/Institute of Mathematics, Kiev |
| Dzhumadail'daev | Askar | Insitute of Mathematics |
| Elashvili | Alexander | Institute of Mathematics Tiblis |
| Faltings | Gerd | MPIM |
| Faustino | Nelson | Universidade de Aveiro |
| Felikson | Anna | MPIM |
| Felshtyn | Alexander | MPIM/University of Szczecin |
| Felshtyn | Marten | |
| Florin | Nicolae | Technische Universität Berlin |
| Frégier | Yael | RMATH Luxembourg |
| Friedrich | Reinhold | MPIM/Universität Bonn |
| Garion | Shelly | MPIM |
| Gashi | Quendrim | MPIM |
| Gatsinzi | Jean Baptiste | MPIM/University of Botswana |
| Grobner | Harald | Universität Wien |
| Grossi-Ferreira | Carlos | MPIM |
| Grunewald | Fritz | Universität Düsseldorf |
| Hambleton | Ian | MPIM/McMaster University Hamilton |
| Hanke | Bernhard | LMU München |
| Harder | Günter | MPIM/Universität Bonn |
| Hegenbarth | Fritz | Universita degli Studi di Milano |
| Heim | Bernhard | MPIM/Universität Mannheim |
| Heinze | Joachim | Springer-Verlag |
| Hirzebruch | Friedrich | MPIM |
| Hitchin | Nigel | University of Oxford |
| Hornbostel | Jens | Universität Bonn |
| Huesmöller | Dale | MPIM |
| Iranmanesh | Ali | Tarbiat Modares Unversity |
| Jacobson | Jeremy | LSU/Universität Bonn |
| Kaiser | Christian | MPIM |
| Kaiser | Uwe | Boise State University |
| Kashani | S.M.B | Unversity of Teheran |
| Katzarkov | Ludmil | Universität Wien |
| Kaufmann | Ralph | Purdue University |
| Keel | Sean | University of Texas, Austin |
| Khimshiashvili | Giorgi | Razmadze Mathematical Institute, Tiblis |

| | | |
|---|---|---|
| Kishore | Marathe | MIS Leipzig |
| Klingenberg | Wilhelm | Durham University |
| Kobayashi | Toshiyuki | MPIM/Tokyo |
| Kolyada | Sergii | MPIM/Institute of Mathematics, Kiev |
| Kontsevich | Maxim | IHES |
| Kounchev | Ognyan | Insitute of Mathematics, Bulgaria |
| Kühnel | Wolfgang | Universität Stuttgart |
| Kurke | Herbert | Humboldt-Universität Berlin |
| Ladkani | Sefi | MPIM |
| Lebedev | Dmitry | Université de Dijon |
| Li | Qin | University of California, Berkeley |
| Licata | Anthony | MPIM/Stanford University |
| Licata | Joan | MPIM/Stanford University |
| Lorscheid | Oliver | MPIM |
| Mahanta | Snigdhayan | IHES |
| Maimanni | Hamidreza | Shahid Rejaee University |
| Makar-Limanov | Leonid | MPIM/Wayne State University |
| Manin | Yuri | MPIM |
| Matveev | Vladimir | MPIM |
| Mellit | Anton | MPIM |
| Mihalache | George | Universität Hamburg |
| Minabe | Satoshi | MPIM |
| Moree | Pieter | MPIM |
| Müller | Jörn | MPIM/Universität Bonn |
| Müller | Werner | Universität Bonn |
| Muro | Fernando | Universitat de Barcelona |
| Mustata | Andrei | MPIM |
| Oort | Frans | University of Utrecht |
| Osipov | Denis | Steklov Mathematical Institute |
| Patnaik | Manish | Harvard University |
| Petit | Francais | Paris VI University |
| Petras | Oliver | HCM Bonn |
| Pevzner | Michael | Université de Reims |
| Pichereau | Anne | MPIM |
| Poletaeva | Elena | University of Texas, Pan American |
| Putzka | Jens | MPIM/HCM Bonn |
| Raulf | Nicole | MPIM/Université de Lille 1 |
| Reizakis | Marina | Springer-Verlag |
| Rovinsky | Marat | MPIM/Steklov Mathematical Institute |
| Schommer-Pries | Christopher | MPIM |
| Seiler | Wolfgang | Universität Mannheim |
| Shatz | Stephen | Pennsylvania State University |
| Shoikhet | Boris | Université du Luxembourg |
| Silhan | Josef | MPIM |
| Smirnov | Maxim | MPIM |
| Snoha | L'ubomir | MPIM/Matj Bel University |
| Somberg | Petr | MFF UK |
| Soucek | Vladimir | Charles University Prague |
| Spirdonov | Vyacheslav | Lab. Of Theor. Physics, JINR, Dubna |
| Strohmaier | Alexander | Loughborough University |
| Strooker | Jan | University of Utrecht |
| Stroppel | Katharina | Universität Bonn |
| Szeczepanski | Andrzej | University at Gdansk |
| Szemberg | Tomas | MPIM |
| Tabachnikov | Sergei | MPIM/Pennsylvania State University |
| Teicher | Mina | Emmy Noether Institute |
| Teichner | Peter | MPIM |
| Terakawa | Hiroyuki | MPIM/Tsuru University |
| Tien | Trinh Duy | Universität Essen-Duisburg |
| Troitsky | Evegenij | MPIM/Moscow State University |

| | | |
|---|---|---|
| Tziolas | Nikolaos | MPIM/University of Cyprus |
| Valdez | Perran | MPIM |
| Van Roosmalen | Adam-Christiaan | MPIM |
| Vdovina | Alina | Newcastle University |
| Virdol | Cristian | MPIM/Columbia University |
| Vlasenko | Maria | MPIM |
| Wang | Mingxi | ETH Zürich |
| Winn | Brian | Loughborough University |
| Xiandong | Ye | MPIM/University of Sci. and Tech. of China |
| Yassemi | Siamak | University of Teheran |
| Yu | Chia-Fu | Inst. of Math. Academica Sinica |
| Zagier | Don | MPIM |
| Zeidler | Eberhard | MIS Leipzig |
| Zharkov | Ilia | Kansas State University |
| Ziller | Wolfgang | University of Pennsylvania |

| | | |
|---|---|---|
| Max-Planck-Institut<br>für Mathematik<br>Vivatsgasse 7<br>53111 Bonn | and | Mathematisches Institut<br>der Universität Bonn<br>Endenicher Allee 60<br>53115 Bonn |

**Program of the Mathematische Arbeitstagung 2009 (I)**

*All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10.*

**Friday, June 05, 2009**

| | |
|---|---|
| 3:30 – 4:15p.m. | Opening and first program discussion |
| 5:00 – 6:00 p.m. | Maxim Kontsevich (IHES)<br>Symplectic geometry of homological algebra |
| 8:00 p.m. | Rector's Party<br>Festsaal der Universität, Hauptgebäude (entrance from "Am Hof" street across from Bouvier bookstore) |

**Saturday, June 06, 2009**

| | |
|---|---|
| 10:15 – 11:15 a.m. | A. Klemm (Univ. Bonn)<br>Duality and Symmetry |
| 12:00 – 1:00 p.m. | W. Ziller (UPenn)<br>Manifolds of Positive Curvature |
| 5:00 – 6:00 p.m. | A. Licata (MPIM)<br>Categorical $SL_2$-Action |

**Sunday, June 07, 2009**

| | |
|---|---|
| 10:15 – 10:30 a.m. | Program discussion (II) |
| 10:30 – 11:30 a.m. | P. Teichner (MPIM / UC Berkeley)<br>The Kervaire Invariant One Problem |
| 12:00 – 1:00 p.m. | F. Oort (Univ. Utrecht)<br>Hecke Orbits |
| 5:00 – 6:00p.m. | G. Harder (MPIM / Univ. Bonn)<br>The Fundamental Lemma |

All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10. There will be *tea breaks* after the morning lecture from 11:15/11:30 a.m. on in Wegelerstrasse and from 3:30 p.m. in the MPI. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 30 Euro. *Lists of participants* and other information will lie out in Wegelerstrasse and in the MPIM. *All* participants are requested to put their name on the list!

**** All Arbeitstagung participants and those accompanying them are invited to the *Rector's Party* on Friday and the *Boat Trip* on Monday (see special announcements) ****

| | | |
|---|---|---|
| Max-Planck-Institut<br>für Mathematik<br>Vivatsgasse 7<br>53111 Bonn | and | Mathematisches Institut<br>der Universität Bonn<br>Endenicher Allee 60<br>53115 Bonn |

**Program of the Mathematische Arbeitstagung 2009 (II)**

*All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10.*

**Monday, June 08, 2009**

| | |
|---|---|
| 10:15 – 11:15 a.m. | N. A'Campo (Univ Basel)<br>Tête-à-Tête twists and geometric Monodromy |
| 12:00 – ca. 7:00 p.m. | Boat Trip<br>See special leaflet |

**Tuesday, June 09, 2009**

| | |
|---|---|
| 10:15 – 10:30 a.m. | Program discussion (III) |
| 10:30 – 11:30 a.m. | A. Mellit (MPIM)<br>Mahler measures and $L$-functions |
| 12:00 – 1:00 p.m. | W. Klingenberg (Univ. Durham)<br>Proof of Carathéodory Conjecture |
| 5:00 – 6:00 p.m. | W. Müller (Univ. Bonn)<br>Analytic Torsion and Cohomology of hyperbolic 3-manifolds |

All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10. Except for Monday, there will be *tea breaks* after the morning lecture from 11:15/11:30 a.m. on in Wegelerstrasse and 3:30 p.m. on in the MPI. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 30 Euro.
*Lists of participants* and other information will lie out in Wegelerstrasse and in the MPI. *All* participants are requested to put their name on the list!

**** All Arbeitstagung participants and those accompanying them are invited to the *Boat Trip* on Monday (see special announcement) ****

| | | |
|---|---|---|
| Max-Planck-Institut<br>für Mathematik<br>Vivatsgasse 7<br>53111 Bonn | and | Mathematisches Institut<br>der Universität Bonn<br>Endenicher Allee 60<br>53115 Bonn |

**Program of the Mathematische Arbeitstagung 2009 (III)**

*All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10.*

**Wednesday, June 10, 2009**

| | |
|---|---|
| 10:15 – 11:15 a.m. | D. Lebedev (ITEP)<br>From Archimedean $L-$factors to Topological Field Theories |
| 12:00 – 1:00 p.m. | G. Mikhalkin (Univ. Genève)<br>Tropical Geometry |
| 5:00 – 6:00 p.m. | T. Kobayashi (Univ. Tokyo)<br>Discontinuous Groups on pseudo-Riemannian Spaces |

**Thursday, June 11, 2009**

| | |
|---|---|
| 10:15 – 11:15 a.m. | M. Belolipetsky (Univ. Durham)<br>Counting Lattices |
| 12:00 – 1:00 p.m. | M. Bourdon (Univ. Lille 1 / MPIM)<br>Quasiconformal Geometry |
| 5:00 – 6:00 p.m. | O. Lorscheid (MPIM)<br>$\mathbb{F}_1-$geometry |

All lectures will take place in the "Großer Hörsaal," Wegelerstraße 10. Except for Monday, there will be *tea breaks* after the morning lecture from 11:15/11:30 a.m. on in Wegelerstrasse and 3:30 p.m. on in the MPI. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 30 Euro.
*Lists of participants* and other information will lie out in Wegelerstrasse and in the MPI.
*All* participants are requested to put their name on the list!

# Symplectic geometry of homological algebra

Maxim Kontsevich

June 10, 2009

## Derived non-commutative algebraic geometry

With any scheme $X$ over ground field $\mathbf{k}$ we can associate a $\mathbf{k}$-linear triangulated category $\mathrm{Perf}(X)$ of perfect complexes, i.e. the full subcategory of the unbounded derived category of quasi-coherent sheaves on $X$, consisting of objects which are locally (in Zariski topology) quasi-isomorphic to finite complexes of free sheaves of finite rank.

The category $\mathrm{Perf}(X)$ is essentially small, admits a natural enhancement to a differential graded (dg in short) category up to a homotopy equivalence, and is Karoubi (e.g. idempotent) closed. The main idea of derived non-commutative algebraic geometry is to treat any Karoubi closed small dg category as the category of perfect complexes on a "space".

By a foundamental result of A. Bondal and M. Van den Bergh, any separated scheme of finite type is *affine* in the derived sense, i.e. $\mathrm{Perf}(X)$ is generated by just one object. Equivalently,

$$\mathrm{Perf}(X) \sim \mathrm{Perf}(A)$$

for some dg algebra $A$, where prefect A-modules are direct summands in the homotopy sense of modules $M$ which are free finitely generated $\mathbb{Z}$-graded $A$-modules, with generators $m_1, \dots, m_N$ of certain degree $\deg(m_i) \in \mathbb{Z}$, such that $dm_i \in \oplus_{j<i} A \cdot m_j$ for all $i$. Algebra $A$ associated with $X$ is not unique, it is defined up to a derived Morita equivalence.

Some basic properties of schemes one can formulate purely in derived terms.

**Definition 1.** *Dg algebra $A$ is called smooth if $A \in \mathrm{Perf}(A \otimes A^{op})$. It is compact if $\dim H^\bullet(A, d) < \infty$. This properties are preserved under the derived Morita equivalence.*

For a separated scheme $X$ of finite type the properties of smoothness and properness are equivalent to the corresponding properties of a dg algebra $A$ with $\mathrm{Perf}(A) \sim \mathrm{Perf}(X)$. Smooth and compact dg algebras are expected to be the "ideal" objects of derived geometry, similar to smooth projective varieties in the usual algebraic geometry. For a smooth algebra $A$ the homotopy category $\mathrm{Fin}(A)$ of dg-modules with finite-dimensional total cohomology is contained in $\mathrm{Perf}(A)$, and for compact $A$ the category $\mathrm{Perf}(A)$ is contained in $\mathrm{Fin}(A)$. One can define two notions of a Calabi-Yau algebra of dimension $D \in \mathbb{Z}$. In the smooth case it says that $A^! := \mathrm{Hom}_{A\otimes A^{op}-mod}(A, A \otimes A^{op})$ is quasi-isomorphic to $A[-D]$ as $A \otimes A^{op}$-module (it corresponds to the triviality of the canonical bundle for smooth schemes). Similarly, in the compact case we demand that $A^* = \mathrm{Hom}_{\mathbf{k}-mod}(A, \mathbf{k})$ is quasi-ismorphic to $A[D]$, as a bimodule (it corresponds for schemes to the condition that $X$ has Gorenstein singularities and the dualizing sheaf is trivial).

The notion of smoothness for dg algebras is itself not perfect, as e.g. it includes somewhat pathological example $\mathbf{k}[x, (1/(x - x_i)_{i\in S}]$ where $S \subset \mathbf{k}$ is an *infinite* subset. It seems that the right analog of smooth shemes (of finite type) is encoded in the following notion of dg algebra of finite type due to B. Toën and M. Vaquie:

**Definition 2.** *A dg algebra $A$ is called of finite type if it is a homotopy retract in the homotopy category of dg algebras of the free finitely generated algbera $\mathbf{k}\langle x_1, \ldots, x_N\rangle$, $\deg(x_i) \in \mathbb{Z}$ with the differential of the form*

$$dx_i \in \mathbf{k}\langle x_1, \ldots, x_{i-1}\rangle,\ \ i = 1, \ldots, N\ .$$

Any dg algebra of finite type is smooth, and any smooth compact dg algebra is of finite type. It is also convenient to replace a free graded algebra in the definition of finite type by the algebra of paths in a finite $\mathbb{Z}$-graded quiver.

A large class of small triangulated categories (including many examples from representation theory) can be interpreted as the categories of perfect complexes on a space of finite type with a given "support". In terms of dg algebras, in order to specify the support one should pick a perfect complex $M \in \mathrm{Perf}(A)$. The corresponding category is the full subcategory of $\mathrm{Perf}(A)$ generated by $M$, and is equivalent to $\mathrm{Perf}(B)$ where $B = \mathrm{End}_{A-mod}(M, M)^{op}$. One can say in non-commutative terms what is the "complement" $X - \mathrm{Supp}(M)$ and the "formal completion" $\widehat{X}_{\mathrm{Supp}\, M}$ of $X$ at $\mathrm{Supp}(M)$. The complement is given by the localization of $\mathrm{Perf}(X) = \mathrm{Perf}(A)$ at $M$, and is again of finite type. By Drinfeld's construction, in terms of dg quivers it means that we add a new free generator $h_M \in$

$\mathrm{Hom}^{-1}(M, M)$ with $dh_M = \mathrm{id}_M$. The formal completion is given by algebra $C = \mathrm{End}_{B-mod}(M, M)^{op}$. E.g. when $A = \mathbf{k}[x]$ and $M = \mathbf{k}$ with $x$ acting trivially, we have $B = H^\bullet(S^1, \mathbf{k})$ (the exterior algebra in one variable in degree $+1$), and $C = \mathbf{k}[[x]]$.

**Examples of categories of finite type**

**Algebraic geometry**: For any smooth scheme $X$ the category $\mathrm{Perf}(X) \simeq D^b(Coh(X))$ is of finite type.

**Topology:** Let $X$ be now a space homotopy equivalent to a finite connected CW complex. Define $A_X := \mathrm{Chains}(\Omega(X, x_0))$, the dg algebra of chains (graded in non-positive degrees) of the monoid of based loops in $X$, with the product induced from the composition of loops. This algebra is of finite type as can be seen directly from the following description of a quasi-isomorphic algebra.

Let us assume for simplicity that $X$ is simplicial subcomplex in a standard simplex $\Delta^K$ for some $K \in \mathbb{Z}_{\geq 0}$. We associate with such $X$ a finite dg quiver $Q_X$. Its vertices are $v_i$, $i = 0, \ldots, K$ for $i \in X$. The arrows are $a_{i_0,\ldots,i_k}$ for $k > 0$, where $(i_0, \ldots, i_k)$ is a face of $X$, and $i_0 < i_1 < \cdots < i_k$. The arrow $a_{i_0,\ldots,i_k}$ has degree $(1 - k)$ and goes from $v_{i_0}$ to $v_{i_k}$. We define the differential in $Q_X$ by

$$da_{i_0,\ldots,i_k} = \sum_{j=1}^{k-1} (-1)^j a_{i_0,\ldots,i_j} \cdot a_{i_j,\ldots,i_k}$$

Then we have to "invert" all arrows of degree 0, i.e. add inverse arrows $a_{i_0,i_1}$ for all egdes $(i_0, i_1)$ in $X$. It can be done either directly (but then we obtain a non-free quiver), or in a more pedantical way which gives a free quiver. In general, if want to invert a arrow $a_{EF}$ in a dg quiver connecting verices $E$ and $F$, with $\deg a_{EF} = 0$ and $da_{EF} = 0$, one can proceed as follows. To say that $a_{EF}$ is an isomorphism is the same as to say that the cone $C := \mathrm{Cone}(a_{EF} : E \to F)$ is zero. Hence we should add an endmorphism $h_C$ of the cone of degree $-1$ whose differential is the identity morphism. Describing $h_C$ as $2 \times 2$ matrix one obtains the following. One has to add 4 arrows

$$h^0_{FE}, h^{-1}_{EE}, h^{-1}_{FF}, h^{-2}_{EF}$$

with degrees indicated by the upper index, with differentials

$$dh^0_{FE} = 0, \ dh^{-1}_{EE} = \mathrm{id}_E - a_{EF} \cdot h^0_{FE},$$

$$dh^{-1}_{FF} = \mathrm{id}_F - h^0_{FE} \cdot a_{EF}, \ dh^{-2}_{EF} = a_{EF} \cdot h^{-1}_{FF} - h^{-1}_{EE} \cdot a_{EF} \, .$$

**Theorem 1.** *The quiver $Q_X$ localized in either way, is dg equivalent to $A_X$.*

In particular, if $X$ is space of type $K(\Gamma, 1)$ then $A_X$ is homotopy equivalent to an ordinary algebra in degree 0, the group ring $\mathbf{k}[\Gamma]$. In particular, such an algebra is of finite type. In the case $\mathrm{char}(\mathbf{k}) = 0$ one can also allow torsion, i.e. consider orbispaces, hence $\Gamma$ can be an arithmetic group, a mapping class group, etc.

The full subcategory of finite-dimensional dg modules $\mathrm{Fin}(A_X) \subset \mathrm{Perf}(A_X)$ is the triangulated category of sheaves whose cohomology are finite rank local systems on $X$. If we invert not all arrows of degree 0 in $Q_X$ for simplicial $X \subset \Delta^K$, we can obtain categories of complexes of sheaves with cohomology constructible with respect to a given CW-stratification, and even more general categories.

**Algebraic geometry II:** The last example of a category of finite type is somewhat paradoxical.

**Theorem 2.** *(V.Lunts) For any separated scheme $X$ of finite type the category $D^b(Coh(X))$ (with its natural dg enhancement) is of finite type.*

Morally one should interpret $\mathrm{Perf}(X)$ as the category of perfect complexes on a smooth derived noncommutative space $Y$ with support on a closed subset $Z$. Then the category $D^b(Coh(X))$ can be thought as the category of perfect complexes on the formal neighborhood $\widehat{X}_Z$. It turns out that for the case of usual schemes this neighbourhood coincides with $Y$ itself. The informal reason is that the "transversal coordinates" to $Z$ in $Y$ are of strictly negative degrees, hence the formal power series coincide with polynomials in $\mathbb{Z}$-graded sense.

### Fukaya categories

Let $(X, \omega)$ be a compact symplectic $C^\infty$ manifold with $c_1(T_X) = 0$

The idea of K. Fukaya is that one should associate with $(X, \omega)$ a compact $A_\infty$ Calabi-Yau category over a non-archimedean field (Novikov ring)

$$Nov := \sum_i a_i T^{E_i}, \ \ a_i \in \mathbb{Q}, \ E_i \in \mathbb{R}, \ \ E_i \to +\infty \ ,$$

where numbers $E_i$ have the meaning of areas of pseudo-holomorphic discs. The objects of $\mathcal{F}(X)$ in the classical limit $T \to 0$ should be oriented Lagrangain spin manifolds $L \subset X$ (maybe endowed with a local system). There are several modifications of the original definition:

- one can allow manifolds with $c_1 \neq 0$ (in this case one get only a $\mathbb{Z}/2\mathbb{Z}$-graded category),
- on can allow $X$ to have a pseudo-convex boundary (see the discussion of the Stein case below),
- (Landau-Ginzburg model), $X$ is endowed with a potential $W : X \to \mathbb{C}$ satisfying some conditions at infinity (then the corresponding Fukaya-Seidel category is not a Calabi-Yau one),
- allow $X$ to have holes inside, then one get so called “wrapped” Fukaya category with infinite-dimensional Hom-spaces.

**Fukaya categories of Stein manifolds**

The simplest and the most important case is when $X$ is compact complex manifold with real boundary such that there exists a strictly plurisubharmonic function $f : X \to \mathbb{R}_{\leq 0}$ with $f_{|\partial X} = 0$ and no critical points on $\partial X$.

Seidel in his book gave a complete definition of the Fukaya category of Stein manifold in terms of Lefschetz fibrations. The additional data necessary for $\mathbb{Z}$-grading is a trivialization of the square of the canonical bundle. One can analyze his construction and associate certain algebra $A$ of finite type (over $\mathbb{Z}$) such that the Fukaya category constructed by Seidel is a full subcategory of $\mathrm{Fin}(A)$. We propose to consider $A$ (or category $\mathrm{Perf}(A)$ and *not* $\mathrm{Fin}(A)$) to be a more foundamental object, and to formulate all the theory in such terms. For example, for $X = T^*Y$ where $Y$ is a compact oriented manifold, the algebra $A$ is $\mathrm{Chains}(\Omega(Y, y_0))$ contains information about the foundamental group of $Y$, whereas the category of finite-dimensional representations could be very poor for non-residually finite group $\pi_1(Y)$.

Also we propose a slightly different viewpoint on $A_X$. Namely, one can make $X$ smaller and smaller without changing $A$, and eventually contract $X$ to a singular Lagrangian submanifold $L \subset X$. Hence we can say that $A = A_L$ depends only on $L$ (up to derived Morita equivalence). One can think for example about $L$ being a 3-valent graph embedded in an open complex curve $X$ as a homotopy retract. If $X$ is endowed with a potential, we should contract $X$ to a noncompact $L$ such that $\mathrm{Re}(W)_{|L} : L \to \mathbb{R}$ is a proper map to $[c, +\infty)$, $c \in \mathbb{R}$, e.g. $L = \mathbb{R}^n$ for $X = \mathbb{C}^n$ with the holomorphic potential $\sum_{i=1}^n z_i^2$.

We expect that $\mathrm{Fin}(A_X)$ is the global category associated with a constructible sheaf (in homotopy sense) $\mathcal{E}_L$ of smooth compact dg categories on

$L$ depending only on the local geometry. In terms of dg algebras, $A_X$ is a homotopy colimit of a finite diagram of local algebras. For example, if $L$ is smooth and oriented and spin, the sheaf $\mathcal{E}_L$ is the constant sheaf of $\mathrm{Perf}(\mathbb{Z})$, and the global algebra is the algebra $\mathrm{Chains}(\Omega(L, x_0))$ considered before.

In terms of topological field theory, the stalks of $\mathcal{E}_L$ are possible boundary terms for the theory of pseudo-holomorphic discs in $X$ with boundary on $L$.

In codimension 1 singular Lagrangian $L$ looks generically as the product of a smooth manifold with the union of three rays $\{z \in \mathbb{C} \,|\, z^3 \in \mathbb{R}_{\geq 0}\}$, endowed with a natural cyclic order. The stalk of the sheaf $\mathcal{E}_L$ at such a point is $\mathrm{Perf}(A_2)$, the category of representations of quiver $A_2$ (two vertices and one directed egde). The symmetry group of $\mathrm{Perf}(A_2)$ after factoring by the central subgroup of shifts by $2\mathbb{Z}$ is equal to $\mathbb{Z}/3\mathbb{Z}$. Explicitly it can be done by the following modification of the quiver at triple points. Namely, consider the quiver with three vertices (corresponding to 3 objects $E, F, G$), a closed arrow $F \to G$ of degree 0, two arrows $E \to F$, $E \to G$ of degreess $-1$ and 0 respectively (with differential saying that we have a morphism $E \to \mathrm{Cone}(F \to G)$. We say that $E$ is quasi-isomorohic to $\mathrm{Cone}(F \to G)$, i.e.

$$\mathrm{Cone}(E \to \mathrm{Cone}(F \to G)) = 0 \ .$$

This can be done explicitly by constructing a homotopy to the identity of the above object, which is a $3 \times 3$-matrix. Combining all equations together we get a quiver with 3 vertices and 12 arrows which gives a heavy but explicit finite type model for exact triangles.

A natural example of a Lagrangian submanifold with triple point singularities comes from any union of transversally intersecting Lagarngain submanifolds $L_i \subset X$, $i = 1, \ldots, k$. For any point $x$ of intersection (or self-intersection) we should remove small discs in two branches of Lagrangian manifolds intersecting at $x$, and glue a small ball with two collars. The set of triple points forms a sphere.

Global algebra $A_L$ of finite type is Calabi-Yau if $L$ is compact, and not Calabi-Yau in general for non-compact $L$. There are many examples of (compact and non-compact) sungular Lagrangian manifolds such that

$$\mathrm{Perf}(A_L) \simeq D^b(Coh(X))$$

for some scheme $X$ of finite type over $\mathbb{Z}$ (maybe singular and/or non-compact). In the pictures at the end we collected several examples of this “limiting mirror symmetry”. Categories of type $A_L$ one can consider as “non-commutative spaces of finite type” defined combinatorially, without parameters. Among other examples one can list toric varieties, maximally degenerate stable curves, etc.

### Deforming degenerate Fukaya categories

Let us assume that $X$ is compact, in fact a complex projective manifold, and take a complement $X^o$ to an ample divisor. New manifold $X^o$ is Stein, and can be contracted to a singular Lagrangian $L \subset X^o$. The advantage of $X^o$ is that is has no continuous parameters as a symplectic manifold. As was advocated by P. Seidel several years ago, one can think of $\mathcal{F}(X)$ as a deformation of $\mathcal{F}(X^o)$. For example, if $X$ is a two-dimensional torus (elliptic curve) and $X^o$ is the complement to a finite set, then $\mathcal{F}(X^o)$ is equivalent to $\mathrm{Perf}(Y_0)$ where $Y_0$ is a degenerate elliptic curve, a chain of copies of $\mathbb{P}^1$.

In algebraic terms, holomorphic discs in $X$ give a solution of the Maurer-Cartan equation

$$d\gamma + [\gamma,\gamma]/2 = 0, \ \ \gamma \in C^\bullet(A_L, A_L)\widehat{\otimes}\mathfrak{m}_{Nov}$$

where $C^\bullet(A_L, A_L) = \mathrm{Cone}(A_L \to \mathrm{Der}(A_L))$ is the cohomological Hochschild complex of smooth algebra $A_L$, and $\mathfrak{m}_{Nov}$ is the maximal ideal in the ring of integers in the Novikov field $Nov$.

Analogy with algebraic geometry suggests that different choices of open $X^o \subset X$ should lead to dg algebras of finite type endowed with deformations over $\mathfrak{m}_{Nov}$ such that algebras became (in certain sense) derived Morita equivalent after the localization to $Nov$. We expect that such a formulation will handle the cases when the deformed Fukaya category is too small, e.g. when the mirror family consists of non-algebraic varieties (e.g. non-algebraic K3 surfaces or complex tori).

### Question about automorphisms

The group of connected components of $X$ (with appropriate modications for the potential/Landau-Ginzburg/wrapped cases) acts by automorphisms of dg category $\mathcal{F}(X)$ over the local field $Nov$. One can ask whetehr this group coincides with the whole group of automorphisms. To our knowledge, there is no counterexamples to it! In principle, one can extend the group by taking the product of $X$ with the Landau-Ginzburg model $(\mathbb{C}^n, \sum_{i=1}^n z_i^2)$ which is undistinguishable categorically from a point. So, a more realistic conjecture is that the automorphism group of Fukaya category coincides with the stabilized symplectomorphism group. Why anything like this should be true?

There is an analogous statement in the (commutative) algebraic geometry. The group of automorphisms of a maximally degenerating Calabi-Yau variety $Y$ over a local non-archimedean field $K$ maps naturally to the group of integral piece-wise linear homeomorphisms of certain polytope (called the skeleton, and usally homeomorphic to a sphere). The skeleton lies intrisically in the Berkovich spectrum $Y^{an}$ where the latter is defined as the colimit of sets of points $X(K')$ over all non-archimedean field extensions $K' \supset K$. The Berkovich spectrum is a very hairy but Hausdorff topological space, and the skeleton is a naturally defined homotopy retract of $Y^{an}$.

We expect that one can define some notion of analytic spectrum for a dg algebra over a non-archimedean field, and its skeleton should be probably a piecewise symplectic manifold (maybe infinite-dimensional). For Fukaya type categories this skeleton should be the original symplectic manifold.

# Dualities and Symmetries in String Theory

Albrecht Klemm
Physikalisches Institut
Universität Bonn
Nussallee 12
53115 Bonn

Mathematische Arbeitstagung June 5-11, 2009
Max-Planck-Institut für Mathematik, Bonn, Germany

## 1 Exposition of the problem

In string theory one considers maps

$$X : \Sigma_g \to \hat{M} \tag{1}$$

from a Riemann surface $\Sigma_g$ to a target space $\hat{M}$. For simplicity we focus on orientable closed Riemann surface of genus $g$. The standard supersymmetric string theory, called type II string, has desirable symmetries at quantum level if $\dim_R(\hat{M}) = 10$. This is called the critical dimension and to describe a four dimensional gravity theory, or more precisely a four dimensional $N = 2$ supergravity theory, one considers $\hat{M} = M \times M_4$. Here $M_4$ is a large space of signature $(3, 1)$, which is to be identified with our universe, while $M$ is a three complex dimensional Calabi-Yau manifold and its typical radii are so small that according to Heisenbergs uncertainty principle one needs higher energy scales then presently explored to detect it directly in experiments. Physical amplitudes are given by variational integrals, the simplest one is the vacuum amplitude

$$Z(M) = \int \mathcal{D}X\mathcal{D}\chi e^{-S(X,\chi,M)} , \tag{2}$$

where the action $S$ is schematically

$$S = \int_\sigma G^{\mu\nu}\partial_\alpha X_\mu \partial^\alpha X_\nu + iB^{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X_\nu + \text{supersymmetric completion} . \tag{3}$$

Here $\chi$ stands for fermionic partners of the bosonic coordinate $X$, which occur in the supersymmetric completion.

Note that the variational integral over the worldsheet metric does not appear since it trivializes due to the special symmetries in the critical dimension.

On the other hand the metric $G^{\mu\nu}$ and the antisymmetric 2-form field $B^{\mu\nu}$ on $M$ are not varied over, so that $Z$ depends on them as well as on other properies of $M$, which determines the nature of physics in $M_4$. The main interest in this talk are the invariances of $Z$ if we modify its argument $M$. These are called spacetime dualities.

Note that the first term in $S$ is equivalent to area of the image curve and the critical sets of $S$ can be identified with the holomorphic maps.

Due to supersymmetric localization there exists a truncation of the theory to these critical bosonic configurations. The truncated theory is called the topological $A$-model. In the truncated theory $Z$ collapses to $Z_A$, which is given by infinite sum over topological sectors labelled by $g$ and the class of the image curve $\beta \in H_2(M,\mathbb{Z})$. The variational integral collapses in each sector into a mathematically welldefined integral over the finite dimensional moduli space of the holomophic maps $\overline{\mathcal{M}_g(M,\beta)}$. The A-model truncation is best decribed by nilpotent BRST operators, which allow to define a cohomological theory whose finite dimensional Hilbert spaces is spanned by states, which are in one to one correspondence with the de Rahm cohomology groups $H^{i,i}(M)$, $i=0,\ldots,3$. Its correlators are the cassical intersections deformed by contributions of the holomorphic maps.

The decisive $A$-model quantity is the free energy

$$F(\lambda,t)=\log(Z_A)=\sum_{g=0}^{\infty}\lambda^{2g-2}F_g(t) \tag{4}$$

with

$$F_g(t)=\text{classical}+\sum_{\beta\in H_2(M\mathbb{Z})} r^g_\beta q^\beta \ . \tag{5}$$

Here

$$r^g_\beta=\int_{\overline{\mathcal{M}_g(M,\beta)}} c_b^{vir}(M,\beta)\in\mathbb{Q} \tag{6}$$

are the Gromov-Witten invariants. They are defined as the integral of a virtual fundamental class over the compactifications of the moduli space of the holomorphic maps. The virtual dimension of the moduli space follows from an index theorem

$$\text{vdim}_{\mathbb{C}}\overline{\mathcal{M}_g(M,\beta)}=\int_\beta c_1+(\dim-3)(1-g) \ . \tag{7}$$

We note that Calabi-Yau threefolds are the critical cases as $\text{vdim}_{\mathbb{C}}\overline{\mathcal{M}_g(M,\beta)}=0$. This implies that generically a point counting problem in a moduli stack yields $r^g_\beta\neq 0$. The variable $q^\beta=\exp(t_\beta)$, where $t_\beta=2\pi i\int_\beta(b+\omega)$ is the complexified Kähler parameter. It is a complex variable build from linear deformation of the 2-form field $b=\delta B$ and the real Kähler form $\omega=i\delta G_{i\bar{\jmath}}\mathrm{d}z^i\wedge\mathrm{d}\bar{z}^{\bar{\jmath}}$. Both take values in $H^{1,1}(M,\mathbb{R})$. We note that $q^\beta\to 0$ in the limit of large volume. I.e. the large volume limit suppresses the contributions of the holomorphic maps. The classical terms are constant map contributiions which are of course independed of the volume. An important feature is, that the $A$-model, does not depend on the pure deformations of the metric $\delta G_{ij}$ and $\delta G_{\bar{\imath}\bar{\jmath}}$, which parametrize the complex structure deformations of $M$.

$F(\lambda,t)$ is a generating function for Gromov-Witten invariants. The problem that we pose here is how to calculate it and the main point of this lecture is to explain how $F(\lambda,t)$ can be reconstructed using dualities and symmetries of (2).

## 2 Other symplectic invariants and integrality conjectures

Before we focus on the main topic we notice that the mathematically well defined rational Gromov-Witten invariants $r^g_\beta$ are conjecturally related to integral BPS invariants $n^g_\beta$, which are physically motivated to be an index on the cohomology of the moduli space of $D2-D0$ branes. The relation between the $n^g_\beta \in \mathbb{Z}$ and the $r^g_\beta$ are defined by

$$Z'_A(Q,q) = \prod_\beta \left[ \left( \prod_{r=1}^\infty (1-Q^r q^\beta)^{rn^0_\beta} \right) \prod_{g=1}^\infty \prod_{l=0}^{2g-2} (1-Q^{g-l-1}q^\beta)^{(-1)^{g+r}\binom{2g-2}{l} n^g_\beta} \right] , \tag{8}$$

where $Q = e^{i\lambda}$ and the prime indicates that we are omitting the constant map contributions.

To get an impression about the key properties of the BPS invariants we listed the complete information up degree $d = 11$ in table 1 for $M$ the quintic hypersurface in $\mathbb{P}^4$. $d \in \mathbb{Z}$ represents $\beta$, in the one dimensional $H_2(M,\mathbb{Z})$ lattice. One important property is that within a fixed class $d$ there is a bound $g_{max}$ on $g$ so that $n^g_d = 0$ for $g \geq g_{max}(d)$. The bound $g_{max}$ growth assymtotically like $g_{max}(d) \propto d^2$. This a simple consequence of the adjunction formula, which implies that there are no embedded curves of genus $g$ if the degree is not high enough. The important difference between $r^g_\beta$ and $n^g_\beta$ is that the latter is a property of the embedded curve in $m$ rather then a property of the map to $M$. Puting it differently all information about the multi covering of the map into a given curve class is encoded in (8).

A simple example of the index definition of $n^g_\beta$ can be stated for smooth curves $C$, where $n^g_\beta = (-1)^{\dim \mathcal{M}_C} e(\mathcal{M}_C)$. Here $\mathcal{M}_C$ is the deformation space. For $d = 5$ and $d = 10$ and maximal genus those smooth curves are complete intersections and a simple calculation of their moduli space yields $n^6_5 = 10$ and $n^{16}_{10} = -50$.

A further relation links the above invariants to the Donaldson-Thomas invariants, which are integrals over the moduli space of ideal sheafs on $M$. Let

$$Z_{DT}(Q,q) = \sum_{\beta,k\in\mathbb{Z}} m^k_\beta Q^k q^\beta \tag{9}$$

define a generating series for the Donaldson-Thomas invariants $m^k_\beta \in \mathbb{Z}$ then the relation is given by

$$Z_{DT}(-Q,q) = Z'_A(Q,q) M(-Q)^{e(M)} , \tag{10}$$

where

$$M(Q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^n} \tag{11}$$

is the McMahon function.

| g | d=1 | d=2 | d=3 | d=4 | d=5 | d=6 |
|---|---|---|---|---|---|---|
| 0 | 2875 | 609250 | 317206375 | 242467530000 | 229305888887625 | 248249742118022000 |
| 1 | 0 | 0 | 609250 | 3721431625 | 12129909700200 | 31147299733286500 |
| 2 | 0 | 0 | 0 | 534750 | 75478987900 | 871708139638250 |
| 3 | 0 | 0 | 0 | 8625 | -15663750 | 3156446162875 |
| 4 | 0 | 0 | 0 | 0 | 49250 | -7529331750 |
| 5 | 0 | 0 | 0 | 0 | 1100 | -3079125 |
| 6 | 0 | 0 | 0 | 0 | 10 | -34500 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 |

| g | d=7 | d=8 | d=9 |
|---|---|---|---|
| 0 | 295091050570845659250 | 375632160937476603550000 | 503840510416985243645106250 |
| 1 | 715784060228807617500 | 1549905417529615684181250 | 3240644643102795856570087500 |
| 2 | 518546255661726962500 | 2251684106310591776675000 | 8146492178683956650256012500 |
| 3 | 111468926053022750 | 1303464598408583455000 | 952321365916921756899150 |
| 4 | 245477430615250 | 255175022548342267500 | 507723496514433561498250 |
| 5 | -1917984531500 | 46569889619570625 | 102807435944931083197500 |
| 6 | 1300955250 | -471852100909500 | 3088416419587021725 |
| 7 | 4874000 | 2876330661125 | -135197508177440750 |
| 8 | 0 | -1670397000 | 1937652290971125 |
| 9 | 0 | -6092500 | -12735865055000 |
| 10 | 0 | 0 | 18763368375 |
| 11 | 0 | 0 | 5502750 |
| 12 | 0 | 0 | 60375 |
| 13 | 0 | 0 | 0 |

| g | d=10 | d=11 |
|---|---|---|
| 0 | 704288164978454686113488249750 | 1017913203569692432490203659468875 |
| 1 | 6628637743914140967424065763000 | 13364420917354630676080163129237500 |
| 2 | 26191063952867325909554513745 | 7757206271485037501990496914497500 |
| 3 | 52939966189791662442040406825 | 245749672908222069996115276347500 |
| 4 | 56466902231186386829298566000 | 448475557200658307168403004753750 |
| 5 | 302653046360802682731297875 | 469508660948449138653717762000 |
| 6 | 694875009474861138496273 | 2677897642168417601686913816250 |
| 7 | 401795199961582390768000 | 73570992429520702387088700000 |
| 8 | -25301032766083303150 | 727426515993680028977012500 |
| 9 | 115559306273927142 | 1409659857957326934400000 |
| 10 | -17976209529424700 | 72285071203117009200 |
| 11 | 150444095741780 | -189989552574821712500 |
| 12 | -454092663150 | 353650228902738500 |
| 13 | 50530375 | -4041708780324500 |
| 14 | -286650 | 22562306494375 |
| 15 | -5700 | -29938013250 |
| 16 | -50 | -7357125 |
| 17 | 0 | -86250 |
| 18 | 0 | 0 |

Table 1: BPS invariants $n^g_\beta$ on the Quintic hypersurface in $\mathbb{P}^4$

# 3 The duality symmetries

## 3.1 Mirror symmetry

Mirror symmetry can be summarized by the statement that

$$\begin{aligned} Z_A(M,\lambda,t) &= Z_B(W,\lambda,\hat{t}), \\ Z_A(W,\lambda,t) &= Z_B(M,\lambda,\hat{t}), \end{aligned} \tag{12}$$

here $(W, M)$ are mirror pairs of manifolds with

$$H^{3-k,p}(M) = H^{k,p}(W), \tag{13}$$

for $k, p = 0, \ldots, 3$. $B$ stands for the topological $B$-model. It emerges by a different localisation of the full variational integral $Z(M)$ to constant maps albeit with a more complicated measure. Mirror symmetry identifies the $A$-model on $M$ with the $B$-model on $W$ and vice versa. The topological states of the $B$ model are in correspondence with the cohomology groups dual (13) to ones which define the states of the $A$-model. The B-model depends only on the complex structure variations $\hat{t}$ of the corresponding manifold. The latter are encoded in period integrals over the holomorphic $(3,0)$-form. Studying the latter at a point of maximal degeneration yields also a concrete expression for the mirror map $\hat{t}(t)$ in (12). It should be noted that (12) is a specialized version of mirror symmetry, which is designed to be mathematically controllable. The physical expectation is simply that string theory on $M$ and on $W$ are indistiguishable.

The construction of mirror manifolds is understood conceptually in symplectic geometry, by the SYZ conjecture, which states that every Calabi-Yau manifold is a (degenerate) Lagrangian $T^3$ fibration over a 3-dim base and that the mirror can be constructed by dualizing the $T^3$ torus fibrewise. Pragmatically thousands of mirror pairs can be easily constructed within the framework of algebraic geometry as anticanonical hypersurfaces in pairs of toric varities defined by pairs of reflexive polyhedra as pointed out by Batyrev

## 3.2 Periods and monodromy

We discuss now the monodromy of one paramter family of mirror quintics $W(\hat{t})$,

$$W(\hat{t}) = \left\{ p = \sum_{i=1}^{5} x_i^5 - 5e^{-\frac{\hat{t}}{5}} \prod_{i=1}^{5} x_i = 0 \text{ in } \mathbb{P}^4 \right\} . \tag{14}$$

It can be obtained as orbifold $M/\mathbb{Z}_5^3$ of the original quintic $M$, where the $\mathbb{Z}_5$'s are generated by phase rotations on the homogeneous coordinates $\mathbb{P}^4$

$$x_i \to \exp(2\pi i g_i^{(\alpha)}/5) x_i, \quad \alpha = 1,2,3, \quad i = 1,\ldots,5 \ , \tag{15}$$

with $g^{(1)} = (1,4,0,0,0)$, $g^{(2)} = (1,0,4,0,0)$ and $g^{(3)} = (1,0,0,4,0)$. We identify $z = e^{\hat{t}}$ and notice that the complex moduli space is parametrized by $z$ as $\mathcal{M} = \mathbb{P} \setminus \{z = 0, 1, \infty\}$.

The holomorphic $(3,0)$-form is locally $\Omega = \frac{z^{-\frac{1}{5}} x_i \wedge_{k\neq i,j} \mathrm{d}x_k}{\partial_j p}$. There is a flat connection on the period vector

$$\Pi = \begin{pmatrix} \int_{A^I} \Omega = X^I \\ \int_{B_I} \Omega = P_I = \frac{\partial F_0}{\partial X^I} \end{pmatrix}, \qquad , I = 0, \ldots, 3 \tag{16}$$

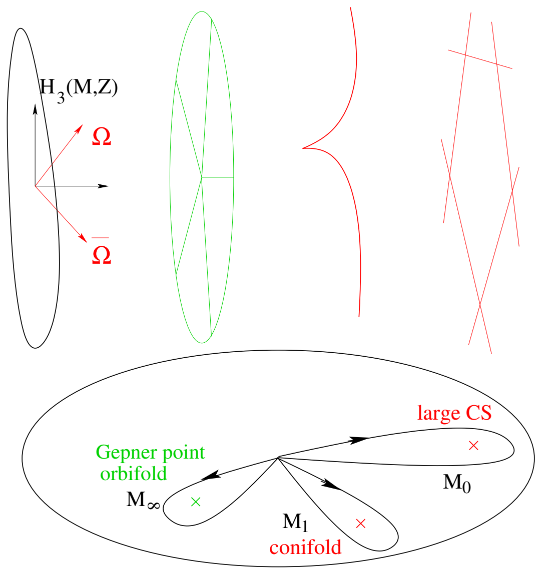
expressed by the PIcard-Fuchs equation

$$[\theta^4 - 5z \prod_{k=1}^{4} (\theta + k)]\Pi(z) = 0, \qquad \theta = z\frac{\mathrm{d}}{\mathrm{d}z}, \tag{17}$$

which undergoes the monodromies $\Pi \mapsto M_i \Pi$ with $M_{z=z_i} \in SP(4, \mathbb{Z})$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & -3 & 1 & -1 \\ -8 & -5 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

generate the monodromy group $\Gamma_M$, where the loops are schematically




Mirror quintic family

### 3.3 $g = 0$

The first sucess of mirror symmetry is that

$$F_0(t) = class. + \sum_{d=1} n_d^0 \mathrm{Li}_3(q^d), \tag{19}$$

where the mirror map at large complex strcuture $(CS)$ $z = 0$ is

$$t = \frac{X^1}{X^0}(z) \ , \tag{20}$$

where $X^0 = 1 + holom$ and $\frac{1}{2\pi i}(X^0 \log(z) + holom.)$ are completly determined from (17).

In the complex moduli space one has special geometry, with Kählerpotential $e^{-K} = i \int \Omega \wedge \bar\Omega$, $C_{ijk} = \int \Omega \partial_i \partial_j \partial_k \Omega = D_i D_j D_K F_0$ and the integrability condition

$$R^i_{k\bar l m} = \delta^i_k g_{\bar l m} + \delta^i_m g_{\bar l k} + C_{kmj} \bar C^{ij}_{\bar l} \tag{21}$$

with $\bar C^{ij}_{\bar l} = \bar C_{\bar l \bar k \bar l} g^{\bar m i} g^{\bar k j} e^{2K}$.

### 3.4 $g = 1$

The genus one amplitude is a Ray-Singer-Torsion family index over $\mathcal{M}$ and fullfills

$$\partial_i \bar\partial_{\bar j} F_1 = \frac{1}{2} \bar C^{mn}_{\bar j} C_{imn} - \left( \frac{e(m)}{24} - 1 \right) g_{i\bar j} \ . \tag{22}$$

It can be fixed by the boundary behaviour $F_1 \sim \frac{1}{12}\log(t_c)$, where $t_c$ is the flat coordinate near the conifold and $F_1 \sim 50\frac{t}{24}$ near large complex strcuture.

### 3.5 $g > 1$

For higher genus the $F_g$ fullfill the holomorphic anomaly equation

$$\partial_{\bar i} F_g = \frac{1}{2} \bar C^{mn}_{\bar i} \left( D_m D_n F_{g-1} + \sum_{r=1}^{g-1} D_m F_r D_n F g - r \right) \tag{23}$$

It has an holomorphic function as an ambiguity. The latter can be fixed by the fact that $F_g$ is modular invariant and physical boundary conditions. The first fact implies that the $F_g$ are finetly generated by a ring which can be viewed as the generalization of the ring of almost holomorphic modular forms from elliptic curves to Calabi-Yau manifolds.

In local flat coordinates the leading behaviour at the boundaries is as follows

- Expansion around the conifold point $z = 1$:

$$\begin{aligned}
F_0^c &= -\tfrac{5}{2}\log(\hat t_c)\hat t_c^2 + \tfrac{5}{12}\,(1 - 6b_1)\,\hat t_D^3 \\
&\quad + \left(\tfrac{5}{12}\,(b_1 - 3b_2) - \tfrac{89}{1440} - \tfrac{5}{4}\,b_1^2\right)\hat t_c^4 + \mathcal{O}(\hat t_c^5) \\
F_1^c &= -\tfrac{\log(\hat t_c)}{12} + \left(\tfrac{233}{120} - \tfrac{113\,b_1}{12}\right)\hat t_c \\
&\quad + \left(\tfrac{233\,b_1}{120} - \tfrac{113\,b_1{}^2}{24} - \tfrac{107 b_2}{12} - \tfrac{2681}{7200}\right)\hat t_c^2 + \mathcal{O}(\hat t_c^3) \\
F_2^c &= \tfrac{1}{240\hat t_c^2} - \left(\tfrac{120373}{72000} + \tfrac{11413 b_2}{144}\right) \\
&\quad + \left(\tfrac{107369}{150000} - \tfrac{120373\,b_1}{36000} + \tfrac{23533\,b_2}{720} - \tfrac{11413 b_1 b_2}{72}\right)\hat t_c + \mathcal{O}(\hat t_c^2) \\
F_3^c &= \tfrac{1}{1008\,\hat t_c^4} - \left(\tfrac{178778753}{324000000} + \tfrac{2287087\,b_2}{43200} + \tfrac{1084235\,b_2{}^2}{864}\right) + \mathcal{O}(\hat t_c) \\
F_4^c &= \tfrac{1}{1440\,\hat t_c^6} - \left(\tfrac{977520873701}{3402000000000} + \tfrac{162178069379\,b_2}{3888000000}\right. \\
&\quad \left. + \tfrac{5170381469\,b_2{}^2}{2592000} + \tfrac{490222589\,b_2{}^3}{15552}\right) + \mathcal{O}(\hat t_c)\ . \\
F_g^{\text{conifold}} &= \tfrac{(-1)^{g-1} B_{2g}}{2g(2g-2)(\hat t_c)^{2g-2}} + \mathcal{O}(\hat t_c^0).
\end{aligned}$$

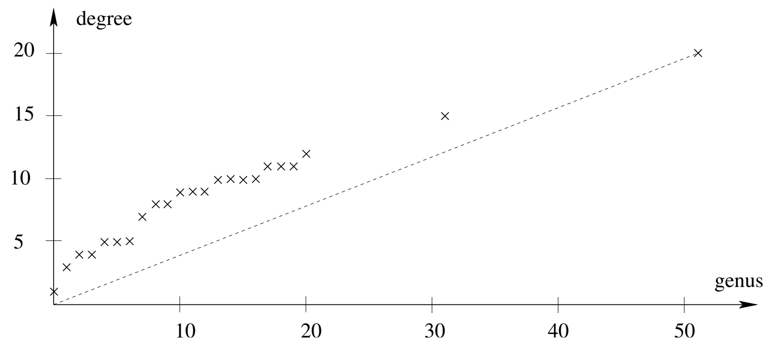
I.e. at the conifold we have the gap condition that the $2g-2$ subleading coefficients are absent.

- Expansions around the orbifold point $\frac{1}{z}=0$

$$
\begin{aligned}
F_0^{\rm o} &= \tfrac{5\,s^3}{6} + \tfrac{5\,s^8}{1008} + \tfrac{5975\,s^{13}}{10378368} + \tfrac{34521785\,s^{18}}{266765571072} + \ldots \\
F_1^{\rm o} &= -\tfrac{s^5}{9} - \tfrac{163\,s^{10}}{18144} - \tfrac{85031\,s^{15}}{46702656} - \tfrac{6909032915\,s^{20}}{20274183401472} + \ldots \\
F_2^{\rm o} &= \tfrac{155\,s^2}{18} - \tfrac{5\,s^7}{864} + \tfrac{585295\,s^{12}}{14370048} + \tfrac{1710167735\,s^{17}}{177843714048} + \ldots \\
F_3^{\rm o} &= \tfrac{488305\,s^4}{9072} - \tfrac{3634345\,s^9}{979776} - \tfrac{1612981445\,s^{14}}{7846046208} - \tfrac{2426211933305\,s^{19}}{116115777662976} + \ldots \\
F_4^{\rm o} &= \tfrac{48550\,s}{567} + \tfrac{36705385\,s^6}{163296} + \tfrac{16986429665\,s^{11}}{603542016} + \tfrac{341329887875\,s^{16}}{70614415872} + \ldots
\end{aligned}
$$

I.e. at the orbifold point we have the constion that $F_g$ behaves regular. The coefficients of the expansion in the flat coordinate $s$ are the orbifold Gromov-Witten invariants and some checks using direct computations of the latter have been made.

It can be shown that these boundary conditions fix $\left[\frac{2g-1}{5}\right]+2g-2$ constant in the holomorphic or modular ambiguity, which is parametrized by $3g-3$ coeffcients. If one uses the fact that $n^g_d=0$ for $g>g_{max}$ one can solve the equation (22) up to genus 51 as can be seen from the follwing figure




## References


[HKQ] M. x. Huang, A. Klemm and S. Quackenbush, "Topological String Theory on Compact Calabi-Yau: Modularity and Boundary Conditions," Lect. Notes Phys. **757** (2009) 45 [arXiv:hep-th/0612125].

# Manifolds of Positive Curvature

**Wolfgang Ziller**

**Arbeitstagung, Bonn, June 2009**

We discussed recent joint work with Karsten Grove and Luigi Verdiani in which we construct a new example with positive sectional curvature in dimension 7:

THEOREM A. *There exists a seven dimensional manifold $P$ with positive sectional curvature with the following properties:*

(a) *$P$ is homeomorphic to the unit tangent bundle of $\mathbb{S}^4$.*
(b) $\mathrm{SO}(4)$ *acts isometrically on $P$ with one dimensional quotient (a so called cohomogeneity one manifold).*
(c) *There exists an orbifold principal fibration* $\mathrm{SU}(2) \to P \to \mathbb{S}^4$, *where* $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ *acts almost freely on $P$.*

We do not know wether the manifold $P$ is diffeomorphic to the unit tangent bundle or not.

The orbifold structure on the base is as follows: The metric is smooth, except along a standard Veronese embedding $\mathbb{RP}^2 \subset \mathbb{S}^4$, where normal to the surface the metric has an angle $2\pi/3$. The quotient is thus homeomorphic to $\mathbb{S}^4$.

The positively curved metric is a Kaluza Klein metric (sometimes called connection metric) in the orbifold principal fibration in (c). It is thus described by a metric on the base and a principal connection. Due to (b) it is sufficient to describe the metric along a geodesic orthogonal to all orbits. Along this geodesic our metric and principal connection is given by piecewise polynomial functions of degree at most 5.

The proof that the metric has positive curvature is obtained by using Thorpe's method. Here one modifies the curvature operator $\hat{R}\colon \Lambda^2 T \to \Lambda^2 T$ with a curvature type endomorphism $\hat{\alpha}\colon \Lambda^2 T \to \Lambda^2 T$ induces by a 4-form $\alpha \in \Lambda^4 T$. If the modified curvature operator $\hat{R} + \hat{\alpha}$ is positive definite, the sectional curvature is positive. We construct an explicit 4-form consisting of piecewise rational functions and combine Sylvester's theorem and Sturm's theorem to show that the minor determinants are all positive.

The example fits into an infinite family of "candidates" coming from the following classification theorem:

THEOREM B (Verdiani, $n$ even, Grove-Wilking-Ziller, $n$ odd). *Let $M$ be a positively curved compact simply connected manifold on which $G$ acts isometrically with* $\dim M/G = 1$. *Then $M^n$ is equivariantly diffeomorphic to one of the following:*

(a) *A rank one symmetric space with a linear action of $G$.*
(b) *One of the normal homogeneous manifolds of positive curvature, or certain positively curved Eschenburg or Bazaikin spaces which admit a cohomogeneity one action.*
(c) *One of the seven dimensional manifolds $P^7_k, Q^7_k$, $k \geq 1$, or $R^7$ with* $G = \mathrm{SO}(4)$.

The manifolds in part (c) are not yet known to admit positive curvature, although $P_1 = \mathbb{S}^7$ and $Q_1 = \mathrm{SU}(3)/\mathrm{S}^1$ (an Aloff-Wallach space) do. Our new example is the manifold $P_2$.

Two theorem's by Grove-Ziller imply that all candidates in (c) admit a $G$ invariant metric with non-negative sectional curvature and one of positive Ricci curvature as well.

The manifolds $P_k$ are 2-connected with $\pi_3(P_k) = \mathbb{Z}_k$, and thus rational homology spheres. A finiteness theorem due to Petrunin-Tuschmann and Fang-Rong implies that the pinching constants $\delta_k$, i.e. $0 < \delta_k \leq \sec \leq 1$, for any positively curved metric on $P_k$ would necessarily go to 0 as $k \to \infty$, and $P_k$ would be the first examples of this type.

For the manifolds in part (c), $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ acts almost freely and they are thus all the total space of principal orbifold bundles over $\mathbb{S}^4$ for $P_k$ and over $\mathbb{CP}^2$ for $Q_k$ and $R$. The total space in case of $P_k$ and $Q_k$ admit so called 3-Sasakian metrics, which already have lots of positive curvature by definition, and the induced metric on the base is the self dual Einstein orbifold metric constructed by Hitchin. The bundle can also be described, up to a 2-fold cover, as the frame bundle of the vector bundle of self dual 2-forms on the base. Our metric on $P = P_2$ is a deformation of the 3-Sasakian metric.

# CATEGORICAL $\mathfrak{sl}_2$ ACTIONS

ANTHONY LICATA

## 1. Introduction

1.1. **Actions of $\mathfrak{sl}_2$ on categories.** A action of $\mathfrak{sl}_2$ on a finite-dimensional $\mathbb{C}$-vector space $V$ consists of a direct sum decomposition $V = \oplus V(\lambda)$ into weight spaces, together with linear maps

$$e(\lambda) : V(\lambda-1) \to V(\lambda+1) \text{ and } f(\lambda) : V(\lambda+1) \to V(\lambda-1)$$

satisfying the condition

$$e(\lambda-1)f(\lambda-1) - f(\lambda+1)e(\lambda+1) = \lambda \mathrm{I}_{V(\lambda)}. \tag{1}$$

Such an action automatically integrates to the group $SL_2$. In particular, the reflection element

$$t = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in SL_2$$

acts on $V$, inducing an isomorphism $V(-\lambda) \to V(\lambda)$.

A first pass at a categorification of this structure involves replacing vector spaces with categories and linear maps with functors. Thus, a naïve categorification of a finite dimensional $\mathfrak{sl}_2$ module consists of a sequence of categories $\mathcal{D}(\lambda)$, together with functors

$$\mathsf{E}(\lambda) : \mathcal{D}(\lambda-1) \to \mathcal{D}(\lambda+1) \text{ and } \mathsf{F}(\lambda) : \mathcal{D}(\lambda+1) \to \mathcal{D}(\lambda-1)$$

between them. These functors should satisfy a categorical version of (1) above,

$$\mathsf{E}(\lambda-1) \circ \mathsf{F}(\lambda-1) \cong \mathrm{I}_{\mathcal{D}(\lambda)}^{\oplus\lambda} \oplus \mathsf{F}(\lambda+1) \circ \mathsf{E}(\lambda+1), \quad \text{for } \lambda \geq 0, \tag{2}$$

and an analogous condition when $\lambda \leq 0$. The sense in which this is naïve is that ideally there should be specified natural transformations which induce the isomorphisms (2).

## 2. Chuang-Rouquier's definition of $\mathfrak{sl}_2$-categorification

In order to get a good theory of $\mathfrak{sl}_2$-categorification, we need to define the algebraic structure arising from natural transformations between various compositions of the functors $\mathsf{E}$ and $\mathsf{F}$. The first such definition, due to Joe Chuang and Raphael Rouquier [CR], is given below. (In the definition, as well as in some later parts of the abstract, we will omit the $\lambda$ from the notation, writing $\mathsf{E}$ and $\mathsf{F}$ instead of $\mathsf{E}(\lambda)$ and $\mathsf{F}(\lambda)$.

**Definition 2.1.** *An $\mathfrak{sl}_2$ categorification consists of a finite length abelian category $\mathcal{A}$, together with exact functors $\mathsf{E}, \mathsf{F} : \mathcal{A} \to \mathcal{A}$ such that:*

(i) *$\mathsf{E}$ is a left and right adjoint to $\mathsf{F}$;*
(ii) *The action of $[\mathsf{E}]$ and $[\mathsf{F}]$ on $V = K_{\mathbb{C}}(\mathcal{A})$ induces a locally finite action of $\mathfrak{sl}_2$;*
(iii) *We have a decomposition $\mathcal{A} = \bigoplus_{\lambda\in\mathbb{Z}} \mathcal{A}_\lambda$ such that $K_{\mathbb{C}}(\mathcal{A}_\lambda) = V_\lambda$ is a weight space of $V$.*

*We also require natural transformations $X : \mathsf{E} \to \mathsf{E}$ and $T : \mathsf{EE} \to \mathsf{EE}$ such that*

(i) *$T^2 = \mathrm{I}_{\mathsf{EE}}$;*
(ii) *$(T\mathrm{I}_{\mathsf{E}}) \circ (\mathrm{I}_{\mathsf{E}}T) \circ (T\mathrm{I}_{\mathsf{E}}) = (\mathrm{I}_{\mathsf{E}}T) \circ (T\mathrm{I}_{\mathsf{E}}) \circ (\mathrm{I}_{\mathsf{E}}T)$ in $\mathrm{End}(\mathsf{E}^3)$;*
(iii) *$T \circ (\mathrm{I}_E X) = (X\mathrm{I}_E) \circ T - \mathrm{I}_{EE}$;*
(iv) *$X_M \in \mathrm{End}(\mathsf{E}M)$ is nilpotent for all objects $M \in \mathcal{A}$.*

It follows that endomorphisms $X$ and $T$ induce an action of the *degenerate affine Hecke algebra* of $GL_n$ on $\mathsf{E}^n$ (and, by adjunction, on $\mathsf{F}^n$.) As a consequence of the definition, Chuang-Rouquier prove that the functor $\mathsf{E}^n$ is isomorphic to the direct sum of $n!$ copies of a single functor $\mathsf{E}^{(n)}$. Similarly, by adjunction, the functor $\mathsf{F}^n$ is isomorphic to $n!$ copies of a single functor $\mathsf{F}^{(n)}$. Thus $\mathsf{E}^{(n)}$ and $\mathsf{F}^{(n)}$ naturally categorify the divided powers $e^{(n)} = \frac{e^n}{n!}$ and $f^{(n)} = \frac{f^n}{n!}$. Chuang-Rouquier then define a complex $\Theta(\lambda)$ of functors, which they call the *Rickard complex*. The terms of the Rickard complex are

$$\Theta(\lambda)_d = \mathsf{E}^{(\lambda+d)}\mathsf{F}^{(d)},$$

and the differential $\delta : \Theta(\lambda)_d \to \Theta(\lambda)_{d-1}$ is built from the adjunction morphism $\mathsf{EF} \to \mathrm{I}$, see [CR].

**Theorem 2.2.** *(Chuang-Rouquier) The functor $\Theta(\lambda)$ defines an equivalence of categories*

$$\Theta(\lambda) : D^b(\mathcal{A}_{-\lambda}) \simeq D^b(\mathcal{A}_\lambda).$$

Futhermore, Chuang and Rouquier construct an explicit $\mathfrak{sl}_2$ categorification using direct summands of induction and restriction functors between symmetric groups. As a corollary of the above theorem, they are then able to prove Broue's abelian defect conjecture for symmetric groups.

## 3. Geometric examples of $\mathfrak{sl}_2$ categorification

There are geometric examples of categorical $\mathfrak{sl}_2$ actions which do not quite satisfy the hypotheses in the Chuang-Rouquier definition above, essentially because the underlying weight space categories are not abelian (though they are triangulated) and the degenerate affine Hecke algebra does not act naturally on $\mathsf{E}^n$ (though the *nil affine Hecke algebra* does.) In these cases, the Chuang-Rouquier definition must be modified slightly.

3.1. **Categorical $\mathfrak{sl}_2$ actions.** We begin by giving a modified definition of $\mathfrak{sl}_2$ categorification which was introduced in joint work with Sabin Cautis and Joel Kamnitzer [CKL1],[CKL2],[CKL3]. Then we will discuss the basic geometric example, which involves cotangent bundles to Grassmanians. Let $\Bbbk$ be a field. We denote by $\mathbb{P}^r$ the projective space of lines in an $r$-dimensional $\mathbb{C}$ vector space, by $\mathbb{G}(r_1, r_1 + r_2))$ the Grassmanian of $r_1$-planes in $r_1 + r_2$ space, and by$H^*(\mathbb{G}(r_1, r_1 + r_2))$ the singular cohomology of the Grassmanian, with it's grading shifted to be symmetric about 0.

A **categorical $\mathfrak{sl}_2$ action** consists of the following data:

- A sequence of $\Bbbk$-linear, $\mathbb{Z}$-graded, additive categories $\mathcal{D}(-N), \dots, \mathcal{D}(N)$ which are idempotent complete. "Graded" means that each category $\mathcal{D}(\lambda)$ has a shift functor $\langle\cdot\rangle$ which is an equivalence.
- Functors
$$\mathsf{E}^{(r)}(\lambda) : \mathcal{D}(\lambda - r) \to \mathcal{D}(\lambda + r) \text{ and } \mathsf{F}^{(r)}(\lambda) : \mathcal{D}(\lambda + r) \to \mathcal{D}(\lambda - r)$$
for $r \geq 0$ and $\lambda \in \mathbb{Z}$.
- Morphisms
$$\eta : \mathrm{I} \to \mathsf{F}^{(r)}(\lambda)\mathsf{E}^{(r)}(\lambda)\langle r\lambda\rangle \text{ and } \eta : \mathrm{I} \to \mathsf{E}^{(r)}(\lambda)\mathsf{F}^{(r)}(\lambda)\langle -r\lambda\rangle$$
$$\varepsilon : \mathsf{F}^{(r)}(\lambda)\mathsf{E}^{(r)}(\lambda) \to \mathrm{I}\langle r\lambda\rangle \text{ and } \varepsilon : \mathsf{E}^{(r)}(\lambda)\mathsf{F}^{(r)}(\lambda) \to \mathrm{I}\langle -r\lambda\rangle.$$
- Morphisms
$$\iota : \mathsf{E}^{(r+1)}(\lambda)\langle r\rangle \to \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \text{ and } \pi : \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \to \mathsf{E}^{(r+1)}(\lambda)\langle -r\rangle.$$
- Morphisms
$$X(\lambda) : \mathsf{E}(\lambda)\langle -1\rangle \to \mathsf{E}(\lambda)\langle 1\rangle \text{ and } T(\lambda) : \mathsf{E}(\lambda+1)\mathsf{E}(\lambda-1)\langle 1\rangle \to \mathsf{E}(\lambda+1)\mathsf{E}(\lambda-1)\langle -1\rangle.$$

On this data we impose the following additional conditions:

- The morphisms $\eta$ and $\varepsilon$ are units and co-units of adjunctions

(i) $\mathsf{E}^{(r)}(\lambda)_R = \mathsf{F}^{(r)}(\lambda)\langle r\lambda\rangle$ for $r \geq 0$
(ii) $\mathsf{E}^{(r)}(\lambda)_L = \mathsf{F}^{(r)}(\lambda)\langle -r\lambda\rangle$ for $r \geq 0$

- $\mathsf{E}$'s compose as
$$\mathsf{E}^{(r_2)}(\lambda + r_1)\mathsf{E}^{(r_1)}(\lambda - r_2) \cong \mathsf{E}^{(r_1+r_2)}(\lambda) \otimes_{\Bbbk} H^*(\mathbb{G}(r_1, r_1 + r_2))$$
For example,
$$\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1) \cong \mathsf{E}^{(2)}(\lambda)\langle -1\rangle \oplus \mathsf{E}^{(2)}(\lambda)\langle 1\rangle.$$
(By adjointness the $\mathsf{F}$'s compose similarly.) In the case $r_1 = r$ and $r_2 = 1$ we also require that the maps
$$\oplus_{i=0}^{r}(X(\lambda + r)^i I) \circ \iota\langle -2i\rangle : \mathsf{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r) \to \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1)$$
and
$$\oplus_{i=0}^{r}\pi\langle 2i\rangle \circ (X(\lambda + r)^i I) : \mathsf{E}(\lambda + r)\mathsf{E}^{(r)}(\lambda - 1) \to \mathsf{E}^{(r+1)}(\lambda) \otimes_{\Bbbk} H^\star(\mathbb{P}^r)$$
are isomorphisms. We also have the analogous condition when $r_1 = 1$ and $r_2 = r$.
- If $\lambda \leq 0$ then
$$\mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1) \cong \mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}).$$
The isomorphism is induced by
$$\sigma + \sum_{j=0}^{-\lambda-1}(IX(\lambda + 1)^j) \circ \eta : \mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{-\lambda-1}) \xrightarrow{\sim} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)$$
where $\sigma$ is the composition of maps
$$\begin{aligned}\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) &\xrightarrow{\eta II} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1)\langle \lambda + 1\rangle \\ &\xrightarrow{IT(\lambda)\mathrm{I}} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1)\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1)\langle \lambda - 1\rangle \\ &\xrightarrow{II\epsilon} \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1).\end{aligned}$$
Similarly, if $\lambda \geq 0$, then
$$\mathsf{E}(\lambda - 1)\mathsf{F}(\lambda - 1) \cong \mathsf{F}(\lambda + 1)\mathsf{E}(\lambda + 1) \oplus \mathrm{I} \otimes_{\Bbbk} H^\star(\mathbb{P}^{\lambda-1}),$$
with the isomorphism induced as above.
- The $X$'s and $T$'s satisfy the nil affine Hecke relations:
(i) $T(\lambda)^2 = 0$
(ii) $(IT(\lambda - 1)) \circ (T(\lambda + 1)I) \circ (IT(\lambda - 1)) = (T(\lambda + 1)I) \circ (IT(\lambda - 1)) \circ (T(\lambda + 1)I)$ as endomorphisms of $\mathsf{E}(\lambda - 2)\mathsf{E}(\lambda)\mathsf{E}(\lambda + 2)$.
(iii) $(X(\lambda + 1)I) \circ T(\lambda) - T(\lambda) \circ (IX(\lambda - 1)) = I = -(IX(\lambda - 1)) \circ T(\lambda) + T(\lambda) \circ (X(\lambda + 1))$ as endomorphisms of $\mathsf{E}(\lambda - 1)\mathsf{E}(\lambda + 1)$.
- For $r \geq 0$, we have $\mathrm{Hom}(\mathsf{E}^{(r)}(\lambda), \mathsf{E}^{(r)}(\lambda)\langle i\rangle) = 0$ if $i < 0$ and $\mathrm{End}(\mathsf{E}^{(r)}(\lambda)) = \Bbbk \cdot \mathrm{I}$.

Given a categorical $\mathfrak{sl}_2$ action, for each $\lambda \geq 0$ we may construct the Rickard complex [CKL2]
$$\Theta_* : \mathcal{D}(\lambda) \to \mathcal{D}(-\lambda).$$
The terms in the complex are
$$\Theta_s = \mathsf{F}^{(\lambda+s)}(s)\mathsf{E}^{(s)}(\lambda + s)\langle -s\rangle,$$
where $s = 0, \ldots, (N - \lambda)/2$. The differential $d_s : \Theta_s \to \Theta_{s-1}$ is given by the composition of maps
$$\mathsf{F}^{(\lambda+s)}\mathsf{E}^{(s)}\langle -s\rangle \xrightarrow{\iota\iota} \mathsf{F}^{(\lambda+s-1)}\mathsf{F}\mathsf{E}\mathsf{E}^{(s-1)}\langle -(\lambda + s - 1) - (s - 1) - s\rangle \xrightarrow{\varepsilon} \mathsf{F}^{(\lambda+s-1)}\mathsf{E}^{(s-1)}\langle -s + 1\rangle.$$
Then we have the following theorem, proved in [CKL2].

**Theorem 3.1.** *Suppose the underlying weight space categories $\mathcal{D}(\lambda)$ are triangulated. Then complex $\Theta_*$ has a unique convolution $\mathsf{T}$, and $\mathsf{T} : \mathcal{D}(-\lambda) \longrightarrow \mathcal{D}(\lambda)$ is an equivalence of triangulated categories.*

3.2. **A Geometric Example.** The basic example of a categorical $\mathfrak{sl}_2$ action comes from Grassmanian geometry, and we refer to [CKL2] for complete details.

Fix $N > 0$. For our weight spaces we will take the derived category of coherent sheaves on the cotangent bundle to the Grassmannian $T^\star\mathbb{G}(k, N)$. We use shorthand $Y(\lambda) = T^\star\mathbb{G}(k, N)$, where $k = (N - \lambda)/2$. These spaces have a particularly nice geometric description,

$$T^\star\mathbb{G}(k, N) \cong \{(X, V) : X \in \mathrm{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } \mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0\},$$

where $\mathrm{End}(\mathbb{C}^N)$ denotes the space of complex $N \times N$ matrices. (The notation $\mathbb{C}^N \xrightarrow{X} V \xrightarrow{X} 0$ means that $X(\mathbb{C}^n) \subset V$ and that $X(V) = 0$.) Forgetting $X$ corresponds to the projection $T^\star\mathbb{G}(k, N) \rightarrow \mathbb{G}(k, N)$ while forgetting $V$ gives a resolution of the variety

$$\{X \in \mathrm{End}(\mathbb{C}^N) : X^2 = 0 \text{ and } \mathrm{rank}(X) \leq \min(k, N - k)\}.$$

On $T^\star\mathbb{G}(k, N)$ we have the tautological rank $k$ vector bundle $V$ as well as the quotient $\mathbb{C}^N/V$.

To describe the kernels $\mathcal{E}$ and $\mathcal{F}$ we will need the correspondences

$$W^r(\lambda) \subset T^\star\mathbb{G}(k + r/2, N) \times T^\star\mathbb{G}(k - r/2, N)$$

defined by

$$\begin{aligned} W^r(\lambda) := \{(X, V, V') :& X \in \mathrm{End}(\mathbb{C}^N), \dim(V) = k + \frac{r}{2}, \dim(V') = k - \frac{r}{2}, \\ & 0 \subset V' \subset V \subset \mathbb{C}^N,\ \ \mathbb{C}^N \xrightarrow{X} V' \text{ ,and } V \xrightarrow{X} 0\}. \end{aligned}$$

(Here, as before, $\lambda$ and $k$ are related by the equation $k = (N - \lambda)/2$).

There are two natural projections $\pi_1 : (X, V, V') \mapsto (X, V)$ and $\pi_2 : (X, V, V') \mapsto (X, V')$ from $W^r(\lambda)$ to $Y(\lambda - r)$ and $Y(\lambda + r)$ respectively. Together they give us an embedding

$$(\pi_1, \pi_2) : W^r(\lambda) \subset Y(\lambda - r) \times Y(\lambda + r).$$

On $W^r(\lambda)$ we have two natural tautological bundles, namely $V := \pi_1^*(V)$ and $V' := \pi_2^*(V)$, where the prime on the $V'$ indicates that the vector bundle is the pullback of the tautological bundle by the second projection. We also have natural inclusions

$$0 \subset V' \subset V \subset \mathbb{C}^N \cong \mathcal{O}_{W^r(\lambda)}^{\oplus N}.$$

We now define the kernel $\mathcal{E}^{(r)}(\lambda) \in D(Y(\lambda - r) \times Y(\lambda + r))$ by

$$\mathcal{E}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(\mathbb{C}^N/V')^{-r} \det(V)^r \{\frac{r(N - \lambda - r)}{2}\}.$$

Similarly, the kernel $\mathcal{F}^{(r)}(\lambda) \in D(Y(\lambda + r) \times Y(\lambda - r))$ is defined by

$$\mathcal{F}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(V'/V)^{\lambda} \{\frac{r(N + \lambda - r)}{2}\}.$$

These kernels define functors (Fourier-Mukai transforms) $\mathsf{E}^{(k)}$ and $\mathsf{F}^{(k)}$, and in [CKL2] we define natural transformations which enhance these functors to a full categorical $\mathfrak{sl}_2$ action.

As a result, we may define the Rickard complex $\Theta$. Convolution with this complex gives new equivalences of triangulated categories between categories corresponding to opposite $\mathfrak{sl}_2$ weight spaces.

**Corollary 3.2.** [CKL3] *The complex $\Theta$ defines an equivalence between derived categories of coherent sheaves of cotangent bundles to dual Grassmanians*

$$\Theta : D(T^*(G(k, N)) \simeq D(T^*(G(N - k, N)).$$

## 4. Further Develepements

The notion of $\mathfrak{sl}_2$-categorification goes back at least to the paper [BFK], which inspired much of the subsequent work on algebraic aspects of categorification. After the seminal contribution [CR], which contains several algebraic examples of $\mathfrak{sl}_2$ categorifications, various geometric aspects of categorical $\mathfrak{sl}_2$ representation theory were develeoped in [CKL1], [CKL2], [CKL3].

On the other hand, it is quite natural to categorify the entire quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, rather than just the finite dimensional representations. This has been accomplished by Rouquier [R], and by Lauda [L]. Moreover, the entire story can be generalized and repeated, with the lead actor $\mathfrak{sl}_2$ replaced by an arbitrary symmetrizable Kac-Moody Lie algbera $\mathfrak{g}$. This is the subject of the significant work of Khovanov-Lauda [KL] and, independently, Rouquier [R].

## References

[BFK] J. Bernstein, I. Frenkel, and M. Khovanov, A categorification of the Temperley-Lieb algebra and Schur quotients of $U(sl_2)$ via projective and Zuckerman functors, *Selecta Math.* (5) (1999), 199–241; math.QA/0002087.

[CR] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, *Ann. of Math.* **167** (2008), no. 1, 245–298; math.RT/0407205.

[CKL1] S. Cautis, J. Kamnitzer and A. Licata, Categorical geometric skew Howe duality, to appear.

[CKL2] S. Cautis, J. Kamnitzer and A. Licata, Derived equivalences for cotangent bundles of Grassmannians via categorical $\mathfrak{sl}_2$ actions, to appear.

[CKL3] S. Cautis, J. Kamnitzer and A. Licata, Coherent Sheaves and Categorical $\mathfrak{sl}_2$ Actions, to appear.

[KL] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, II, and III; math.QA/0803.4121, math.QA/0804.2080, and math.QA/0807.3250.

[L] A. Lauda, A categorification of quantum $\mathfrak{sl}_2$, arXiv:0803.3652v2.

[R] R. Rouquier, 2-Kac-Moody algebras; math.RT/0812.5023.

# The Kervaire invariant problem, after Mike Hill (Virginia), Mike Hopkins (Harvard) and Doug Ravenel (Rochester)

Peter Teichner

June 11, 2009

The authors recently proved that $\theta_j$ does not exist for $j > 6$. Here $\theta_j$ is a hypothetical element of order 2 in the stable homotopy groups of spheres in dimension $2^{j+1} - 2$.
In 1960, Kevaire defined a $\mathbb{Z}/2$-valued invariant for closed, smooth manifolds with a stable framing. In geometric terms, the above result means that the only possible dimensions for such manifolds with nontrivial Kervaire invariant are

$$2, 6, 14, 30, 62, 126$$

The first 5 dimensions were previously known to be realized, the first 3 by $S^j \times S^j$ for $j = 1, 2, 3$. The status of $\theta_6$ (in dimension 126) remains open.
The theorem implies that the kernel and cokernel of the Kervair-Milnor map

$$\Theta_n \to \pi_n^{st}/im(J)$$

are completely known finite abelian groups. Here $\Theta_n$ is the group of exotic smooth structures on $S^n$ and the map associates to it the underlying framed manifold. The image of $J : KO_{n+1} \to \pi_n^{st}$ realizes the different choices of framings on such homotopy spheres.

For further details see:
`http://www.math.rochester.edu/u/faculty/doug/kervaire.html`

# Hecke orbits

Frans Oort

June 2009

**Arbeitstagung Bonn, 2009**

This is a report on work, mostly *joint with Ching-Li Chai.*

**1) Moduli spaces and Hecke orbits.** We write $\mathcal{A}_g \to \mathrm{Spec}(\mathbb{Z})$ for the moduli space of polarized abelian varieties. However from §3 we will write $\mathcal{A}_g$ instead of $\mathcal{A}_g \otimes \mathbb{F}_p$, the moduli space of polarized abelian varieties in characteristic $p$.

Let $[(A, \mu)] = x \in \mathcal{A}_g$. We say that $[(B, \nu)] = y$ is in the Hecke orbit of $x$, notation $y \in \mathcal{H}(x)$, if there exists a diagram

$$(A, \mu)_\Omega \xleftarrow{\varphi} (C, \zeta) \xrightarrow{\psi} (B, \nu)_\Omega;$$

here $A$ and $B$ are in the same characteristic, $\Omega$ is some algebraically closed field, and $\varphi : C \to A$ and $\psi : C \to B$ are isogenies such that

$$\varphi^*(\mu) = \zeta = \psi^*(\nu).$$

If moreover the degrees of $\varphi$ and $\psi$ are both some power of a prime number $\ell$, different from a given $p$, we write $y \in \mathcal{H}_\ell(x)$.

If $A$ and $B$ are both in characteristic $p$ and $\varphi$ and $\psi$ are both of $\alpha_p$-coverings, then we write $y \in \mathcal{H}_\alpha(x)$.

If $A$ and $B$ are both in characteristic $p$ and $\varphi$ and $\psi$ have degrees not divisible by $p$ we write $y \in \mathcal{H}^{(p)}(x)$.

**Question.** Given $(A, \mu)$; what is the Zariski closure of the Hecke orbit $\mathcal{H}(x)$?

**2) Over $\mathbb{C}$.** In case $(A, \mu)$ is defined over $\mathbb{C}$, it is easy to see that $\mathcal{H}(x)$ is classically everywhere dense in $\mathcal{A}_g(\mathbb{C})$; hence

$$\overline{\mathcal{H}(x)} = \mathcal{A}_g \otimes \mathbb{C}.$$

**3) A theorem by Ching-Li Chai in 1995.** From now on we work in characteristic $p$. We say an abelian variety $A$ of dimension $g$ is *ordinary* if $A(k)[p] \cong (\mathbb{Z}/p)^g$. We say an elliptic curve is *supersingular* if it is not ordinary. The following facts are not difficult to prove / well known.

**(3a)** For an *ordinary* elliptic curve $E$ its moduli point $x$ has a Hecke orbit which is everywhere dense in $\mathcal{A}_1$. In this case even $\mathcal{H}_\ell(x)$ is everywhere dense in $\mathcal{A}_1$ for every prime number $\ell \neq p$.

**(3b)** For a *supersingular* elliptic curve its moduli point $x \in \mathcal{A}_{1,1}$ has a Hecke orbit which is nowhere dense in $\mathcal{A}_1$. In fact, $\mathcal{H}(x) \cap \mathcal{A}_{1,1}$ is finite.

We see that in general, and in contrast with characteristic zero, a Hecke orbit need not be dense in the moduli space. What can we expect? What is the Zariski closure of a Hecke orbit?

**(3b)** **Theorem,** Chai, 1995, see [1]. *For an ordinary abelian variety $A$ the Hecke orbit of $(A, \mu)$ is everywhere Zariski dense in the moduli space.*

This is a deep result. The proof uses various methods, the most crucial being showing that the closure of the Hecke orbit in $\mathcal{A}_{g,1}$ contains, the "cusp at infinity". A tricky computation then shows that around this point the Hecke orbit is dense.

**(3c)** In this paper by Chai we find the following remark by M. Larsen. *Let $(E, \lambda)$ be an ordinary elliptic curve with its principal polarization. It is not difficult to show that the Hecke orbit of $(A, \mu) := (E, \lambda)^g$ is everywhere dense in the moduli space.*

**4) Methods and ideas.** We like to determine the Zariski closure of every Hecke orbit in positive characteristic. Perhaps the question is not so interesting, but we will see that methods developed in order to answer this question give insight into structure of $\mathcal{A}_g \otimes \mathbb{F}_p$.

- Structure of $A[p^\infty]$ carries information about $A$.
- This is used to define *two stratifications* and *two foliations* of $\mathcal{A}_g$. E.g. see [8], [12] and [14]. Interplay between these will provide useful information.
- Note that this information is typical for characteristic $p$ geometry. We do not have "continuous" paths, nor complex uniformization, but we do have quite a lot of other structure, which enables us to study properties in characteristic $p$.
- We use "interior boundaries": instead of degenerating the abelian varieties, we can "make the $p$-structure more special".
- At ordinary points we have Serre-Tate canonical coordinates. These can be generalized to "central leaves" of $\mathcal{A}_g$.
- Every abelian variety over a finite field admits sufficiently many Complex Multiplications (as Tate showed). However a new notion "hypersymmetric abelian varieties" is more restrictive, see [4]. Such cases can be considered as analogous to abelian varieties of CM-type in characteristic zero.
- As in [1] the method of Hilbert Modular Varieties will be of technical importance.

**5) Newton polygons.** A Newton polygon for an abelian variety is a polygon

- starting at $(0, 0)$, and ending at $(h = 2g, d = g)$,
- lower convex,
- with breakpoints in $\mathbb{Z} \times \mathbb{Z}$,

- and slopes $\beta$ wit $0 \leq \beta \leq 1$.
- A NP is called symmetric if the slopes $\beta$ and $1-\beta$ appear with the same multiplicity.

Every $p$-divisible group in characteristic $p$ determines a Newton Polygon; basically its slopes are given as "the $p$-adic values of the eigenvalues of the Frobenius morphism". This statement is correct over $\mathbb{F}_p$. In general more theory is necessary in order to give the definition of the NP of a $p$-divisible group. See [9]. For an abelian variety one defines the Newton Polygon $\mathcal{N}(A)$ to be the NP of $A[p^\infty]$; the NP of an abelian variety is symmetric (Manin, FO).

A theorem by Diedonné en Manin says that over an *algebraically closed field $k$ isogeny classes* of $p$-divisible groups are classified by Newton Polygons. See [9].

An example: we write $\sigma$ for the NP where all slopes are equal to $1/2$. This is called the supersingular Newton polygon. A non-trivial fact (Tate, FO, Shioda, Deligne): $\mathcal{N}(A) = \sigma$ if and only if $A \otimes k \sim E^g$, where $E$ is a supersingular elliptic curve.

**6) Stratifications and foliations.**

**6a) NP**: $A[p^\infty]$ up to $\sim_k$.
We write:

$$\mathcal{W}^0_\xi(\mathcal{A}_g) = \{[(A,\mu)] \mid \mathcal{N}(A) = \xi\}.$$

Here $\xi$ is a symmetric NP. These are called te open Newton Polygon strata.

**Theorem** (Grothendieck, Katz), see [8].

$$\mathcal{W}^0_\xi(\mathcal{A}_g) \subset \mathcal{A}_g$$

*is localy closed.*

The "interior boundary" of $\mathcal{W}^0_\xi(\mathcal{A}_{g,1})$ was predicted by a conjecture, the "principally polarized version" of a conjecture by Grothendieck. For proofs see [11], and [13].

**6b) Fol** $A[p^\infty]$ up to $\cong_k$.
For $x = [(A,\mu)]$ we write

$$\mathcal{C}(x) = \{[(B,\nu)] \mid \exists \Omega \ \ (A,\mu)[p^\infty]_\Omega \cong (B,\nu)[p^\infty]_\Omega, \quad T_\ell(A,\mu)_\Omega \cong T_\ell(B,\nu)_\Omega \ \ \forall \ell \neq p\}.$$

Here $\Omega$ is some algebraically closed field. This is called "the central leaf through $x$".

**Theorem.** *For $x \in \mathcal{W}^0_{\mathcal{A}_g}$:*

$$\mathcal{C}(x) \subset \mathcal{W}^0_{\mathcal{A}_g}$$

*is closed.*
See [14]. This uses the notion of "slope filtrations" as developed by T. Zink, and a theorem in [16].

An obvious remark, which will be of use later:

*if $y \in \mathcal{C}(x)$, say $y, x$ both defined over the same perfect field, then $\mathcal{C}(y) = \mathcal{C}(x)$.*

**Remark.** The "interior boundaries" of central leaves are mysterious, although S. Harashita and I have a conjecture how they should look like.

**6c) EO** $A[p]$ up to $\cong_k$.
For $(A, \mu)$, where $\mu$ is a principal polarization, we write $\varphi$ for the isomorphism class of $(A, \mu)[p] \otimes k$.

$$S_\varphi = \{[(B, \nu)] \mid \exists \Omega \ \ (A, \mu)[p]_\Omega \cong (B, \nu)[p]_\Omega\}.$$

**Theorem.**

$$S_\varphi \subset \mathcal{A}_{g,1}$$

*is a localy closed subset. Every stratum $S_\varphi$ is quasi-affine.*
See [12]. These strata are called EO-strata, where the E refers to T. Ekedahl. The "interior boundaries" of these strata are determined in [12]. Note that if the *dimension of $S[\varphi]$ is positive* then its closure has extra points inside $\mathcal{A}_{g,1}$, i.e. the "interior boundary"

$$\partial(S_\varphi) := \overline{S_\varphi} \quad - \quad S_\varphi \quad \text{is not empty.}$$

**7) The Hecke Orbit conjecture.**
**HO** **Conjecture** (FO, 1995), **theorem** (Chai & FO, manuscript in preparation).

$$\forall x \in \mathcal{W}^0_\xi(\mathcal{A}_g) \quad \mathcal{H}(x) \textit{ is dense in } \mathcal{W}^0_\xi(\mathcal{A}_g).$$

See [10], [14]. A detailed proof will be given in [7]. For a preliminary survey of a proof see [3].

**8) The almost-product-structure.** Let $W$ be an irreducible component of $\mathcal{W}^0_\xi(\mathcal{A}_g \otimes k)$ and let $x \in W$. For the notion of an "isogeny leaf" $I(x)$, the smallest connected subset of $\mathcal{H}_\alpha(x)$ containing $x$, see [14]. This is also constructed as part of the mod $p$ reduction of a Rapoport-Zink space.

*There exist reduced, irreducible schemes $T$ and $J$ and a finite surjective morphism*

$$\Phi : T \times J \to W$$

*such that for every $t \in T$, we have that*

$$\Phi(\{t\} \times J) \textit{ is a irreducible component of an isogeny leaf inside } W$$

*and for every $j \in J$, we have that*

$$\Phi(T \times \{j\}) \textit{ is an irreducible component of a central leaf.}$$

I.e. "Central leaves and isogeny leaves give, up to a finite map, a product structure on every component of a Newton Polygon stratum".

**9) Reductions.**
**9a)** We write $\mathbf{HO}_\ell$ for the conjecture that for every $x$ the Hecke-$\ell$-orbit $\mathcal{H}_\ell$ is dense in the central leaf $\mathcal{C}(x)$. Analogous definition for $\mathbf{HO}^{(p)}$.

In fact, what can be proved:

$$(\mathbf{HO}_\ell \text{ for at least one } \ell \neq p) \quad \Longleftrightarrow \quad \mathbf{HO}^{(p)}$$

By the almost-product-structure we see that

$$\mathbf{HO}_\ell \text{ for at least one } \ell \neq p \Longleftrightarrow \mathbf{HO}^{(p)} \Longrightarrow \mathbf{HO}.$$

**9b)** In order to show **HO** for every $x$ it suffices to show **HO** for every $x \in \mathcal{A}_g(\mathbb{F})$, where $\mathbb{F} = \overline{\mathbb{F}_p}$.

**9c)** We write $\mathbf{HO}_{\ell,\mathrm{discrete}}$ for:
*For every non-supersingular $x \in \mathcal{A}_g$ the central leaf $\mathcal{C}(x)$ is absolutely irreducible.*

**9d)** We write $\mathbf{HO}_{\ell,\mathrm{contin}}$ for:
*For every non-supersingular $x \in \mathcal{A}_g$ the Zariski closure of the Hecke orbit $\mathcal{H}_\ell(x)$ contains an irreducible component of the same dimension as $\mathcal{C}(x)$; i.e. $\mathcal{H}_\ell(x)$ is dense in at least one irreducible component of $\mathcal{C}(x)$.*

**9e)** We conclude:

$$\mathbf{HO}_{\ell,\mathrm{discrete}} \quad + \quad \mathbf{HO}_{\ell,\mathrm{contin}} \Longrightarrow \mathbf{HO}.$$

**9f)** For any $y \in \mathcal{H}(x)$ there is a finite-to-finite (Hecke) correspondence

$$\mathcal{C}(x)_k \quad \longleftarrow \quad T \quad \longrightarrow \quad \mathcal{C}(y)_k.$$

**9f)** We conclude that we need only show **HO** for moduli points over $\mathbb{F}$ and their central leaves inside $\mathcal{A}_{g,1} \otimes \mathbb{F}$.

**10) Hypersymmetric abelian varieties.**
Note that Tate showed that for any abelian variety $A$ over a finite field the natural maps

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{End}(T_\ell(A)),$$

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}(A[p^\infty])$$

are isomorphisms.

**Definition.** *An abelian variety over $\mathbb{F} := \overline{\mathbb{F}_p}$ is said to be hypersymmetric if the natural map*

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}(A[p^\infty])$$

*is an isomorphism.*

It is not difficult to prove that for any $p$ and for any symmetric Newton polygon there exists a hypersymmetric abelian variety having that Newton Polygon. For details see [4].

Here is a fact which will be used.

*For every $x \in \mathcal{A}_g$ the central leaf $\mathcal{C}(x)$ contains a hypersymmetric point.*

Sketch of a proof. One shows that any central leaf admits a Hecke correspondence with a central leaf inside $\mathcal{A}_{g,1}$, use (9f). Hence we assume that $x \in \mathcal{A}_{g,1}$. As every supersingular abelian variety is hypersymmetric we are done in that case. Assume that $x \in \mathcal{W}^0_\xi$, with $\xi \neq \sigma$. In that case $W_\xi := \mathcal{W}^0_\xi(\mathcal{A}_{g,1})$ is a geometrically irreducible. Hence $W_\xi$ contains a hypersymmetric point. By the almost-product-structure, see (8), there is a $\mathrm{H}_\alpha$-action moving that point into a given central leaf. □

We give some ideas leading to a proof of the Hecke Orbit conjecture (apologies, many details are missing in this description).

**11) "Shaken not stirred".**
**11a) Theorem.** *Every non-supersingular* $W^0_\xi := \mathcal{W}^0\xi(\mathcal{A}_{g,1})$ *is geometrically irreducible.*
Note the amusing fact that $W^0_\sigma$ has"many components"' for $p \gg 0$, but all other $W_\xi$ are irreducible.

This theorem I conjectured long ago, see [10]. A proof uses "interior boundaries": results in [11], [12], [13], and a description of moduli spaces of supersingular abelian varieties (Tadao Oda-FO, K.-Z.Li-FO); from these results one concludes that Hecke-$\ell$ operates transitively on the set of geometrically irreducible components of $W^0_\xi$; then one concludes using [2]. For details see [5].

**11b) Theorem.** *For every non-supersingular* $x \in \mathcal{A}_g$ *the central leaf* $\mathcal{C}(x)$ *is geometrically irreducible.*
Note that this also works for non-principal polarizations. For details see [5].
**Conclusion.** $\mathbf{HO}_{\ell,\mathrm{discrete}}$ holds.

**11c)** We say that a principally abelian variety $(B, \nu)$ over $k$ is *split* if there is an isogeny

$$(B, \nu) \sim (B_1, \nu_1) \times \cdots \times (B_r, \nu_r),$$

where the Newton polygon of each of these factors has at most two slopes.

**11d) "The Hilbert trick."** Note that any abelian variety $A$ over a fintie field has smCM. Hence there exists a commutative, totally real algebra $E$ of rank over $\mathbb{Q}$ equal to the dimension of $A$ such that $E \subset \mathrm{End}^0(A)$. This proves that through any point of $\mathcal{A}_{g,1}(\mathbb{F})$ we can choose the image of a Hilbert Modular variety. For details see [1], and especially see [6], Section 9.

**11e)** For HMV various strata were studied. Results by Goren-FO, Andreatta-Goren. Finally Chia-Fu Yu showed the discrete **HO** problem for Hilbert Modular Varieties, [17].

**11f)** Using EO-strata we show that any component of the image of a Hilbert Modular variety contains supersingular points. Here we make essential use of the idea of "interior boundaries".

**11g)** Write

$$Z(x) = \overline{\mathcal{H}^{(p)}(x)}.$$

Collecting all information obtained up to now one shows:

*for every* $x \in \mathcal{A}_{g,1}$ *there exists a point* $y \in Z(x) \cap \mathcal{C}(x)$ *which is hypersymmetric and split.*

(This is one of the most difficult and tricky parts of the proof.)

**11h)** *For a hypersymmetric and split point* $\mathbf{HO}^{(p)}_{\text{contin}}$ *holds.*
Here we see the idea by M. Larsen, already mentioned in [1], see (3c). One ingredient is a generalization of Serre-Tate coordinates to the case of any central leaf, completed at any point.

**11i)** We see:

$$\mathcal{C}(y) = Z(y) \cap \mathcal{C}(y) \quad \subset \quad Z(x) \cap \mathcal{C}(x) \quad \subset \quad \mathcal{C}(x) = \mathcal{C}(y).$$

Indeed, as $Z(x) \cap \mathcal{C}(x)$ is $\mathcal{H}^{(p)}$-stable, the first inclusion follows. This proves $\mathbf{HO}^{(p)}_{\text{contin}}$ for every $x \in \mathcal{A}_{g,1}(\mathbb{F})$. Hence, using reduction steps, this proves **HO**. □

**12) Analogies: three conjectures.**
Here are conjectures / theorems, where the basic structure are are quite similar. However methods of proof are very different.

**Geometry**: a variety $V$ over some field $K$, of finite type over its prime field.

**Arithmetic**: a subset $\Gamma$ of $V$. Typically the points of $\Gamma$ are not all defined over some fixed finite extension of $K$.

**Question.** What is the closure of $\Gamma$ inside $V$? In all three problem we first predict what $\overline{\Gamma}$ should be, and then (try to) prove this to be true.

**12a) The Manin-Mumford conjecture.** Here $V = A$ is an abelian variety over a field $K$ of characteristic zero. The set $\Gamma$ is some subset of $\text{Tors}(A)$, a set of torsion points.

*The closure of* $\Gamma$ *is a finite union of translates of abelian subvarieties of* $A$*.*

This was first proved by M. Raynaud.

**12b) The André-Oort conjecture.** Here $V = S$ is a Shimura variety over a field $K$ of characteristic zero. The set $\Gamma$ is some subset of $\text{Spec}(S)$, a set of "special points"; in case $S$ is a moduli scheme of abelian varieties (possibly with some extra structure), a special point is defined by an abelian variety with sufficiently many complex multiplications.

*The closure of* $\Gamma$ *should be a finite union of special subvarieties, Hecke translates of Shimura subvarieties.*

It seems that this conjecture has been proved, assuming the generalized RH, by Yafaev - Klingler - Ullmo (using ideas by Edixhoven and Clozel).

**12c) The Hecke Orbit conjecture.** Here $V = \mathcal{A}_g \otimes \mathbb{F}_p$, and $\Gamma = \mathcal{H}(x)$. See above.

# References


[1] C.-L. Chai – *Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli space.* Invent. Math. **121** (1995), 439–479.

[2] C.-L. Chai – *Monodromy of Hecke-invariant subvarieties.* Pure Appl. Math. Quaterly **1** (Special issue: in memory of Armand Borel), 291–303.

[3] C.-L. Chai –*Hecke orbits on Siegel modular varieties.* Progress in Mathematics **235**, Birkhäuser, 2004, pp. 71–107.

[4] C.-L. Chai & F. Oort – *Hypersymmetric abelian varieties.* Quaterly J. Pure Applied Math. **2** (Coates Special Issue) (2006), 1–27.

[5] C.-L. Chai & F. Oort – *Monodromy and irreducibility of leaves.* [To appear]

[6] C.-L. Chai and F. Oort – *Moduli of abelian varieties and p-divisible groups: density of Hecke orbits and a conjecture by Grothendieck.* Summer School on arithmetic geometry, Göttingen July/August 2006. To appear: Clay Mathematics Proceedings. Arithmetic geometry, Proceedings of the Clay Summer School Gttingen 2006, (Editors: Y. Tschinkel, H. Darmon and B. Hassett).[To appear]

[7] Ching-Li Chai and Frans Oort – *Hecke orbits.* [In preparation]

[8] N. M. Katz – *Slope filtration of F–crystals.* Journ. Géom. Alg. Rennes, Vol. I, Astérisque **63** (1979), Soc. Math. France, 113 - 164.

[9] Yu. I. Manin – *The theory of commutative formal groups over fields of finite characteristic.* Usp. Math. **18** (1963), 3-90; Russ. Math. Surveys **18** (1963), 1-80.

[10] F. Oort – *Some questions in algebraic geometry,* preliminary version. Manuscript, June 1995.
See: `http://www.math.uu.nl/people/oort/` http://www.math.uu.nl/people/oort/

[11] F. Oort – *Newton polygons and formal groups: conjectures by Manin and Grothendieck.* Ann. Math. **152** (2000), 183 - 206.

[12] F. Oort – *A stratification of a moduli space of polarized abelian varieties.* In: *Moduli of abelian varieties.* (Ed. C. Faber, G. van der Geer, F. Oort). Progress Math. 195, Birkhäuser Verlag 2001; pp. 345–416.

[13] F. Oort – *Newton polygon strata in the moduli space of abelian varieties.* In: *Moduli of abelian varieties.* (Ed. C. Faber, G. van der Geer, F. Oort). Progress Math. 195, Birkhäuser Verlag 2001; pp. 417–440.

[14] F. Oort – *Foliations in moduli spaces of abelian varieties.* Journ. A. M. S. **17** (2004), 267–296.

[15] F. Oort – *Foliations in moduli spaces of abelian varieites and dimension of leaves.* [To appear]

[16] F. Oort & T. Zink – *Families of p-divisible groups with constant Newton polygon.* Documenta Mathematica **7** (2002), 183–201.

[17] C.-F. Yu – [A paper on the Lie stratification of HB varieties; in preparation.]

Frans Oort
Mathematisch Instituut
P.O. Box. 80.010
NL - 3508 TA Utrecht
The Netherlands
email: f.oort@uu.nl

# The Fundamental Lemma

**Günter Harder**

The fundamental lemma is a celebrated result in the theory of automorphic forms, it is the key result that is needed for the stabilization of the trace formula. It has been formulated by Langlands and Diana Shelstad and it has been proved recently by Ngo Bau Chau, the final proof is based on the work of many other mathematicians.

My aim in this talk was to explain the meaning of the fundamental lemma to general audience.

In a certain sense the fundamental lemma was explained by Langlands in a talk at the Arbeitstagung in the early 1970-th. In this talk Langlands reported on his joint paper with Labesse with title "$L$-indistinguishability for SL(2)." They discovered the phenomenon that two automorphic representations which have the same $L$ function may occur with different multiplicities in the space of automorphic forms.

At the same Arbeitstagung Hirzebruch proved the following theorem:

*Let $p$ be a prime which is $3 \bmod 4$ let $F = \mathbb{Q}[\sqrt{p}]$. Let $\mathcal{O}$ be its ring of integers, then the group $Sl_2(\mathcal{O})$ acts on the product of two upper half planes $H^+ \times H^+$, the compactification of the quotient $Sl_2(\mathcal{O})\backslash H^+ \times H^+$ yields a Hilbert-Blumenthal surface $S^{++}$. But it also acts on $H^+ \times H^-$ and we get a second such surface $S^{+-}$. Then we have a discrepancy between the spaces of holomorphic 2-forms:*

$$\dim H^0(S^{++}, \Omega^2) - \dim H^0(S^{+-}, \Omega^2)) = h(\sqrt{-p})$$

*where the number on the right is the class number of $\mathbb{Q}(\sqrt{-p})$.*

The elements in these spaces provide $L$-indistinguishabie automorphic forms.

In their paper Labesse and Langlands used a baby version of the fundamental lemma for the group $Sl_2$. This was not so difficult to prove, but after that it turned out to be incredibly difficult to prove generalizations of this fundamental lemma for other reductive groups.

The fundamental lemma was formulated as a very precise conjecture and looks like that (here $G$ is a reductive group over $\mathbb{Q}$)

$$\sum_{\xi_p} \int_{Z_{\gamma\cdot\xi_p}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} f_p(x_p^{-1}\gamma \cdot \xi_p x_p)\kappa_p(\xi_p)\epsilon(\xi_p)dx_p =$$

$$\Delta_p(\gamma, \kappa) \int_{Z_{\gamma H^\kappa}(\mathbb{Q}_p)\backslash H^\kappa(\mathbb{Q}_p)} f_p^{H^\kappa}(y_p^{-1}\gamma y_p)dy_p$$

here $\kappa$ is -under certain conditions a character on a finite abelian group from which the $\xi_p$ are taken. The $H^\kappa$ is the so called endoscopic group attached to $\kappa$, the factor in front is the transfer factor. The integrals on the left are the $\kappa$ orbital integrals, the integral for the trivial character $\kappa = 1$ is called the stable orbital integral for the group $G$ and the fundamental lemma says that a $\kappa$ orbital integral -which now is unstable if $\kappa \neq 1$- is up to the tranfer factor equal to a stable orbital integral on the endoscopic group.

The fundamental lemma enters the stage if we apply the trace formula (Arthur-Selberg or topological trace formula) to compute the trace of a Hecke operator $T_h$ on a certain space $X$ of functions or on some cohomology groups. Then we encounter certain sums of orbital integrals which can be manipulated to become a sum of $\kappa$ -orbital integrals. Eventually we get

$$tr(T_h|H) = \sum_{\kappa} tr(T_{h^\kappa}^{H^\kappa}|X^{H^\kappa}) = T_h^G|X + \sum_{\kappa\neq 1} tr(T_{h^\kappa}^{H^\kappa}|X^{H^\kappa})$$

where the terms on the right hand side are stable.

If our group is $G/\mathbb{Q} = R_{F/\mathbb{Q}}(Sl_2/F)$ as above then the torus $H/\mathbb{Q}$ of norm one elements in $F^\times$ is endoscopic and the resulting term on the right side explains the class number in the difference of dimensions.

Ngo Bao Chau: Le lemme fondamental pour les algebres de Lie, arXiv:0801.0446

# Tête-à-tête twists and geometric monodromy.

Norbert A'Campo

**Introduction**. Let $(\Sigma, \Gamma)$ be a pair consisting of a a compact connected oriented surface $\Sigma$ with non empty boundary $\partial\Sigma$ and a finite graph $\Gamma$ that is embedded in the interior of $\Sigma$. We assume that the surface $\Sigma$ is a regular neihborhood of the graph $\Gamma$ and that the embedded graph has the tête-à-tête property, which property we will define later in this paper. Moreover, we will construct for each pair $(\Sigma, \Gamma)$ with the tête-à-tête property a mapping classe $T_\Gamma$ on $(\Sigma, \partial\Sigma)$. We call the mapping classes resulting from this construction tête-à-tête twists.

A surface of genus $g$ and with $r$ boundary components carries up to congruence by homeomorphism of the surface only finite many graphs with the tête-à-tête property and hence for fixed $(g, r)$ there are only finite many mapping classes, which are tête-à-tête twists.

The main theorem of this paper asserts:

**Theorem.** *The geometric monodromy diffeomorphism of a plane curve singularity is a tête-à-tête twist.*

As a corollary, we obtain a very strong topological restriction for mapping classes, that are geometric monodromies of plane curve singularities.

**Section 1. Tête-à-tête twist.**

Let $\Gamma$ be a finite connected metric graph with $e(\Gamma)$ edges and no vertices of valency 1. We assume, that the edges are parametrized by continuous bijective maps $E_e : [0, L_e] \to \Gamma, L_e > 0, e = 1, \cdots, e(\Gamma)$, such that the distance from $E_e(t)$ to $E_e(s)$ is $|t - s|, t, s \in [0, L_e]$.

Let $\Sigma$ be a smooth, connected and oriented surface with non empty boundary $\partial\Sigma$. We say, that a map $\pi$ of $\Gamma$ into $\Sigma$ is regular if $\pi$ is continuous, injective, $\pi(\Gamma) \cap \partial\Sigma = \emptyset$, the compositions $\pi \circ E_e, e = 1, \cdots, e(\Gamma)$, are smooth regular embeddings of intervals and moreover, at each vertex $v$ of $\Gamma$ all outgoing speed vectors of $\pi \circ E_e, v = E_e(0)$ or $v = E_e(L_e)$ are distinct.

We denote by abuse of language by the pair $(\Sigma, \Gamma)$ the pair $(\Sigma, \pi(\Gamma))$.

A safe walk along $\Gamma$ is a continuous injective path $\gamma : [0, 2] \to \Sigma$ with following properties:

– $\gamma(t) \in \Gamma, t \in [0, 2]$,

– the speed, measured with the parametrization $E_e$ at $t \in [0, 2]$ equals $\pm 1$ if $\gamma(t)$ is in the interior of edge $e$,

– if the path $\gamma$ runs at $t \in (0, 2)$ into the vertex $v$, the path $\gamma$ makes the a sharpest possible right turn, i.e. the oriented angle at $v = \gamma(t) \in \Sigma$ in between the speed vectors $-\dot{\gamma}(t_-)$ and $\dot{\gamma}(t_+)$ is smallest possible.

It follows, that a save walk $\gamma$ is determined by its starting point $\gamma(0)$ and its starting speed vector $\dot{\gamma}(0)$. Futhermore, if the metric graph $\Gamma \subset \Sigma$ is without cycles of length less are equal 2, from each interior point of an edge start two distinct save walks.

**Definition:** *Let $(\Sigma, \Gamma)$ be the pair of a surface and regular embedded metric graph. We say that the tête-à-tête tête-à-tête property holds for the the pair if*

– *the graph $\Gamma$ has no cycles of length $\leq 2$,*

– *the graph $\Gamma$ is a regular retract of the surface $\Sigma$,*

– *for each point $p \in \Gamma$, $p$ not being a vertex, the two distinct safe walks $\gamma_p^+, \gamma_p^- : [0, 2] \to \Sigma$ with $p = \gamma_p^+(0) = \gamma_p^-(0)$ satisfy to $\gamma_p^+(2) = \gamma_p^-(2)$.*

It follows that the underlying metric graph of a pair $(\Sigma, \Gamma)$ with tête-à-tête property is the union of its cycles of length 4.

We give basic examples of pairs $(\Sigma, \Gamma)$ with tête-à-tête property:

— the surface is the cylinder $[-1, 1] \times S^1$ and the graph $\Gamma$ is the cycle $\{0\} \times S^1$ subdivided by 4 vertices in edges of equal length. Here we think $S^1$ as a circle of length 4.

— the surface $\Sigma_{1,1}$ is of genus 1 with 1 boundary component and the metric graph $\Gamma \subset \Sigma$ is the biparted complet graph $K_{3,2}$.

— for $p, q \in \mathbf{N}, p > 0, q > 0$, the biparted complet graph $K_{p,q}$ is the spine of a surface $S_{g,r}, g = 1/2(p-1)(q-1), r = (p, q)$, such that the tête-à-tête property holds. For instance, let $P$ and $Q$ be two parallel lines in the plane and draw $p$ points on $P$, $q$ points on $Q$. We add $pq$ edges and get a planar projection of the graph $K_{p,q}$. The surface $S_{g,r}$ is a regular thickening of that projection.

Let $(\Sigma, \Gamma)$ a pair of a surface and graph with tête-à-tête property. Our purpose is to construct for this pair a well defined element $T_\Gamma$ in the relative mapping class group of the surface $\Sigma$. For each edge $e$ of $\Gamma$ we embed relatively a copy $(I_e, \partial I_e)$ of the interval $[-1, 1]$ into $(\Sigma, \partial\Sigma)$ such that alle copies are pairwise disjoint and such that each copy $I_e$ intersects in its midpoint $0 \in I_e$ the graph $\Gamma$ transversally in one point which is the midpoint of the edge $e$. We call $I_e$ the dual arc of the edge $e$. Let $\Gamma_e$ be the union of $\Gamma \cup I_e$. We consider $\Gamma_e$ also as a metric graph. The graph $\Gamma_e$ has 2 terminal vertices $a, b$.

Let $w_a, w_b : [-1, 2] \to \Gamma_e$ be the only save walks along $\Gamma_e$ with $w_a(-1) = a, w_b(-1) = b$. We displace by a small isotopy the walks $w_a, w_b$ to smooth injektive path $w'_a, w'_b$, that keeps the points $w_a(-1), w_b(-1)$ and $w_a(2), w_b(2)$ fixed, such that $w'_a(t) \notin \Gamma_e$ for $t \in (-1, 2)$. The walks $w_a, w_b$ meet each other in the midpoint of the edge $e$. Hence by the tête-à-tête property we have $w_a(2) = w_b(2)$. Let $w_e$ the juxtaposition of the pathes $w'_a$ and $-w'_b$. We may assume that the path $w_e$ is smooth and intersects $\Gamma$ transversally. Let $I'_e$ the image of the path $w_e$. We now claim that there exits up to isotopy a unic relative diffeomorphism $\phi_\Gamma$ of $\Sigma$ with $\phi_\Gamma(I_e) = I'_e$. We define the tête-à-tête twist $T_\Gamma$ as the class of $\phi_\Gamma$.

For our first basic example we obtain back the classical right Dehn twist. The second example has as tête-à-tête twist the geometric monodromy of the plane curve singularity $x^3 - y^2$. The twist of the example $(S_{g,r}, K_{p,q})$ computes the geometric monodromy of for the singularity $x^p + y^q$.

**Section 2. Relative tête-à-tête retracts.**

We prepare material, that will allow us to glue the previous examples. Let $S$ be a connected compact surface with boundary $\partial S$. The boundary $\partial S = A \cup B$ is decomposed as a partition of boundary components of the surface $S$. We assume $A \neq \emptyset, B \neq \emptyset$.

**Definition.** *A relative tête-à-tête graph $(S, A, \Gamma)$ in $(S, A)$ is an embedded metric graph $\Gamma$ in $S$ with $A \subset \Gamma$. Moreover, the following properties hold:*

*– the graph $\Gamma$ has no cycles of length $\leq 2$,*

*– the graph $\Gamma$ is a regular retract of the surface $\Sigma$,*

*– for each point $p \in \Gamma \setminus A$, $p$ not being a vertex, the two distint safe walks $\gamma_p^+, \gamma_p^- : [0, 2] \to \Sigma$ with $p = \gamma_p^+(0) = \gamma_p^-(0)$ satisfy to $\gamma_p^+(2) = \gamma_p^-(2)$.*

*— for each point $p \in A$, $p$ not being a vertex, the only save walk $\gamma_p^+$ satisfies $\gamma_p^+(2) \in A$.*

We call the subset $A$ the boundary of the relative tête-à-tête graph $(S, A, \Gamma)$. This boundary carries a self map $p \in A \mapsto \gamma_p^+(2) \in A$, which we call the boundary walk $w$.

We now give a family of examples of relative tête-à-tête graphs.

— Consider the previous example $(S_{g,r}, K_{p,q}), g = 1/2(p-1)(q-1), r = (p, q)$. We blow up in the real oriented sense the $p$ vertices of valency $q$, so we replace such a vertex $v_i, 1 \leq i \leq p$ by a circle $A_i$ and attach the edges of $K_{p,q}$ that are incident with $v_i$ to the circle in the cyclic order given by the embedding of $K_{p,q}$ in $S_{g,r}$. We get a surface $S_{g,r+p}$ and its boundary is partitioned in $A := \cup A_i$ and $B = \partial S_{g,r}$. The new graph is the union of $A$ with the strict transform of $K_{p,q}$. So the new graph is in fact the total transform $K'_{p,q}$. We think this graph as a metric graph. The metric will be such that all edges have a positive length and that the tête-à-tête property remains for all points of $K'_{p,q} \setminus A$. We achieve this by giving the edges of $A$ the length $2\epsilon, \epsilon > 0, \epsilon$ small and by giving the edges of $K'_{p,q} \setminus A$ the length $1 - \epsilon$. The boundary walk is an interval exchange map from $w : A \to A$. We denote by the pair $(S_{g,r+p}, K'_{p,q})$ this relative tête-à-tête graph together with its boundary walk.

**Section 3. Gluing and closing of relative tête-à-tête graphs**.

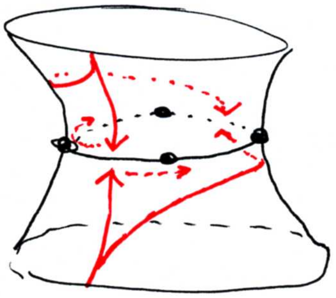
First we discribe the procedure of closing. We do it by an example. Consider $(S_{6,1+2}, K'_{2,13})$. We have two $A$ boundary components $A_1$ and $A_2$. In oder to close the $A$ components, we choose a piece-wise linear orientation reversing selfmap $s_1 : A_1 \to A_1$ of order 2. The boundary component $A_1$ will be closed if we identify the pieces using the map $s_1$. In order to get the tête-à-tête property we do the same with the component $A_2$, but we take care such that the involution $s_2 : A_2 \to A_2$ is equivariant via the boundary walk $w$ to the involution $s_1$. Hence we take $p \in A_2 \mapsto s_2(p) := w \circ s_1 \circ w^{-1}(p) \in A_2$. More concretely, we can choose for $s_1 : A_1 \to A_1$ an involution that exchange in an orientation reversing way the opposite edges of an hexagon. If we do so, we get a surface $S_{8,1}$ with tête-à-tête graph. The corresponding twist is the geometric monodromy of the singularity $(x^3 - y^2)^2 - x^5 y$. If we make our choices generically, the resulting graph will have 51 vertices, 36 edges, 6 vertices of valency 2, 45 vertices of valency 3.

Now an example of gluing. We glue in an walk equivariant way to copies of $(S_{2,1}, K'_{2,5})$. We get a tête-à-tête graph on the surface $S_{5,2}$. The corresponding twist is the monodromy of the singularity $(x^3 - y^2)(x^2 - y^3)$.

This is work in progress. A futher constuction for isolated singularities $f : \mathbf{C}^{n+1} \to \mathbf{C}$ provides its Milnor fiber with a spine, that consists of lagrangian strata. Again the monodromy is concentrated at the spine. The monodromy diffeomorphism is a generalized tête-à-tête twist. The case of plane curves is already interesting for we are aiming progress in restricting the adjacency tables. Thanks for your interest.

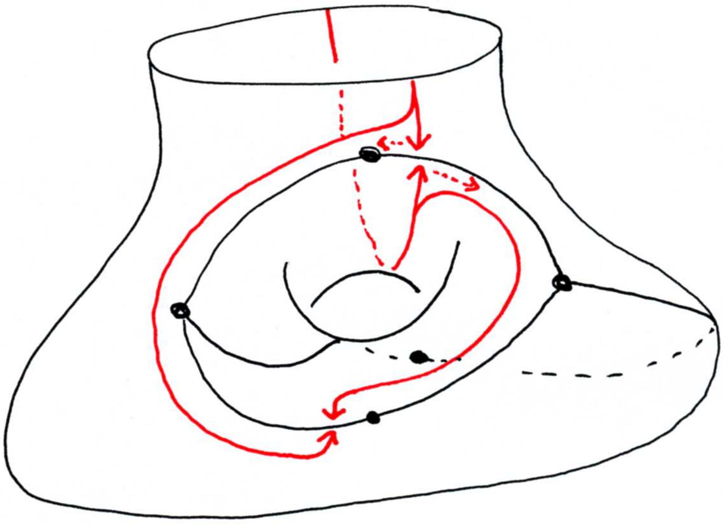
University of Basel, Rheinsprung 21, CH-4051 Basel.

Example of pairs $(S, \Gamma)$ with tête-à-tête property



$x^2 - y^2$

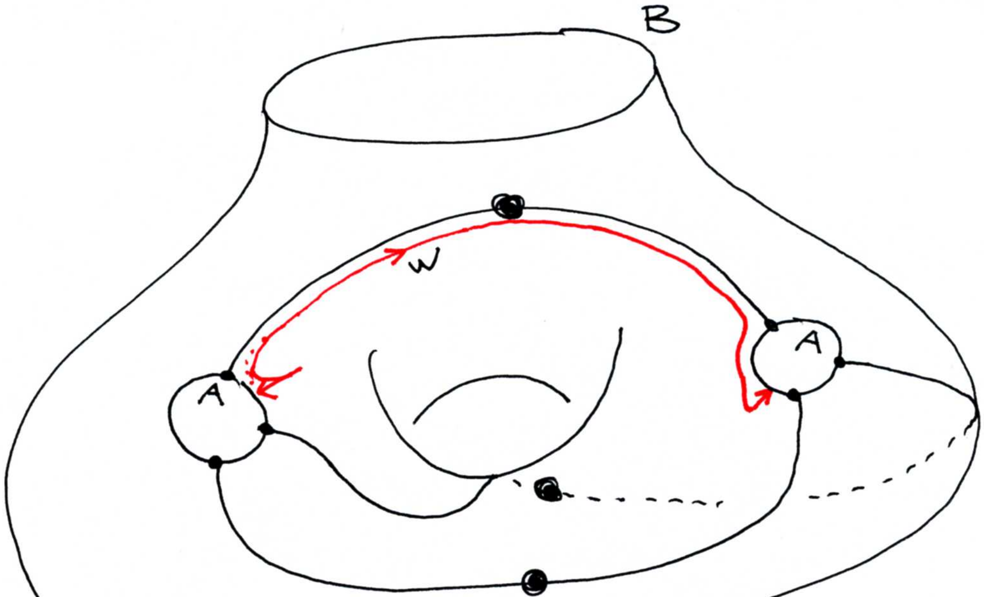
$(S_{0,2}, K_{2,2})$



$x^3 - y^2$

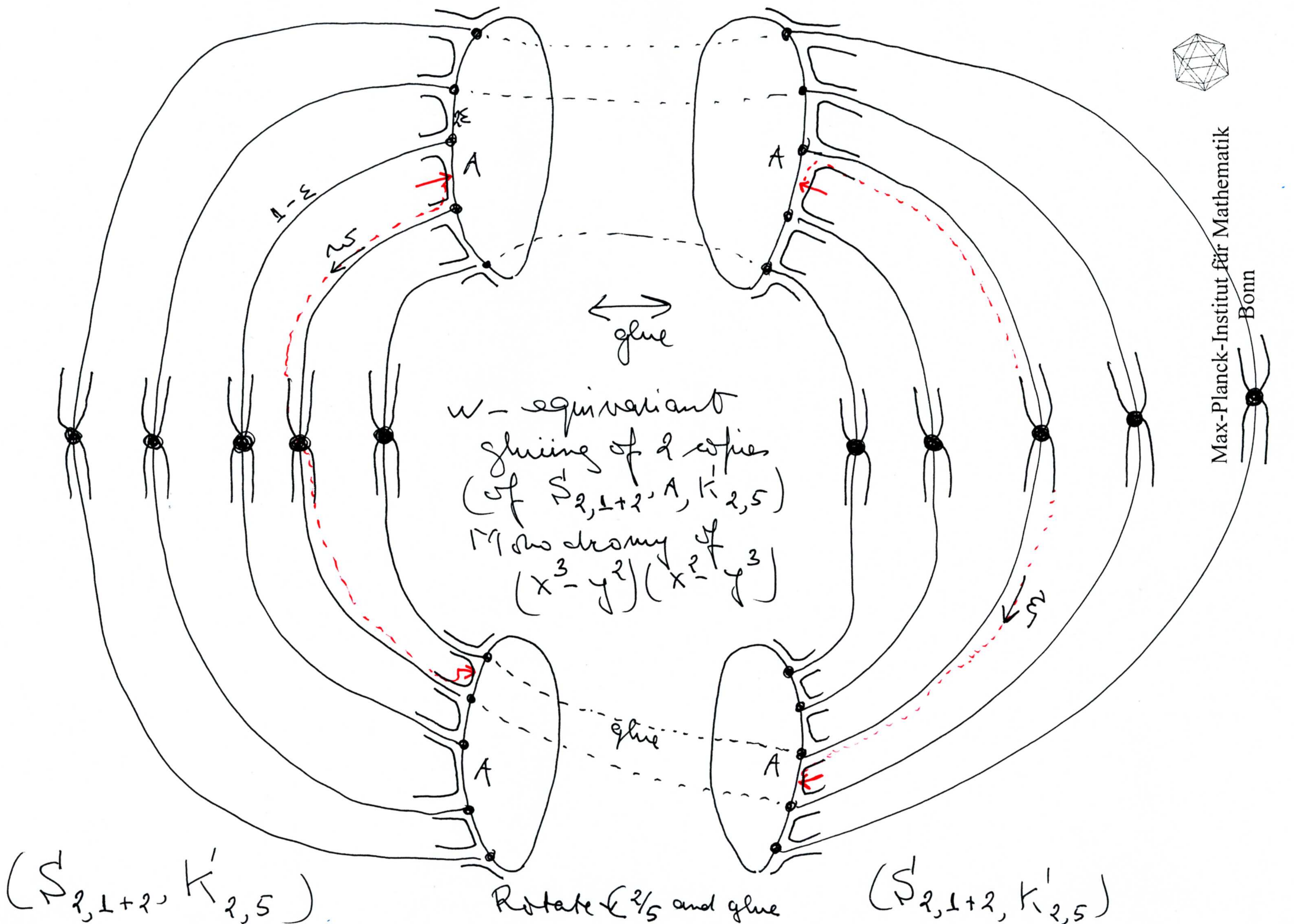
$(S_{1,1}, K_{2,3})$

Example of a relative pair $(S, A, \Gamma)$




$(S_{1,1+2}, A, K'_{2,3})$

(Length ——) $= 2\varepsilon$, (Length ●——●) $= 1-\varepsilon$


glue
w-equivariant
gluing of 2 copies
(of $S'_{2,1+2}, A, K'_{2,5}$)
Monodromy of
$(x^3-y^2)(x^2-y^3)$
A
A
A
A
$1-\varepsilon$
$\sim\varepsilon$
$\sim\varepsilon$
glue
$(S_{2,1+2}, K'_{2,5})$
Rotate $\angle\, 2/5$ and glue
$(S'_{2,1+2}, K'_{2,5})$

# Elliptic dilogarithms and parallel lines

Anton Mellit *

June 10, 2009

**Abstract**

We prove Boyd's conjectures relating Mahler's measures and values of L-functions of elliptic curves in the cases when the corresponding elliptic curve has conductor 14.

## 1 Boyd's conjectures

Rogers provided a table of relations between Mahler's measures and values of L-functions of elliptic curves of low conductors 11, 14, 15, 20, 24, 27, 32, 36 in [Rog08]. Among these relations some had been proved and some had not. According to Rogers, those relations which involve curves with complex multiplication (conductors 27, 32, 36) were all proved. Except those, only a relation with curve of conductor 11 was proved. Let us list the relations with curves of conductor 14.

Let $P \in \mathbb{C}[y,z]$. Then Mahler's measure of $P$ is defined as

$$m(P) := (2\pi \mathrm{i})^{-2} \int_{|y|=|z|=1} \log|P(y,z)| \frac{dy}{y} \frac{dz}{z}.$$

Denote

$$n(k) := m(y^3 + z^3 + 1 - kyz),$$
$$g(k) := m((1+y)(1+z)(y+z) - kyz).$$

Let $E$ be the elliptic curve of conductor 14 with Weierstrass form $y^2 + yx + y = x^3 + 4x - 6$. It is isomorphic to the modular curve $X^0(14)$ with the pullback of the Néron differential $\frac{dx}{2y+x+1}$ given by the eta-product [MO97]

$$f := \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau).$$

*mellit@gmail.com

Then $L(E,s) = L(f,s)$ and the relations listed by Rogers are

$$n(-1) = \frac{7}{\pi^2} L(f,2), \tag{1}$$

$$n(5) = \frac{49}{2\pi^2} L(f,2), \tag{2}$$

$$g(1) = \frac{7}{2\pi^2} L(f,2), \tag{3}$$

$$g(7) = \frac{21}{\pi^2} L(f,2), \tag{4}$$

$$g(-8) = \frac{35}{\pi^2} L(f,2). \tag{5}$$

# 2 The regulator

Fix a smooth projective curve $C/\mathbb{C}$. An element $\sum_i \{f_i, g_i\} \in \Lambda^2\mathbb{C}(C)^\times$ will be denoted simply by $\{f,g\}$ and we will omit the corresponding "$\sum_i$" sign in expressions below to soften the notation. The regulator of $\{f,g\} \in K_2(C)$ is defined as $r_C(\{f,g\}) \in H^1(C,\mathbb{R})$ whose value on $[\gamma] \in H_1(C,\mathbb{Z})$ is

$$r_C(\{f,g\})([\gamma]) = \int_\gamma \log|f| d\arg g - \log|g| d\arg f.$$

Let $\omega$ be a holomorphic 1-form on $C$. The value of the regulator on $\omega$ is defined as follows:

$$\langle r_C(\{f,g\}), \omega\rangle = \langle r_C(\{f,g\}) \cap \omega, [C]\rangle = 2\int_C \log|f| d\arg g \wedge \omega. \tag{6}$$

Denote by $\mathscr{K}_n$ (resp. $\mathscr{K}_g$) the set of values of the function $\frac{y^3+z^3+1}{yz}$ (resp. $\frac{(1+y)(1+z)(y+z)}{yz}$) on the torus $|y| = |z| = 1$. Then by a theorem of Deninger [Den97] for $k \notin \mathscr{K}_n$ (resp. $k \notin \mathscr{K}_g$) one can express $n(k)$ (resp. $g(k)$) as $\frac{1}{2\pi} r_C(\{y,z\})([\gamma])$ for a certain $[\gamma] \in H_1(C,\mathbb{Z})$, where $C$ is the projective closure of the equation $y^3+z^3+1-kyz$ (resp. $(1+y)(1+z)(y+z)-kyz$). When $k$ is on the boundary of $\mathscr{K}_n$ (resp. $\mathscr{K}_g$) Deninger's result still applies by continuity.

# 3 Elliptic dilogarithm

Let $E/\mathbb{C}$ be an elliptic curve. Define a map from $\Lambda^2\mathbb{C}(E)^\times$ to $\mathbb{Z}[E(\mathbb{C})]^-$ by

$$\{f,g\} \to (f) * (g)^-$$

where "$*$" and "$-$" mean the convolution and the antipode operations on divisors of an elliptic curve. Fix an isomorphism $E \cong \mathbb{C}/\langle 1,\tau\rangle$ for $\tau \in \mathfrak{H}$. Let $u$ be the coordinate on $\mathbb{C}$. Let $x \in E(\mathbb{C})$, $x = a\tau + b$ for $a, b \in \mathbb{R}$. As in [Zag90] (it seems that the sign there is wrong) put

$$R(\tau, x) = -\frac{\mathrm{i}}{\pi}(\mathrm{Im}\,\tau)^2 \sum_{(m,n)\neq(0,0)} \frac{\sin(2\pi(na-mb))}{(m\tau+n)^2(m\overline{\tau}+n)}.$$

We have

$$\langle r_E(\{f,g\}), du\rangle = R(\tau, (f) * (g)^-).$$

For a holomorphic 1-form $\omega$ on $E$ put

$$R_{E,\omega}(x) = \frac{\omega}{du} R(\tau, x).$$

Then $R_{E,\omega}$ does not depend on the choice of the isomorphism $E \cong \mathbb{C}/\langle 1, \tau\rangle$ and we call $R_{E,\omega}$ the elliptic dilogarithm, while usually people call elliptic dilogarithm the real part of $R(\tau, x)$.

When $E$ is defined over $\mathbb{R}$ and an orientation on $E(\mathbb{R})$ is chosen there is a canonical choice of the isomorphism above and we will write $R_E(x)$ for the "old dilogarithm" $R(\tau, x) = R_{E,du}(x)$.

By linearity we extend $R_{E,\omega}$ to the odd part of the group of divisors $\mathbb{Z}[E(\mathbb{C})]^-$.

The function $R_{E,\omega}(x)$ satisfies the following properties:

(i) For any $\lambda \in \mathbb{C}$ $R_{E,\lambda\omega}(x) = \lambda R_{E,\omega}(x)$.

(ii) For an isogeny $\varphi : E' \to E$ and $x \in E(\mathbb{C})$

$$R_{E,\omega}(x) = \sum_{x' \in \varphi^{-1}(x)} R_{E',\varphi^*\omega}(x'). \tag{7}$$

(iii) For a function $f \in \mathbb{C}(E)^\times$, $f \neq 1$, one has $R_{E,\omega}((f) * (1-f)^-) = 0$.

The second property is called the distribution relation, the third one is the Steinberg relation.

We expect that any algebraic relation between $R_{E,\omega}(x)$ where $E, \omega, x$ are defined over $\overline{\mathbb{Q}}$ follows from the relations listed above.

# 4 Beilinson's theorem for $\Gamma_0(N)$

Let $N$ be a squarefree integer with prime decomposition $N = p_1, \ldots, p_n$. Let $f = \sum a(n)q^n$ be a newform for $\Gamma_0(N)$ of weight 2. Let $W$ be the group of Atkin-Lehner involutions. This is a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. For $m > 0$, $m|N$ denote by $w_m$ the Atkin-Lehner involution corresponding to $m$. Any cusp of $\Gamma_0(N)$ is given by $w(\infty)$ for a unique $w \in W$. The width of $w_m(\infty)$ is $m$. It is known that for a prime $p|N$ we have $f|_2 w_p = -a(p)f$.

Let $\mathbb{Q}[W]_0$ be the augmentation ideal of $\mathbb{Q}[W]$. For any $\alpha \in \mathbb{Q}[W]_0$, $\alpha = \sum_{w\in W} \alpha_w [w]$ consider $F_\alpha \in \mathbb{C}(X_0(N))^\times \otimes \mathbb{Q}$ such that $(F_\alpha) = \sum_{w \in W} \alpha_w [w(\infty)]$.

Let $\gamma : W \to \{\pm 1\}$ be such that $f|_2 w = \gamma(w) f$ for all $w \in W$. Let $d = \sum_{m|N} m w_m$,

$$d^{-1} = \prod_{k=1}^{n} \frac{1 - p_k w_{p_k}}{1 - p_k^2}.$$

Let $\gamma^*$ be the involution of $\mathbb{Q}[W]$ which sends $w$ to $\gamma(w)w$ for $w \in W$. Put $\alpha' = d^{-1}\alpha$, $\beta' = d^{-1}\beta$ for $\alpha, \beta \in \mathbb{Q}[W]_0$. Let $\varepsilon : \mathbb{Q}[W] \to \mathbb{Q}$ be a linear map such that $\varepsilon(w_m)$ is 0 for $m \neq 1$ and 1 for $m = 1$. Then

$$\langle r_{X_0(N)}(\{F_\alpha, F_\beta\}), 2\pi \mathrm{i} f(\tau)d\tau\rangle = -\frac{144N}{\pi}\varepsilon(w_N\alpha'\gamma^*(\beta'))L(f,1)L(f,2). \quad (8)$$

# 5 Parallel lines

By results stated above both sides of the conjectured identities are reduced to relations between values of the elliptic dilogarithm. To prove relations between elliptic dilogarithms one usually tries to construct rational functions $f$ such that divisors of both $f$ and $1-f$ are supported on a given set of points.

Let $E/\mathbb{C}$ be an elliptic curve and $Z \subset E(\mathbb{C})$ be a finite subgroup. Let us realize $E$ as a plane cubic with equation $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{C}$. For each triple $p, q, r \in Z \setminus \{0\}$ such that $p + q + r = 0$ consider the line $l_{p,q,r}$ passing through $p$, $q$, $r$ with equation $y + s_{p,q,r}x + t_{p,q,r} = 0$. Suppose $s_{p,q,r} = s_{p',q',r'}$ for another triple of points, which is equivalent to the lines $l_{p,q,r}$ and $l_{p',q',r'}$ being parallel. Then from equations of these lines one can obtain two functions $f$, $g$ on $E$ such that $f + g = 1$ and divisors of $f$ and $g$ are supported on $Z$. Thus we obtain (hopefully a non-trivial) relation between values of the elliptic dilogarithm at points of $Z$.

I propose to search for parallel lines as above in two ways. The first way, dubbed "breadth-first search", is to fix $Z = \mathbb{Z}/m \times \mathbb{Z}/m'$ and consider the moduli space of elliptic curves $E$ with embedding $Z \to E$. Then for any two triples $p, q, r$ and $p', q', r'$ the difference $s_{p,q,r} - s_{p',q',r'}$ is a function on the moduli space, which can be found explicitly, and at the points where the function is zero we obtain a relation.

Another approach, which I call "depth-first search", is to fix a curve $E$ and consider some large subgroup $Z$ hoping that when $Z$ is large enough some parallel lines will appear. However, this seems to work only for some "nice" curves.

In the proof of (1) - (5) we use identities found by the two approaches on the curve $Y^2 + YX + Y = X^3 - X$, and obtain results for isogenous curves by the distribution relation.

Finally let us mention an interesting propery of the slopes $s_{p,q,r}$.

**Proposition.** *There exists a unique map from $Z \setminus \{0\}$ to $\mathbb{C}$, denoted $p \to z_p \in \mathbb{C}$, such that*

*(i)* $z_p + z_{-p} = 0$ *for all* $p$,

*(ii)* $z_p + z_q + z_r = s_{p,q,r}$ *for all* $(p, q, r)$ *with* $p + q + r = 0$,

*moreover, we have ($x_p$ is the $x$-coordinate of $p$)*

*(iii)* $x_p + x_q + x_r = s^2_{p,q,r}$ *for all* $(p, q, r)$ *with* $p + q + r = 0$.

In fact these $z_p$ are related to Eisenstein series of weight 1 and they satisfy a certain distribution relation.

# 6 Acknowledgements

The author would like to thank W. Zudilin for bringing his attention to the problem. He is also grateful to M. Rogers, H. Gangl, A. Levin, F. R. Villegas, A. Goncharov and D. Zagier for interesting discussions and to the Max Planck Institute for Mathematics in Bonn for its hospitality and stimulating environment.

# References

[Bei85] A.A. Beilinson. Higher regulators and values of $L$-functions. *J. Sov. Math.*, 30:2036–2070, 1985.

[Blo00] Spencer J. Bloch. *Higher regulators, algebraic $K$-theory, and zeta functions of elliptic curves.* CRM Monograph Series. 11. Providence, RI: American Mathematical Society (AMS). ix, 97 p., 2000.

[Den97] Christopher Deninger. Deligne periods of mixed motives, $K$-theory and the entropy of certain $\mathbb{Z}^n$-actions. *J. Am. Math. Soc.*, 10(2):259–281, 1997.

[MO97] Yves Martin and Ken Ono. Eta-quotients and elliptic curves. *Proc. Am. Math. Soc.*, 125(11):3169–3176, 1997.

[Rog08] Mathew D. Rogers. Hypergeometric formulas for lattice sums and mahler measures. 2008.

[RV99] F. Rodriguez Villegas. Modular Mahler measures. I. Ahlgren, Scott D. (ed.) et al., Topics in number theory. In honor of B. Gordon and S. Chowla. Proceedings of the conference, Pennsylvania State University, University Park, PA, USA, July 31-August 3, 1997. Dordrecht: Kluwer Academic Publishers. Math. Appl., Dordr. 467, 17-48 (1999)., 1999.

[SS88] Norbert Schappacher and Anthony J. Scholl. Beilinson's theorem on modular curves. Beilinson's conjectures on special values of L-functions, Meet. Oberwolfach/FRG 1986, Perspect. Math. 4, 273-304 (1988)., 1988.

[Zag90] Don Zagier. The Bloch-Wigner-Ramakrishnan polylogarithm function. *Math. Ann.*, 286(1-3):613–624, 1990.

**Holomorphic Discs in the Space of Oriented Lines via Mean Curvature Flow and Applications**

Wilhelm Klingenberg, Durham University
(joint work with Brendan Guilfoyle, IT Tralee)

We introduce a metric $\mathbb{G}$ on the space $\mathbb{L}$ of oriented geodesics in Euclidean 3-space. It is of index $(2,2)$. Together with the natural complex structure $\mathbb{J}$ due to Nigel Hitchin and the classical symplectic structure $\Omega$ on this space, this endows $\mathbb{L}$ with the structure of a neutral Kähler surface. The geometry of this space captures the geometry of $C^1$-smooth surfaces in Euclidean 3-space via a correspondance that associates with $S$ the family $\Sigma \subset \mathbb{L}$ of oriented Euclidean-normal lines to $S$.

Our results are as follows.

1. We establish long-time existence for those solutions of mean curvature flow for spacelike surfaces in $(\mathbb{L}, \mathbb{G})$ that remain in a fixed compact subset of $\mathbb{L}$.

2. Given a Lagrangian surface $\Sigma \subset \mathbb{L}$, we consider mean curvature flow for spacelike surfaces $\Sigma_t$ in $\mathbb{L}$ subject to three boundary conditions:

a) $\partial\Sigma_t \subset \Sigma$ for all $t$,
b) the angle between $\Sigma_t$ and $\Sigma$ remains constant in time,
c) $T\partial\Sigma_t$ is holomorphic as $t \to \infty$.

In this situation, we prove that there exist times $t_j \to \infty$ such that $\Sigma_{t_j}$ converges to a holomorphic curve $\Sigma_{t_\infty}$.

3. We prove that the existence of $\Sigma_{t_\infty}$ as above implies a bound on the relative first chern class of the pair $(\mathbb{L}, \Sigma)$ along the boundary of $\Sigma_{t_\infty}$. This in turn implies a local index bound on the index of an isolated umbilic point of $S$. Here $S$ arises as an integral surface of the family of lines that correspond to $\Sigma$ in Euclidean 3-space.

# ANALYTIC TORSION AND COHOMOLOGY OF HYPERBOLIC 3-MANIFOLDS

WERNER MÜLLER

## 1. Introduction

In this talk we discuss the connection between the Ray-Singer analytic torsion of hyperbolic 3-manifolds and the torsion of the integer cohomology of arithmetic hyperbolic 3-manifolds.

**1. Analytic torsion.** Let $X$ be a compact Riemannian manifold of dimension $n$. Let $\rho\colon \pi_1(X) \to \mathrm{GL}(V_\rho)$ be a finite-dimensional representation of the fundamental group of $X$ and let $E_\rho \to X$ be the associated flat vector bundle. Then the Ray-Singer analytic torsion $T_X(\rho)$ attached to $\rho$ is defined as follows. Pick a Hermitian fiber metric $h$ in $E_\rho$ and let

$$\Delta_p(\rho)\colon \Lambda^p(X, E_\rho) \to \Lambda^p(X, E_\rho)$$

be the Laplacian on $E_\rho$-valued $p$-forms w.r.t. the hyperbolic metric $g$ on $X$ and the fibre metric $h$ in $E_\rho$. Then $\Delta_p(\rho)$ is a non-negative self-adjoint operator whose spectrum consists of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots$ of finite multiplicities. Let

$$\zeta_p(s;\rho) = \sum_{\lambda_i > 0} \lambda_i^{-s}, \quad \mathrm{Re}(s) > n/2.$$

be the zeta function of $\Delta_p(\rho)$. It is well known that $\zeta_p(s;\rho)$ admits a meromorphic extension to $s \in \mathbb{C}$ which is regular at $s = 0$. Then the regularized determinant $\det \Delta_p(\rho)$ of $\Delta_p(\rho)$ is defined as

$$\det \Delta_p(\chi) = \exp\left(-\frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}\right).$$

The analytic torsion is defined as the following weighted product of regularized determinants

$$T_X(\rho; h) = \prod_{p=0}^{n} \left(\det \Delta_p(\rho)\right)^{(-1)^p p/2}.$$

By definition it depends on $h$. However, if $n$ is odd and $\rho$ is acyclic, i.e., $H^*(X, E_\rho) = 0$, then $T_X(\rho; h)$ is independent of $h$ (see [Mu1]) and we denote it by $T_X(\rho)$.

The representation $\rho$ is called unimodular, if $|\det \rho(\gamma)| = 1$, $\forall \gamma \in \Gamma$. Let $\rho$ be a unimodular, acyclic representation. Then the Reidemeister tosion $\tau_X(\rho)$ is defined [Mu1, section 1]. It is defined combinatorially in terms of a smooth triangulation of $X$ and we have $T_X(\rho) = \tau_X(\rho)$ [Ch], [Mu2], [Mu1].

*Date*: June 10, 2009.

**2. Hyperbolic 3-manifolds.** Let $X$ be a compact oriented 3-dimensional hyperbolic manifold. Then there exists a discrete, torsion free, co-compact subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ such that $X = \Gamma\backslash\mathbb{H}^3$, where $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ is the 3-dimensional hyperbolic space.

For $m \in \mathbb{N}$ let $\rho_m\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(S^m(\mathbb{C}^2))$ be the standard irreducible representation of dimension $m+1$ acting in the space homogeneous polynomials $S^m(\mathbb{C}^2)$ of degree $m$. By restriction of $\rho_m$ to $\Gamma$ we obtain a representation of $\Gamma$ which we continue to denote by $\rho_m$. It follows from [BW, Theorem 6.7, Chapt. VII] that $\rho_m$ is acyclic. Since $\mathrm{SL}(2,\mathbb{C})$ is semisimple, it follows that $\det \rho_m(g) = 1$ for all $g \in \mathrm{SL}(2,\mathbb{C})$. Therefore the Reidemeister torsion $\tau_X(\rho_m)$ of $X$ with respect to $\rho_m|_\Gamma$ is well defined. Our main result determines the asymptotic bahavior of $\tau_X(\rho_m)$ as $m \to \infty$.

**Theorem 1.** *Let $X$ be a closed, oriented hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$. Then*

$$-\log \tau_X(\rho_m) = \frac{1}{\pi}\mathrm{vol}(X)m^2 + O(m)$$

*as $m \to \infty$.*

**3. Arithmetic groups.**

Let $F \subset \mathbb{C}$ be an imaginary quadratic field. Let $\mathcal{H} = \mathcal{H}(a,b;F)$ be a quaternion algebra over $F$, $a,b \in F^\times$. Then $\mathcal{H}$ splits over $\mathbb{C}$

$$\varphi\colon \mathcal{H} \otimes_F \mathbb{C} \cong M(2,\mathbb{C}).$$

Let $\mathfrak{R}$ be an order in $\mathcal{H}$ and let $\mathfrak{R}^1 = \{x \in \mathfrak{R}\colon N(x) = 1\}$. Let $\Gamma = \varphi(\mathfrak{R}^1)$. Then $\Gamma$ is a lattice in $\mathrm{SL}(2,\mathbb{C})$. Moreover $\Gamma$ is co-compact, if and only if $\mathcal{H}$ is a skew field. The norm 1 elements of $\mathcal{H}$ act by conjugation on the trace zero elements. In this way we get a $\Gamma$-invariant lattice $\Lambda \subset S^2(\mathbb{C}^2)$. Taking symmetric powers, it induces a $\Gamma$-invariant lattice in all even symmetric powers $S^{2m}(\mathbb{C}^2)$. So the integer cohomology $H^*(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})$ is defined. These are finite abelian groups. Denote by $|H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})|$ the order of $H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})$. Then we have

$$\tau_{\Gamma\backslash\mathbb{H}^3}(\rho_{2m}) = \prod_{p=1}^{3} |H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})|^{(-1)^{(p+1)}}. \tag{1}$$

Combining this result with Theorem 1, we get

**Theorem 2.** *Let $\Gamma$ be a co-compact, arithmetic lattice. Then*

$$\sum_{p=1}^{3}(-1)^p \log |H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})| = \frac{4}{\pi}\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)m^2 + O(m)$$

*as $m \to \infty$.*

**4. Ruelle zeta function.** The proof of Thoerem 1 is based on the study of the twisted

Ruelle zeta function $R(s, \rho)$ attached to a finite-dimensional representation $\rho$ of $\Gamma$. In a half-plane $\mathrm{Re}(s) \gg 0$ it is defined by the following infinite product

$$R(s, \rho) = \prod_{\substack{[\gamma]\neq 1 \\ prime}} \det\left(\mathrm{I} - \rho(\gamma) e^{-s\ell(\gamma)}\right),$$

where the product runs over all non-trivial prime conjugacy classes in $\Gamma$ and $\ell(\gamma)$ denotes the length of the corresponding closed geodesic. It admits a meromomorphic extension to $s \in \mathbb{C}$ [Fr2, p.181]. It follows from the main result of [Wo] that $R(s, \rho_m)$ is holomorphic at $s = 0$ and

$$|R(0, \rho_m)| = T_{\Gamma\backslash\mathbb{H}^3}(\rho_m)^2. \tag{2}$$

The corresponding result for unitary representations $\rho$ was proved by Fried [Fr1]. Now the proof of Theorem 1 is reduced to the study of the asymptotic behavior of $|R(0, \rho_m)|$ as $m \to \infty$. The volume appears through the functional equation satisfied by $R(s, \rho_m)$.

## References


[BW] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Second edition. Mathematical Surveys and Monographs, 67. Amer. Math. Soc., Providence, RI, 2000.

[Ch] J. Cheeger, *Analytic torsion and the heat equation.* Ann. of Math. (2) 109 (1979), no. 2, 259–322.

[Fr1] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. math. **84** (1986), 523–540.

[Fr2] D. Fried, *Meromorphic zeta functions of analytic flows*, Commun. Math. Phys. **174** (1995), 161 - 190.

[Mu1] W. Müller, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), 721–753.

[Mu2] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds.* Adv. in Math. **28** (1978), no. 3, 233–305.

[Wo] A. Wotzke, *The Ruelle zeta function and analytic torsion of hyperbolic manifolds*, dissertation, Bonn, 2008.



Universität Bonn, Mathematisches Institut, Beringstrasse 1, D – 53115 Bonn, Germany

*E-mail address*: mueller@math.uni-bonn.de

# From Archimedean $L$-factors to Topological Field Theories *

Anton Gerasimov, Dimitri Lebedev and Sergey Oblezin

## Introduction

Archimedean local $L$-factors were introduced to simplify functional equations of global $L$-functions. From the point view of arithmetic geometry these factors complete the Euler product representation of global $L$-factors by taking into account Archimedean places of the compactified spectrum of the global field. A construction of non-Archimedean local $L$-factors is rather transparent and uses characteristic polynomial of the image of the Frobenius homomorphism in finite-dimensional representations of the local Weil-Deligne group closely related to the local Galois group. On the other hand, Archimedean $L$-factors are expressed through products of $\Gamma$-functions and thus are analytic objects avoiding simple algebraic interpretation. Moreover, Archimedean Weil-Deligne groups are rather mysterious objects in comparison with their non-Archimedean counterparts. In a series of papers [GLO1], [GLO2], [GLO3], [GLO4] we approach the problem of the proper interpretation of Archimedean $L$-factors using various methods developed to study quantum integrable systems and low-dimensional topological field theories. As a result we produce several interesting explicit representations for Archimedean $L$-factors and related special functions revealing some hidden structures that might be relevant to the Archimedean (also known as $\infty$-adic ) algebraic geometry. Some of our considerations are close to the approach advocated by Deninger [D1], [D2]. Also equivariant symplectic volumes of the space of maps of a disk into symplectic manifolds were previously discussed in [Gi1], [Gi2] in connection with the Gromov-Witten theory.

## 1 Archimedean Hecke algebra

Let $K$ be a maximal compact subgroup of $G = GL(\ell+1,\mathbb{R})$. Define spherical Hecke algebra $\mathcal{H}_{\mathbb{R}} = \mathcal{H}(GL(\ell+1,\mathbb{R}),K)$ as an algebra of $K$-biinvariant functions on $G$, $\phi(g) = \phi(k_1 g k_2)$, $k_1, k_2 \in K$ with the multiplication given by

$$\phi * f(g) = \int_G \phi(g\tilde{g}^{-1})\, f(\tilde{g})d\tilde{g}. \tag{1.1}$$

To ensure the convergence of the integrals one usually imposes the condition of compact support on $K$-biinvariant functions. We will consider a more general class of exponentially decaying functions.

By the multiplicity one theorem for principle series representations of $GL(\ell+1,\mathbb{R})$ there is a unique smooth spherical vector $\langle k|$ in a principal series irreducible representation $\mathcal{V}_{\underline{\lambda}} = \mathrm{Ind}_{B_-}^{G}\, \chi_{\underline{\lambda}}$ where $\chi_{\underline{\lambda}}$ is a character of a Borel subgroup $B_-$. The action of a $K$-biinvariant function $\phi$ on the spherical vector $\langle k|$ in $\mathcal{V}_{\underline{\lambda}}$ is reduced to multiplication by a character $\Lambda_\phi$ of the Hecke algebra:

$$\phi * \langle k| \equiv \int_G dg\phi(g^{-1})\, \langle k|\pi_{\underline{\lambda}}(g) = \Lambda_\phi(\underline{\lambda})\langle k|, \qquad \phi \in \mathcal{H}_{\mathbb{R}}. \tag{1.2}$$

*Talk given by the second author at Arbeitstagung 2009, MPIM, Bonn.

Define $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}$ as a matrix element in a principle series irreducible representation $\mathcal{V}_{\underline{\lambda}}$ satisfying the covariance property

$$\Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(kan) = \chi_N(n)\, \Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(a), \tag{1.3}$$

where $kan \in KAN_- \to G$ is the Iwasawa decomposition. We parametrize the representations $\mathcal{V}_{\underline{\lambda}}$ of $GL(\ell+1,\mathbb{R})$ by vectors $\underline{\lambda} = (\lambda_1, \cdots, \lambda_{\ell+1})$ in $\mathbb{C}^{\ell+1}$. Whitaker functions play an important role in the theory of quantum integrable systems providing explicit solutions of quantum Toda chains. Let us define a related function

$$\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) = e^{-\langle \rho, \underline{x}\rangle}\ \Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}), \tag{1.4}$$

where $\underline{x} = (x_1, \ldots, x_{\ell+1}) \in \mathbb{R}^{\ell+1}$, $\rho \in \mathbb{R}^{\ell+1}$ , with $\rho_j = \frac{\ell}{2} + 1 - j,\ \ j = 1, \ldots, \ell+1$ and we use the standard orthogonal pairing $\langle\, ,\, \rangle$ on $\mathbb{R}^{\ell+1}$. The functions (1.4) are common eigenfunctions of a ring of commuting differential operators generated by coefficients of a polynomial

$$t^{\mathfrak{gl}_{\ell+1}}(\lambda) = \sum_{j=1}^{\ell+1} (-\imath)^j \lambda^{\ell+1-j} \mathcal{H}_j^{\mathfrak{gl}_{\ell+1}}(x, \partial_x), \tag{1.5}$$

where the first two operators are given by

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}} = -\imath \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_i}, \qquad \mathcal{H}_2^{\mathfrak{gl}_{\ell+1}} = -\frac{1}{2}(\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}})^2 - \frac{1}{2}\sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial {x_i}^2} + \sum_{i=1}^{\ell} e^{x_i - x_{i+1}}. \tag{1.6}$$

The last differential operator is a quantum Hamiltonian operator of $\mathfrak{gl}_{\ell+1}$-Toda chain. Commuting differential operators (1.5) provide an action of the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_{\ell+1})$ on the matrix elements satisfying (1.3). We have

$$t^{\mathfrak{gl}_{\ell+1}}(\lambda)\ \ \Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) = \prod_{j=1}^{\ell+1} (\lambda - \lambda_j)\ \ \Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}). \tag{1.7}$$

The following version of the Givental integral representation [Gi3] for $\mathfrak{gl}_{\ell+1}$-Whittaker function was proposed in [GKLO].

**Theorem 1.1** *The following integral recursive representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions holds*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(\underline{x}_{\ell+1}) = \int_{\mathbb{R}^\ell} \prod_{i=1}^{\ell} dx_{\ell,i}\ Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1}, \underline{x}_\ell|\lambda_{\ell+1}) \Psi^{\mathfrak{gl}_\ell}_{\lambda_1,\ldots,\lambda_\ell}(\underline{x}_\ell), \tag{1.8}$$

$$Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1}, \underline{x}_\ell|\lambda_{\ell+1}) = \exp\left\{ \lambda_{\ell+1}\Big(\sum_{i=1}^{\ell+1} x_{\ell+1,i} - \sum_{i=1}^{\ell} x_{\ell,i}\Big) - \sum_{i=1}^{\ell}\Big(e^{x_{\ell+1,i}-x_{\ell,i}} + e^{x_{\ell,i}-x_{\ell+1,i+1}}\Big)\right\},$$

*where $\underline{x}_k = (x_{k,1}, \ldots, x_{k,k})$ and we assume that $Q^{\mathfrak{gl}_1}_{\mathfrak{gl}_0}(x_{11}|\lambda_1) = e^{\lambda_1 x_{1,1}}$.*

Note that due to (1.2) any left $K$-invariant matrix element is an eigenfunction with respect to the action of any $\phi \in \mathcal{H}_{\mathbb{R}}$. Thus we have for the Whittaker function

$$\phi * \Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(g) = \Lambda_\phi(\underline{\lambda})\Phi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(g), \qquad \phi \in \mathcal{H}_{\mathbb{R}}, \tag{1.9}$$

**Theorem 1.2** *Let $\phi_{\mathcal{Q}_B(\lambda)}(g)$ be a $K$-biinvariant function on $G = GL(\ell+1,\mathbb{R})$ given by*

$$\phi_{\mathcal{Q}_B(\lambda)}(g) = 2^{\ell+1}|\det g|^{\lambda+\frac{\ell}{2}} e^{-\pi \mathrm{Tr} g^t g}. \tag{1.10}$$

*Then, the action of $\phi_{\mathcal{Q}_0(\lambda)}$ on the Whittaker function $\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g)$ (defined by (1.3)) descends to the action of an integral operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_B(\lambda)$ with the kernel*

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_B(\underline{x},\underline{y}|\lambda) = 2^{\ell+1}\exp\Big\{\sum_{j=1}^{\ell+1}(\lambda+\rho_j)(x_j-y_j) - \pi\sum_{k=1}^{\ell}\Big(e^{2(x_k-y_k)}+e^{2(y_k-x_{k+1})}\Big) - \pi e^{2(x_{\ell+1}-y_{\ell+1})}\Big\},$$

*where $\underline{x} = (x_1,\ldots,x_{\ell+1})$ and $\underline{y} = (y_1,\ldots,y_{\ell+1})$. The corresponding eigenvalue*

$$\big(\phi_{\mathcal{Q}_B(\lambda)} * \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}\big)(g) = L_{\mathbb{R}}(\lambda|\underline{\lambda})\ \Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \tag{1.11}$$

*is given by*

$$L_{\mathbb{R}}(\lambda|\underline{\lambda}) = \prod_{j=1}^{\ell+1}\pi^{-\frac{\lambda-\lambda_j}{2}}\Gamma\Big(\frac{\lambda-\lambda_j}{2}\Big). \tag{1.12}$$

The integral operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_B(\lambda)$ is an example of the Baxter operator which provides a key tool to solve quantum integrable systems. Its construction for quantum $\mathfrak{gl}_{\ell+1}$-Toda chains and its interpretation as an element of a spherical Hecke algebra $\mathcal{H}_{\mathbb{R}}$ was given in [GLO1].

The eigenvalues (1.12) can be considered as elementary building blocks from which general Whittaker functions can be constructed via Mellin-Barnes representations. Consider a simple example of the degenerate Whittaker function for which an analog of the Givental representation is given by

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(x) = \int\limits_{\mathbb{R}^\ell}\prod_{k=1}^{\ell}dx_{k,1}\quad e^{\mathcal{F}(x_{1,1},\ldots,x_{\ell,1},x_{\ell+1,1})}, \tag{1.13}$$

where $x := x_{\ell+1,1}$ and

$$\mathcal{F}(t) = \lambda_1 x_{11} + \sum_{k=1}^{\ell}\lambda_{k+1}(x_{k+1,1}-x_{k,1}) - e^{x_{11}} - \sum_{k=1}^{\ell}e^{x_{k+1,1}-x_{k,1}}.$$

The degenerate Whittaker function satisfies the following differential equation

$$\Big\{\prod_{k=1}^{\ell+1}\Big(-\frac{\partial}{\partial x}+\lambda_k\Big) - e^x\Big\}\Psi_{\underline{\lambda}}(x) = 0. \tag{1.14}$$

Besides the Givental representation there exists a representation of the Mellin-Barnes type

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(x) = \int\limits_{\sigma-\imath\infty}^{\sigma+\imath\infty}d\lambda\ e^{\lambda x}\prod_{k=1}^{\ell+1}\Gamma(\lambda_k-\lambda), \tag{1.15}$$

where $\sigma$ is such that $\sigma < \min\{\mathrm{Re}\,\lambda_j, j=1,\ldots,\ell+1\}$. Thus, basically, the degenerate Whittaker function is given by an action of integral projection operator on a product of eigenvalues (1.12).

There is a $p$-adic analog $\mathcal{H}_p = \mathcal{H}(GL(\ell+1,\mathbb{Q}_p), GL(\ell+1,\mathbb{Z}_p))$ of the Hecke algebra $\mathcal{H}_{\mathbb{R}}$. One can define a $\mathcal{H}_p$-valued function of an axillary variable such that its action by convolution on the $p$-adic analog [CS] of the Whittaker function is given by the multiplication on a local non-Archimedean

$L$-factor $L_p(s)$. In [GLO1] we argue that (1.10) should be considered as an Archimedean analog of the $\mathcal{H}_p$-valued function in non-Archimedean case. In particular the corresponding eigenvalues (1.12) are given by real Archimedean $L$-factors

$$L_{\mathbb{R}}(s|V,\Lambda) = \det_V \pi^{-\frac{s-\Lambda}{2}}\,\Gamma\left(\frac{s-\Lambda}{2}\right), \tag{1.16}$$

where $V=\mathbb{C}^{\ell+1}$, $s=\lambda$ and $\Lambda$ is diagonal matrix with the diagonal entries $\Lambda_j=\lambda_j$. In the next Section we provide a functional integral representation of the Archimedean $L$-factors (1.16). Taking into account that general Whittaker functions can be constructed form $L$-factors this leads to a functional integral representation of general Whittaker functions.

## 2 $L$-factors via equivariant topological linear sigma model

In this Section we demonstrate how local Archimedean $L$-factors (1.16) can be described in the framework of the two-dimensional topological field theory. Precisely, we consider equivariant version of type A topological linear sigma model on a disk $D=\{z|\,|z|\leq 1\}$ with non-compact target space $X=\mathbb{C}^{\ell+1}$. The vector space $\mathbb{C}^{\ell+1}$ is supplied with a Kähler form and a Kähler metric given in local complex coordinates $(\varphi^j,\varphi^{\bar{j}})$ by

$$\omega=\frac{\imath}{2}\sum_{j=1}^{\ell+1} d\varphi^j\wedge\bar{\varphi}^j,\qquad g=\frac{1}{2}\sum_{j=1}^{\ell+1}(d\varphi^j\otimes d\bar{\varphi}^j+d\bar{\varphi}^j\otimes d\varphi^j). \tag{2.1}$$

We also supply the disk $D$ with the flat metric $d^2s=dzd\bar{z}=dr^2+r^2d\sigma^2$, $z=r\,e^{\imath\sigma}$. Let $K$ and $\bar{K}$ be canonical and anti-canonical bundles on $D$. Let $\mathrm{Map}(D,\mathbb{C}^{\ell+1})$ be the space of maps $\Phi: D\to X$ of the disk $D$ to $\mathbb{C}^{\ell+1}$. Let $T_{\mathbb{C}}X=T^{1,0}\mathbb{C}^{\ell+1}\oplus T^{0,1}\mathbb{C}^{\ell+1}$ be a decomposition of the complexified tangent bundle of $\mathbb{C}^{\ell+1}$. Now let us specify the field content of the topological sigma model for $X=\mathbb{C}^{\ell+1}$. We define commuting fields $F$ and $\bar{F}$ as sections of $K\otimes\Phi^*(T^{0,1}X)$ and of $\bar{K}\otimes\Phi^*(T^{1,0}X)$ correspondingly. The anticommuting fields $\chi$, $\bar{\chi}$ are sections of the bundles $\Phi^*(\Pi T^{1,0}X)$, $\Phi^*(\Pi T^{0,1}X)$ and anticommuting fields $\psi$, $\bar{\psi}$ are sections of the bundles $K\otimes\Phi^*(\Pi T^{0,1}X)$, $\bar{K}\otimes\Phi^*(\Pi T^{1,0}X)$. Here $\Pi\mathcal{E}$ denotes the vector bundle $\mathcal{E}$ with the reverse parity of the fibres. Denote by $\langle,\rangle$ a natural Hermitian pairing on the spaces of sections of various bundles involved. We have the standard action of $U_{\ell+1}$ on $V=\mathbb{C}^{\ell+1}$ and an action of $S^1$ on $D$ by rotations $\sigma\to\sigma+\alpha$. The action of $G=S^1\times U_{\ell+1}$ lifts naturally to the action on the fields $(F,\bar{F},\varphi,\bar{\varphi},\psi,\bar{\psi},\chi,\bar{\chi})$. Let $\Lambda$ be an image of an element of $\mathfrak{u}_{\ell+1}$ in the representation $\mathbb{C}^{\ell+1}$. Let $\hbar$ be a generator of $S^1$, $v_0=\partial_\sigma$ be a corresponding vector field on $S^1$ and $\mathcal{L}_{v_0}$ be the Lie derivative along $v_0$.

Consider $G$-equivariant type A topological linear sigma model on $D$ with the target space $X=\mathbb{C}^{\ell+1}$ described by a $G$-invariant action functional

$$S_D=\imath\int_\Sigma d^2z\,\Big(\langle F,\overline{\partial}\varphi\rangle+\langle\bar{F},\partial\bar{\varphi}\rangle+\langle\bar{\psi},\partial\bar{\chi}\rangle+\langle\psi,\overline{\partial}\chi\rangle\Big), \tag{2.2}$$

The action is also invariant with respect to an odd transformation $\delta_G$

$$\begin{aligned}&\delta_G\varphi=\chi, &&\delta_G\chi=-(\imath\Lambda\varphi+\hbar\,\mathcal{L}_{v_0}\varphi), &&\delta_G\psi=F, &&\delta_GF=-(\imath\Lambda\psi+\hbar\,\mathcal{L}_{v_0}\psi),\\ &\delta_G\bar{\varphi}=\bar{\chi}, &&\delta_G\bar{\chi}=-(-\imath\Lambda\bar{\varphi}+\hbar\,\mathcal{L}_{v_0}\bar{\varphi}), &&\delta_G\bar{\psi}=\bar{F}, &&\delta_G\bar{F}=-(-\imath\Lambda\bar{\psi}+\hbar\,\mathcal{L}_{v_0}\bar{\psi}).\end{aligned} \tag{2.3}$$

Let us remark that $\delta_G$ can be considered as an infinite-dimensional analog of the de Rham differential in the Cartan model for equivariant cohomology. Observables in the topological sigma model are given by $\delta_G$-closed $G$-invariant functionals of the fields.

**Theorem 2.1** *Let $V=\mathbb{C}^{\ell+1}$ be a standard representation of $U_{\ell+1}$, $\Lambda$ be the image of an element $u\in\mathfrak{u}_{\ell+1}$ in End$(V)$. Then the following identity holds*

$$\left\langle e^{\mu\,\mathcal{O}_{\Lambda,\hbar}}\right\rangle_D \,=\, \hbar^{-\frac{\ell+1}{2}}\,\det_V\,\left(\frac{2}{\mu\hbar}\right)^{-\Lambda/\hbar}\,\Gamma(\Lambda/\hbar), \tag{2.4}$$

*where $\mathcal{O}_{\Lambda,\hbar}$ is given by*

$$\mathcal{O}_{\Lambda,\hbar}=\frac{\imath}{2}\int_0^{2\pi}d\sigma\,(-\langle\chi(re^{\imath\sigma}),\chi(re^{\imath\sigma})\rangle+\langle\varphi(re^{\imath\sigma}),(\imath\Lambda+\hbar\mathcal{L}_{v_0})\varphi(re^{\imath\sigma})\rangle)|_{r=1}. \tag{2.5}$$

*The functional integral in the $S^1\times U_{\ell+1}$-equivariant type A topological linear sigma model* (2.2) *in the l.h.s. of* (2.4) *is defined using $\zeta$-function regularization of Gaussian integrals.*

Taking $\mu=2/\pi$, $\hbar=1$ and making the change of variables $\Lambda\to(s\,\cdot\mathrm{id}-\Lambda)/2$ the correlation function (2.4) turns into local Archimedean $L$-factor (1.16). Let us note that the correlation function (2.4) for arbitrary $\mu$ and $\hbar$ can be considered as an Archimedean $L$-factor taking into account freedom to redefine $\epsilon$-factor in the functional equation for global $L$-functions.

The functional integral (2.4) can be interpreted as a $S^1\times U_{\ell+1}$-equivariant symplectic volume of the space of holomorphic maps of the disk $D$ to $\mathbb{C}^{\ell+1}$. Let $M$ be a $2(\ell+1)$-dimensional symplectic manifold with a symplectic form $\omega$. Let $G$ be a compact Lie group acting on $(M,\omega)$ and the action is Hamiltonian with the momentum map $H:\,M\to\mathfrak{g}^*$ to the dual $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ of $G$. Then $G$-equivariant symplectic volume of $M$ is defined as an the following integral

$$Z(M,\lambda)=\int_M e^{\omega+\langle\lambda,H\rangle}=\int_M\frac{\omega^{\ell+1}}{(\ell+1)!}\;e^{\langle\lambda,H\rangle},\qquad\lambda\in\mathfrak{g}, \tag{2.6}$$

where $\langle\,,\,\rangle$ is the paring between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The integral (2.6) is a finite-dimensional analog of the functional integral in the l.h.s. of (2.4) where the observable (2.5) plays the role of the equivariant symplectic form $\omega_G=\omega+\langle\lambda,H\rangle$.

# 3 $q$-version of $\mathfrak{gl}_{\ell+1}$-Whittaker function

Any local non-Archimedean factor $L_p(s)$ can be represented as a trace of Frobenius homomorphism acting in the direct sum of symmetric powers $S^*V$ of some fixed representation $V$ of the Galois group. Similar representation of a non-Archimedean Whittaker function as a trace of Frobenius homomorphism in finite-dimensional representations of Galois group is given in [CS]. These representations provides an arithmetic interpretation of local non-Archimedean $L$-factors/Whittaker functions. On the other hand Archimedean $L$-factors/Whittaker functions are analytic objects avoiding an analog of such interpretation. To make the corresponding structure in Archimedean case visible one can use a $q$-deformation of $L$-factors/Whittaker functions interpolating between non-Archimedean ($q=0$) and Archimedean ($q\to1$) cases. In this Section we recall a construction [GLO3] of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p}_{\ell+1})$ defined on the lattice $\underline{p}_{\ell+1}=(p_{\ell+1,1},\ldots,p_{\ell+1,\ell+1})\in\mathbb{Z}^{\ell+1}$. The $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions are common eigenfunctions of $q$-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\,=\,(\sum_{I_r}\prod_{i\in I_r}z_i)\;\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}), \tag{3.1}$$

where

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\,=\,\sum_{I_r}(X_{i_1}^{1-\delta_{i_2-i_1,1}}\cdot\ldots\cdot X_{i_{r-1}}^{1-\delta_{i_r-i_{r-1},1}}\cdot X_{i_r}^{1-\delta_{i_{r+1}-i_r,1}})T_{i_1}\cdot\ldots\cdot T_{i_r}. \tag{3.2}$$

Here the sum is over ordered subsets $I_r = \{i_1 < i_2 < \ldots < i_r\} \subset \{1, 2, \ldots, \ell+1\}$, $i_{r+1} := \ell + 2$. We use the following notations

$$T_i f(\underline{p}_{\ell+1}) = f(\widetilde{\underline{p}}_{\ell+1}), \qquad \widetilde{p}_{\ell+1,k} = p_{\ell+1,k} + \delta_{k,i}, \qquad i,k = 1, \ldots, \ell+1,$$

$$X_i = 1 - q^{p_{\ell+1,i} - p_{\ell+1,i+1}+1}, \quad i = 1, \ldots, \ell,$$

and $X_{\ell+1} = 1$. We also assume $q \in \mathbb{C}^*, |q| < 1$. For example, the first nontrivial Hamiltonian has the following form:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \sum_{i=1}^{\ell}(1 - q^{p_{\ell+1,i} - p_{\ell+1,i+1}+1})T_i \;+\; T_{\ell+1}. \tag{3.3}$$

The main result of [GLO3] is a construction of common eigenfunctions of quantum Hamiltonians (3.2) satisfying the "class one" condition (important for arithmetic interpretations [CS]). Thus one shall have

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p}_{\ell+1}) = 0, \tag{3.4}$$

outside dominant domain $p_{\ell+1,1} \geq \ldots \geq p_{\ell+1,\ell+1}$. Denote by $\mathcal{P}^{(\ell+1)} \subset \mathbb{Z}^{\ell(\ell+1)/2}$ a subset of integers $p_{n,i}$, $n = 1, \ldots, \ell+1$, $i = 1, \ldots, n$ satisfying the Gelfand-Zetlin conditions $p_{k+1,i} \geq p_{k,i} \geq p_{k+1,i+1}$ for $k = 1, \ldots, \ell$. In the following we use the standard notation $(n)_q! = (1-q)...(1-q^n)$.

**Theorem 3.1** *Let $\mathcal{P}_{\ell+1,\ell}$ be a set of $\underline{p}_\ell = (p_{\ell,1}, \ldots, p_{\ell,\ell})$ satisfying the conditions $p_{\ell+1,i} \geq p_{\ell,i} \geq p_{\ell+1,i+1}$. The following recursive relation holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \sum_{\underline{p}_\ell \in \mathcal{P}_{\ell+1,\ell}} \Delta(\underline{p}_\ell)\; z_{\ell+1}^{\sum_i p_{\ell+1,i} - \sum_i p_{\ell,i}}\; Q_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_\ell|q)\Psi^{\mathfrak{gl}_\ell}_{z_1,\ldots,z_\ell}(\underline{p}_\ell),$$

*where*

$$Q_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_\ell|q) \;=\; \frac{1}{\prod\limits_{i=1}^{\ell}(p_{\ell+1,i} - p_{\ell,i})_q!\;(p_{\ell,i} - p_{\ell+1,i+1})_q!},$$
$$\Delta(\underline{p}_\ell) = \prod_{i=1}^{\ell-1}(p_{\ell,i} - p_{\ell,i+1})_q!\;. \tag{3.5}$$

The representation (3.5) is a $q$-analog of Givental's integral representation of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function given in Theorem 1.1 and turns into (1.8) after taking appropriate limit $q \to 1$.

**Proposition 3.1** *There exists a $\mathbb{C}^* \times GL(\ell+1,\mathbb{C})$-module $V$ such that the common eigenfunction constructed in Theorem 3.1 allows the following representation for $p_{\ell+1,1} \leq p_{\ell+1,2} \leq \ldots p_{\ell+1,\ell+1}$:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{p}_{\ell+1}) \;=\; \mathrm{Tr}_{\,V}\, q^{L_0} \textstyle\prod_{i=1}^{\ell+1} q^{\lambda H_i}, \tag{3.6}$$

*Here $z_j = q^{\lambda_j}$, $H_i$, $i = 1, \ldots, \ell+1$ are Cartan generators of $\mathfrak{gl}_{\ell+1} = \mathrm{Lie}(GL(\ell+1,\mathbb{C}))$ and $L_0$ is a generator of $\mathrm{Lie}(\mathbb{C}^*)$.*

Define a degenerate $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a specialization of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n,k) \;:=\; \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n+k, k, \ldots, k). \tag{3.7}$$

This degenerate $q$-Whittaker function is an analog of the calssical degeenrate Whittake function (1.13) and has explicit representations analogous to (1.13) and (1.15)

$$\begin{aligned}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n,k) \,&=\, \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \sum_{n_1+\ldots+n_{\ell+1}=n} \frac{z_1^{n_1}}{(n_1)_q!}\cdot\ldots\cdot\frac{z_{\ell+1}^{n_{\ell+1}}}{(n_{\ell+1})_q!}\,,\\ &=\Big(\prod_{i=1}^{\ell+1} z_i^k\Big)\oint\limits_{t=0}\frac{dt}{2\pi\imath\, t}\,t^{-n}\prod_{i=1}^{\ell+1}\Gamma_q(z_it),\end{aligned} \tag{3.8}$$

for $n\geq 0$ and $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n,k) \,=\, 0$ for $n<0$. Here we use a $q$-version of $\Gamma$-function

$$\Gamma_q(x) \,=\, \prod_{n=0}^{\infty}\frac{1}{1-q^nx}=\sum_{n=0}^{\infty}\frac{t^n}{(n)_q!}.$$

Simialrly to (1.15) the $q$-version of degenerate Whittaker function is expressed through the $q$-versions of a local $L$ factor

$$L_q(s|V)=\det_V\Gamma_q(q^{s-\Lambda}), \tag{3.9}$$

where $V\,=\,\mathbb{C}^{\ell+1}$ and $\Lambda\,=\,(\Lambda_1,\ldots,\Lambda_{\ell+1})$. Thus defined $L_q$-factors allow a representation as a trace analogous to the representation (3.6) for Whittaker functions. The representation (3.6) can be considered as $q$-version of the Shintani-Casselman-Shalika formula [CS] representing non-Archimedean Whittaker function as trace of Frobenius over a finite-dimensional representation of the local Galois group. Indeed in the limit $q\to 0$ the Whittaker given in Theorem 3.1 reduces to a character of an irreducible finite-dimensional representations of $GL_{\ell+1}$ corresponding to a partition $p_{\ell+1,1}\leq\cdots\leq p_{\ell+1,\ell+1}$

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{p}_{\ell+1}) \,=\, \chi^{\mathfrak{gl}_{\ell+1}}_{\underline{p}_{\ell+1}}(\underline{z}) := \sum_{p_{k,i}\in\mathcal{P}^{\ell+1}}\prod_{k=1}^{\ell+1} z_k^{(\sum_{i=1}^k p_{k,i}-\sum_{i=1}^{k-1}p_{k-1,i})}\,, \tag{3.10}$$

where we set $z_i\,=\,q^{\lambda_i}$, $i\,=\,1,\ldots,\ell+1$ and the notation $\underline{z}\,=\,(z_1,z_2,\ldots,z_{\ell+1})$ is used. Thus for $q=0$ (3.6) reproduces the non-Archimedean expression [CS]. In the next Sections we elucidate the nature of the $\mathbb{C}^*\times GL_{\ell+1}$-modules $V$ apearing in (3.6).

## 4 $q$-Whittaker function and spaces of quasimaps

In this Section we provide an interpretation of the trace type representation (3.6) for the degenerate $q$-Whittaker function (3.7) and an analog of (3.6) for $L_q$-factors (3.9). Consider the space $\mathcal{M}_d(\mathbb{P}^\ell)$ of holomorphic maps of $\mathbb{P}^1$ to $\mathbb{P}^\ell$ of degree $d$. Explicitly, it can be described as a set of collections of $(\ell+1)$ relatively prime polynomials of degree $d$, up to a common constant factor. The space $\mathcal{M}_d(\mathbb{P}^\ell)$ allows a compactification by the space of quasi-maps $\mathcal{QM}_d(\mathbb{P}^\ell)=\mathbb{P}^{(\ell+1)(d+1)-1}$ defined as a set of collections of $(\ell+1)$ polynomials of degree $d$, up to a common constant factor. On the space $\mathcal{QM}_d(\mathbb{P}^\ell)$ there is a natural action of the group $\mathbb{C}^*\times GL_{\ell+1}$ (and, thus, of its maximal compact subgroup $S^1\times U_{\ell+1}$) where the action of $GL_{\ell+1}$ is induced by the standard action on $\mathbb{P}^\ell$ and the action of $\mathbb{C}^*$ is induced by the action of $\mathbb{C}^*$ on $\mathbb{P}^1$. The space of sections of the line bundle $\mathcal{O}(n)$ on $\mathcal{QM}_d(\mathbb{P}^\ell)$ is naturally a $\mathbb{C}^*\times GL_{\ell+1}$-module. Let $T\in GL_{\ell+1}$ be a Cartan torus, $H_1,\ldots,H_{\ell+1}$ be a basis in Lie$(T)$, and $L_0$ be a generator of Lie$(\mathbb{C}^*)$. Let $\mathcal{L}_k$ be a one-dimensional $GL_{\ell+1}$-module such that $H_i\mathcal{L}_k=k\mathcal{L}_k$, for $i=1,\ldots,\ell+1$. Cohomology groups $H^*(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{O}(n))\otimes\mathcal{L}_k$ have a natural structure of $\mathbb{C}^*\times GL_{\ell+1}(\mathbb{C})$-module. Let $\mathcal{M}_d(\mathbb{C},\mathbb{C}^{\ell+1})$ be a space of holomorphic maps of $\mathbb{C}$ to $\mathbb{C}^{\ell+1}$ defined as a set of collections of $(\ell+1)$ polynomials of degree $d$ and let $\mathcal{W}_d$ be a space of polynomial functions on $\mathcal{M}_d(\mathbb{C},\mathbb{C}^{\ell+1})$.

**Proposition 4.1** *For the $\mathbb{C}^*\times GL_{\ell+1}$-character of the module $\mathcal{V}_{n,k,d} = H^0(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{L}_k\otimes\mathcal{O}(n))$, $n\geq 0$ the following integral representation holds*

$$\mathrm{Tr}\,_{\mathcal{V}_{n,k,d}}\, q^{L_0}\, e^{\sum\lambda_i H_i} \;=\; \Big(\prod_{i=1}^{\ell+1} z_i^k\Big)\oint_{t=0}\frac{dt}{2\pi\imath\, t^{n+1}}\prod_{m=1}^{\ell+1}\prod_{j=0}^{d}\frac{1}{(1-tq^j z_m)}, \tag{4.1}$$

*where $\underline{z}=(z_1,\ldots,z_{\ell+1})$, $z_m=e^{\lambda_m}$, $H_i$, $i=1,\ldots,\ell+1$ are Cartan generators of $\mathfrak{gl}_{\ell+1}=\mathrm{Lie}(GL(\ell+1,\mathbb{C}))$ and $L_0$ is a generator of $\mathrm{Lie}(\mathbb{C}^*)$.*

Let us remark that the r.h.s. can be interpreted as a Riemann-Roch-Hirzebruch formula for $G$-equivariant holomorphic Euler characteristic of the line bundle $\mathcal{L}_k(n)=\mathcal{L}_k\otimes\mathcal{O}(n)$

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{L}_k(n)) \;=\; \big\langle \mathrm{Ch}_G(\mathcal{L}_k(n))\,\mathrm{Td}_G(\mathcal{T}\mathcal{QM}_d(\mathbb{P}^\ell)),[\mathcal{QM}_d(\mathbb{P}^\ell)]\big\rangle. \tag{4.2}$$

using the standard model for equivariant K-theory of projective spaces

$$K(\mathbb{P}^N)=\mathbb{C}[t,t^{-1}]/(1-t)^{N+1},\qquad K_{U_{N+1}}(\mathbb{P}^N)=\mathbb{C}[t,t^{-1},z,z^{-1}]\Big/\prod_{j=1}^{N+1}(1-tz_j)\,. \tag{4.3}$$

Using this Proposition, $q$-deformed degenerate $\mathfrak{gl}_{\ell+1}$-Whittaker functions can be expressed in terms of holomorphic sections of line bundles on a space $\mathcal{L}\mathbb{P}^\ell_+$ defined as an appropriate limit of $\mathcal{QM}_d(\mathbb{P}^\ell)$ when $d\to+\infty$. Geometrically $\mathcal{L}\mathbb{P}^\ell_+$ should be considered as a space of algebraic disks in $\mathbb{P}^\ell$.

**Theorem 4.1** *(i) Let $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k)$ be a degenerate Whittaker function* (3.7)*. Then the following holds*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k) \;=\; \lim_{d\to\infty}\,\mathrm{Tr}\,_{\mathcal{V}_{n,k,d}}\,q^{L_0}\,e^{\sum\lambda_iH_i} \;=\; \Big(\prod_{j=1}^{\ell+1}z_j^k\Big)\oint_C\frac{dt}{2\pi\imath\,t^{n+1}}\prod_{i=1}^{\ell+1}\Gamma_q(tz_i), \tag{4.4}$$

*where the integration contour $C$ encircles all poles except $t=0$.*

*(ii) The following expression for a $q$-version of the local $L$-factor* (3.9) *holds*

$$L_q(s|V)\;:=\;\det_V\Gamma_q(q^{s-\Lambda})=\lim_{d\to\infty}\;\mathrm{Tr}\,_{\mathcal{W}_d}\,q^{L_0}\,q^{\sum\lambda_iH_i}, \tag{4.5}$$

*where $V=\mathbb{C}^{\ell+1}$ and $\Lambda=(\Lambda_1,\ldots,\Lambda_{\ell+1})$, $\Lambda_j=s-\lambda_j$.*

Taking a limit $d\to\infty$ at the level of underlying vector spaces $\mathcal{V}_{n,k,d}$ and $\mathcal{W}_d$ can be naturally understood in terms of topological field theory interpretation of representation given in Theorem 4.1. In the following Section we provide such interpretation for $q$-deformed $L$-function (4.5).

## 5 $\Gamma_q$-function via equivariant linear sigma model on $D\times S^1$

In Section 2 we describe functional integral representation of a $\Gamma$-function as an equivariant symplectic volume of the space of holomorphic maps $D\to\mathbb{C}$. According to the standard Correspondence Principle in quantum/statistical mechanics such equivariant volumes provide asymptotics of the partition functions of quantum theories. Applying this reasoning to the equivariant volume considered in Section 2 and using the standard path integral interpretation of quantum mechanics we obtain the following functional representation of the $q$-version of $\Gamma$-function.

**Theorem 5.1** *Consider a three-dimensional topological linear sigma model on $N = S^1 \times D$ with the action*

$$S = S_0 + \mathcal{O}, \tag{5.1}$$

*where*

$$S_0 = \imath \int_{S^1\times D} d^2z\, d\tau\, \left(\partial_{\bar z}\chi\bar\psi_z + \bar F_z\partial_{\bar z}\varphi + \partial_z\bar\chi\psi_{\bar z} + F_{\bar z}\partial_z\bar\varphi\right), \tag{5.2}$$

*and*

$$\mathcal{O} = \frac{\imath}{2}\beta \int_{\partial N = S^1\times S^1} d\tau\, d\sigma\, (\bar\chi\chi + \bar\varphi(\hbar\partial_\sigma + 2\pi\imath\beta^{-1}\partial_\tau + \imath\lambda)\varphi). \tag{5.3}$$

*Then the functional integral with free boundary conditions defined using $\zeta$-function regularization is equal to*

$$Z(t,q) = \prod_{n=0}^{+\infty} \frac{1}{1 - tq^n} = \Gamma_q(t), \tag{5.4}$$

*where $t = e^{-\beta\lambda}$, $q = e^{-\beta\hbar}$.*

Note that similar to the two-dimensional topological theory considered in Section 2 this three-dimensional theory is also invariant with respect to odd transformations

$$\delta_{G_0}\varphi = \chi, \qquad \delta_{G_0}\chi = -(\hbar\partial_\sigma + 2\pi\imath\beta^{-1}\partial_\tau + \imath\lambda)\varphi,$$

$$\delta_{G_0}\psi_{\bar z} = F_{\bar z}, \qquad \delta_{G_0}F_{\bar z} = -(\hbar\partial_\sigma + 2\pi\imath\beta^{-1}\partial_\tau + \imath\lambda)\psi_{\bar z}.$$

Finally note the the functional integral (5.1) defined using $\zeta$-function regularization gives a proper interpretation of the $d \to \infty$ limit considered in the previous Section.

## 6 Concluding remarks

The construction of the functional integral representation of local Archimedean $L$-factors uses an integral representation of the $\Gamma$-function. This functional integral representation should be compared with the standard Euler integral representation. One can show that the Euler integral representation naturally arises as a disk partition function in the equivariant type $B$ topological Landau-Ginzburg model on a disk with the target space $\mathbb{C}$ and the superpotential $W(\xi) = e^\xi + \lambda\xi$, $\xi \in \mathbb{C}$. This result is not surprising in view of a mirror symmetry between type A and type B topological sigma model. Thus we have two integral representations of $\Gamma$-function, one is in terms of an infinite-dimensional equivariant symplectic volume and another is given by a finite-dimensional complex integral. Taking into account the mirror symmetry relating the two underlying topological theories, the two integral representations should be considered on equal footing. These two integral representations of $\Gamma$-functions are similar to two different constructions (arithmetic and automorphic) of local Archimedean $L$-factors. The equivalence of the resulting $L$-factors is a manifestation of local Archimedean Langlands correspondence. The analogy between mirror symmetry and local Archimedean Langlands correspondence looks not accidental and can eventually imply that local Archimedean Langlands correspondence follows from the mirror symmetry.

## References


[CS] W. Casselman, J. Shalika, *The unramified principal series of p-adic groups II. The Whittaker function*. Comp. Math. **41**, pp. 207–231 (1980).

[D1] C. Deninger, *On the* $\Gamma$*-factors attached to motives*, Invent. Math. 104 (1991), pp. 245–261.

[D2] C. Deninger, *Local L-factors of motives and regularized determinants*, Invent. Math. 107 (1992), pp. 135–150

[GLO1] A. Gerasimov, D. Lebedev, S. Oblezin, *Baxter operator and Archimedean Hecke algebra*, Comm. Math. Phys. DOI 10.1007/s00220-008-0547-9; [arXiv:0706.347], 2007.

[GLO2] A. Gerasimov, D. Lebedev, S. Oblezin, *Baxter Q-operators and their Arithmetic implications*, Lett. Math. Phys. DOI 10.1007/911005-008-0285-0; [arXiv:0711.2812].

[GLO3] A. Gerasimov, D. Lebedev, S. Oblezin, *On q-deformed* $\mathfrak{gl}_{\ell+1}$*-Whittaker functions I,II,III*, [arXiv:0803.0145], [arXiv:0803.0970], [arXiv:0805.3754].

[GLO4] A. Gerasimov, D. Lebedev, S. Oblezin, *Archimedean L-factors and Topological Field Theories*, [arXiv:0906.1065].

[GKLO] A. Gerasimov, S. Kharchev, D. Lebedev, S. Oblezin, *On a Gauss-Givental representation of quantum Toda chain wave function*, Int. Math. Res. Notices, (2006), Article ID 96489, [arXiv:0505310].

[Gi1] A. Givental, *Homological geometry I. Projective hypersurfaces*, Selecta Mathematica, New Series Volume 1, **2**, pp. 325–345, 1995.

[Gi2] A. Givental, *Equivariant Gromov - Witten Invariants*, Internat. Math. Res. Notices 1996, no. 13, pp. 613–663, [arXiv:9603021].

[Gi3] A. Givental, *Stationary Phase Integrals, Quantum Toda Lattices, Flag Manifolds and the Mirror Conjecture*. Topics in Singularity Theory, Amer. Math. Soc. Transl. Ser., 2 **180**, American Mathematical Society, Providence, Rhode Island, 1997, pp. 103–115 [arXiv:9612001].

[GiL] A. Givental, Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. Math. **151** (2003), pp. 193–219; [arXiv:0108105].



**A.G.** *Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia;*
*School of Mathematics, Trinity College, Dublin 2, Ireland;*
*Hamilton Mathematics Institute, Trinity College, Dublin 2, Ireland;*

**D.L.** *Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia*;
*E-mail address*: lebedev@itep.ru

**S.O.** *Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia*;
*E-mail address*: Sergey.Oblezin@itep.ru

# TROPICAL GEOMETRY

GRIGORY MIKHALKIN

This Arbeitstagung 2009 talk surveys the current state of development of Tropical Geometry. We do not make an attempt to make an exhausting survey, but rather choose some particular topics to make it a collection of some short stories from the area.

## 1. Introduction and a (basic) example

Recall that as Mathematics operates with rather abstract notions, many notions may admit several different-looking (and perhaps still sufficiently abstract) *realizations*.

For example, let us consider (algebro-geometric) curves. These are 1-dimensional algebraic varieties. Their classical realization (XIX century) is provided by Riemann surfaces, i.e. smooth 2-dimensional manifolds with a choice of complex structure in their tangent bundle. The story generalizes to higher-dimensional algebraic varieties, but it is especially easy is dimension 1. In this dimension the complex structure is given by an endomorphism $J$ in every tangent space with the property that $J^2 = -1$ (i.e. an almost complex structure). Furthermore, a complex structure on a Riemann surface may be described by a metric of constant curvature. Projective curves correspond to compact surfaces. The genus of a curve is one half of its first Betti number (i.e. the number of cycles). It can also be computed as the dimension of the space of holomorphic 1-forms on the surface.

Compact tropical curves can be realized as so-called metric graphs (considered up to an equivalence). These are finite graphs where the interior of each edge is enhanced with an inner metric. We impose the requirement that the length of an edge adjacent to a 1-valent vertex must be infinite. Such an edge is called a *leaf edge*. The genus of a tropical curve is the number of cycles. It can be also computed as the dimension of the space of tropical 1-forms on the graph.

To get tropical curves we consider metric graphs equivalent if one can obtained from the other by contracting a leaf edge. Clearly genus depends only on an equivalence class. Note that all genus 0 curves are equivalent. Thus the tropical rational curve is unique just as in the classical case. Curves of positive genus $g > 0$ admit unique minimal

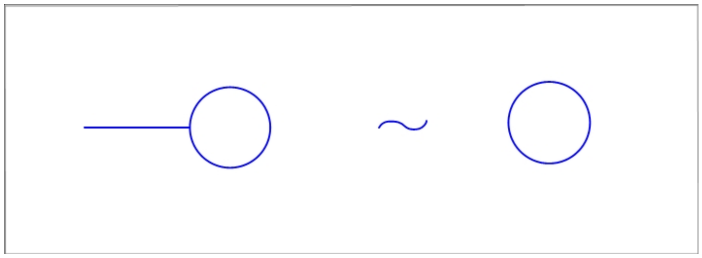


FIGURE 1. Equivalent elliptic curves

models – graphs without leaves. Generically such graphs are 3-valent and have $3g-3$ edges of some length. Thus the space of all curves of genus $g>1$ is $(3g-3)$-dimensional.

It is easy to see that tropical curves possess many other properties that we can expect from projective curves. In particular, any curve of genus smaller than 3 admits a hyperelliptic involution. In the same time a generic genus 3 curve is not hyperellptic, but trigonat, etc. To a tropical curve we may associate its Picard group, its jacobian varieties. Many classical 19th century theorems about Riemann surfaces (such as Abel-Jacobi, Riemann-Roch, the Riemann theorem on $\Theta$-divisor, etc) admit straightforward and easy-to-visualize tropical counterparts, cf. [9].

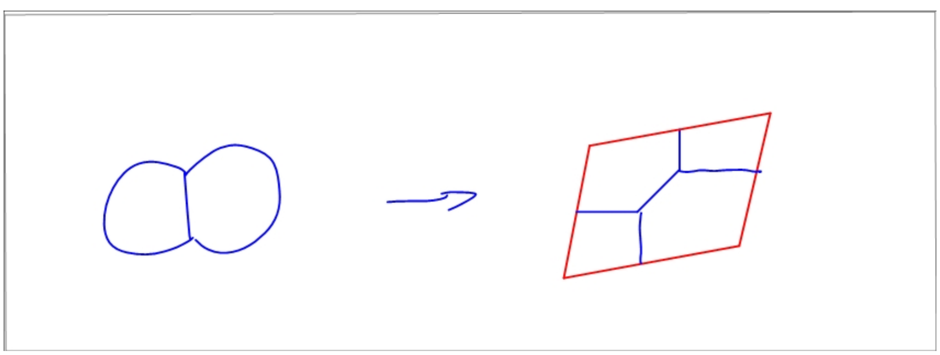


FIGURE 2. A genus 2 curve as a $\Theta$-divisor in its Jacobian variety

## 2. Tropical varieties and morphisms, the balancing condition

As a set tropical numbers $\mathbb{T}$ coincide with the half-open real line $[-\infty,+\infty)$. There are two tropical arithmetic operations (which we denote in quotation marks to distinguish them from standard arithmetic operations): tropical addition “$x+y$” $= \max\{x,y\}$ and tropical multiplication “$xy$” $= x+y$. Clearly we get tropical division “$x/y$” $= x-y$. However there is no chance for tropical subtraction as tropical addition is idempotent: $x+x=x$. Actually in most geometric constructions we can easily avoid using arithmetics at all.

Let us consider the affine $n$-space $\mathbb{T}^n$ and the $n$-torus $(\mathbb{T}^\times)^n = \mathbb{R}^n$. Here $\mathbb{T}^\times = \mathbb{T} \smallsetminus \{0_\mathbb{T}\} = \mathbb{R}$ as the neutral element under addition is $0_\mathbb{T} = -\infty$. Tropical structure in these spaces is given by the sheaf of tropical regular functions that are obtained from tropical rational functions by restricting them to open sets where they are convex. The geometric structure that encodes such a sheaf is the integer-affine structure on $\mathbb{R}^n$. Thus tropical varieties can be thought as polyhedral complexes enhanced with an integer-affine structure.

There are local and global conditions on such an enriched polyhedral complex $(X,\mathcal{O})$. Locally we require that $(X,\mathcal{O})$ is equivalent to $(\mathbb{T}^n, \mathcal{O}_{\mathbb{T}^n})$. Equivalence here is generated by smooth divisors, i.e. those that are themselves smooth $(n-1)$-dimensional tropical varieties. Globally we require a certain finite type condition. The resulting object is a (smooth) tropical manifold. Tropical manifolds come with (equivalent) local embeddings to $\mathbb{T}^N$, $N \geq n$, that exhibit them as piecewise-linear polyhedral complexes $Q \subset \mathbb{R}^N$ (or, rather their closures in $\mathbb{T}^N \supset \mathbb{R}^N$). By a piecewise-linear polyhedral complex we mean a union of convex polyhedra in $\mathbb{R}^N$. Furthermore, we require that the slope of each face $E$ is integer, i.e. the vector subspace $V_E \subset \mathbb{R}^N$ parallel to $E$ is generated by integer vectors.

Any local model polyhedron complex $Q \subset \mathbb{R}^n$ is *balanced*. This is a property at $(n-1)$-dimensional faces of $Q$. Let $E \subset Q$ be an $(n-1)$-face and $F_1, \dots, F_k$ be the $n$-facets adjacent to $Q$. Each $F_j$ defines a vector $v_j$ in the quotient vector space $\mathbb{R}^{N-n} = \mathbb{R}^N/V_E$,namely a primitive integer vector parallel to the image of $F_j$ in the projection. The balancing condition is formulated as

$$\sum_j v_j = 0 \in \mathbb{R}^N/V_E.$$

It is always satisfied if $Q$ is locally equivalent to $\mathbb{T}^n$. Furthermore we have some additional (finer) properties at faces of codimension greater than 1.

Alternatively, we may define a class of tropical $n$-spaces where we only impose the balancing condition at the faces of codimension 1 and no additional conditions at higher codimensions. Furthermore, at the $n$-faces we may put integer weights. These are the so-called tropical cycles. A cycle is *effective* if the weights are positive. We may define

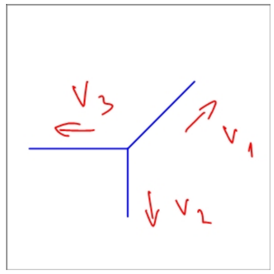



Figure 3. Balancing condition

positive *multiplicities* at the points of such cycles. If all these multiplicities equal to one then the cycle is called a homological tropical variety (or a pseudomanifold). Such spaces are locally given by matroids and their local realizability by complex effective cycles depends on the realizability of the corresponding matroid, cf. [7].

All morphisms between homological varieties are given by integer affine-linear maps of the ambient varieties. Morphisms between smooth tropical manifolds are more restricted, they are given by regular functions. E.g. scaling by 2 of all the edges is induced by an integer affine-linear map of the ambient $\mathbb{R}^2$, but is not an endomorphism of a tripod graph (as a smooth tropical 1-manifold). Note that the number of critical points of this would-be endomorphism is negative and thus it is never approximated by a complex map.

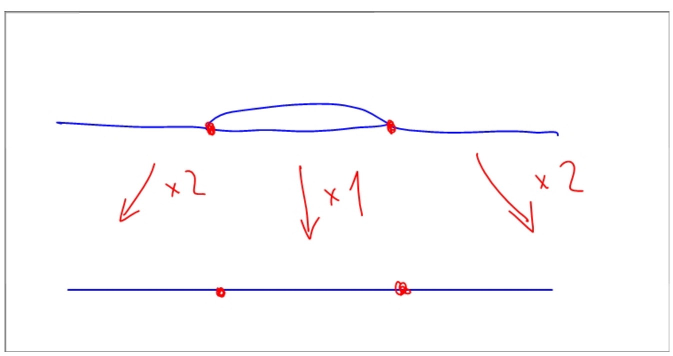



Figure 4. A (realizable) degree 2 map from an elliptic curve to $\mathbb{TP}^1$.

## 3. Interactions between tropical and classical worlds

Connection between complex and tropical numbers is provided by $\log_t : \mathbb{C} \to \mathbb{T},\ z \mapsto \log_t |z|$. When $t \to \infty$ the map becomes more and more homomorphism-like. Images of complex affine varieties under the map

$$\mathrm{Log}_t : \mathbb{C}^n \to \mathbb{T}^n$$

obtained by coordinate-wise application of $\log_t$ are called *amoebas* and carry significant information about geometry of complex varieties. Even better picture is obtained after consideration of images of families $V_t \subset \mathbb{C}^n$ under $\mathrm{Log}_t$ when $t \to \infty$. The limits of these images are (perhaps singular) tropical varieties.

More generally, tropical varieties $X$ sometimes can be obtained as a result of collapse $\lambda_t : \mathcal{X}_t \to X$ of families of complex varieties $\mathcal{X}_t$. Such a collapse is easy to produce in the case when $X$ is a tropical curve (with the help of decomposition into pairs-of-pants) or if $X$ is a smooth tropical complete intersection (by tropicalizing the defining equations). Tropical varieties may be enhanced with *phases* responsible for gluing data. The phase-tropical structure can also be included in the approximation data.

For curves the phase data amount to the twist for gluing pairs-of-pants. If the curve is given by a 3-valent graph and we fix a cyclic orientation for the edges adjacent to every 3-valent vertex we have a canonical (untwisted) choice of gluing. E.g. if we have a plane curve $h : C \to \mathbb{TP}^2$ the cyclic order is given by the ambient plane. The untwisted phase-tropical curves give the so-called simple Harnack curves, cf. [5].

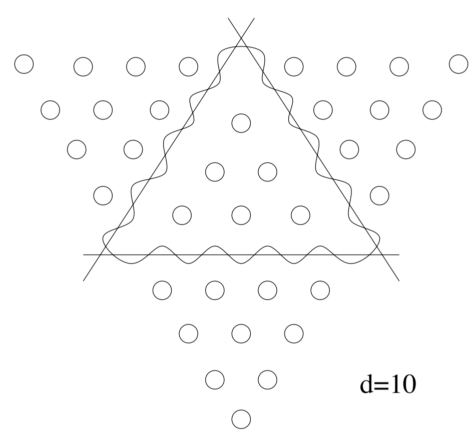



FIGURE 5. A Harnack curve of degree 10.

Suppose that $h : C \to X$ is a tropical morphism, where $C$ is a curve and $X$ is a complete intersection. We may approximate $C$ by a complex family $\mathcal{C}_t$ and $X$ with a complex family $\mathcal{X}_t$. But can we approximate $h$ with a family of holomorphic maps $H_t : \mathcal{C}_t \to \mathcal{X}_t$. It turns out that it is not always so. Nevertheless the following theorem provides a criterion for such realizability.

It can be shown (with the help of the tropical Riemann-Roch theorem) that any tropical curve $h : C \to X$ in $X$ has a deformation space of dimension at least $-K_X.[h(C)] + (1-g)(\dim X - 3)$.

**Definition 3.1.** A tropical map $h : C \to X$ is called *regular* if the dimension of the deformation space of $h$ is $-K_X.[h(C)]+(1-g)(\dim X - 3)$. Otherwise $h$ is called *superabundant*.

**Theorem 1** ([6]). *A regular tropical morphism $h : C \to X$ is approximable by a family of holomorphic maps $H_t : \mathcal{C}_t \to \mathcal{X}_t$.*

There are many examples of non-realizable superabundant curves. For example a map $h : C \to \mathbb{TP}^1$ from an elliptic curve depicted on Figure 6 is realizable only if the lengths $a$ and $b$ are equal. This is a special case of a realizability of genus 1 curves found by David Speyer [10].

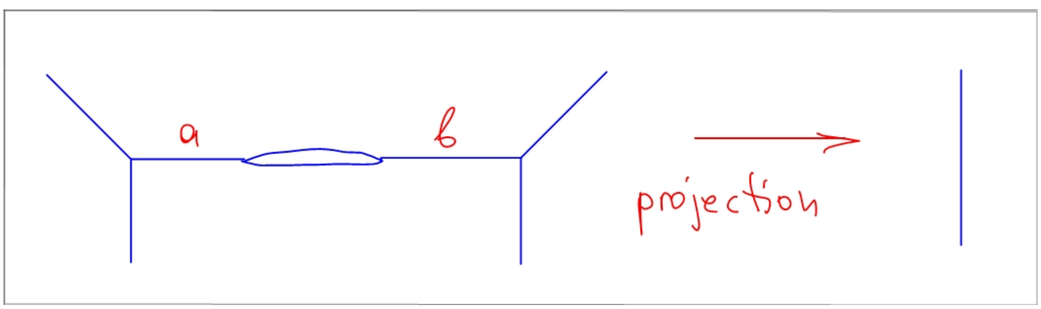



FIGURE 6. A non-realizable superabundant map from an elliptic curve to $\mathbb{TP}^1$.

## 4. APPLICATIONS TO COMPLEX AND REAL ENUMERATIVE GEOMETRY

Theorem 1 allows to replace certain (regular) enumerative problems in classical (complex and real) geometry with the corresponding tropical problems. Often the latter problems are much more manageable combinatorially.

For example, consider the problem of finding the number of complex (or real) curves of degree $d$ and genus $g$ passing through $3d - 1 +$

$g$ generic points in $\mathbb{P}^2$ or $2d$ points in $\mathbb{P}^3$. In the case of the real enumerative problems the curves have to be counted with signes defined by Welschinger [12], [13] in the case of genus 0 (in the case of positive genus we do not consider the real case at all as at the moment there is no corresponding real invariant defined).

Theorem 1 may be used to reduce both complex and real problem to enumeration of tropical curve passing through the corresponding collection of points in $\mathbb{TP}^2$ or $\mathbb{TP}^3$. Each such tropical curve acquires a multiplicity that might be different for the instances of real and complex enumeration.

In the corresponding tropical enumerative problem we may choose the points to be well stretched vertically. Tropical curves passing through such points are described by the so-called *floor diagrams*, see [1]. Every floor diagram (with marking) encodes a unique tropical curve. Without the marking the floor diagram is an even better-looking combinatorial object. As it was shown in [2] in the planar genus 0 case it corresponds to a tree on $d$ vertices, so there is $d^{d-2}$ of them. Thus the number of corresponding complex and real curve (the genus 0 Gromov-Witten and Welschinger numbers for $\mathbb{P}^2$) can be interpreted as two (different) statistics on trees. Both of this statistics are non-negative and coincide on trees corresponding to floor diagrams where the weight of all edges are equal to 1 (otherwise they differ by scaling depending on these weights). In particular, this implies the results of Itenberg-Kharlamov-Shustin [4] on logarithmic asymptotics of the Welschinger invariants.

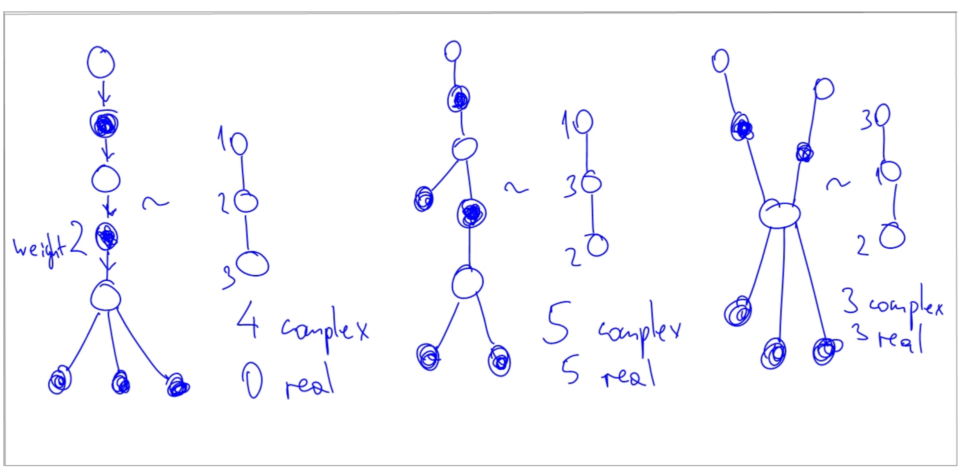



FIGURE 7. Floor diagrams computing the number of complex and real rational cubic curves through 8 generic points in $\mathbb{P}^2$.

## 5. Patchworking of real varieties

Yet another direction of applications of tropical geometry is based on interpretation of Theorem 1 as a generalization of Viro's patchworking theorem [11]. Recall that the Viro theorem allows to find real curves embedded to the plane with controlled topology in the context of the first part of Hilbert's 16th problem. Theorem 1 allows to generalize this construction to immersed curves in the plane as well as to algebraic knots and links in $\mathbb{RP}^3$.

To illustrate what happens with the analogue of Hilbert's question in dimension 3 (particularly in the positive genus case) we list a classification of smooth curves of degree 5 and genus 1 in $\mathbb{RP}^3$ recently obtained by Mikhalkin and Orevkov [8]. All topological types in this case are depicted on Figure 8.

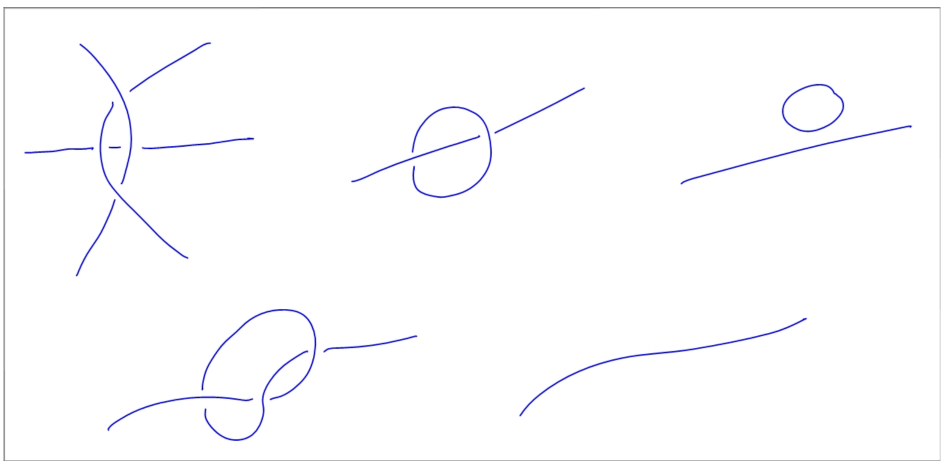


Figure 8. Topological types of degree 5 genus 1 knots in $\mathbb{RP}^3$.

As it was shown by Harnack [3] the number of components of a real curve of genus $g$ can not exceed $g+1$. The following theorem comes as an application of Theorem 1 and allows to represent any projective link in $\mathbb{RP}^3$ by an algebraic curve of the minimal possible genus (without specifying the degree).

**Theorem 2.** *Let $L \subset \mathbb{RP}^3$ be a link in $g+1$ components (i.e. a smoothly embedded disjoint union of $g+1$ circles). There exists a smooth algebraic curve of genus $g$ isotopic to $L$.*

Clearly this theorem provides a generalization for the well-known theorem that any knot can be approximated by a rational curve. Finding the minimal degree of an algebraic realization for most simple knots and links in $\mathbb{RP}^3$ is a challenging question.

I would like to finish this talk with the question on the knot type of rational curves passing through $2d$ points in $\mathbb{RP}^3$. A rational curve of odd degree in $\mathbb{RP}^3$ is homologous to $[\mathbb{RP}^1] \in H_1(\mathbb{RP}^3)$. We say that it is *knotted* if it is not isotopic to $\mathbb{RP}^1 \subset \mathbb{RP}^3$.

**Question 1.** *Suppose that $d$ is a large odd degree. Is it true that for any generic collection of $2d$ points in $\mathbb{RP}^3$ there exists a knotted rational curve passing through the points. Are there any knot types that are forced to appear in such enumeration?*

In this question we restrict to the case odd degree as 3-dimensional Welschinger invariant is non-trivial then. (An easy symmetry considerarion shows that it is zero if $d$ is even.) Perhaps a similar question is also sensible for the even degree.

## References


[1] E. Brugallé, G. Mikhalkin, *Enumeration of curves via floor diagrams* , 1. C. R. Math. Acad. Sci. Paris **345** (2007), 329-334.

[2] S. Fomin, G. Mikhalkin, *Labeled floor diagrams*, to appear.

[3] A. Harnack, *Uber Vieltheiligkeit der ebenen algebraischen Curven,* Math. Ann. **10** (1876), 189-199.

[4] I. Itenberg, V. Kharlamov, and E. Shustin, *Welschinger invariant and enumeration of real rational curves*, Int. Math. Res. Not. 2003 no. 49 (2003), 26392653.

[5] G. Mikhalkin, *Real algebraic curves, moment map and amoebas,* Ann. of Math. (2) **151** (2000), no. 1, 309-326.

[6] G. Mikhalkin, *Tropical Geometry and its application*, Proceedings of the ICM 2006 Madrid, Spain, 827-852.

[7] G. Mikhalkin, *What are tropical counterparts of algebraic varieties?*, Oberwolfach Report 26/2008, 36–38, http://www.mfo.de/programme/schedule/2008/24/OWR_2008_26.pdf.

[8] G. Mikhalkin, S. Orevkov, *Topology of algebraic curves of degree 5 in* $\mathbb{RP}^3$, to appear.

[9] G. Mikhalkin, I. Zharkov, *Tropical curves, their Jacobians and Theta functions*, in "Curves and Abelian Varieties" (V. Alexeev, A. Beauville, H. Clemens and E. Izadi (Eds.)), Contemporary Math 465 (2008), AMS, 203–230.

[10] D. Speyer, *Uniformizing Tropical Curves I: Genus Zero and One,* Preprint arXiv:0711.2677v1 [math.AG](2007).

[11] O. Ya. Viro, *Real plane algebraic curves: constructions with controlled topology,* Leningrad Math. J. **1** (1990), no. 5, 1059-1134.

[12] J.-Y. Welschinger. *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry,* Invent. Math. **162** (2005), 195 - 234.

[13] J.-Y. Welschinger, *Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants,* Duke Math. J. **127** (2005), 89 - 121.

# Discontinuous Groups on pseudo-Riemannian Spaces

Mathematische Arbeitstagung 2009 at MPI Bonn

5–11 June 2009

Toshiyuki Kobayashi

(the University of Tokyo)

http://www.ms.u-tokyo.ac.jp/˜toshi/



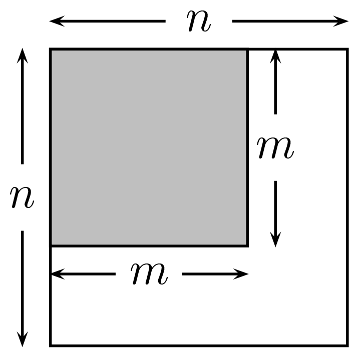
# Compact quotients $\Gamma\backslash SL(n)/SL(m)$

Problem (Existence problem for uniform lattice):

Does there exist compact Hausdorff quotients of

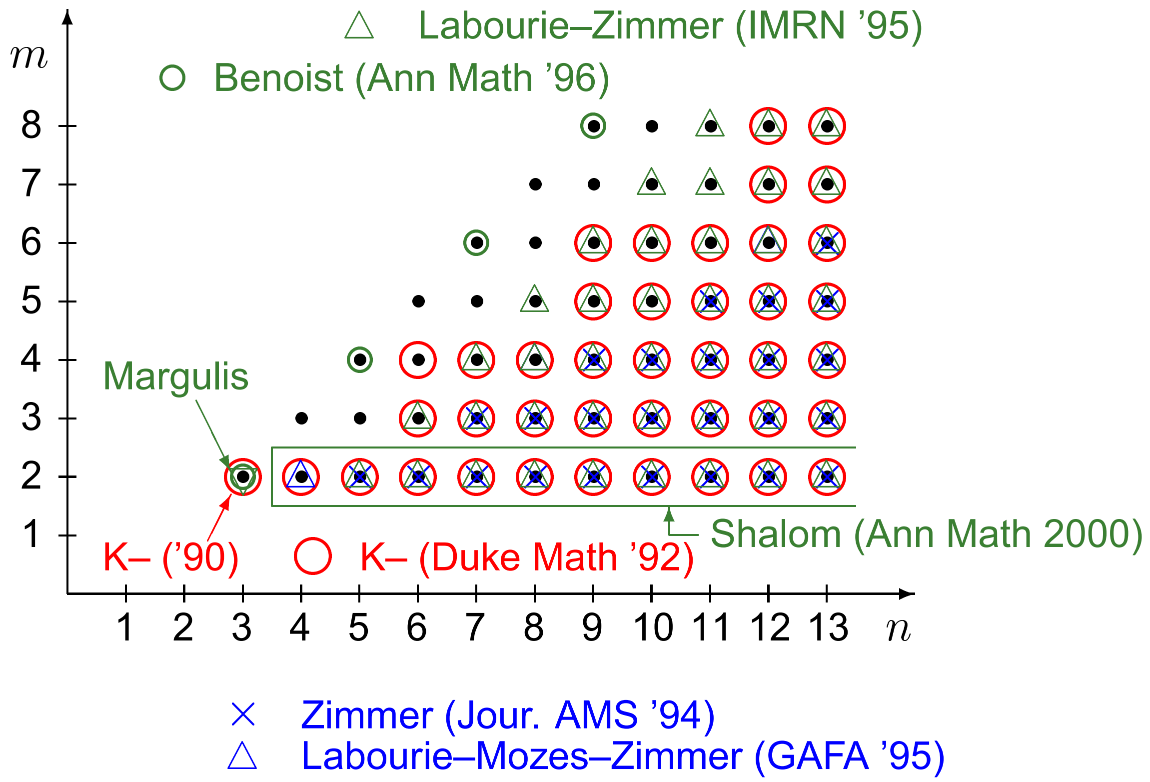
$$SL(n,\mathbb{F})/SL(m,\mathbb{F}) \quad (n > m,\ \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$$

by discrete subgps $\Gamma$ of $SL(n,\mathbb{F})$?



# Compact quotients for $SL(n)/SL(m)$

Uniform lattice does not exist for the following $(n, m)$:






# $SL(n)/SL(m)$ case

> <u>Conjecture</u> For any $n > m > 1$, there does not exist uniform lattice for $SL(n)/SL(m)$.

Affirmative results:

| | | |
|---|---|---|
| K– | criterion of proper actions | $\frac{n}{3} > [\frac{m+1}{2}]$ |
| Zimmer | orbit closure thm (Ratner) | $n > 2m$ |
| Labourier–Mozes–Zimmer | | |
| | ergodic action | $n \geq 2m$ |
| Benoist | criterion of proper actions | $n = m + 1,\ m$ even |
| Margulis | unitary representation | $(n \geq 5, m = 2)$ |
| Shalom | unitary representation | $n \geq 4, m = 2$ |

# Non-Riemannian homo. spaces

Discrete subgp $\not\Rightarrow$ $\Leftarrow$ Discontinuous gp

for non-Riemannian homo. spaces

General Problem

How does a local geometric structure affect the global nature of manifolds?

New phenomena & methods?



# 2. Complex symmetric structure

$G/K$: Riemannian symmetric space

$\Downarrow$ complexification

$G_{\mathbb{C}}/K_{\mathbb{C}}$: complex symmetric space

Fact (Borel 1963) Compact quotients exist for $^{\forall}$Riemannian symm sp. $G/K$.

Conj. Compact quotients exist for $G_{\mathbb{C}}/K_{\mathbb{C}}$
$\Longleftrightarrow G_{\mathbb{C}}/K_{\mathbb{C}} \approx S^7_{\mathbb{C}}$ or complex group mfd

$\Leftarrow$ proved by K–Yoshino 05,
$\Rightarrow$ remaining case $S^{4k-1}_{\mathbb{C}}$, $k \geq 3$ (Benoist, K– )

# Space forms (examples)

Space form $\cdots$ $\begin{cases} \text{Signature } (p,q) \text{ of pseudo-Riemannian metric } g \\ \text{Curvature } \kappa \in \{+,0,-\} \end{cases}$

E.g. $q = 0$ (Riemannian mfd)

| sphere $S^n$ | $\mathbb{R}^n$ | hyperbolic sp |
|---|---|---|
| $\kappa > 0$ | $\kappa = 0$ | $\kappa < 0$ |

E.g. $q = 1$ (Lorentz mfd)

| de Sitter sp | Minkowski sp | anti-de Sitter sp |
|---|---|---|
| $\kappa > 0$ | $\kappa = 0$ | $\kappa < 0$ |



# Space form problem

Space form problem for pseudo-Riemannian mfds

Local Assumption

signature $(p,q)$, curvature $\kappa \in \{+,0,-\}$

$\Downarrow$

Global Results

- Do compact forms exist?
- What groups can arise as their fundamental groups?

# Compact space forms

$(p, q)$: signature of metric, curvature $\kappa \in \{+, 0, -\}$

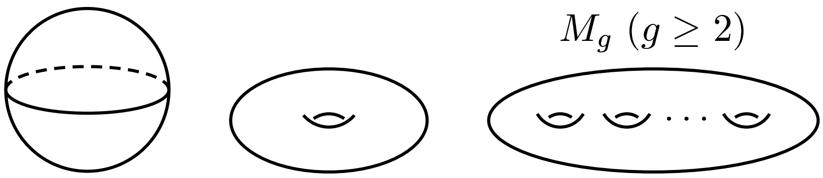
Assume $p \geq q$ (without loss of generality).

- $\kappa > 0$: Calabi–Markus phenomenon
  (Calabi, Markus, Wolf, Wallach, Kulkarni, K–)
- $\kappa = 0$: Auslander conjecture
  (Bieberbach, Auslander, Milnor, Margulis, Goldman, Abels, Soifer, ...)
- $\kappa < 0$: Existence problem of compact forms



# 2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2, 0)$)




curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Lorentz case ($\Longleftrightarrow$ signature $(1, 1)$)
compact forms do NOT exist
for $\kappa > 0$ and $\kappa < 0$

# Compact space forms ($\kappa < 0$)

- Riemannian case $\cdots$ hyperbolic space

Compact quotients

$\iff$ Cocompact discont. gp for $O(n,1)/O(n) \times O(1)$

$\iff$ Cocompact discrete subgp of $O(n,1)$

(uniform lattice)

Exist by $\underbrace{\text{Siegel, Borel–Harish-Chandra, Mostow–Tamagawa}}_{\text{arithmetic}}$,

$\underbrace{\text{Vinberg, Gromov–Piateski-Shapiro}}_{\text{non-arithmetic}} \cdots$



# Existence of compact forms

- For pseudo-Riemannian mfd of signature $(p,q)$

<u>Thm</u> <u>Conjecture</u> Compact space forms of $\kappa < 0$ exist

$\Longleftarrow$ ① $q$ any, $p = 0$ ($\leftrightarrow \kappa > 0$)

$\Longrightarrow$ ② $q = 0$, $p$ any (hyperbolic sp)

③ $q = 1$, $p \equiv 0 \mod 2$

④ $q = 3$, $p \equiv 0 \mod 4$ (pseudo-Riemannian)

⑤ $q = 7$, $p = 8$

$\Longleftarrow$ True (Proved (1950–2005))

(①② (Riemmanian) ; ③④⑤ (pseudo-Riemannian) Kulkarni, K– )

$\Longrightarrow$ Partial answers:

$q = 1$, $p \leq q$, or $pq$ is odd

Hirzebruch's proportionality principle (K–Ono)

# Infinitesimal approximation

$$G = K \exp \mathfrak{p} \implies G_\theta = K \ltimes \mathfrak{p}$$ (Cartan motion gp)

$$G/H = O(p, q+1)/O(p,q) \implies G_\theta / H_\theta$$

Thm (K–Yoshino, 2005)

Compact forms of $G_\theta/H_\theta$ exist $\iff p \equiv 0 \mod 2^{\varphi(q)}$

Here, $\varphi(q) = \left[\frac{q}{2}\right] + \begin{cases} 0 & (q \equiv 0, 6, 7 \mod 8) \\ 1 & (q \equiv 1, 2, 3, 4, 5 \mod 8) \end{cases}$

| E.g. | | | |
|---|---|---|---|
| | $q = 0$ | | $p$ any |
| | $q = 1$ | $\varphi(1) = 1$ | $p \equiv 0 \mod 2$ |
| | $q = 3$ | $\varphi(3) = 2$ | $p \equiv 0 \mod 4$ |
| | $q = 7$ | $\varphi(7) = 3$ | $p \equiv 0 \mod 8$ |



# Radon–Hurwitz number (1922)

Def. (Radon–Hurwitz number)

$\rho(p) := 8\alpha + 2^\beta$

if $p = u \cdot 2^{4\alpha+\beta}$ ($u$: odd, $0 \le \beta \le 3$)

$$p \equiv 0 \mod 2^{\varphi(q)} \iff q < \rho(p)$$

Radon–Hurwitz number (1922)

⇓

Adams: vector fields on sphere (1962)

⇓

Uniform lattice for $G_\theta/H_\theta$ (2005)

# General idea: Compact-like actions

Non-compact Lie groups

occasionally behave nicely
when embedded in $\infty$-dim groups
as if they were

compact groups
(very nice behaviours)



# Compact-like linear/non-linear actions

$L \curvearrowright \mathcal{H}$ (linear)

Unitarizability
$=$ existence of $L$-invariant inner product

Discrete decomposability
$=$ no continuous spectrum
in the $L$-irreducible decomposition

$L \curvearrowright M$ (non-linear)

Proper acions/properly discontinuous actions
$=$ The action map $\begin{matrix} L \times M \to M \times M \\ (g, x) \mapsto (x, g \cdot x) \end{matrix}$ is proper.

# Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space, unitary reprn.
$L \curvearrowright \mathcal{H}$ discrete decomposability

$\cdots$ $L$ behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

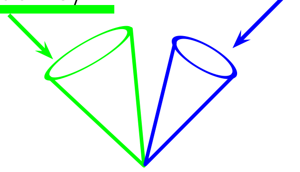
$M$: topological space
$L \curvearrowright M$ proper actions

$\cdots$ $L$ behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group



# Criterion of admissible restriction

Theorem A (Criterion) (K– Ann Math '98, Progr Math '05)
Let $G' \underset{\text{reductive}/\mathbb{R}}{\subset} G$ and $\pi \in \widehat{G}$. If



$(\star)$ $\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$ in $\sqrt{-1}\mathfrak{t}^*$, ($\mathbb{R}^n = \sqrt{-1}\mathfrak{t}^*$)

$\Longleftrightarrow$ $\pi|_{K'}$ is $K'$-admissible.

In particular, the restriction $\pi|_{G'}$ is $G'$-admissible.
(discretely decomposable & of finite multiplicities)

Proof uses micro-local analysis.

# $\pitchfork$ and $\sim$ (definition)

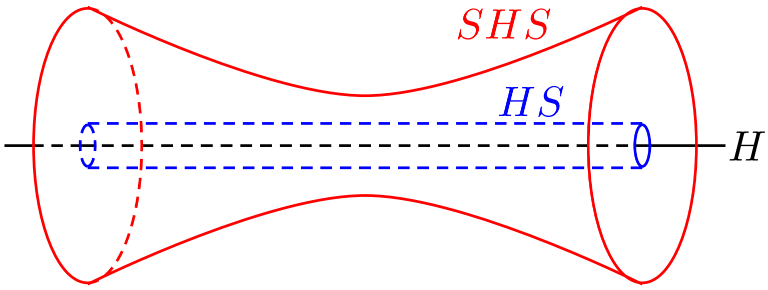
$$L \quad \subset \quad G \quad \supset \quad H$$

Idea: forget even that $L$ and $H$ are group

Def. (K– )

1) $L \pitchfork H \iff \overline{L \cap SHS}$ is compact for $^\forall$ compact $S \subset G$

2) $L \sim H \iff {}^\exists$ compact $S \subset G$ s.t. $L \subset SHS$ and $H \subset SLS$.






# $\pitchfork$ and $\sim$

$$L \quad \subset \quad G \quad \supset \quad H$$

Forget even that $L$ and $H$ are group

1) $L \pitchfork H \iff$ generalization of proper actions

2) $L \sim H \iff$ economy in considering

Meaning of $\pitchfork$:

$$L \pitchfork H \iff L \curvearrowright G/H \text{ proper action}$$

for closed subgroups $L$ and $H$

$\sim$ provides economies in considering $\pitchfork$

$$H \sim H' \implies \boxed{H \pitchfork L \iff H' \pitchfork L}$$

# Criterion for $\pitchfork$ and $\sim$

$G$: real reductive Lie group
$G = K\exp(\mathfrak{a})K$: Cartan decomposition
$\nu$: $G \to \mathfrak{a}$: Cartan projection (up to Weyl gp.)

Thm B (K– , Benoist)
1) $L \sim H$ in $G \iff \nu(L) \sim \nu(H)$ in $\mathfrak{a}$.
2) $L \pitchfork H$ in $G \iff \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$.

abelian

Special cases include

(1)'s $\Rightarrow$ : Uniform bounds on errors in eigenvalues when a matrix is perturbed.
(2)'s $\Leftrightarrow$ : Criterion for properly discont. actions.



# Criterion for compact-like actions

$G$ : reductive Lie group $\supset K$
$\cup$ $\cup$
$G'$ : reductive subgp $\supset K'$
$\mu$ : $T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*$ momentum map

Thm A $\pi \in \widehat{G}$, $G' \subset G$
$\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$
$\Longrightarrow \pi|_{G'}$ is discrete decomposable.

$G$ : reductive Lie gp, $G \supset L, H$ (subsets)
$\nu : G \to \mathfrak{a}$ (Cartan projection)

Thm B (proper action)
$L \pitchfork H$ in $G \iff \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$

# Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space
$L \curvearrowright \mathcal{H}$ discrete decomposability
$\cdots$ $L$ behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

$M$: topological space
$L \curvearrowright M$ proper actions
$\cdots$ $L$ behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group



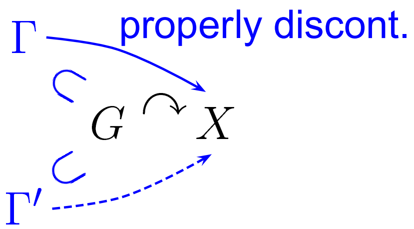
# Local $\Longrightarrow$ Global

$G \supset H$ reductive Lie groups
$\Longrightarrow$ $G/H$ pseudo-Riemannian homo. sp

Cor (Criterion for the Calabi–Markus phenomenon)
Any discont. gp for $G/H$ is finite
$\Longleftrightarrow \mathrm{rank}_{\mathbb{R}}\, G = \mathrm{rank}_{\mathbb{R}}\, H$

Application (space form of signature $(p, q)$, $\kappa < 0$)
Exists a space form $M$ s.t. $|\pi_1(M)| = \infty$
$\Longleftrightarrow p > q$ or $(p, q) = (1, 1)$
(Calabi, Markus, Wolf, Kulkarni, Wallach)
$p > q + 1 \Longrightarrow {}^{\exists}M$ with free non-commutative $\pi_1(M)$

# Rigidity, stability, and deformation




Suppose $\Gamma'$ is 'close to' $\Gamma$

| | |
|---|---|
| (R) (local rigidity) | $\Gamma' = g\Gamma g^{-1}$ $({}^\exists g \in G)$ |
| (S) (stability) | $\Gamma' \curvearrowright X$ properly discont. |

In general,

- (R) $\Rightarrow$ (S).
- (S) may fail (so does (R)).



# Local rigidity and deformation

$\Gamma \subset G \curvearrowright X = G/H$ cocompact, discontinuous gp

<u>General Problem</u>

1. When does local rigidity (R) fail?
2. Does stability (S) still hold?

<u>Point</u>: for non-compact $H$

1. (good aspect) There may be large room for deformation of $\Gamma$ in $G$.
2. (bad aspect) Properly discontinuity may fail under deformation.

# Rigidity Theorem

① $\Gamma \curvearrowright G/\{e\} \iff (\Gamma \times 1) \curvearrowright (G \times G)/\Delta G$ ②

$\Gamma \subset G$ simple Lie gp

Fact (Selberg–Weil's local rigidigy, 1964)
$^\exists$uniform lattice $\Gamma$ admitting continuous deformations for ①
$\iff G \approx SL(2, \mathbb{R})$ (loc. isom).

Thm (K– )
$^\exists$uniform lattice $\Gamma$ admitting continuous deformations for ②
$\iff G \approx SO(n+1, 1)$ or $SU(n, 1)$ $(n = 1, 2, 3, ...)$.

Local rigidity (R) may fail. Stability (S) still holds.

$\cdots$ Solution to Goldman's stability conjecture (1985), 3-dim case



# Compact-like linear/non-linear actions

$\mathcal{H} = L^2(G/H), L^2(G/\Gamma)$: Hilbert space
$L \curvearrowright \mathcal{H}$ discrete decomposability
$\cdots$ $L$ behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group

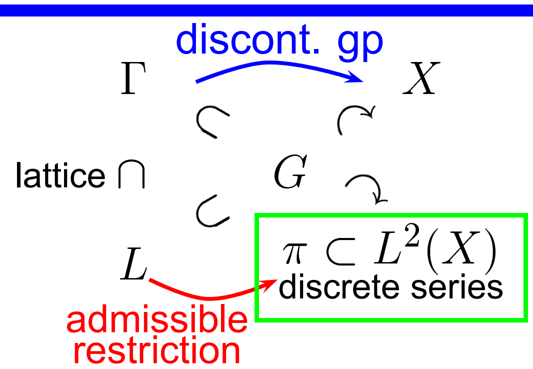
$M = G/H$: topological space
$L \curvearrowright M$ proper actions
$\cdots$ $L$ behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group

# Interacting example

$$(G, L, H) = (SO(4,2),\ SO(4,1),\ U(2,1))$$

Tessellation of pseudo-Riemannian mfd $X$

$X = SO(4,2)/U(2,1)$ $\quad (\underset{\text{open}}{\subset} \mathbb{P}^3\mathbb{C})$




$\pi$: discrete series of $G$ with GK-dim 5

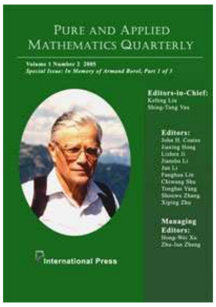
(quarternionic discrete series)

$\Longrightarrow \pi|_L$ is $L$-admissible



# References


1) Pure & Appl. Math. Quarterly **1** (2005) Borel Memorial Volume




2) Sugaku Expositions, Amer. Math. Soc. (2009)
translated by Miles Reid

3) Contemp. Math., Amer. Math. Soc., (2009), pp. 73–87.

4) Invent. Math. (1994), Ann. Math. (1998), Invent. Math. (1998)


For more references:
http://www.ms.u-tokyo.ac.jp/~toshi

# Counting Lattices

Arbeitstagung, Bonn, June 2009

Mikhail Belolipetsky

Let $H$ be a non-compact simple Lie group endowed with a fixed Haar measure $\mu$. Let $\mathrm{L}_H(x)$ (resp. $\mathrm{AL}_H(x)$) denote the number of conjugacy classes of lattices (resp. arithmetic lattices) in $H$ of covolume at most $x$.

A classical theorem of Wang [W] asserts that if $H$ is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{L}_H(x)$ is finite for every $x$. This is also true for $\mathrm{AL}_H(x)$ even for $H = \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ by a result of Borel [Bo].

Recent years there has been a growing interest in the asymptotic behavior of these functions.

In [BGLM] the rate of growth of torsion-free lattices was determined for $H = \mathrm{SO}(n, 1)$, $n \geq 4$; it is super-exponential. The lower bound there is already obtained by considering a suitable single lattice in $\mathrm{SO}(n, 1)$ and its finite index subgroups. The upper bound is proved by geometric methods.

In [BGLS] we give a very precise super-exponential estimate for $\mathrm{AL}_H(x)$ for $H = \mathrm{SL}_2(\mathbb{R})$. Our main result states that $\lim_{x\to\infty} \frac{\log \mathrm{AL}_H(x)}{x \log x} = \frac{1}{2\pi}$. Here again the full rate of growth is already obtained by considering the finite index subgroups of a single lattice — the main challenge is in proving the upper bound.

In [GLP] and [LN] (see also [GLNP]) precise asymptotic estimates were given for the growth rate of the number of congruence subgroups in a fixed lattice $\Lambda$ in $H$. (Some of the results there are conditional on the GRH). That rate of growth turns out to depend only on $H$ and not on $\Lambda$.

All this suggested that the rate of growth of the finite index subgroups within one lattice is the main contribution to $\mathrm{L}_H(x)$. This led to the following *conjecture* (see e.g. [GLNP]):

Let $H$ be a non-compact simple Lie group of real rank at least 2. Then

$$\lim_{x\to\infty} \frac{\log \mathrm{L}_H(x)}{(\log x)^2/\log\log x} = \gamma(H), \quad \textit{with} \;\; \gamma(H) = \frac{(\sqrt{h(h+2)} - h)^2}{4h^2},$$

where $h$ is the Coxeter number of the (absolute) root system corresponding to $H$ (i.e. the root system of the split form of $H$).

In [B] it is shown that the growth rate of the maximal arithmetic lattices in $H$ is very small (conjecturally polynomial, and indeed a polynomial bound is given there for the maximal non-uniform lattices and a slightly weaker bound of the form $x^{(\log x)^\epsilon}$ is proved for all maximal lattices). This gave a further support to the conjecture.

In [BL2] we show that the conjecture is essentially true for non-uniform lattices but in [BL1] we prove, somewhat surprisingly, that it is false in general. In fact, we discover here a new phenomenon: the main contribution to the growth of uniform lattices in $H$ does not come from subgroups of a single lattice. As it will be explained below, it comes from a "diagonal counting" when we run through different arithmetic groups $\Gamma_i$ defined over number fields $k_i$ of different degrees $d_i$, and for each $\Gamma_i$ we count some of its subgroups. The difference between the uniform and non-uniform cases relies on the fact that all non-uniform lattices in $H$ are defined over number fields of a bounded degree over $\mathbb{Q}$. On the other hand, uniform lattices may come from number fields $k_i$ of arbitrarily large degrees, i.e., $d_i \to \infty$.

We now briefly describe the line of the argument. If $\Gamma$ is an arithmetic lattice, there exists a number field $k$ with ring of integers $\mathcal{O}$ and the set of archimedean valuations $V_\infty$, an absolutely simple, simply connected $k$-group G and an epimorphism $\phi : G = \prod_{v \in V_\infty} \mathrm{G}(k_v)^o \to H$, such that $\mathrm{K}er(\phi)$ is compact and $\phi(\mathrm{G}(\mathcal{O}))$ is commensurable with $\Gamma$. G. Prasad [P] gave an explicit formula for the covolume of such $\phi(\mathrm{G}(\mathcal{O}))$ in $H$. The analysis of this formula and also the growth of the low-index congruence subgroups of $\phi(\mathrm{G}(\mathcal{O}))$ shows that we can expect fast subgroup growth if we consider groups over fields of growing degree with relatively slow growing discriminant $\mathcal{D}_k$. More precisely, we can combine this two entities together into the so-called root-discriminant $rd_k = \mathcal{D}_k^{1/\deg k}$ and then look for a sequence of number fields $k_i$ with degrees growing to $\infty$ but with bounded $rd_{k_i}$. In a seminal work Golod and Shafarevich [GS] came up with a construction of infinite class field towers. It is such a tower of number fields $k_i$ that we use to define our arithmetic subgroups $\Gamma_i$. Galois cohomology methods show the existence of suitable $k_i$-algebraic groups $\mathrm{G}_i$ which give rise to arithmetic lattices $\Gamma_i = \mathrm{G}_i(\mathcal{O}_i)$ in $H$ whose covolume is bounded exponentially in $d_i = \deg k_i$. We then present $c^{d_i^2}$ congruence subgroups of $\Gamma_i$ whose covolume is still bounded exponentially in $d_i$. Using the theory of Bruhat-Tits buildings we can show that sufficiently many of such congruence subgroups are not conjugate to each other in $H$. This completes the proof of the lower bound $\log \mathrm{L}_H(x) \geq a(\log x)^2$ for some positive constant $a = a(H)$ at least for most real simple Lie groups $H$. The remaining cases require further consideration: for example, if $H$ is a complex Lie group, the fields $k_i$ should be replaced by suitable extensions obtained via the help of the theory of Pisot numbers. These fields do not form a class field tower any more but still have bounded root discriminant.

The proof of the upper bound $\log \mathrm{L}_H(x) \leq b(\log x)^2$ for groups $H$ of real rank at least 2 which satisfy Serre's congruence subgroup conjecture in [BL1] presents a new type of difficulty: this time we need to obtain a uniform upper bound on growth which does not depend on the degrees of the defining fields. (This is what makes the growth rate $x^{\log x}$ instead of $x^{\log x/\log\log x}$.) The new bound requires some new "subgroup growth" methods which we develop in [BL1]. A key ingredient of the proof is an important theorem of Babai, Cameron and Pálfy (see [LS, Theorem 4, p. 339]) which bounds the size of permutation groups with restricted Jordan-Holder components. We are taking advantage of the fact that this restriction applies uniformly for the profinite completions of all the lattices in a given group $H$.

On the other hand, the result of [BL2] shows that if one restricts attention only to non-uniform lattices then the original conjecture is true for most higher rank simple groups $H$ (and conjecturally for all). Thus, let us assume that if $G$ is a split form of $H$, then the center of the simply connected cover of $G$ is a 2-group, and that $H$ is not a triality. This is the case for most $H$'s. In fact, it says that $H$ is not of type $\mathrm{E}_6$ or $\mathrm{D}_4$, and if it is of type $\mathrm{A}_n$, then $n$ is of the form $n = 2^\alpha - 1$ for some $\alpha \in \mathbb{N}$. For such $H$ we can show that $\lim_{x\to\infty} \frac{\log \mathrm{L}_H^{nu}(x)}{(\log x)^2/\log\log x} = \gamma(H)$, where $\gamma(H)$ is defined as above and $\mathrm{L}_H^{nu}(x)$ denotes the number of conjugacy classes of non-uniform lattices in $H$ of covolume at most $x$.

The proof of of this result uses Gauss's Theorem which gives a bound for the 2-rank of the class groups of quadratic extensions. In order to be able to extend the result to all simple groups $H$ we would need similar bounds for $l$-ranks for $l > 2$. In fact, we show in [BL2] that it is essentially equivalent to such bounds.

## References


[B] M. Belolipetsky, Counting maximal arithmetic subgroups (with an appendix by J. Ellenberg and A. Venkatesh), *Duke Math. J.* **140** (2007), no. 1, 1–33.

[BGLS] M. Belolipetsky, T. Gelander, A. Lubotzky, A. Shalev, Counting arithmetic lattices and surfaces, *preprint* arXiv:0811.2482v1 [math.GR].

[BL1] M. Belolipetsky, A. Lubotzky, Counting manifolds and class field towers, *preprint* arXiv:0905.1841v1 [math.GR].

[BL2] M. Belolipetsky, A. Lubotzky, Counting non-uniform lattices, *in preparation.*

[Bo] A. Borel, Commensurability classes and volumes of hyperbolic 3-manifolds, *Ann. Scuola Norm. Sup. Pisa (4)* **8** (1981), 1–33.

[BGLM] M. Burger, T. Gelander, A. Lubotzky, S. Mozes, Counting hyperbolic manifolds, *Geom. Funct. Anal.* **12** (2002), 1161–1173.

[GLNP] D. Goldfeld, A. Lubotzky, N. Nikolov, L. Pyber, Counting primes, groups and manifolds, *Proc. of National Acad. of Sci.* **101** (2004), 13428–13430.

[GLP] D. Goldfeld, A. Lubotzky, L. Pyber, Counting congruence subgroups, *Acta Math.* **193** (2004), 73–104.

[GS] E. S. Golod, I. P. Shafarevich, On the class field tower, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 261–272 [Russian].

[LN] A. Lubotzky, N. Nikolov, Subgroup growth of lattices in semisimple Lie groups, *Acta Math.* **193** (2004), 105–139.

[LS] A. Lubotzky, D. Segal, *Subgroup growth*, Progr. Math. **212**, Birkhäuser Verlag, Basel, 2003.

[P] G. Prasad, Volumes of $S$-arithmetic quotients of semi-simple groups, *Inst. Hautes Études Sci. Publ. Math.,* **69** (1989), 91–117.

[W] H. C. Wang, Topics on totally discontinuous groups, in *Symmetric spaces (St. Louis, Mo., 1969–1970)*, Pure Appl. Math. **8**, Dekker, New York, 1972, 459–487.



Department of Mathematical Sciences, Durham University, Durham DH1 3LE, UK;
Sobolev Institute of Mathematics, Koptyuga 4, 630090 Novosibirsk, Russia
E-mail address: `mikhail.belolipetsky@durham.ac.uk`

# Quasi-conformal geometry and word hyperbolic Coxeter groups

Marc Bourdon (joint work with Bruce Kleiner)

Arbeitstagung, 11 june 2009

In [6] J. Heinonen and P. Koskela develop the theory of (analytic) modulus in metric spaces, and introduce the notion of Loewner space. They establish that many results concerning the classical quasi-conformal geometry on Euclidean spaces are valid for on Loewner spaces. In geometric group theory the regularity properties of quasi-symmetric homeomorphisms between Loewner spaces are responsible for several rigidity phenomena generalizing Mostow's rigidity. Otherwise only few examples of Loewner spaces are known, these include the boundaries of rank one symmetric spaces, the boundaries of some fuchsian buildings, and some exotic self-similar spaces.

Cannon's conjecture states that every word hyperbolic group whose boundary is homeomorphic to the 2-sphere acts by isometries properly discontinuously and cocompactly on the real hyperbolic space $\mathbb{H}^3$. It can be seen as a group theoretical analogue of Thurton's hyperbolization conjecture recently solved by G. Perelman. As a tool to approach Cannon's conjecture, various notions of combinatorial modulus have been developed by several authors (*e.g.* [3], [4], [1], [5])).

This talk will focus on the combinatorial modulus. It reports on a recent joint work with B. Kleiner [2]. A combinatorial version of the Loewner property, called the *combinatorial Loewner property*, is presented. It is weaker than Heinonen-Koskela's, indeed if $Z$ is a $Q$-Loewner space then every metric space quasi-symmetrically homeomorphic to $Z$ satisfies the combinatorial $Q$-Loewner property. We suspect that in most of the interesting cases – like the boundaries of word hyperbolic groups – a converse is also true, namely that if a metric space admits the combinatorial $Q$-Loewner property then it is quasi-symmetrically homeomorphic to a $Q$-Loewner space.

Our main results concern the combinatorial modulus on boundaries of word hyperbolic Coxeter groups. We obtain a sufficient condition for such a boundary to satisfy the combinatorial Loewner property, and use this to exhibit a number of examples, some old and some new. As an application of our techniques we obtain a proof of Cannon's conjecture in the special case of Coxeter groups. This result was essentially known. Our view is that the principal value of the proof is that it illustrates the feasibility of the asymptotic approach (using the ideal boundary and modulus estimates), and it may suggest ideas for attacking the general case. It gives also a new proof of the Andreev's theorem about the Coxeter hyperbolic polytopes in $\mathbb{H}^3$, in the case when the prescribed dihedral angles are submultiples of $\pi$.

## References


[1] M. Bonk, B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres,* Invent. Math., 150 (2002), 1247–1287.

[2] M. Bourdon, B. Kleiner, *Combinatorial modulus, combinatorial Loewner property and Coxeter groups,* Preprint (2009).

[3] J. W. Cannon, *The combinatorial Riemann mapping theorem,* Acta Math. **173** (1994), no. 2, 155–234.

[4] J. W. Cannon, W.J. Floyd, W.R. Parry, *Sufficiently rich families of planar rings,* Ann. Acad. Sci. Fenn. Math. **24** (1999), no. 2, 265–304.

[5] P. Haïssinsky, *Empilements de cercles et modules combinatoires,* Annales de l'Institut Fourier, to appear.

[6] J. Heinonen, P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry,* Acta Math. **181** (1998), 1-61.



Marc Bourdon, Université Lille 1, Département de mathématiques, Bat. M2, 59655 Villeneuve d'Ascq, France, bourdon@math.univ-lille1.fr .

# GEOMETRY OVER THE FIELD WITH ONE ELEMENT

OLIVER LORSCHEID

## 1. MOTIVATION

Two main sources have led to the development of several notions of $F_1$-geometry in the recent five years. We will concentrate on one of these, which originated as remark in a paper by Jacques Tits ([10]). For a wide class of schemes $X$ (including affine space $\mathbb{A}^n$, projective space $\mathbb{P}^n$, the Grassmannian $\mathrm{Gr}(k,n)$, split reductive groups $G$), the function

$$N(q) \ = \ \#X(\mathbb{F}_q)$$

is described by a polynomial in $q$ with integer coefficients, whenever $q$ is a prime power. Taking the value $N(1)$ sometimes gives interesting outcomes, but has a $0$ of order $r$ in other cases. A more interesting number is the lowest non-vanishing coefficient of the development of $N(q)$ around $q-1$, i.e. the number

$$\lim_{q\to 1} \ \frac{N(q)}{(q-1)^r},$$

which Tits took to be the number $\#X(\mathbb{F}_1)$ of "$\mathbb{F}_1$-points" of $X$. The task at hand is to extend the definition of the above mentioned schemes $X$ to schemes that are "defined over $\mathbb{F}_1$" such that their set of $\mathbb{F}_1$-points is a set of cardinality $\#X(\mathbb{F}_1)$. We describe some cases, and suggest an interpretation of the set of $\mathbb{F}_1$-points:

- $\#\mathbb{P}^{n-1}(\mathbb{F}_1) = n = \#M_n$ with $M_n := \{1,\dots,n\}$.
- $\#\,\mathrm{Gr}(k,n)(\mathbb{F}_1) = \binom{n}{k} = \#M_{k,n}$ with $M_{k,n} = \{$subsets of $M_n$ with $k$ elements$\}$.
- If $G$ is a split reductive group of rank $r$, $T \simeq \mathbb{G}_m^r \subset G$ is a maximal torus, $N$ its nomalizer and $W = N(\mathbb{Z})/T(\mathbb{Z})$, then the Bruhat decomposition $G(\mathbb{F}_q) = \coprod_{w\in W} BwB(\mathbb{F}_q)$ (where $B$ is a Borel subgroup containing $T$) implies that $N(q) = \sum_{w\in W}(q-1)^r q_w^d$ for certain $d_w \geq 0$. This means that $\#G(\mathbb{F}_1) = \#W$.

In particular, it is natural to ask whether the group law $m : G\times G \to G$ of a split reductive group may be defined as a "morphism over $\mathbb{F}_1$". If so, one can define "group actions over $\mathbb{F}_1$". The limit as $q\to 1$ of the action

$$\mathrm{GL}(n,\mathbb{F}_q)\times \mathrm{Gr}(k,n)(\mathbb{F}_q) \longrightarrow \mathrm{Gr}(k,n)(\mathbb{F}_q)$$

induced by the action on $\mathbb{P}^{n-1}(\mathbb{F}_q)$ should be the action

$$S_n \times M_{k,n} \longrightarrow M_{k,n}$$

induced by the action on $M_n = \{1,\dots,n\}$.

The other, more lofty motivation for $\mathbb{F}_1$-geometry stems from the search for a proof of the Riemann hypothesis. In the early 90s, Deninger gave criteria for a category of motives that would provide a geometric framework for translating Weil's proof of the Riemann hypothesis for global fields of positive characteristic to number fields. In particular, the Riemann zeta function $\zeta(s)$ should have a cohomological interpretation, where an $H^0$, an $H^1$ and an $H^2$-term are involved. Manin proposed in [7] to interpret the $H^0$-term as the zeta function of the "absolute point" $\mathrm{Spec}\,\mathbb{F}_1$ and the $H^2$-term as the zeta function of the "absolute Tate motive" or the "affine line over $\mathbb{F}_1$".

## 2. OVERVIEW OVER RECENT APPROACHES

We give a rough description of the several approaches towards $\mathbb{F}_1$-geometry, some of them looking for weaker structures than rings, e.g. monoids, others looking for a category of schemes with certain additional structures. In the following, a *monoid* always means a abelian mutliplicative semi-group with 1. A *variety* is a scheme $X$ that defines, via base extension, a variety $X_k$ over any field $k$.

2.1. **Soulé, 2004 (**[9]**).** This is the first paper that suggests a candidate of a category of varieties over $\mathbb{F}_1$. Soulé consideres schemes together with a complex algebra, a functor on finite rings that are flat over $\mathbb{Z}$ and certain natural transformations and a universal property that connects the scheme, the functor and the algebra. Soulé could prove that smooth toric varieties provide natural examples of $\mathbb{F}_1$-varieties. In [6] the list of examples was broadened to contain models of all toric varieties over $\mathbb{F}_1$, as well as split reductive groups. However, it seems unlikely that Grassmannians that are not projective spaces can be defined in this framework.

2.2. **Connes-Consani, 2008 (**[1]**).** The approach of Soulé was modified by Connes and Consani in the following way. They consider the category of scschemes together with a functor on finite abelian groups, a complex variety, certain natural transformations and a universal property analogous to Soulé's idea. This category behaves only slightly different in some subtle details, but the class of established examples is the same (cf. [6]).

2.3. **Deitmar, 2005 (**[3]**).** A completely different approach was taken by Deitmar who uses the theory of prime ideals of monoids to define spectra of monoids. A $\mathbb{F}_1$-scheme is a topological space together with a sheaf of monoids that is locally isomorphic the spectrum of a ring. This theory has the advantage of having a very geometric flavour and one can mimic algebraic geometry to a large extent. However, Deitmar has shown himself in a subsequent paper that the $\mathbb{F}_1$-schemes whose base extension to $\mathbb{Z}$ are varieties are nothing more than toric varieties.

2.4. **Toën-Vaquié, 2008 (**[11]**).** Deitmar's approach is complemented by the work of Toën and Vaquié, which proposes locally representable functors on monoids as $\mathbb{F}_1$-schemes. Marty shows in [8] that the Noetherian $\mathbb{F}_1$-schemes in Deitmar's sense correspond to the Noetherian objects in Toën-Vaquié's sense. We raise the question: is the Noetherian condition necessary?

2.5. **Borger, in progress.** The category investigated by Borger are schemes $X$ together with a family of morphism $\{\psi_p : X \to X\}_{p \text{ prime}}$, where the $\psi_p$'s are lifts of the Frobenius morphisms $\mathrm{Frob}_p : X \otimes \mathbb{F}_p \to X \otimes \mathbb{F}_p$ and all $\psi_p$'s commute with each other.

There are further approaches by Durov ([4], 2007) and Haran ([5], 2007), which we do not describe here. In the following section we will examine more closely a new framework for $\mathbb{F}_1$-geometry by Connes and Consani in spring 2009.

## 3. $\mathbb{F}_1$-SCHEMES À LA CONNES-CONSANI AND TORIFIED VARIETIES

The new notion of an $\mathbb{F}_1$-scheme due to Connes and Consani ([2]) combines the earlier approaches of Soulé and of themselves with Deitmar's theory of spectra of monoids and Toën-Vaquié's functorial viewpoint. First of all, Connes and Consani consider monoids with $0$ and remark that the spaces that are locally isomorphic to spectra of monoids with $0$, called $M_0$*-schemes*, are the same as locally representable functors of monoids with $0$. (Note that they do not make any Noetherian hypothesis). There is a natural notion of morphism in this setting. The base extension is locally given by taking the semi-group ring, i.e. if $A$ is a monoid with zero $0_A$ and $X = \mathrm{Spec}\, A$ is its spectrum, then

$$X_{\mathbb{Z}} \;:=\; X \otimes_{\mathbb{F}_1} \mathbb{Z} \;:=\; \mathrm{Spec}\big(\mathbb{Z}[A]/(1 \cdot 0_A - 0_{\mathbb{Z}[A]})\big).$$

An $\mathbb{F}_1$-scheme is a triple $(\tilde{X}, X, e_X)$, where $\tilde{X}$ is an $M_0$-scheme, $X$ is a scheme and $e_X : \tilde{X}_{\mathbb{Z}} \to X$ is a morphism such that $e_X(k) : \tilde{X}_{\mathbb{Z}}(k) \xrightarrow{\sim} X(k)$ is a bijection for all fields $k$.

Note that an $M_0$-scheme $\tilde{X}$ defines the $\mathbb{F}_1$-scheme $(\tilde{X}, \tilde{X}_{\mathbb{Z}}, \mathrm{id}_{\tilde{X}_{\mathbb{Z}}})$. We give first examples of $\mathbb{F}_1$-schemes of this kind. The affine line $\mathbb{A}^1_{\mathbb{F}_1}$ is the spectrum of the monoid $\{T^i\}_{i\in\mathbb{N}} \amalg \{0\}$ and, indeed, we have $\mathbb{A}^1_{\mathbb{F}_1} \otimes_{\mathbb{F}_1} \mathbb{Z} \simeq \mathbb{A}^1$. The multiplicative group $\mathbb{G}_{m,\mathbb{F}_1}$ is the spectrum of the monoid $\{T^i\}_{i\in\mathbb{Z}} \amalg \{0\}$, which base extends to $\mathbb{G}_m$ as desired. Both examples can be extended to define $\mathbb{A}^n_{\mathbb{F}_1}$ and $\mathbb{G}^n_{m,\mathbb{F}_1}$ by considering multiple variables $T_1, \dots, T_n$. More generally, all $\mathbb{F}_1$-schemes in the sense of Deitmar deliver examples of $M_0$ and thus $\mathbb{F}_1$-schemes in this new sense. In particular, toric varieties can be realized.

To obtain a richer class of examples, we recall the definition of a torified variety as given in a joint work with Javier López Peña ([6]). A *torified variety* is a variety $X$ together with morphism $e_X : T \to X$ such that $T \simeq \coprod_{i\in I} \mathbb{G}_m^{d_i}$, where $I$ is a finite index set and $d_i$ are non-negative integers and such that for every field $k$, the morphism $e_X$ induces a bijection $T(k) \xrightarrow{\sim} X(k)$. We call $e_X : T \to X$ a *torification* of $X$.

Note that $T$ is isomorphic to the base extension $\tilde{X}_{\mathbb{Z}}$ of the $M_0$-scheme $\tilde{X} = \coprod_{i\in I} \mathbb{G}^{d_i}_{m,\mathbb{F}_1}$. Thus every torified variety $e_X : T \to X$ defines an $\mathbb{F}_1$-scheme $(\tilde{X}, X, e_X)$.

In [6], a variety of examples are given. Most important for our purpose are toric varieties, Grassmannians and split reductive groups. If $X$ is a toric variety of dimension $n$ with fan $\Delta = \{\text{cones } \tau \subset \mathbb{R}^n\}$, i.e. $X = \mathrm{colim}_{\tau\in\Delta} \operatorname{Spec} \mathbb{Z}[A_\tau]$, where $A_\tau = \tau^\vee \cap \mathbb{Z}^n$ is the intersection the dual cone $\tau^\vee \subset \mathbb{R}^n$ with the dual lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Then the natural morphism $\coprod_{\tau\in\Delta} \operatorname{Spec} \mathbb{Z}[A_\tau^\times] \to X$ is a torification of $X$.

The Schubert cell decomposition of $\mathrm{Gr}(k,n)$ is a morphism $\coprod_{w\in M_{k,n}} \mathbb{A}^{d_w} \to \mathrm{Gr}(k,n)$ that induces a bijection of $k$-points for all fields $k$. Since the affine spaces in this decomposition can be further decomposed into tori, we obtain a torification $e_X : T \to \mathrm{Gr}(k,n)$. Note that the lowest-dimensional tori are $0$-dimensional and the number of $0$-dimensional tori is exactly $\# M_{k,n}$.

Let $G$ be a split reductive group of rank $r$ with maximal torus $T \simeq \mathbb{G}_m^r$, normalizer $N$ and Weyl group $W = N(\mathbb{Z})/T(\mathbb{Z})$. Let $B$ be a Borel subgroup containing $T$. The Bruhat decomposition $\coprod_{w\in W} BwB \to G$, where $BwB \simeq \mathbb{G}_m^r \times \mathbb{A}^{d_w}$ for some $d_w \geq 0$, yields a torification $e_G : T \to G$ analogously to the case of the Grassmannian. This defines a model $\mathcal{G} = (\tilde{G}, G, e_G)$ over $\mathbb{F}_1$. Note that in this case the lowest-dimensional tori are $r$-dimensional and that the number of $r$-dimensional tori is exactly $\# W$.

Clearly, there is a close connection between torified varieties and the $\mathbb{F}_1$-schemes in the sense of Connes and Consani with the idea that Tits had in mind. However, the natural choice of morphism in this category is a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ of $M_0$-schemes together with a morphism $f : X \to Y$ of schemes such that

$$\begin{array}{ccc} \tilde{X}_{\mathbb{Z}} & \xrightarrow{\tilde{f}_{\mathbb{Z}}} & \tilde{Y}_{\mathbb{Z}} \\ \downarrow{\scriptstyle e_X} & & \downarrow{\scriptstyle e_X} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Unfortunately, the only reductive groups $G$ whose group law $m : G \times G \to G$ extends to a morphism $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ in this sense such that $(\mathcal{G}, \mu)$ becomes a group object in the category of $\mathbb{F}_1$-schemes are algebraic groups of the form $G \simeq \mathbb{G}_m^r \times (\text{finite group})$. In the following section we will show how to modify the notion of morphism to realize Tits' idea.

## 4. STRONG MORPHISMS

Let $\mathcal{X} = (\tilde{X}, X, e_X)$ and $\mathcal{Y} = (\tilde{Y}, Y, e_Y)$ be $\mathbb{F}_1$-schemes. Then we define the *rank of a point* $x$ in the underlying topological space $\tilde{X}$ as $\operatorname{rk} x := \operatorname{rk} \mathcal{O}^\times_{X,x}$, where $\mathcal{O}_{X,x}$ is the stalk (of monoids) at $x$ and $\mathcal{O}_{X,x}^\times$ denotes its group of invertible elements. We define the *rank of* $X$ as $\operatorname{rk} X := \min_{x\in\tilde{X}}\{\operatorname{rk} x\}$ and we let

$$\tilde{X}^{\mathrm{rk}} \ := \coprod_{\operatorname{rk} x = \operatorname{rk}\tilde{X}} \operatorname{Spec}\mathcal{O}_{X,x}^\times,$$

which is a sub-$M_0$-scheme of $\tilde{X}$. A *strong morphism* $\varphi:\mathcal{X}\to\mathcal{Y}$ is a pair $\varphi=(\tilde{f},f)$, where $\tilde{f}:\tilde{X}^{\mathrm{rk}}\to\tilde{Y}^{\mathrm{rk}}$ is a morphism of $M_0$-schemes and $f:X\to Y$ is a morphism of schemes such that

$$\begin{array}{ccc} \tilde{X}^{\mathrm{rk}}_{\mathbb{Z}} & \xrightarrow{\tilde{f}_{\mathbb{Z}}} & \tilde{Y}^{\mathrm{rk}}_{\mathbb{Z}} \\ \downarrow{\scriptstyle e_X} & & \downarrow{\scriptstyle e_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

This notion comes already close to achieving our goal. In the category of $\mathbb{F}_1$-schemes together with strong morphisms, the object $(\operatorname{Spec}\{0,1\},\operatorname{Spec}\mathbb{Z},\mathrm{id}_{\operatorname{Spec}\mathbb{Z}})$ is the terminal object, which we should define as $\operatorname{Spec}\mathbb{F}_1$. We define

$$\mathcal{X}(\mathbb{F}_1) \ := \ \operatorname{Hom}_{\text{strong}}(\operatorname{Spec}\mathbb{F}_1,\mathcal{X}),$$

which equals the set of points of $\tilde{X}^{\mathrm{rk}}$ as every strong morphism $\operatorname{Spec}\mathbb{F}_1\to\mathcal{X}$ is determined by the image of the unique point $\{0\}$ of $\operatorname{Spec}\{0,1\}$ in $\tilde{X}^{\mathrm{rk}}$. We see at once that $\#\mathcal{X}(\mathbb{F}_1)=\#M_{k,n}$ if $\mathcal{X}$ is a model of the Grassmannian $\operatorname{Gr}(k,n)$ as $\mathbb{F}_1$-scheme and that $\#\mathcal{G}(\mathbb{F}_1)=\#W$ if $\mathcal{G}=(\tilde{G},G,e_G)$ is a model of a split reductive group $G$ with Weyl group $W$.

Furthermore, if the Weyl group can be lifted to $N(\mathbb{Z})$ as group, i.e. if the short exact sequence of groups

$$1\longrightarrow T(\mathbb{Z})\longrightarrow N(\mathbb{Z})\longrightarrow W\longrightarrow 1$$

splits, then from the commutativity of

$$\begin{array}{ccc} \tilde{G}^{\mathrm{rk}}_{\mathbb{Z}}\times\tilde{G}^{\mathrm{rk}}_{\mathbb{Z}} & \xrightarrow{\tilde{m}_{\mathbb{Z}}} & \tilde{G}^{\mathrm{rk}}_{\mathbb{Z}} \\ \downarrow{\scriptstyle (e_G,e_G)} & & \downarrow{\scriptstyle e_G} \\ G\times G & \xrightarrow{m} & G \end{array}$$

we obtain a morphism $\tilde{m}:\tilde{G}^{\mathrm{rk}}\times\tilde{G}^{\mathrm{rk}}\to\tilde{G}^{\mathrm{rk}}$ of $M_0$-schemes such that $\mu=(\tilde{m},m):\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is a strong morphism that makes $\mathcal{G}$ into a group object.

However, $\mathrm{SL}(n)$ provides an example where the Weyl group cannot be lifted. This leads us, in the following section, to introduce a second kind of morphisms.

## 5. Weak morphisms

The morphism $\operatorname{Spec}\mathcal{O}_{X,x}^\times\to *_{M_0}$ to the terminal object $*_{M_0}=\operatorname{Spec}\{0,1\}$ in the category of $M_0$-schemes induces a morphism

$$\tilde{X}^{\mathrm{rk}} \ = \coprod_{x\in\tilde{X}^{\mathrm{rk}}}\operatorname{Spec}\mathcal{O}_{X,x}^\times \ \longrightarrow \ *_{\mathcal{X}} \ := \coprod_{x\in\tilde{X}^{\mathrm{rk}}} *_{M_0}.$$

Given $\tilde{f}:\tilde{X}^{\mathrm{rk}}\to\tilde{Y}^{\mathrm{rk}}$, there is a unique morphism $*_{\mathcal{X}}\to *_{\mathcal{Y}}$ such that

$$\begin{array}{ccc} \tilde{X}^{\mathrm{rk}} & \xrightarrow{\tilde{f}} & \tilde{Y}^{\mathrm{rk}} \\ \downarrow & & \downarrow \\ *_{\mathcal{X}} & \longrightarrow & *_{\mathcal{Y}} \end{array}$$

commutes. Let $X^{\mathrm{rk}}$ denote the image of $e_X:\tilde{X}^{\mathrm{rk}}\to X$. A *weak morphism* $\varphi:\mathcal{X}\to\mathcal{Y}$ is a pair $\varphi=(\tilde{f},f)$, where $\tilde{f}:\tilde{X}^{\mathrm{rk}}\to\tilde{Y}^{\mathrm{rk}}$ is a morphism of $M_0$-schemes and $f:X\to Y$

is a morphism of schemes such that

$$\begin{array}{ccccccc}
\tilde{X}^{\mathrm{rk}}_{\mathbb{Z}} & & \xrightarrow{\quad \tilde{f}_{\mathbb{Z}} \quad} & & \tilde{Y}^{\mathrm{rk}}_{\mathbb{Z}} & & \\
& \searrow & & & & \searrow & \\
& & (*_{\mathcal{X}})_{\mathbb{Z}} & \longrightarrow & & & (*_{\mathcal{Y}})_{\mathbb{Z}} \\
& \nearrow & & & & \nearrow & \\
X^{\mathrm{rk}} & & \xrightarrow{\quad f \quad} & & Y^{\mathrm{rk}} & &
\end{array}$$

commutes.

The key observation is that a weak morphism $\varphi = (\tilde{f}, f) : \mathcal{X} \to \mathcal{Y}$ has a base extension $f : X \to Y$ to $\mathbb{Z}$, but also induces a morphism $\tilde{f}_* : \mathcal{X}(\mathbb{F}_1) \to \mathcal{Y}(\mathbb{F}_1)$. With this in hand, we yield the following results.

## 6. Algebraic groups over $\mathbb{F}_1$

The idea of Tits' paper is now realized in the following form.

**Theorem 6.1.** *Let $G$ be a split reductive group with group law $m : G \times G \to G$ and Weyl group $W$. Let $\mathcal{G} = (\tilde{G}, G, e_G)$ be the model of $G$ as described before as $\mathbb{F}_1$-scheme. Then there is morphism $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ of $M_0$-schemes such that $\mu = (\tilde{m}, m)$ is a weak morphism that makes $\mathcal{G}$ into a group object. In particular, $\mathcal{G}(\mathbb{F}_1)$ inherits the structure of a group that is isomorphic to $W$.*

We have already seen that $\mathcal{X}(\mathbb{F}_1) = M_{k,n}$ when $\mathcal{X}$ is a model of $\mathrm{Gr}(n, k)$ as $\mathbb{F}_1$-scheme. Furthermore, we have the following.

**Theorem 6.2.** *Let $\mathcal{G}$ be a model of $G = \mathrm{GL}(n)$ as $\mathbb{F}_1$-scheme and let $\mathcal{X}$ be a model of $X = \mathrm{Gr}(k, n)$ as $\mathbb{F}_1$-scheme. Then the group action*

$$f : \ \mathrm{GL}(n) \times \mathrm{Gr}(k, n) \ \longrightarrow \ \mathrm{Gr}(k, n),$$

*induced by the action on $\mathbb{P}^{n-1}$, can be extended to a strong morphism $\varphi : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ such that the group action*

$$\varphi(\mathbb{F}_1) : \ S_n \times M_{k,n} \ \longrightarrow \ M_{k,n},$$

*of $\mathcal{G}(\mathbb{F}_1) = S_n$ on $\mathcal{X}(\mathbb{F}_1) = M_{k,n}$ is induced by the action on $M_n = \{1, \ldots, n\}$.*

## References

[1] A. Connes, C. Consani. *On the notion of geometry over* $\mathbb{F}_1$. arXiv: 0809.2926 [math.AG], 2008.
[2] A. Connes, C. Consani. *Schemes over* $\mathbb{F}_1$ *and zeta functions*. arXiv:0903.2024v2 [math.AG], 2009.
[3] A. Deitmar. *Schemes over* $\mathbb{F}_1$. Number fields and function fields—two parallel worlds, Progr. Math., vol. 239, 2005.
[4] N. Durov. *A New Approach to Arakelov Geometry*. arXiv: 0704.2030v1 [math.AG], 2007.
[5] S. M. J. Haran. *Non-additive geometry*. Compositio Math. Vol.143 (2007) 618–688.
[6] J. López Peña, O. Lorscheid. *Torified varieties and their geometries over* $\mathbf{F}_1$. arXiv:0903.2173 [math.AG], 2009.
[7] Y. Manin. *Lectures on zeta functions and motives (according to Deninger and Kurokawa)*. Astérisque No. 228 (1995), 4, 121–163.
[8] F. Marty. *Relative Zariski open objects*. arXiv:0712.3676 [math.AG], 2007.
[9] C. Soulé. *Les variétés sur le corps à un élément*. Mosc. Math. J. 4 (2004), 217–244.
[10] J. Tits. *Sur les analogues algébriques des groupes semi-simples complexes*. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956 (1957), pp. 261–289.
[11] B. Toën and M. Vaquié. *Au-dessous de* $\mathit{Spec}\,\mathbb{Z}$. Journal of K-Theory (2008) 1–64.

Max-Planck Institut für Mathematik, Vivatsgasse, 7. D-53111, Bonn, Germany
*E-mail address*: oliver@mpim-bonn.mpg.de