The Kervaire invariant problem, after Mike Hill (Virginia), Mike Hopkins (Harvard) and Doug Ravenel (Rochester)

Peter Teichner

June 11, 2009

The authors recently proved that θ_j does not exist for $j > 6$. Here θ_j is a hypothetical element of order 2 in the stable homotopy groups of spheres in dimension $2^{j+1} - 2$.
In 1960, Kevaire defined a $\mathbb{Z}/2$-valued invariant for closed, smooth manifolds with a stable framing. In geometric terms, the above result means that the only possible dimensions for such manifolds with nontrivial Kervaire invariant are

$$2, 6, 14, 30, 62, 126$$

The first 5 dimensions were previously known to be realized, the first 3 by $S^j \times S^j$ for $j = 1, 2, 3$. The status of θ_6 (in dimension 126) remains open.
The theorem implies that the kernel and cokernel of the Kervair-Milnor map

$$\Theta_n \to \pi_n^{st}/im(J)$$

are completely known finite abelian groups. Here Θ_n is the group of exotic smooth structures on S^n and the map associates to it the underlying framed manifold. The image of $J : KO_{n+1} \to \pi_n^{st}$ realizes the different choices of framings on such homotopy spheres.

For further details see:
`http://www.math.rochester.edu/u/faculty/doug/kervaire.html`