THE ERDŐS–MOSER EQUATION $1^k + 2^k + \cdots + (m-1)^k = m^k$ REVISITED USING CONTINUED FRACTIONS

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Abstract. If the equation of the title has an integer solution with $k \geq 2$, then $m > 10^{9.3 \cdot 10^6}$. This was the current best result and proved using a method due to L. Moser (1953). This approach cannot be improved to reach the benchmark $m > 10^{10^7}$. Here we achieve $m > 10^{10^9}$ by showing that $2k/(2m-3)$ is a convergent of $\log 2$ and making an extensive continued fraction digits calculation of $(\log 2)/N$, with N an appropriate integer. This method is very different from that of Moser. Indeed, our result seems to give one of very few instances where a large scale computation of a numerical constant has an application.

1. Introduction

In this note we are interested in non-trivial integer solutions, that is, solutions with $k \geq 2$, of the equation

$$1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = m^k. \tag{1}$$

Conjecturally such solutions do not exist. For $k = 1$ one has clearly the solution $1 + 2 = 3$ (and no further ones). From now on we will assume that $k \geq 2$. Moser [28] showed in 1953 that if (m, k) is a solution of (1), then $m > 10^{10^6}$ and k is even. His result has since then been improved on. Butske et al. [6] have shown by computing, rather than estimating, certain quantities in Moser's original proof that $m > 1.485 \cdot 10^{9\,321\,155}$. By proceeding along these lines this bound cannot be improved on substantially. Butske et al. [6, p. 411] expressed the hope that new insights will eventually make it possible to reach the more natural benchmark 10^{10^7}.

Using that $\Sigma_k(m) = 1^k + 2^k + \cdots + m^k \leq \int_1^m t^k \mathrm{d}t$ and $\Sigma_k(m+1) > \int_0^m t^k \mathrm{d}t$ we obtain that $k + 1 < m < 2(k+1)$. This shows that the ratio k/m is bounded. By a more elaborate reasoning along these lines Krzysztofek [20] obtained that $k + 2 < m < \frac{3}{2}(k+1)$. This implies that $k \geq 4$ and hence

$$k + 2 < m < 2k. \tag{2}$$

Dividing both sides of (1) by m^k one sees that for every integer $m \geq 2$, (1) has precisely one *real* solution k. It is known that $\lim_{m\to\infty} k/m = \log 2$ and we show here that in fact the behaviour of k as a function of m can be determined in a much more explicit way (Theorem 1 and Section 2).

Moree et al. [27], using properties of the Bernoulli numbers and polynomials (an approach initiated in Urbanowicz [30]), showed that $N_1 = \mathrm{lcm}(1, 2, \ldots, 200) \mid k$.

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Kellner [19] in 2002 showed that also all primes $200 < p < 1000$ have to divide k. Actually, Moree et al. [27, p. 814] proved a slightly stronger result and on combining this with Kellner's, one obtains that $N_2 \mid k$ with

$$N_2 = 2^8 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot \prod_{23 \le p \le 997} p > 5.7462 \cdot 10^{427}.$$

For some further references and info on the Erdős–Moser equation we refer to the book by Guy [14, D7].

In this note we attack (1) using the theory of continued fractions. This approach was first explored in 1976 by Best and te Riele [3] in their attempt to solve the related conjecture of Erdős [11] that there are infinitely many pairs (m, k) such that $\Sigma_k(m) \ge m^k$ and $2(m-1)^k < m^k$. In this context they also gave the following variant of one of their results (without proof), namely, (3) with $O(m^{-2})$ replaced with $o(m^{-1})$. The proof we give here uses the same circle of ideas as used by Best and te Riele. It seems that after their work continued fractions in the Erdős–Moser context have been completely ignored. We hope the present paper makes clear that this is unjustified.

Theorem 1. *For integer $m > 0$ and* real $k > 0$ *satisfying equation* (1)*, we have the asymptotic expansion*

$$k = \log 2\left(m - \frac{3}{2} - \frac{c_1}{m} + O\left(\frac{1}{m^2}\right)\right) \quad \text{as } m \to \infty, \tag{3}$$

with $c_1 = \frac{25}{12} - 3\log 2 \approx 0.00389\dots$. Moreover, if $m > 10^9$ then

$$\frac{k}{m} = \log 2\left(1 - \frac{3}{2m} - \frac{C_m}{m^2}\right), \qquad \text{where} \quad 0 < C_m < 0.004. \tag{4}$$

Corollary 1. *If (m, k) is a solution of* (1) *with $k \ge 2$, then $2k/(2m-3)$ is a convergent p_j/q_j of* $\log 2$ *with j even.*

Corollary 2. *The number of solutions $m \le x$ of* (1)*, as x tends to infinity, is at most $O(\log x)$.*

The equation (1) seems to be a sole example of an exponential Diophantine equation in just two unknowns for which even the finiteness of solutions is not yet established. The best result in this direction is given by Corollary 2, which is an immediate consequence of the exponential growth of p_j as a function of j and Corollary 1.

Corollary 1 is not the only result which relates convergents to solutions of Diophantine equations. For example, if (x_0, y_0) is a positive solution to Pell's equation $x^2 - dy^2 = \pm 1$, with d a positive square-free integer, then x_0/y_0 is a convergent of the continued fraction expansion of $\sqrt{d}$. On the other hand, in our situation the number in question, $\log 2$, is transcendental and its continued fraction expansion is expected to be sufficiently 'generic' (unlike that of quadratic irrationals).

Corollary 1 naturally leads us to investigate common factors of k and $2m - 3$. This can be done using the method of Moser, but is not in the literature, as before there was no special reason for considering $2m - 3$.

A key role in this arithmetic study is played by the congruence

$$\frac{\sum_{j=1}^{l-1} j^r}{y} = \begin{cases} 0 \pmod{\frac{1}{2}} & \text{if } r > 1 \text{ is odd;} \\ -\sum_{p|l,\, p-1|r} \frac{1}{p} \pmod{1} & \text{otherwise.} \end{cases} \tag{5}$$

This identity can be proved using the Von Staudt–Clausen theorem; for alternative proofs see, e.g., Carlitz [7] or Moree [25]. Its relevance for the study of (1) was first pointed out by Moree [26].

Given $N \geq 1$, put

$$\mathcal{P}(N) = \{p : p-1 \mid N\} \cup \{p : 3 \text{ is a primitive root modulo } p\}.$$

By a classical result of Hooley [16] it follows, assuming the Generalized Riemann Hypothesis (GRH), that $\mathcal{P}(N)$ has a natural density A, with $A = 0.3739558136\ldots$ the Artin constant, in the set of primes. If $2k/(2m-3) = p_j/q_j$ is a convergent of $\log 2$ arising in Corollary 1, then it can be shown that $(q_j, 6) = 1$ and, if $p \in \mathcal{P}(N_2)$ and p divides q_j, then

$$\nu_p(q_j) = \nu_p(3^{p-1}-1) + 1 \geq 2,$$

where we write $\nu_p(n) = a$ if $p^a \mid n$ and $p^{a+1} \nmid n$. All primes $p \leq 2017$ are in $\mathcal{P}(N_2)$. For $p \neq 3$ we have $\nu_p(3^{p-1}-1) = 1$ unless $3^{p-1} \equiv 1 \pmod{p^2}$, that is, p is a Mirimanoff prime. (It is known that the only Mirimanoff primes $p < 10^{14}$ are 11 and 1006003.)

The main idea of this paper is, in essence, to make use of the fact that the convergents p_j/q_j of $\log 2$ have no reason to also satisfy $N_2 \mid p_j$. The first piece of information comes from asymptotic analysis and the latter piece from arithmetic. Analysis and arithmetic give rise to conditions on the solutions that 'do not feel each other' and this is exploited in our main result:

Theorem 2. *Let $N \geq 1$ be an arbitrary integer. Let*

$$\frac{\log 2}{2N} = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

be the (regular) continued fraction of $(\log 2)/(2N)$, with $p_i/q_i = [a_0, a_1, \ldots, a_i]$ its i-th partial convergent.

Suppose that the integer pair (m, k) with $k \geq 2$ satisfies (1) with $N \mid k$. Let $j = j(N)$ be the smallest integer such that:

(a) *j is even;*
(b) *$a_{j+1} \geq 180N - 2$;*
(c) *$(q_j, 6) = 1$; and*
(d) *$\nu_p(q_j) = \nu_p(3^{p-1}-1) + \nu_p(N) + 1$ for all primes $p \in \mathcal{P}(N)$ dividing q_j.*

Then $m > q_j/2$.

Computing many partial quotients (that is, continued fraction digits) of $\log 2$ is closely related to computing $\log 2$ with many digits of accuracy. Indeed, it is a well-known result of Lochs that for a generic number knowing it accurately up to n decimal digits implies that we can compute about $0.97n$ (where $0.97 \approx 6(\log 2)(\log 10)/\pi^2$) continued fraction digits accurately. For example, knowing 1000 decimal digits of π allows one to compute 968 continued fraction digits.

It seems a hopeless problem to prove anything about $\mathsf{E}(\log q_{j(N)})$, the expected value of $\log q_{j(N)}$ produced by the result. However, metric theory of continued fractions offers some hope of proving a non-trivial lower bound for $\mathsf{E}(\log q_{j(N)}(\xi))$, where we require conditions (a), (b), (c) and (d) to be satisfied but replace $(\log 2)/(2N)$ by a 'generic' $\xi \in [0,1] \setminus \mathbb{Q}$. In this context recall the result of Lévy [21] that, for such a ξ,

$$\lim_{j\to\infty} \frac{\log q_j(\xi)}{j} = \frac{\pi^2}{12\log 2} \approx 1.18. \tag{6}$$

The Gauss–Kuz'min statistics asserts that, for a generic ξ, the probability that a given term in its continued fraction expansion is at least b, equals $\log_2(1+1/b)$. This allows one to deal with the case where we only have condition (b). Likewise a result of Moeckel [23], reproved in a very different way a few years later by Jager and Liardet [17], allows one to deal with the case where we only focus on condition (c). Their result says that for a generic $\xi \in [0,1] \setminus \mathbb{Q}$ we have

$$\lim_{n\to\infty} \frac{\{1 \le m \le n : q_m(\xi) \equiv a \pmod d\}}{n} = \frac{d}{J(d)}\,\frac{\varphi((a,d))}{(a,d)},$$

where φ denotes Euler's totient function, $J(m) = m^2\prod_{p|m}(1-1/p^2)$ Jordan's totient and (a,m) the greatest common divisor of a and m. This result shows that $(q_j,6)=1$ with probability 1/2 (note that a natural number is coprime to 6 with probability 1/3). P. Liardet communicated to us that methods of his paper [22] can be used to take into account both conditions (a) and (c); also the authors of [15] claim that this can be done. We expect that there is a positive constant c_1 such that for a generic ξ satisfying conditions (a), (b) and (c), we have $\mathsf{E}(\log q_{j(N)}(\xi)) \sim c_1 N$ as N tends to infinity. Furthermore, we expect that for a generic ξ satisfying conditions (a), (b), (c) and (d), $\mathsf{E}(\log q_{j(N)}(\xi)) \sim c_2 N \log^\beta N$ for some positive constants c_2 and β; condition (b) is responsible for N, condition (d) for $\log^\beta N$, while conditions (a) and (c) affect c_2. We are definitely not experts in metric aspects of number theory, thus leave this problem to the interested reader acquainted with the subject. Indeed, we even expect that going beyond computing the expected value of $\log q_{j(N)}(\xi)$ is possible, and a probability distribution for $\log q_{j(N)}(\xi)$ can be obtained.

Using the above results from metric theory of continued fractions and some heuristics we are led to believe that roughly speaking we can get

$$m > 10^{257N}$$

from Theorem 2. Being able to compute the convergents of $(\log 2)/(2N)$ arbitrarily far, we would expect (taking $N = N_2$) to show that $m > 10^{10^{400}}$. With the current computer technology computing sufficiently many convergents is the bottleneck. Taking this into consideration we would expect to get

$$m > 10^{0.515r},$$

from Theorem 2, where r is the number of convergents we can compute accurately and 0.515 is the base 10 logarithm of Lévy's constant (6). Note that the fact that N_2 has many divisors gives us some flexibility and increases the likelihood of the heuristics to be applicable. Indeed, our numerical experimenting agrees well with our heuristic considerations (see Section 4). Early 2009, A. Yee and R. Chan [31] reached $r > 31 \cdot 10^9$ for $\log 2$. On the other hand, Y. Kanada and his team [18] computed π to over 1.24 trillion decimal digits already in 2002, using formulae of the same complexity as those used for the computation of $\log 2$ (see [2, Chapter 3] for details). Thus, given the present computer (im)possibilities, one could hope to show (with a lot of effort!) that $m > 10^{10^{12}}$.

Applying Theorem 2 with $N = 2^8 \cdot 3^5 \cdot 5^3$ or $N = 2^8 \cdot 3^5 \cdot 5^4$, and invoking the result of Moree et al. [27] that $N \mid k$, we obtain the following

Theorem 3. *If an integer pair (m, k) with $k \geq 2$ satisfies* (1)*, then*

$$m > 2.7139 \cdot 10^{1\,667\,658\,416}.$$

As an application we can show that $\omega(m-1) \geq 33$, this improves on the result of Brenton and Vasiliu [5], who have shown that $\omega(m-1) \geq 26$, where ω denotes the number of distinct prime divisors; see Section 5.1 for further details.

The fact $N_2 \mid k$ naively implies that k is of size 10^{427} (at least), which is much smaller than Moser's 10^{10^6}. However, in this paper we show that the fact actually yields that $k > 10^{10^9}$ (and likely even $k > 10^{10^{400}}$) — a modestly small number dividing k leads to a huge lower bound for k. Thus, on revisiting [27] after 16 years, its main result is seen to be far more powerful than the second author thought at that time.

In the three following sections we prove Theorems 1, 2 and 3, respectively. Our final Section 5 is devoted to discussing some problems related to the Erdős–Moser equation.

2. Asymptotic dependence of k in terms of m

Our proof of Theorem 1 makes use of the following lemma.

Lemma 1. *For any real $k > 0$, we have*

$$(1-y)^k = e^{-ky}\left(1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \frac{k(5k-6)}{30}y^5 + O(y^6)\right) \quad \text{as } y \to 0. \quad (7)$$

Moreover, for $k > 8$ and $0 < y < 1$, the inequality

$$e^{-ky}\left(1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \frac{k(5k-6)}{30}y^5 - \frac{k^3}{6}y^6\right)$$
$$< (1-y)^k < e^{-ky}\left(1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \frac{k^2}{2}y^5\right) \quad (8)$$

holds.

Proof. As for the asymptotic relation in (7), we simply develop the Taylor expansion of $(1-y)^k e^{ky}$ up to y^5. Unfortunately, estimates coming from the classical forms for the remainder are not sufficient to derive a sharp dependence on k as in (8) for the last term. Therefore, we need more drastic methods to quantify the asymptotics in (7) when $0 < y < 1$.

First note that

$$\begin{aligned}(1-y)e^y &= 1 - \sum_{n=2}^{\infty} \frac{n-1}{n!} y^n \\ &= 1 - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{8} - \frac{y^5}{30} - \cdots, \qquad 0 < y < 1.\end{aligned}$$

Since all coefficients, starting from $n = 2$, in this power series are negative and their sum is exactly -1, for these values of y we have the inequality

$$1 - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{8} - \frac{y^5}{30} - \frac{y^6}{120} < (1-y)e^y < 1 - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{8}. \tag{9}$$

The quantities

$$x_1 = \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{8} \quad \text{and} \quad x_2 = \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{8} + \frac{y^5}{30} + \frac{y^6}{120}, \tag{10}$$

which appear in (9), lie between 0 and 1 for $0 < y < 1$.

Our next ingredient is Gerber's generalization of the Bernoulli inequality [12] (see also Alzer [1]). It states that the remainder after k terms of the (possibly divergent) binomial series for $(1 + x)^a$ (a, x real with $-1 < x$) has the same sign as the first neglected term. In particular we have for real $k > 2$ and $0 < x < 1$,

$$(1-x)^k < 1 - kx + \frac{k(k-1)}{2} x^2, \tag{11}$$

and for real $k > 3$ and $0 < x < 1$,

$$(1-x)^k > 1 - kx + \frac{k(k-1)}{2} x^2 - \frac{k(k-1)(k-2)}{6} x^3. \tag{12}$$

Using the right inequality in (9) and taking $x = x_1$ in (11) we obtain, for $k > 2$,

$$\begin{aligned}(1-y)^k e^{ky} &< 1 - k\left(\frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{8}\right) + \frac{k(k-1)}{2}\left(\frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{8}\right)^2 \\ &= 1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 \\ &\qquad + k(k-1)y^5\left(\frac{1}{6} + \frac{17}{144}y + \frac{1}{24}y^2 + \frac{1}{128}y^3\right) \\ &< 1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \frac{385}{1152}k(k-1)y^5 \end{aligned} \tag{13}$$

implying the upper estimate in (8). In the same vein, the application of the left identity in (9) and of (12) with $x = x_2$ results, for $k > 3$, in

$$\begin{aligned}(1-y)^k e^{ky} &> 1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \frac{k(5k-6)}{30}y^5 \\ &\qquad - ky^6 \sum_{n=0}^{12} (a_n k^2 + b_n k + c_n) y^n, \end{aligned} \tag{14}$$

where the polynomials $p_n(k) = a_n k^2 + b_n k + c_n$, $n = 0, 1, \dots, 12$, all have positive leading coefficients a_n; moreover, $p_n(k) > 0$ for $k > 3$ and $n = 2, 3, \dots, 12$, $p_1(k) = \frac{1}{24}k^2 - \frac{11}{60}k + \frac{17}{120} > 0$ for $k > 4$, and $p_0(k) = \frac{1}{48}k^2 - \frac{13}{72}k + \frac{121}{720} > 0$ for $k > 8$. Using this positivity of the polynomials we can continue the inequality in (14) for $k > 8$ as follows:

$$\begin{aligned}(1-y)^k e^{ky} &> 1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \frac{k(5k-6)}{30}y^5 \\ &\quad - ky^6 \sum_{n=0}^{12}(a_n k^2 + b_n k + c_n) \\ &= 1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \frac{k(5k-6)}{30}y^5 \\ &\quad - ky^6\left(\frac{1}{6}k^2 - \frac{17}{24}k + \frac{11}{20}\right), \end{aligned} \tag{15}$$

from which we deduce the left inequality in (8), and the lemma follows. □

Proof of Theorem 1. The original equation (1) is equivalent to

$$1 = \sum_{j=1}^{m-1}\left(1 - \frac{j}{m}\right)^k. \tag{16}$$

Applying to each term on the right-hand side the inequality from (8) we obtain

$$\begin{aligned} S_0 - \frac{k}{2m^2}S_2 - \frac{k}{3m^3}S_3 + \frac{k(k-2)}{8m^4}S_4 + \frac{k(5k-6)}{30m^5}S_5 - \frac{k^3}{6m^6}S_6 \\ < \sum_{j=1}^{m-1}\left(1 - \frac{j}{m}\right)^k < S_0 - \frac{k}{2m^2}S_2 - \frac{k}{3m^3}S_3 + \frac{k(k-2)}{8m^4}S_4 + \frac{k^2}{2m^5}S_5, \end{aligned} \tag{17}$$

with the notation

$$S_n = \sum_{j=1}^{m-1} j^n e^{-kj/m} = \sum_{j=1}^{m-1} j^n z^j \bigg|_{z=e^{-k/m}}.$$

By (2) we have $e^{-1} < z < e^{-1/2}$, where $z = e^{-k/m}$, and hence $1/(1-z) < 1/(1-e^{-1/2}) < 3$, and in the closed-form expression of the sum

$$S_0 = \sum_{j=1}^{m-1} z^j = \frac{z}{1-z} - \frac{z^m}{1-z},$$

the second term as well as its $z\frac{\mathrm{d}}{\mathrm{d}z}$-derivatives are bounded:

$$0 < \frac{z^m}{1-z}\bigg|_{z=e^{-k/m}} < 3e^{-k} \quad \text{and}$$

$$0 < \left(\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \frac{z^m}{1-z}\right)\bigg|_{z=e^{-k/m}} < 3^{n+1}m^n e^{-k}, \quad \text{for } n = 1, 2, \dots .$$

Therefore, we can write the inequality in (17) as

$$\begin{aligned} S_0' - \frac{k}{2m^2}S_2' - \frac{k}{3m^3}S_3' + \frac{k(k-2)}{8m^4}S_4' + \frac{k(5k-6)}{30m^5}S_5' - \frac{k^3}{6m^6}S_6' - \left(\frac{3^3k}{2} + \frac{3^4k}{3} + \frac{3^7k^3}{6}\right)e^{-k} \\ < \sum_{j=1}^{m-1}\left(1-\frac{j}{m}\right)^k \\ < S_0' - \frac{k}{2m^2}S_2' - \frac{k}{3m^3}S_3' + \frac{k(k-2)}{8m^4}S_4' + \frac{k^2}{2m^5}S_5' + \left(3 + \frac{3^5k(k-2)}{8} + \frac{3^6k^2}{2}\right)e^{-k} \end{aligned}$$

implying

$$\begin{aligned} S_0' - \frac{k}{2m^2}S_2' - \frac{k}{3m^3}S_3' + \frac{k(k-2)}{8m^4}S_4' + \frac{k(5k-6)}{30m^5}S_5' - \frac{k^3}{6m^6}S_6' - 500k^3e^{-k} \\ < \sum_{j=1}^{m-1}\left(1-\frac{j}{m}\right)^k < S_0' - \frac{k}{2m^2}S_2' - \frac{k}{3m^3}S_3' + \frac{k(k-2)}{8m^4}S_4' + \frac{k^2}{2m^5}S_5' + 500k^2e^{-k}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} S_n' = \sum_{j=1}^{\infty} j^n z^j\Big|_{z=e^{-k/m}} &= \left(\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \frac{z}{1-z}\right)\Big|_{z=e^{-k/m}} \\ &= (-1)^n\left(\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \frac{1}{z-1}\right)\Big|_{z=e^{k/m}} \quad \text{for } n = 0, 1, \dots; \end{aligned}$$

in particular,

$$\begin{aligned} S_0' &= \frac{1}{z-1}, \quad S_2' = \frac{z+z^2}{(z-1)^3}, \quad S_3' = \frac{z+4z^2+z^3}{(z-1)^4}, \quad S_4' = \frac{z+11z^2+11z^3+z^4}{(z-1)^5}, \\ S_5' &= \frac{z+26z^2+66z^3+26z^4+z^5}{(z-1)^6}, \quad S_6' = \frac{z+57z^2+302z^3+302z^4+57z^5+z^6}{(z-1)^7} \end{aligned}$$

with $z = e^{k/m}$. Since $500k^3e^{-k} < (2k)^{-3} < m^{-3}$ for $k > m/2 > 30$, using our equation (16) we can write the estimates (18) as

$$\begin{aligned} \frac{k(5k-6)}{30m^5}S_5' - \frac{k^3}{6m^6}S_6' - \frac{1}{m^3} \\ < 1 - S_0' + \frac{k}{2m^2}S_2' + \frac{k}{3m^3}S_3' - \frac{k(k-2)}{8m^4}S_4' < \frac{k^2}{2m^5}S_5' + \frac{1}{m^3}. \end{aligned} \tag{19}$$

Noting that $e^{1/2} < z = e^{k/m} < e$, we find

$$\begin{aligned} 0 < S_5' &< \frac{e+26e^2+66e^3+26e^4+e^5}{(e^{1/2}-1)^6} < 41438, \\ 0 < S_6' &< \frac{e+57e^2+302e^3+302e^4+57e^5+e^6}{(e^{1/2}-1)^7} < 658544, \end{aligned}$$

we continue (19) as follows:

$$\left|1-\frac{1}{z-1}+\frac{k}{2m^2}\frac{z+z^2}{(z-1)^3}+\frac{k}{3m^3}\frac{z+4z^2+z^3}{(z-1)^4}\right.$$
$$\left.-\frac{k(k-2)}{8m^4}\frac{z+11z^2+11z^3+z^4}{(z-1)^5}\right|<\frac{110000}{m^3}, \quad (20)$$

where $z=e^{k/m}$.

We already know that k/m is bounded as $m\to\infty$; making the ansatz $k/m = c+O(1/m)$, hence $z=e^{k/m}=e^c+O(1/m)$, we find from (20) that

$$1-\frac{1}{e^c-1}=O\left(\frac{1}{m}\right) \quad \text{as } m\to\infty,$$

hence $e^c=2$ and $c=\log 2$. Now we take

$$\frac{k}{m}=\log 2+\frac{a}{m}+\frac{b}{m^2}+O\left(\frac{1}{m^3}\right) \quad \text{as } m\to\infty,$$

hence

$$z=e^{k/m}=2+\frac{2a}{m}+\frac{a^2+2b}{m^2}+O\left(\frac{1}{m^3}\right) \quad \text{as } m\to\infty.$$

Substituting these formulas into (20) results in

$$\begin{aligned}
O\left(\frac{1}{m^3}\right) &= 1-\frac{1}{1+2a/m+(a^2+2b)/m^2+O(m^{-3})}\\
&\quad+\frac{\log 2+a/m+O(m^{-2})}{2m}\,\frac{6+10a/m+O(m^{-2})}{1+6a/m+O(m^{-2})}\\
&\quad+\frac{\log 2+O(m^{-1})}{3m^2}\,\frac{26+O(m^{-1})}{1+O(m^{-1})}\\
&\quad-\frac{\log^2 2+O(m^{-1})}{8m^2}\,\frac{150+O(m^{-1})}{1+O(m^{-1})}+O\left(\frac{1}{m^3}\right)\\
&=\frac{2a+3\log 2}{m}-\frac{3a^2-3a+13a\log 2-2b+\frac{75}{4}\log^2 2-\frac{26}{3}\log 2}{m^2}+O\left(\frac{1}{m^3}\right),
\end{aligned}$$

hence $a=-\frac{3}{2}\log 2$, $b=(3\log 2-\frac{25}{12})\log 2$ and, finally, we get the asymptotic formula (3).

To quantify this asymptotic expansion, we introduce the function

$$f_m(C)=\left(1-\frac{1}{z-1}+\frac{\lambda}{2m}\frac{z+z^2}{(z-1)^3}+\frac{\lambda}{3m^2}\frac{z+4z^2+z^3}{(z-1)^4}\right.$$
$$\left.\left.-\frac{\lambda(\lambda-2/m)}{8m^2}\frac{z+11z^2+11z^3+z^4}{(z-1)^5}\right)\right|_{z=e^\lambda},$$

where

$$\lambda=\lambda(C)=\log 2\left(1-\frac{3}{2m}-\frac{C}{m^2}\right)$$

agrees with our k/m up to $O(m^{-2})$. Direct computation then shows that

$$f_m(0) > 0.005m^{-2} - 100m^{-3} \quad \text{and} \quad f_m(0.004) < -0.00015m^{-2} + 100m^{-3}$$

for $m \geq 100$. Therefore, $f_m(0) > 110000/m^3$ for $m > 2202 \cdot 10^4$ and $f_m(0.004) < -110000/m^3$ for $m > 734 \cdot 10^6$, so that $|f_m(C)| < 110000/m^3$ is possible only if $0 < C < 0.004$. Comparing this result with (20) we conclude that, for k and $m > 10^9$ satisfying (16), we necessarily have

$$\frac{k}{m} = \log 2\left(1 - \frac{3}{2m} - \frac{C_m}{m^2}\right)$$

with $0 < C_m < 0.004$. □

Clearly, the strategy to deduce further terms in the expansion (3) remains the same, but in order to achieve precision $O(m^{-n})$ for an integer $n \geq 2$ we have to use the Taylor expansion of $(1-y)^k e^{ky}$ up to y^{2n+1} (each new term in (3) requires two extra terms in the expansion of $(1-y)^k e^{ky}$). In this way we get

$$\begin{aligned} k &= cm - \frac{3}{2}c - \left(\frac{25}{12}c - 3c^2\right)m^{-1} + \left(-\frac{73}{8}c + \frac{61}{2}c^2 - 25c^3\right)m^{-2} \\ &\quad + \left(-\frac{41299}{720}c + \frac{657}{2}c^2 - 598c^3 + \frac{1405}{4}c^4\right)m^{-3} + O(m^{-4})\Bigg) \\ &\approx 0.69314718m - 1.03972077 - 0.00269758m^{-1} + 0.00323260m^{-2} \\ &\quad + 0.00217182m^{-3} + O(m^{-4}), \end{aligned} \tag{21}$$

where $c = \log 2$. However, we do not possess any clear general strategy to quantify such expansions. Already proving a sharp dependence on k for the remainder of the n-th truncation of the Taylor expansion of $(1-y)^k e^{ky}$ (like we do for $n = 4$ in Lemma 1) seems to be a difficult task. We discuss related problems in Section 5.

Proof of Corollary 1. Let (m, k) be a non-trivial integer solution of (1). By Moser's result we know that $m > 10^9$. It follows from Theorem 1 that

$$0 < \log 2 - \frac{2k}{2m-3} < \frac{0.0111}{(2m-3)^2}. \tag{22}$$

By Legendre's theorem, $|\log 2 - p/q| < 1/(2q^2)$ implies that p/q is a convergent of $\log 2$, while $\log 2 > p/q$ insures that the index of the convergent is even. Thus, $2k/(2m-3)$ is a convergent p_j/q_j of the continued fraction of $\log 2$ with j even. □

3. The proof of the main theorem

In this section we prove Theorem 2. The restrictions on the prime factorization of q_j in that result are established using an argument in the style of Moser given in the proof of the following lemma.

Lemma 2. *Let (m, k) be a solution of (1) with $k \geq 2$. Let p be a prime divisor of $2m-3$. If $p - 1 \mid k$, then*

$$\nu_p(2m-3) = \nu_p(3^{p-1} - 1) + \nu_p(k) + 1 \geq 2.$$

If 3 is a primitive root modulo p, then $p - 1 \mid k$.

Proof. Using that k must be even, we find that

$$\sum_{j=1}^{2m-4} j^k \equiv \sum_{j=1}^{m-1} j^k + \sum_{j=1}^{m-3}(2m-3-j)^k \equiv \sum_{j=1}^{m-1} j^k + \sum_{j=1}^{m-3} j^k \pmod{2m-3}$$
$$\equiv m^k + m^k - (m-1)^k - (m-2)^k \equiv 2(3^k-1)(m-1)^k \pmod{2m-3},$$

where we used that $m^k \equiv (2m-3+m)^k \equiv 3^k(m-1)^k \pmod{2m-3}$ and $(m-2)^k \equiv (2m-3-m+1)^k \equiv (m-1)^k \pmod{2m-3}$. On applying (5) with $l = 2m-3$ and $r = k$ we then obtain that

$$\frac{2(3^k-1)(m-1)^k}{2m-3} \equiv -\sum_{\substack{p \mid 2m-3 \\ p-1 \mid k}} \frac{1}{p} \pmod{1}. \tag{23}$$

If $p \mid 2m-3$ and $p-1 \mid k$, the p-order of the right-hand side is -1. The p-order of the left-hand side must also be -1, that is, we must have

$$\nu_p(2m-3) = \nu_p(3^k-1) + k\nu_p(m-1) + 1 = \nu_p(3^{p-1}-1) + \nu_p(k) + 1,$$

where we used that $m-1$ and $2m-3$ are coprime. Now suppose that $p \mid 2m-3$ and 3 is a primitive root modulo p (thus $p \mid 3^k - 1$ implies $p-1 \mid k$). If $p-1 \nmid k$, the p-order of the left-hand side is ≤ -1 and > -1 on the right-hand side. Thus, we infer that $p-1 \mid k$. $\square$

This completes the required ingredients needed in order to prove the main result.

Proof of Theorem 2. Since by assumption $N \mid k$, we can write $k = Nk_1$ and thus rewrite (22) as

$$0 < \frac{\log 2}{2N} - \frac{k_1}{2m-3} < \frac{0.0111}{2N(2m-3)^2}. \tag{24}$$

We infer that $k_1/(2m-3) = p_j/q_j$ is a convergent to $(\log 2)/(2N)$ with j even. Since $p \mid m$ implies $p-1 \nmid k$ (see, e.g., Moree [26, Proposition 9]), we have $(6, q_j) = 1$. We rewrite (24) as

$$0 < \frac{\log 2}{2N} - \frac{p_j}{q_j} < \frac{0.0111}{2Nd^2q_j^2},$$

with d the greatest common divisor of k_1 and $2m-3$. On the other hand,

$$\frac{\log 2}{2N} - \frac{p_j}{q_j} > \frac{1}{(a_{j+1}+2)q_j^2},$$

hence $(a_{j+1}+2)^{-1} < 0.0111/(2Nd^2)$, from which the result follows on also noting that $2m-3 \geq q_j$ and invoking Lemma 2 (note that if $\nu_p(q_j) \geq 1$, then $\nu_p(q_j) = \nu_p(2m-3) - \nu_p(k_1)$). $\square$

To prove that $p \mid m$ implies $p-1 \nmid k$ one uses that k must be even and takes $l = m$ in (5), showing that $\sum_{p \mid m,\, p-1 \mid k} \frac{1}{p}$ must be an integer. Since a sum of reciprocals of distinct primes can never be an integer, the result follows.

4. Computation of the continued fractions

We make use of conditions (a), (b), (c) of Theorem 2. We recall that we expect $\mathsf{E}(\log q_{j(N)}(\xi)) \sim c_1 N$ for a generic $\xi \in [0,1]$ satisfying these conditions. Indeed, on the basis of theoretical results, heuristics and numerical experiments, we conjecture that $c_1 = 60\pi^2$.

N	$j = j(N)$	a_{j+1}	q_j (rounded down)	q_j mod 6	$p = p(q_j)$
1	642	764	$2.383153 \cdot 10^{330}$	−1	149
2	664	1 529	$2.383153 \cdot 10^{330}$	−1	149
2^2	1 254	21 966	$1.132014 \cdot 10^{638}$	+1	5
2^3	1 264	43 933	$1.132014 \cdot 10^{638}$	+1	5
2^4	1 280	87 866	$1.132014 \cdot 10^{638}$	+1	5
2^5	1 294	175 733	$1.132014 \cdot 10^{638}$	+1	5
2^6	8 950	26 416	$3.458446 \cdot 10^{4\,589}$	−1	
2^7	8 926	52 834	$3.458446 \cdot 10^{4\,589}$	−1	
2^8	119 476	122 799	$1.374540 \cdot 10^{61\,317}$	+1	
$2^8 \cdot 3$	119 008	368 398	$1.374540 \cdot 10^{61\,317}$	+1	
$2^8 \cdot 3^2$	139 532	782 152	$9.351282 \cdot 10^{71\,882}$	+1	56 131
$2^8 \cdot 3^3$	6 168 634	1 540 283	$8.220719 \cdot 10^{3\,177\,670}$	+1	
$2^8 \cdot 3^4$	22 383 618	5 167 079	$5.128265 \cdot 10^{11\,538\,265}$	+1	17
$2^8 \cdot 3^5$	155 830 946	31 664 035	$2.257099 \cdot 10^{80\,303\,211}$	−1	
$2^8 \cdot 3^5 \cdot 5$	351 661 538	85 898 211	$9.729739 \cdot 10^{181\,214\,202}$	−1	
$2^8 \cdot 3^5 \cdot 5^2$	1 738 154 976	1 433 700 727	$1.594940 \cdot 10^{895\,721\,905}$	+1	5
	1 977 626 256	853 324 651	$1.196828 \cdot 10^{1\,019\,133\,881}$	−1	
$2^8 \cdot 3^5 \cdot 5^3$	2 015 279 170	4 388 327 617	$5.565196 \cdot 10^{1\,038\,523\,018}$	−1	19
	3 236 170 820	2 307 115 390	$5.427815 \cdot 10^{1\,667\,658\,416}$	+1	
$2^8 \cdot 3^5 \cdot 5^4$	2 015 385 392	21 941 638 090	$5.565196 \cdot 10^{1\,038\,523\,018}$	−1	19
	3 236 257 942	11 535 576 954	$5.427815 \cdot 10^{1\,667\,658\,416}$	+1	

Table 1. Smallest integers j satisfying conditions (a), (b) and (c) of Theorem 2

The computation of $(\log 2)/(2N)$ is done in two steps. First, we generate d digits of $\log 2$. For this we use the γ-cruncher [31]. With this program, A. Yee and R. Chan computed 31 billion decimal digits of $\log 2$ in about 24 hours. Second, we set a rational approximation of $(\log 2)/(2N)$ with a relative error bounded by 10^{-d}. Then partial quotients of the continued fraction of $(\log 2)/(2N)$ are computed: about $0.97d$ of them can be evaluated, with safe error control [4] (cf. the result of Lochs mentioned in Section 1). We maintain a floating point approximation of numbers q_j

(rounded down) and residues of $q_j \pmod 6$ by the formula $q_{i+1} = a_{i+1}q_i + q_{i-1}$ for $i \geq 0$, where $q_0 = 1$ and $q_{-1} = 0$.

Table 1 was created with the 'basic method' of [4] for $N \leq 2^8 \cdot 3^4$. It was fast enough to reach the benchmark $m > 10^{10^7}$ in four days with $50 \cdot 10^6$ digits of $\log 2$. Bit-complexity of this algorithm (or of the indirect or direct methods [4]) is quadratic and reaching the $m > 10^{10^{10}}$ milestone would take centuries.

Some subquadratic GCD algorithms were discovered that have asymptotic running time $O(n(\log n)^2 \log\log n)$ [24]. A faster version of the program was written: this time a recursive HGCD method is applied. It is adapted for computing a continued fraction by using Lemma 3 of [4] (which is similar to Algorithm 1.3.13 of [8]) for error control. With it the program leaps over 10^{10^8} in just about one hour. Finally, the new benchmark $m > 10^{10^9}$ is established in no more than 10 hours with $3 \cdot 10^9$ digits of $\log 2$, $N = 1555200$ and condition (d): the first found solution fits conditions (a)–(c), but not (d). With $N = 7776000$, $m > 10^{10^9}$ is achieved for the smallest j. See Table 1: in the last column, p is a prime such that $p \in \mathcal{P}(N)$ and $\nu_p(q_j) = 1$, that is, such that condition (d) of Theorem 2 is violated.

Now, computation time is not a problem to achieve the $m > 10^{10^{10}}$ milestone, a few days will be sufficient on a computer with a large amount of memory. We remark that the complexity and hardware requirement for computation of the digits of $\log 2$, respectively for computation of its continued fraction expansion, are similar.

5. Miscellaneous

5.1. `The number of distinct prime factors of` $m-1$**.** There is a different application of Theorem 3 suggested by the work of Brenton and Vasiliu [5], to factorization properties of the number $m-1$ coming from a non-trivial solution (m, k) of (1). A result of Moser [28] (which can also be deduced from the key identity (5), cf. the proof of Lemma 2 above) asserts that

$$\sum_{p \mid m-1} \frac{1}{p} + \frac{1}{m-1} \in \mathbb{Z}; \tag{25}$$

in particular, the number $m-1$ is square-free. Since the sum of reciprocals of the first 58 primes is less than 2, we conclude that either $\omega(m-1) \geq 58$ or the integer in (25) is equal to 1. In the latter case, we can apply Curtiss' bound [9] for positive integer solutions of Kellogg's equation

$$\sum_{i=1}^{n} \frac{1}{x_i} = 1,$$

namely, $\max_i\{x_i\} \leq A_n - 1$, where the Sylvester sequence $\{A_n\}_{n\geq 1} = \{2, 3, 7, 43, \dots\}$ is defined by the recurrence $A_n = 1 + \prod_{i=1}^{n-1} A_i$ (for some further info, see e.g. Odoni [29]). From this result and the estimate $A_n < (1.066 \cdot 10^{13})^{2^{n-7}}$, we infer

$$m < (1.066 \cdot 10^{13})^{2^{\omega(m-1)-6}},$$

which together with the lower bound on m from Theorem 3 yields $\omega(m-1) \geq 33$. A similar estimate on the basis of another (25)-like identity of Moser implies that $\omega(m+1) \geq 32$.

5.2. **Generalized EM equation.** The method we use in Section 2 for deriving the asymptotics of k in terms of m works for the more general equation

$$1^k + 2^k + \cdots + (m-1)^k = tm^k, \tag{26}$$

with $t \in \mathbb{N}$ fixed, as well. Indeed, the coefficients in the Taylor series expansion

$$(1-y)^k e^{ky} = 1 - \frac{k}{2}y^2 - \frac{k}{3}y^3 + \frac{k(k-2)}{8}y^4 + \cdots = \sum_{n=0}^{\infty} g_n(k) y^n \tag{27}$$

are polynomials satisfying

$$g_0(k) = 1, \quad g_1(k) = 0, \qquad \text{and} \qquad \deg_k g_n(k) = \left[\frac{n}{2}\right], \quad g_n(0) = 0 \quad \text{for } n \geq 2; \tag{28}$$

the latter follows from raising the series $(1-y)e^y = 1 - y^2/2 - y^3/3 - \cdots$ to the power k. In these settings, equation (26) becomes

$$\begin{aligned} t &= \sum_{j=1}^{m-1}\left(1 - \frac{j}{m}\right)^k = \sum_{j=1}^{m-1} e^{-kj/m} \sum_{n=0}^{\infty} g_n(k)\left(\frac{j}{m}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{g_n(k)}{m^n} \sum_{j=1}^{m-1} j^n e^{-jk/m} \end{aligned}$$

(since $\sum_{j=m}^{\infty} j^n e^{-jk/m} = O(m^n e^{-k})$)

$$\begin{aligned} &\sim \sum_{n=0}^{\infty} \frac{g_n(k)}{m^n} \sum_{j=1}^{\infty} j^n e^{-jk/m} = \sum_{n=0}^{\infty} \frac{g_n(k)}{m^n} \left(\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \frac{z}{1-z}\right)\bigg|_{z=e^{-k/m}} \\ &= \sum_{n=0}^{\infty} \frac{g_n(k)}{m^n} (-1)^n \left(\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \frac{1}{z-1}\right)\bigg|_{z=e^{k/m}}, \end{aligned}$$

hence in the notation $\lambda = k/m$ and $x = 1/m$ we have

$$t = \sum_{n=0}^{\infty} g_n\left(\frac{\lambda}{x}\right)(-x)^n \left(\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \frac{1}{z-1}\right)\bigg|_{z=e^{\lambda}}. \tag{29}$$

Searching λ in the form $\lambda = c_0 + c_1 x + c_2 x^2 + \cdots$, we find successively

$$c_0 = c(t) = \log\left(1 + \frac{1}{t}\right) = \log\frac{t+1}{t}, \qquad c_1 = -\left(t + \frac{1}{2}\right)c,$$

$$c_2 = \left(t + \frac{1}{2}\right)^3 c^2 - \left(t + \frac{1}{2}\right)^2 c - \frac{1}{4}\left(t + \frac{1}{2}\right)c^2 + \frac{c}{6},$$

and so on. Note that $c_n(-(t+1)) = (-1)^{n+1}c_n(t)$ for $n = 0, 1, 2, \ldots$; this reflects the equivalence of equation (26) and

$$1^k + 2^k + \cdots + (m-1)^k + m^k = (t+1)m^k. \tag{30}$$

From this asymptotics we see that

$$\frac{2k}{2m - t_1} = c + \frac{t_1^3c^2 - 2t_1^2c - t_1c^2 + 4c/3}{2(2m - t_1)^2} + O\biggl(\frac{1}{(2m - t_1)^3}\biggr), \tag{31}$$

where $t_1 = 2t+1$ and $c = \log(1+1/t)$. It can be checked that for all positive integers t we have the inequality

$$-0.22 < t_1^3c^2 - 2t_1^2c - t_1c^2 + \frac{4c}{3} < 0,$$

and hence $2k/(2m - 2t - 1)$ is a convergent (with even index) of this logarithm $c = \log(1 + 1/t)$ for m large enough.

5.3. `Saddle-point method.` A different approach to treat the asymptotic behaviour of k in terms of m for k and m satisfying (1) (or, more generally, (26)) is based on the integral representation

$$1^k + 2^k + \cdots + (m-1)^k = \frac{\Gamma(k)}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{e^{mz}}{(e^z - 1)z^{k+1}}\,\mathrm{d}z,$$

where C is an arbitrary positive real number (cf. [10, p. 273]). On noting that

$$\frac{e^{mz}}{e^z - 1} = \frac{e^{(m-1)z}}{1 - e^{-C}}\biggl(1 + \frac{1 - e^{z-C}}{e^z - 1}\biggr)$$

one obtains, on taking $C = (k+1)/(m-1)$ and after invoking some rather trivial estimates, that

$$1^k + 2^k + \cdots + (m-1)^k = \frac{(m-1)^k}{1 - e^{-(k+1)/(m-1)}}\bigl(1 + \rho_k(m)\bigr), \tag{32}$$

with

$$|\rho_k(m)| < \frac{\sqrt{2(k+1)}C}{\sqrt{\pi}(k-1)(e^C - 1)}.$$

(This part of the argument is due to Delange; for more details see [10, pp. 273–274].) By (2), C is bounded and we infer that $|\rho_k(m)| = O(k^{-1/2}) = O(m^{-1/2})$. On putting m^k on the left-hand side of (32) and using $(1 - 1/m)^m = \exp(-1 + O(m^{-1}))$, we immediately conclude that, as $m \to \infty$,

$$\frac{k}{m} = \log 2 + O\biggl(\frac{1}{\sqrt{m}}\biggr),$$

where the implied constant is absolute. A more elaborate analysis, using the saddle-point method, will very likely allow one as many terms in the latter expansion as required.

5.4. `Experimental asymptotics.` It is worth mentioning a fast experimental approach of doing asymptotics like (21). Given numerically a few hundred terms of a sequence $s = \{s_n\}_{n\ge1}$ that one believes has an asymptotic expansion in inverse powers of n, one can try to apply the $\mathtt{asymp}_k$ trick, a simple but often powerful method to numerically determine the coefficients in the ansatz

$$s_n \sim c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots .$$

As a second step one tries to identify the so-found coefficients with (linear combinations of) known constants. Thus, one arrives at a conjecture that hopefully can be turned into a proof. For more details and some 'victories' achieved by the $\mathtt{asymp}_k$ method, see Grünberg and Moree [13].

D. Zagier has applied this trick to the sequence of $k = k(m)$ obtained from (1) on letting m run through the first thousand values. Excellent agreement with our theoretical results was obtained in this way.

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