Remarks on endomorphisms and rational points

E. Amerik, F. Bogomolov, M. Rovinsky

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1 Introduction

Let X be an algebraic variety defined over a number field K. One says that the rational points are potentially dense in X, or that X is potentially dense, if there is a finite extension L of K, such that $X(L)$ is Zariski-dense. For instance, unirational varieties are obviously potentially dense. A well-known conjecture of Lang affirms that a variety of general type cannot be potentially dense; more recently, the question of geometric characterisation of potentially dense varieties has been raised by several mathematicians, for example, by Abramovich and Colliot-Thélène and especially by Campana [C]. According to their points of view, one expects that the varieties with trivial canonical class should be potentially dense. This is well-known for abelian varieties, but the simply-connected case remains largely unsolved.

Bogomolov and Tschinkel [BT] proved potential density of rational points for $K3$ surfaces admitting an elliptic pencil, or an infinite automorphisms group. Hassett and Tschinkel [HT] did this for certain symmetric powers of general $K3$-surfaces with a polarization of a suitable degree. The key observation of their work was that those symmetric powers are rationally fibered in abelian varieties over a projective space, and, as the elliptic $K3$ surfaces of [BT], they admit a potentially dense multisection which one can translate by suitable fiberwise rational self-maps to obtain the potential density of the ambient variety.

More recently, Amerik and Voisin [AV] gave a proof of potential density for the variety of lines of a sufficiently general cubic fourfold defined over a number field. Such a variety X is an irreducible holomorphic symplectic fourfold with cyclic Picard group, in fact the only (up to now) example of a simply-connected variety, defined over a number field, with trivial canonical class and cyclic Picard group where the potential density is established. The starting idea is similar: as noticed in [V1], X admits a rational self-map of degree 16. Moreover, X carries a two-parameter family $\Sigma_b, b \in B$ of surfaces birational to abelian surfaces. It is proved in [AV] that under certain genericity conditions on the pair (X, b), satisfied by many pairs defined over a number field, the iterates $f^n(\Sigma_b), n \in \mathbb{N}$ are Zariski-dense in X. Since rational points are potentially dense on abelian varieties, this clearly implies that X is potentially dense. The proof is rather involved: even the proof of the fact that the number of iterates is infinite for some Σ_b defined over a number field is highly non-trivial, using, for instance, l-adic Abel-Jacobi invariants in the continuous étale cohomology.

The purpose of this article is to further investigate the connection between the existence of "sufficiently nontrivial" rational self-maps and the potential density of rational points. In the first part we prove, among other facts, that if the differential of a rational self-map $f : X \dashrightarrow X$ at a non-degenerate fixed point $q \in X(\bar{\mathbb{Q}})$ has multiplicatively independent eigenvalues, then the rational points are potentially dense on X. More precisely, we show that under this condition, one can find a point $x \in X(\bar{\mathbb{Q}})$ such that the set of its iterates is Zariski-dense. Note that this remains unknown for X and f as in [AV], though the question has been raised in [AC] where it is shown that the map $f : X \dashrightarrow X$ does not preserve any rational fibration and therefore the set of the iterates of a general complex point of X is Zariski-dense.

Unfortunately, it seems to be difficult to find interesting examples with multiplicatively independent eigenvalues of the differential at a fixed point. There is certainly plenty of such self-maps on rational varieties, but since for those the potential density is obvious, we cannot consider their self-maps as being "interesting". In the case of [AV], the eigenvalues of Df_q at a fixed point q are far from being multiplicatively independent (lemma 3.3). We do not know whether the multiplicative independence could hold for a fixed point of a power of f.

Nevertheless, even the independence of certain eigenvalues gives interesting new information. To illustrate this, we exploit this point of view in the second part, where a simplified proof of the potential density of the variety of lines of the cubic fourfold is given.

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2 Invariant neighbourhoods

Let X be a smooth projective variety of dimension n and let $f : X \dashrightarrow X$ be a rational self-map, both defined over a "sufficiently large" number field K. We assume that f has a fixed point $q \in X(K)$. This assumption is not restrictive if, for example, f is a regular polarized (that is, such that $f^*L = L^{\otimes k}$ for a certain ample line bundle L and a positive integer k) endomorphism: indeed, in this case the set of periodic points in $X(\bar{\mathbb{Q}})$ is even Zariski-dense [F], so replacing f by a power and taking a finite extension of K if necessary, we find a fixed point.

For a number field K we denote by $\mathcal{O}_K$ the ring of integers of K; for a point $\mathfrak{p}$, i.e. an equivalence class of valuations of K, $K_\mathfrak{p}$ denotes the corresponding completion, $\mathcal{O}_\mathfrak{p}$ the ring of integers in $K_\mathfrak{p}$.

Our starting point is that, for any fixed point $q \in X(K)$ and a suitable prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, we can find a "$\mathfrak{p}$-adic neighbourhood" $q \in O_{\mathfrak{p},q} \subset X(K_\mathfrak{p})$, on which f is defined and which is f-invariant.

More precisely, choose an affine neighbourhood $U \subset X$ of q, such that the restriction of f to U is regular. By Noether normalisation lemma, there is a finite K-morphism $\pi = (x_1, \ldots, x_n) : U \longrightarrow \mathbb{A}^n_K$ to the affine space, which is étale at q and which maps q to 0. Then the K-algebra $\mathcal{O}(U)$ is integral over $K[x_1, \ldots, x_n]$, i.e., it is generated over $K[x_1, \ldots, x_n]$ by some regular functions $x_{n+1}, \ldots, x_m$ integral over $K[x_1, \ldots, x_n]$. The coordinate ring of U is included into the local ring of q and the latter is included into its completion: $\mathcal{O}(U) \subset \mathcal{O}_{U,q} \subset \widehat{\mathcal{O}}_{U,q} = K[[x_1, \ldots, x_n]]$. In particular, $x_{n+1}, \ldots, x_m$ become elements of $K[[x_1, \ldots, x_n]]$. As f^* defines an endomorphism of the ring $\mathcal{O}_{U,q}$ and of its completion, the functions $f^*x_1, \ldots, f^*x_m$ become power series in x_i with coefficients in K.

We use the following well-known result (a stronger version for $n = 1$ goes back to Eisenstein) to deduce that the coefficients of the power series $x_{n+1}, \ldots, x_m, f^*x_1, \ldots, f^*x_m$ are $\mathfrak{p}$-integral for almost all primes $\mathfrak{p}$, i.e.,

$$x_{n+1}, \ldots, x_m, f^*x_1, \ldots, f^*x_m \in \mathcal{O}_K[1/N][[x_1, \ldots, x_n]]$$

for some integer $N \geq 1$:

Lemma 2.1 *Let k be a field of characteristic zero and let $\phi \in k[[x_1, \ldots x_n]]$ be a function algebraic over $k(x_1, \ldots, x_d)$. Then $\phi \in A[[x_1, \ldots x_n]]$, where A is a finitely generated $\mathbb{Z}$-algebra.*

Proof. Let F be a minimal polynomial of ϕ over $k[x_1, \ldots, x_n]$, so $F(\phi) = 0$ and $F'(\phi) \neq 0$. Then $F'(\phi) \in \mathfrak{m}^s \smallsetminus \mathfrak{m}^{s+1}$ for some $s \geq 0$, where $\mathfrak{m}$ is the maximal ideal in $k[[x_1, \ldots, x_n]]$.

Denote by ϕ_d the only polynomial of degree $< d$ congruent to ϕ modulo $\mathfrak{m}^d$. For a formal series Φ in x and an integer m denote by $\Phi_{(m)}$ the homogeneous part of Φ of degree m. Clearly, $F'(\phi_d)_{(m)}$ is independent of d for $d > m$.

We are going to show by induction on d that the coefficients of the homogeneous component of ϕ of degree d belong to the $\mathbb{Z}$-subalgebra in k generated by coefficients of F (as a polynomial in $n+1$ variables), by coefficients of ϕ_{s+1} and by the inverse of a certain (non-canonical) polynomial D in coefficients of F and in coefficients of ϕ_{s+1}. To define D, choose a discrete valuation v of the field $k(x_1, \ldots, x_n)$ of rank n trivial on k, say such that $v(x_i) > v(k(x_1, \ldots, x_{i-1})^\times)$ for all $1 \leq i \leq n$ (equivalently,

$0 < v(x_1^{m_1}) < \cdots < v(x_n^{m_n})$ for all $m_1, \ldots, m_n > 0$). Then D is the coefficient of the monomial in $F'(\phi_{s+1})_{(s)}$ with the minimal valuation.

For $d \leq s$ there is nothing to prove, so let $d > s$. Let $\Delta_d := \phi - \phi_d$, so $\Delta_d \in \mathfrak{m}^d$. Then $0 = F(\phi_d + \Delta_d) \equiv F(\phi_d) + F'(\phi_d)\Delta_d \pmod{\Delta_d^2}$, so in particular, $F(\phi_d)_{(d+s)} + F'(\phi_d)_{(s)}(\Delta_d)_{(d)} = 0$, or equivalently, $F(\phi_d)_{(d+s)} + F'(\phi_{s+1})_{(s)}\phi_{(d)} = 0$, and thus, $\phi_{(d)} = -\frac{F(\phi_d)_{(d+s)}}{F'(\phi_{s+1})_{(s)}}$.

The field of rational functions $k(x_1, \ldots, x_n)$ is embedded into its completion with respect to v, and by our choice of v this completion can be identified with the field of iterated Laurent series $k((x_1)) \cdots ((x_n))$. In particular, $(F'(\phi_{s+1})_{(s)})^{-1}$ becomes an iterated Laurent series, whose coefficients are polynomials over $\mathbb{Z}$ in the coefficients of $F'(\phi_{s+1})_{(s)}$ and in D^{-1} (write $F'(\phi_{s+1})_{(s)}$ as a product of its minimal valuation monomial and a rational function, then write the inverse of the rational function as a geometric series). Then, by induction assumption, the coefficients of $\phi_{(d)}$ are polynomials over $\mathbb{Z}$ in coefficients of F, coefficients of ϕ_{s+1} and in D^{-1}.

Therefore, for almost all primes $\mathfrak{p} \subset \mathcal{O}_K$, the coefficients of the power series $x_{n+1}, \ldots, x_m, f^*x_1, \ldots f^*x_n$ are integral in $K_\mathfrak{p}$. Choose a $\mathfrak{p}$ satisfying this condition and such that, moreover, the irreducible polynomials $P_i(X) = P_i(x_1, \ldots, x_n; X) \in K[x_1, \ldots, x_n; X]$ which are minimal monic polynomials of x_i for $n < i \leq m$, have $\mathfrak{p}$-integral coefficients and the elements $P_i'(0, \ldots, 0; x_i(q))$ are invertible in $\mathcal{O}_\mathfrak{p}$ for each $n < i \leq m$ (this last condition holds for almost all primes $\mathfrak{p}$ since the morphism x is étale at q).

Define the system of $\mathfrak{p}$-adic neighbourhoods $O_{\mathfrak{p},q,s}$, $s \geq 1$ of the point q as follows:

$$O_{\mathfrak{p},q,s} = \{t \in U(K_\mathfrak{p}) | x_i(t) \equiv x_i(q) \pmod{)}\mathfrak{p}^s \ for\ 1 \leq i \leq m\}.$$

We set $O_{\mathfrak{p},q} := O_{\mathfrak{p},q,1}$.

Proposition 2.2 *(1) The functions $x_1, \ldots x_n$ give a bijection between $O_{\mathfrak{p},q,s}$ and the n-th cartesian power of $\mathfrak{p}^s$.*

(2) The set $O_{\mathfrak{p},q}$ contains no indeterminacy points of f.

(3) $f(O_{\mathfrak{p},q,s}) \subset O_{\mathfrak{p},q,s}$ for $s \geq 1$. Moreover, $f : O_{\mathfrak{p},q,s} \xrightarrow{\sim} O_{\mathfrak{p},q,s}$ is bijective if $\det Df_q$ is invertible in $\mathcal{O}_\mathfrak{p}$.

(4) The $\bar{\mathbb{Q}}$-points are dense in $O_{\mathfrak{p},q,s}$.

Proof: These properties are clear from the definition and the inclusion of the elements $x_1, \ldots, x_m, f^*x_1, \ldots, f^*x_m$ into $\mathcal{O}_\mathfrak{p}[[x_1, \ldots, x_n]]$.

1. The map x from $O_{\mathfrak{p},q,s}$ to the n-th cartesian power of $\mathfrak{p}^s$ is injective, since the coordinates $x_{n+1}, \ldots, x_m$ of a point t are determined uniquely by the coordinates $x_1, \ldots, x_n$ and the condition $t \in O_{\mathfrak{p},q}$. Let $P_i(x_i) = P_i(x_1, \ldots, x_n; x_i) = 0$ be the minimal monic polynomial of x_i for $n < i \leq m$. For fixed values of $x_1, \ldots, x_n \in \mathfrak{p}$ this equation has precisely $\deg P_i$ solutions (with multiplicities) in $\bar{K}_\mathfrak{p}$. As $P_i'(x_i(q)) \in \mathcal{O}_\mathfrak{p}^\times$, $x_i(t) \equiv x_i(q) \pmod{\mathfrak{p}}$ is a simple root of P_i

(mod $\mathfrak{p}$), and thus, any root congruent to $x_i(q)$ modulo $\mathfrak{p}$ is not congruent to any other root modulo $\mathfrak{p}$.

It is surjective, since $x_{n+1}, \dots, x_m$ are convergent series on $\mathfrak{p}^s$ with constant values modulo $\mathfrak{p}^s$.

2. The functions f^*x_i, $1 \leq i \leq m$, are convergent series on $O_{\mathfrak{p},q}$.

3. The functions f^*x_i on the n-th cartesian power of $\mathfrak{p}^s$, $1 \leq i \leq m$, are constant modulo $\mathfrak{p}^s$. This shows that $f(O_{\mathfrak{p},q,s}) \subseteq O_{\mathfrak{p},q,s}$. If $\det Df_q \neq 0$ then the inverse map f^{-1} is well-defined in a neighbourhood of 0. If $\det Df_q \in \mathcal{O}_{\mathfrak{p}}^{\times}$ then f^{-1} is defined by series in $\mathcal{O}_{\mathfrak{p}}[[x_1, \dots, x_n]]$, i.e., it is well-defined on $O_{\mathfrak{p},q}$.

4. The $\bar{\mathbb{Q}}$-points are dense in the n-th cartesian power of $\mathfrak{p}^s$ and they lift uniquely to $O_{\mathfrak{p},q,s}$.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the tangent map Df_q. We assume that q is a non-degenerate fixed point of f, so that $\lambda_i \neq 0$. Note that the λ_i are algebraic numbers. Extending, if necessary, the field K, we may assume that $\lambda_i \in K$. The following is a consequence of the p-adic versions of several well-known results in dynamics and number theory:

Proposition 2.3 *Assume that $\lambda_1, \dots, \lambda_n$ are multiplicatively independent. Then in some $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},q,s}$, the map f is equivalent to its linear part Λ (i.e. there exists a formally invertible n-tuple of formal power series $h = (h^{(1)}, \dots, h^{(n)})$ in n variables $(x_1, \dots, x_n) = x$ convergent together with its formal inverse on a neighbourhood of zero such that $h(\lambda_1 x_1, \dots, \lambda_n x_n) = f(h(x_1, \dots, x_n))$.*

Proof: It is well-known that in absence of relations

$$\lambda_1^{m_1} ... \lambda_n^{m_n} = \lambda_j,\ 1 \leq j \leq n,\ m = \sum m_i \geq 2,\ m_i \geq 0$$

("resonances"), there is a unique formal linearization of f, obtained by formally solving the equation $f(h(x)) = h(\Lambda(x))$; the expressions $\lambda_1^{m_1} ... \lambda_n^{m_n} - \lambda_j$ appear in the denominators of the coefficients of h (see for example [Arn]). The problem is of course whether h has non-zero radius of convergence, that is, whether the denominators are "not too small". By Siegel's theorem (see [HY]) for its p-adic version) this is the case as soon as the numbers λ_i satisfy the diophantine condition

$$|\lambda_1^{m_1} ... \lambda_n^{m_n} - \lambda_j|_p > Cm^{-\alpha}$$

for some C, α. By [Yu] , this condition is always satisfied by algebraic numbers.

When the fixed point q is not isolated, the eigenvalues λ_i are always resonant. However, as follows from the results proved in the Appendix, if "all resonances come from the fixed subvariety", the linearization is still possible.

More precisely, extending K if necessary, we may choose $q \in X(K)$ which is a smooth point of the fixed point locus of f. Let F be the irreducible component of this locus containing q. Let $r = \dim F$. By a version of Noether's normalization lemma [E, Theorem 13.3 & geometric interpretation on p.284], we may assume that our finite morphism $\pi = (x_1, \dots, x_n) : U \longrightarrow \mathbb{A}^n_K$ which is étale at q, maps $F \cap U$ onto the coordinate plane $\{x_{r+1} = \cdots = x_n = 0\}$.

Proposition 2.4 *Let q be a general fixed point of f as above. Suppose that the tangent map Df_q is semisimple and that its eigenvalues $\lambda_1, \dots, \lambda_n$ satisfy the condition $\lambda_{r+1}^{m_{r+1}} \cdots \lambda_n^{m_n} \neq \lambda_i$ for all integer $m_{r+1}, \dots, m_n \geq 0$ with $m_{r+1} + \cdots + m_n \geq 2$ and all i, $r < i \leq n$ (and $\lambda_1 = \cdots = \lambda_r = 1$).*

Suppose that the eigenvalues of Df_q do not vary with q. Then, for each $\mathfrak{p}$ as above, the map f can be linearized in some $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},q,s}$ of q, i.e., there exists a formally invertible n-tuple of formal power series $h = (h^{(1)}, \dots, h^{(n)})$ in n variables $(x_1, \dots, x_n) = x$ convergent together with its formal inverse on a neighbourhood of zero such that $h(\lambda_1 x_1, \dots, \lambda_n x_n) = f(h(x_1, \dots, x_n))$.

Proof: Taking into account that the λ_i are algebraic numbers and so the diophantine condition

$$|\lambda_{r+1}^{m_{r+1}} \cdots \lambda_n^{m_n} - \lambda_j|_p > C(\sum_{r<i\leq n} m_i)^{-\alpha}$$

is automatically satisfied for some $C, \alpha > 0$, whenever $\sum_{r<i\leq n} m_i \geq 2$, this is just Theorem 4.1.

In order to apply these propositions to the study of iterated orbits of algebraic points, we need the following lemma.

Lemma 2.5 *Let $a_1, a_2, \dots$ be a sequence in $K_{\mathfrak{p}}^{\times}$ tending to 0. Let $b_1, b_2, \dots$ be a sequence of pairwise distinct elements in $\mathcal{O}_{\mathfrak{p}}^{\times}$ generating a torsion-free subgroup. Then any infinite subset $S \subset \mathbb{N}$ contains an element s such that $\sum_{i\geq 1} a_i b_i^s \neq 0$.*

Proof: Renumbering if necessary, we may suppose that $|a_1| = |a_2| = \cdots = |a_N| > |a_i|$ for any $i > N$. Suppose that $\sum_{i\geq 1} a_i b_i^s = 0$ for every $s \in S$.

First assume that $S = \mathbb{N}$. It follows that for any polynomial P, $\sum_{i\geq 1} a_i P(b_i) = 0$. By the triangular inequality, we'll get a contradiction as soon as we find a polynomial P such that $|P(b_k)| < |P(b_1)|$ for $2 \leq k \leq N$ and $|P(b_k)| \leq |P(b_1)|$ for $k > N$. To construct such a P, choose an ideal $\mathfrak{q} = \mathfrak{p}^s$ such that the b_i are different modulo $\mathfrak{q}$ for $i = 1, \dots, N$ and let

$$P(x) = \prod_{b \in \mathcal{O}_{\mathfrak{p}}/\mathfrak{q},\ b_1 \notin b} (x - \bar{b}),$$

where $\bar{b}$ denotes any representative of the class b. An easy check gives that $|P(x)| = |P(b_1)|$ when $x \equiv b_1 \pmod{\mathfrak{q}}$ and $|P(x)| < |P(b_1)|$ otherwise, so P has the required properties.

(This polynomial P has been indicated to us by A. Chambert-Loir.)

Now let $S \subset \mathbb{N}$ be arbitrary. Take an integer M such that $|b_i^M - 1| < |p|^{\frac{1}{p-1}}$ for all i (this is possible since b_i are in $\mathcal{O}_{\mathfrak{p}}^{\times}$ and the residue field is finite), then $c_i = \log b_i^M$ is defined for all i and $\exp(c_i) = b_i^M$. Since the subgroup generated by the b_i is torsion-free, all c_i are different. We claim that for certain coefficients a'_i, we can write identities $\sum_{i \geq 1} a'_i c_i^m = 0$ for all $m \in \mathbb{N}$, so this case reduces to that of $S = \mathbb{N}$.

Indeed, consider S as a subset of $\mathbb{Z}_p$; it has a limit point s_0. For a sequence of m-tuples $j_1 > j_2 > \cdots > j_m$ in S such that $j_1 \equiv j_2 \equiv \cdots \equiv j_m \pmod{M}$ and $\lim j_1 = \lim j_2 = \cdots = \lim j_m = s$, and any analytic function g on $\mathbb{Z}_p$, one has

$$\frac{g^{(m)}(s_0)}{m!} = \lim \sum_{l=1}^{m} \frac{g(j_l)}{\prod_{k \neq l}(j_l - j_k)}$$

(Newton interpolation formula).

Pick a class modulo M which contains a sequence in S converging to s_0. There is an analytic function g_i such that $g_i(k) = b_i^k$ when k is in this class. By the formula above, we have

$$g_i^{(m)}(s_0) = \frac{c_i^m}{M^m} g_i(s_0) = m! \lim \sum_{l=1}^{m} \frac{g(j_l)}{\prod_{k \neq l}(j_l - j_k)},$$

where all the j_l are in the same class modulo M. This gives

$$\sum_{i=1}^{\infty} a_i g_i(s_0) c_i^m = M^m m! \lim \sum_{l=0}^{m} \frac{1}{\prod_{k \neq l}(j_l - j_k)} \sum_{i=1}^{\infty} a_i g_i(j_l) = 0$$

since the j_l are in S, q.e.d.

From now on, we assume that $\mathfrak{p}$ is chosen such that all λ_i belong to $\mathcal{O}_{\mathfrak{p}}^{\times}$ (this is of course the case for almost all $\mathfrak{p}$.

The first part of the lemma (case $S = \mathbb{N}$) immediately implies the following corollary:

Corollary 2.6 *If $\lambda_1, \ldots, \lambda_n$ are multiplicatively independent, the rational points on X are potentially dense.*

Proof: since algebraic points are dense in $O_{\mathfrak{p},q,s}$, we can find a point $x \in X(\bar{\mathbb{Q}})$ which is contained in $O_{\mathfrak{p},q,s}$, away from the coordinate hyperplanes in the local coordinates $(y_1, \ldots y_n)$ linearizing f. We claim that the iterated orbit of this point is Zariski dense in X. Indeed, if not, there is a regular function G on U vanishing on $f^i(x)$ for all i; in the local linearizing coordinates on $O_{\mathfrak{p},q,s}$, G becomes a convergent power series $G = \sum_I a_I y^I$. If $x = (x_1, \ldots x_n)$, we get $\sum_I a_I x^I (\lambda^I)^i = 0$ for all $i \in \mathbb{N}$. Since the λ_i are multiplicatively independent, the numbers λ^I are distinct, contradicting lemma 2.5.

Another useful version of this corollary is the following:

Corollary 2.7 *Let n be a natural number, T be the n-th cartesian power of $\mathcal{O}_{\mathfrak{p}}^{\times}$ and $\Lambda = (\lambda_1, \dots, \lambda_n) \in T$, where $\lambda_1, \dots, \lambda_n$ are multiplicatively independent. Let $S \subset \mathbb{N}$ be an infinite subset. Then the set $\{\Lambda^i \mid i \in S\}$ is "analytically dense" in T, i.e., for any non-zero Laurent series $F = \sum_I a_I y^I$ convergent on T there is $s \in S$ such that $F(\Lambda^s) \neq 0$.*

Proof. Otherwise, after a renumbering of a_I as a_i and setting $b_i = \lambda^I$ for i corresponding to I, we get $\sum_{i \geq 1} a_i b_i^s = 0$ for all $s \in S$. This contradicts Lemma 2.5.

The case $S \neq \mathbb{N}$ yields the following remark which might be useful in the study of the case when the fixed point q is not isolated (as in the next section).

Corollary 2.8 *Under assumptions of Proposition 2.4, let $Y \subset U$ be such an irreducible subvariety that, possibly after a finite field extension, $Y(K_{\mathfrak{p}})$ meets a sufficiently small $\mathfrak{p}$-adic neighbourhood of q. Suppose that the multiplicative subgroup $H \subset \mathcal{O}_{\mathfrak{p}}^{\times}$ generated by $\lambda_1, \dots, \lambda_n$ is torsion-free. Let $S \subset \mathbb{N}$ be an infinite subset. Then the Zariski closure of the union $\bigcup_{i \in S} f^{\circ i}(Y)$ is independent of S, and therefore, is irreducible.*

Proof. To check the independence of S, let us show that any regular function F on U vanishing on $\bigcup_{i \in S} f^{\circ i}(Y)$ vanishes also on $\bigcup_{i \geq 1} f^{\circ i}(Y)$.

Let $y_1, \dots, y_n$ be local coordinates at q linearizing and diagonalizing f on a neighbourhood $O_{\mathfrak{p},q,s}$. Then for any choice of a sufficiently large number field K, $f^{\circ i}(Y)(K_{\mathfrak{p}}) \cap O_{\mathfrak{p},q,s}$ contains $f^{\circ i}(Y(K_{\mathfrak{p}}) \cap O_{\mathfrak{p},q,s})$, and therefore, each irreducible component of the Zariski closure of $\bigcup_{i \in S} f^{\circ i}(Y)$ meets $O_{\mathfrak{p},q,s}$.

This implies that we can work in $O_{\mathfrak{p},q,s}$, where F becomes a convergent power series $\sum_I a_I y^I$. For any point $t = (t_1, \dots, t_n) \in Y(K_{\mathfrak{p}}) \cap O_{\mathfrak{p},q,s}$ the series F vanishes at the points $f^{\circ i}(t)$ for all $i \in S$ (i.e., $\sum_I a_I \lambda^{iI} t^I = 0$ for all $i \in S$) and it remains to show that F vanishes also at the points $f^{\circ i}(t)$ *for all* $i \geq 1$.

Let us work only with multi-indices I such that t^I is non-zero. The set of such multi-indices splits into the following equivalence classes: $I \sim I'$ if $\lambda^I = \lambda^{I'}$. By Lemma 2.5, the sum of $a_I t^I$ over each equivalence class is zero, and thus, $\sum_I a_I \lambda^{iI} t^I = 0$ for all $i \geq 1$.

Finally, the following is an obvious generalization of 2.6:

Corollary 2.9 *Under the assumptions of Proposition 2.4, let t be a sufficiently general algebraic point of U in a sufficiently small $\mathfrak{p}$-adic neighbourhood of q and H be the multiplicative group generated by $\lambda_1, \dots, \lambda_n$. Then the dimension of the Zariski closure of the f-orbit $\{f^{\circ i}(t) \mid i \in \mathbb{N}\}$ is greater than or equal to $r = rank(H)$.*

Proof. Let $y_1, \dots, y_n$ be local coordinates at q linearizing and diagonalizing f in a neighbourhood $O_{\mathfrak{p},q,s}$. Replacing f by a power, we may assume that the eigenvalues λ_i generate a torsion-free group. Take a point $t = (t_1, \dots, t_n) \in O_{\mathfrak{p},q,s}$ away from the coordinate hyperplanes and consider the natural embedding of the n-th cartesian power $\mathcal{T}$ of $\mathcal{O}_{\mathfrak{p}}^{\times}$:

$$t : \mathcal{T} \to O_{\mathfrak{p},q,s}, \ (\mu_1, \dots, \mu_n) \mapsto (\mu_1 t_1, \dots, \mu_n t_n).$$

On the image of $\mathcal{T}$, we can find a monomial coordinate change such that in the new coordinates $y_1', \dots, y_n'$ one has

$$f(y_1', \dots, y_n') = (\lambda_1' y_1', \dots, \lambda_r' y_r', y_{r+1}', \dots y_n')$$

and λ_i', $1 \le i \le r$ are multiplicatively independent (Euclid's algorithm). Now apply Corollary 2.7 (notice that since it is about Laurent series, we can allow monomial coordinate changes as above).

3 Variety of lines of the cubic fourfold

The difficulty in using the results of the previous section to prove potential density of rational points is that it can be hard to find an interesting example such that the eigenvalues of the tangent map at some fixed point are multiplicatively independent. For instance if f is an automorphism and X is a projective $K3$ surface, or, more generally, an irreducible holomorphic symplectic variety, then the product of the eigenvalues is always a root of unity, as noticed for instance in [Bv].

So even when a linearization in the neighbourhood of a fixed point is possible, the orbit of a general algebraic point may be contained in a relatively small analytic subvariety of the neighbourhood (of course this subvariety does not have to be algebraic, but it is unclear how to prove that it actually is not). Nevertheless, with some additional geometric information, one can still follow this approach to prove the potential density.

In the rest of this note, we illustrate this by giving a simplified proof of the potential density of the variety of lines of a cubic fourfold, which is the main result of [AV]. The proof uses several ideas from [AV], but we think that certain aspects become more transparent thanks to the introduction of the "dynamical" point of view and the use of $\mathfrak{p}$-adic neighbourhoods.

We recall the setting of [AV] (the facts listed below are taken from [V1] and [A]). Let V be a general smooth cubic in $\mathbb{P}^5$ and let $X \subset G(1,5)$ be the variety of lines on V. This is an irreducible holomorphic symplectic fourfold: $H^{2,0}(X)$ is generated by a nowhere vanishing form σ. For $l \subset V$ general, there is a unique plane P tangent to X along l (consider the Gauss map, it sends l to a conic in the dual projective space). The map f maps l to the residual line l'. It multiplies the form σ by -2; in particular, its degree is 16. The indeterminacy locus S consists of points such that the image of the corresponding line by the Gauss map is a line (and the mapping is 2:1). This is a smooth surface of general type, resolved by a single blow-up. For a general X, the Picard group is cyclic and thus the Hodge structure on $H^2(X)^{prim}$ is irreducible (thanks to $h^{2,0}(X) = 1$); the space of algebraic cycles is generated by $H^2 = c_1^2(U^*)$ and $\Delta = c_2(U^*)$, where U is the restriction of $U_{G(1,5)}$, the universal rank-two bundle on $G(1,5)$. By Terasoma's theorem [T], these conditions are satisfied by a "sufficiently general" X defined over a number field, in fact even over $\mathbb{Q}$; "sufficiently general" meaning "outside of a thin subset in the parameter space". One computes that the cohomology class of S is $5(H^2 - \Delta)$ to conclude that S is irreducible and non-isotropic with respect to σ.

3.1 Fixed points and linearization

The fixed point set F of our rational self-map $f : X \dashrightarrow X$ is the set of points such that along the corresponding line l, there is a tritangent plane to V. Strictly speaking, this is the closure of the fixed point set, since some of such points are in the indeterminacy locus; but for simplicity we shall use the term "fixed point set" as far as there is no danger of confusion.

Proposition 3.1 *The fixed point set F of f is an isotropic surface of general type.*

Proof: It is clear from $f^*\sigma = -2\sigma$ that F is isotropic. Let $I \subset G(1,5) \times G(2,5)$ with projections p_1, p_2 be the incidence variety $\{(l,P) | l \subset P\}$ and let $\mathcal{F} \subset I \times \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ denote the variety of triples $\{(l,P,V) | V \cup P = 3l\}$. This is a projective bundle over I, so $\mathcal{F}$ is smooth and thus its fiber F'_V over a general $V \in \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ is also smooth. This fiber clearly projects generically one-to-one on the corresponding $F = F_V$, since along a general line $l \subset V$ there is only one tangent plane, and a fortiori only one tritangent plane if any; so $F' = F'_V$ is a desingularization of F. Since $dim(I) = 11$ and since intersecting the plane P along the triple line l imposes 9 conditions on a cubic V, we conclude that F' and F are surfaces.

To compute the canonical class, remark that F' is the zero locus of a section of a globally generated vector bundle on I. This vector bundle is the quotient of $p_2^* S^3 U^*_{G(2,5)}$ (where $U_{G(2,5)}$ denotes the tautological subbundle on $G(2,5)$) by a line subbundle $\mathcal{L}_3$ whose fiber at (l,P) is the space of degree 3 homogeneous polynomials on P with zero locus l. One computes that the class of $\mathcal{L}_3$ is three times the difference of the inverse images of the Plücker hyperplane classes on $G(2,5)$ and $G(1,5)$, and it follows that the canonical class of F is $p_2^*(3c_1(U^*))$, which is ample (we omit the details since an analogous computation is given in [V], and a more detailed version of it in [Pc]).

Remark 3.2 *Since F is isotropic and S is not, S cannot coincide with a component of F. In fact, dimension count shows that $F \cap S$ is a curve.*

Proposition 3.3 *For a general (that is, non-singular and out of the indeterminacy locus) point $q \in F$, the tangent map Df_q is diagonalized with eigenvalues* $1, 1, -2, -2$.

Proof: This follows from the fact that $f^*\sigma = -2\sigma$. and the fact that the map is the identity on the lagrangian plane $T_pF \subset T_pX$. Let e_1, e_2, e_3, e_4 be the Jordan basis with $e_1, e_2 \in T_pF$. There is no Jordan cell corresponding to the eigenvalue 1, since in this case e_4 would be an eigenvector with eigenvalue 4, but then $\sigma(e_1, e_4) = \sigma(e_2, e_4) = \sigma(e_3, e_4) = 0$, contradicting the fact that σ is non-degenerate. By the same reason, the eigenvalues at e_3 and e_4 are both equal to ± 2. Suppose that Df_q is not diagonalized, so sends e_3 to $\pm 2e_3$ and e_4 to $e_3 \pm 2e_4$. In both cases $\sigma(e_3, e_4) = 0$. If e_3 goes to $2e_3$, we immediately see that $e_3 \in Ker(\sigma)$, a contradiction. Finally, if $Df_q(e_3) = -2e_3$ and $Df_q(e_4) = e_3 - 2e_4$, we have

$$-2\sigma(e_1, e_4) = \sigma(e_1, e_3) - 2\sigma(e_1, e_4),$$

so that $\sigma(e_1, e_3) = 0$, but by the same reason $\sigma(e_2, e_3) = 0$, again a contradiction to non-degeneracy of σ.

Proposition 3.4 *(1) Let $q \in X(K)$ be a general fixed point of f as above and let $O_{\mathfrak{p},q}$ be its $\mathfrak{p}$-adic neighbourhood for a suitable $\mathfrak{p}$, as in the previous section. Then f is equivalent to its linear part in a sufficiently small subneighbourhood $O_{\mathfrak{p},q,s}$; that is, there exist power series $h = h_q$ in four variables $(t_1, t_2, t_3, t_4) = t$ such that $h(t_1, t_2, -2t_3, -2t_4) = f \circ h(t)$, convergent together with its inverse in some neighbourhood of zero.*

(2) In the complex setting, the analogous statements are true. Moreover, the maps h_{t_1,t_2}, where $h_{t_1,t_2}(x, y) = h(t_1, t_2, x, y)$ extend to global meromorphic maps from $\mathbb{C}^2$ to X.

Proof: 1) Since the couple of non-trivial eigenvalues $(-2, -2)$ is non-resonant, this is just the Proposition 2.4.

2) In the complex case, the linearization is a variant of a classical result due to Poincare [P]. One writes the formal power series in the same way as in the p-adic setting, thanks to the absence of the resonances; but it is much easier to prove its convergence thanks to the fact that now $\lambda_3 = \lambda_4 = -2$ and $|-2| > 1$, and so the absolute values of the denominators which appear when one computes the formal power series are bounded from below (these denominators are in fact products of the factors of the form $\lambda_3^{m_3}\lambda_4^{m_4} - \lambda_i$ for $m_3 + m_4 \geq 2$, $m_3, m_4 \geq 0$). For the sake of brevity, we refer to [Rg] which proves the analogue of Theorem 4.1, case (2), in the complex case and under a weaker diophantine condition on the eigenvalues (Rong assumes moreover that $|\lambda_i| = 1$, but as we have just indicated, in our case all estimates only become easier, going back to [P]).

To extend the maps h_{t_1,t_2} to $\mathbb{C}^2$, set

$$h_{t_1,t_2}(x) = f^k(h_{t_1,t_2}((-2)^r x)),$$

where $(-2)^r x$ is sufficiently close to zero; one checks that this is independent of choices.

We immediately get the following corollary (which follows from the results of [AV], but for which there was as yet no elementary proof):

Corollary 3.5 *There exist points in $X(\bar{\mathbb{Q}})$ which are not preperiodic for f.*

Proof: Indeed, $\bar{\mathbb{Q}}$-points are dense in $O_{\mathfrak{p},q}$. Take one in a suitable invariant subneighbourhood and use the linearization given by the proposition above.

Remark 3.6 *If f were regular, this would follow from the theory of canonical heights; but this theory does not seem to work sufficiently well for polarized rational self-maps.*

3.2 Non-preperiodicity of certain surfaces

The starting point of [AV] was the observation that X is covered by a two-parameter family $\Sigma_b, b \in B$ of birationally abelian surfaces, namely, surfaces parametrizing lines contained in a hyperplane section of V with 3 double points. On a general X, a general such surface has cyclic Neron-Severi group ([AV]); moreover, many of those surfaces Σ defined over a number field have the same property, as shown by an argument similar to that of Terasoma [T]. In fact, given a “general” X defined over a number field, the set of such surfaces on X whose Neron-Severi group is not cyclic is thin.

In [AV], it is shown that the iterates of a suitable Σ defined over a number field and with cyclic Neron-Severi group are Zariski-dense in X. The first step is to prove non-preperiodicity, that is, the fact that the number of $f^k(\Sigma)$, $k \in \mathbb{N}$, is infinite. Already at this stage the proof is highly non-trivial, using the l-adic Abel-Jacobi invariant in the continuous étale cohomology.

In this subsection, we give an elementary proof of the non-preperiodicity of a suitable Σ, which is based on proposition 3.4. Moreover, this works without an assumption on its Néron-Severi group, and also for any X, not only for a “general” one.

Lemma 3.7 *The surface Σ is never invariant by f.*

Proof: The surface Σ is the variety of lines contained in the intersection $Y = V \cap H$, where H is a hyperplane in $\mathbb{P}^5$ tangent to V at exactly three points. For a general line l corresponding to a point of Σ, there is a unique plane P tangent to V along l, and the map f sends l to the residual line l'. If Σ is invariant, l' and therefore P lie in H, and P is tangent to Y along l. But this means that l is ”of the second type” on Y in the sense of Clemens-Griffiths (i.e. the Gauss map of $Y \subset H = \mathbb{P}^4$ maps l to a line in $(\mathbb{P}^4)^*$ as a double covering), see [CG]. At the same time it follows from the results of [CG] that a general line on a cubic threefold with double points is ”of the first type” (mapped bijectively onto a conic by the Gauss map), a contradiction.

Passing to the p-adic setting and taking a Σ meeting a small neighbourhood of a general fixed point q of f, we see by corollary 2.8 that the Zariski closure of $\cup_k f^k(\Sigma)$

is irreducible. Since Σ cannot be f-invariant by the lemma above, this means that Σ is not preperiodic and so the Zariski closure D of $\cup_k f^k(\Sigma)$ is at least a divisor.

Coming back to the complex setting and taking a Σ passing close to q in both p-adic and complex topologies, let us make a few remarks on the geometry of D.

In a neighbourhood of our fixed point q, the intersections of D with the images of h_{t_1,t_2} are f-invariant analytic subsets. From the structure of f as in Proposition 3.4 we deduce that such a subset is either the whole image of h_{t_1,t_2}, or a finite union of "lines through the origin" (that is, images of such lines by h_{t_1,t_2}). If the last case holds generically, D must contain F by dimension reasons. If the generic case is the first one, D might only have a curve in common with F.

To sum up, we have the following

Theorem 3.8 *The Zariski closure D is of dimension at least three. If it is of dimension three, this is an irreducible divisor which either contains the surface of fixed points F, or has a curve in common with F. In this last case, D contains correspondent "leaves" (images of $\mathbb{C}^2$ from 3.4) through the points of this curve.*

3.3 Potential density

In this subsection, we exclude the case when D is a divisor.

Let $\mu : \tilde{D} \to X$ denote a desingularization; $\tilde{D}$ is equipped with a rational self-map $\tilde{f}$ satisfying $\mu\tilde{f} = f\mu$.

Our proof is a case-by-case analysis on the Kodaira dimension of D. In [AV] , we already have simple geometric arguments ruling out the cases of $\kappa(D) = -\infty$ and $\kappa(D) = 0$. The case $\kappa(D) = -\infty$ is especially simple since then the holomorphic 2-form would be coming from the rational quotient of D, but Σ obviously must dominate the rational quotient and this cannot be isotropic. The case $\kappa(D) = 0$ is less easy and uses the fact that $Pic(X) = \mathbb{Z}$ or, equivalently, that the Hodge structure $H^2(X)^{prim}$ is irreducible of rank 22. Namely, an argument using Minimal Model theory and the existence of an holomorphic 2-form on D gives that D must be rationally dominated by an abelian threefold or by a product of a K3 surface with an elliptic curve. But the second transcendental Betti number of those varieties cannot exceed 21, which contradicts the fact that $\tilde{D}$ carries an irreducible Hodge substructure of rank 22; see [AV] for details.

Let us deal with the case $\kappa(D) = 2$. We need the following lemma:

Lemma 3.9 *On a general X, the points of period 3 with respect to f form a curve.*

Proof: Let l_1 be (a line corresponding to) such a point, $l_2 = f(l_1)$, $l_3 = f^2(l_1)$, so that $f(l_3) = l_1$. There are thus planes P_1, P_2, P_3, such that P_1 is tangent to V along l_2 and contains l_3, etc. Clearly, $P_1 \neq P_2 \neq P_3$. The span of the planes P_j is a projective 3-space Q. Let us denote the two-dimensional cubic, intersection of V and Q, by W. We can choose the coordinates $(x : y : z : t)$ on Q such that l_1 is given by $y = z = 0$, etc. Then the intersection of W and P_1 is given by the equation $z^2y = 0$, etc. The only other monomial from the equation of W, up to a constant, can be xyz, since it has to be divisible by the three coordinates. Therefore W is a cone (with vertex at 0) over the cubic given by the equation

$$ax^2y + by^2z + cy^2z + dxyz = 0$$

in the plane at infinity. Now a standard dimension count ([A]) shows that a general cubic admits a one-parameter family of two-dimensional linear sections which are cones. Each cone on V gives rise to a plane cubic on X. This cubic is invariant under f, and f acts by multiplication by -2 (for a suitable choice of zero point). The points of period 3 with respect to f lie on such cubic and are their points of 9-torsion.

Remark 3.10 *In fact the lemma says slightly more: it applies to the indeterminacy points which are "3-periodic in the generalized sense", that is, points appearing if one replaces the condition "$f(l_1) = l_2$, $f(l_2) = l_3$, $f(l_3) = l_1$" by "$l_2 \in f(l_1)$, $l_3 \in f(l_2)$, $l_1 \in f(l_3)$"; here by $f(l_1)$ we mean the rational curve which is the image of l_1 by the correspondence which is the graph of f (equivalently, $l_2 \in f(l_1)$ says that for some plane P_3 tangent to V along l_1, the residual line in $P_3 \cap V$ is l_2).*

By blowing-up $\tilde{D}$, we may assume that the Iitaka fibration $\tilde{D} \to B$ is regular. Its general fiber is an elliptic curve. By [NZ], the rational self-map $\tilde{f}$ descends to B and induces a transformation of finite order, so the elliptic curves are invariant by a power of $\tilde{f}$. From proposition 3.4, we obtain that they are in fact invariant by $\tilde{f}$ itself: indeed, locally in a neighbourhood of a fixed point, the curves invariant by f are the same as the curves invariant by its power. On a general elliptic curve, there is a finite (non-zero) number of points of period three, since $\tilde{f}$ acts as multiplication by -2. We have two possibilities:

1) These are mapped to points of period three (in the "generalized sense" as in the 3.10) on X (or the surface they form is contracted to any other curve on X). Then any preimage of our surface by an iteration of $\tilde{f}$ is contracted as well, but since there are infinitely many of them, this is impossible.

2) This surface dominates a component of the surface of fixed points of f. In this case, several points of period three must collapse to the same fixed point p. But then the resulting branches of each elliptic curve near the generic fixed point are interchanged by f, which contradicts the local description of f in 3.4.

This rules out the possibility $\kappa(D) = 2$.

Finally, let us consider the case $\kappa(D) = 1$. The Iitaka fibration $\tilde{D} \to C$ maps $\tilde{D}$ to a curve C and the general fiber U is of Kodaira dimension 0. As before, by [NZ] $\tilde{f}$ induces a finite order automorphism on C, and one deduces from 3.4 that this is in fact the identity. We have two possible cases:

Case 1: U is not isotropic with respect to the holomorphic 2-form σ. We use the idea from [AV] as in the case $\kappa(D) = 0$. Namely, since X is generic, the Hodge structure $H^2_{prim}(X,\mathbb{Q})$ is simple.Since the pull-back of σ to $\tilde{D}$ is non-zero, $H^2(\tilde{D},\mathbb{Q})$ carries a simple Hodge substructure of rang 22. Since U is non-isotropic, the same is true for U, but a surface of Kodaira dimension zero never satisfies this property.

Case 2: U is isotropic with respect to σ. The kernel of the pull-back σ_D of σ to $\tilde{D}$ gives a locally free subsheaf of rank one in the tangent bundle $T_{\tilde{D}}$, which is in fact a subsheaf of T_U since U is isotropic. There is thus a foliation in curves on U, and this foliation has infinitely many algebraic leaves (these are intersections of U with the iterates of our original surface Σ). By Jouanolou's theorem, this is a fibration. In other words, D is fibered over a surface T in integral curves of the kernel of σ_D, and U project to curves. These cannot be rational curves since the surface T is not uniruled (indeed, the form σ_D must be a lift of a holomorphic 2-form on T). Therefore these are elliptic curves, and since $\kappa_U = 0$, so are the fibers of $\pi : D \to T$.

Recall from 3.4 that either D contains F, or it contains a curve on F; and in this last case, locally near generic such point, D is a fibration in (isotropic) two-dimensional disks over a curve; in particular, such a point is a smooth point of D. If D contains F, we get a contradiction with 3.1: indeed, F must be dominated by a union of fibers of π, but F is of general type and the fibers are elliptic. If D contains a curve on F, then we look at the "leave" (image of $\mathbb{C}^2$ from the Proposition 3.4) at a general point q of this curve. Its intersection with the image of U is an invariant curve, that is, the image of a line through the origin. Since U and the leave are both isotropic, this must be an integral curve of the kernel of the restriction of σ to D. But U varies in a family, and this implies that the restriction of σ to D is zero at q, a contradiction since σ is non-degenerate.

We thus come to a conclusion that D cannot be a divisor, so $D = X$.

Remark 3.11 *Here, unlike in the proof of non-preperiodicity, we do assume that X is "sufficiently general" and so $Pic(X) = \mathbb{Z}$. It would be interesting to check whether one can modify the argument to get rid of this assumption.*

4 Appendix: a version of Siegel's theorem

In this appendix we explain how to modify the proof of Siegel's theorem on linearization of p-adic diffeomorphisms given in [HY, Theorem 1, §4, p.423] in order to adapt it to the situation where the fixed point is not isolated.

Let k be a complete non-archimedian field and $n > r \geq 0$ be integers.

Theorem 4.1 *Let $f = (f^{(1)}, \ldots, f^{(n)})$ be an analytic diffeomorphism of an n-dimensional domain. Assume that the fixed set of f is r-dimensional and that the tangent maps of f are semisimple at all fixed points of f. Suppose, moreover, that the eigenvalues of df distinct from 1 (at fixed points of f) are either*

1. *equal and are not roots of unity at general fixed point, or*

2. *constant and, if denoted by $\lambda_{r+1},\dots,\lambda_n$, satisfy the following bad diophantine approximation property: $|\lambda_{r+1}^{i_{r+1}}\cdots\lambda_n^{i_n}-\lambda_j|_k \geq C(i_{r+1}+\cdots+i_n)^{-\beta}$ for some $C>0$ and $\beta\geq 0$, any $r<j\leq n$ and $(i_{r+1},\dots,i_n)\in\mathbb{Z}_{\geq 0}^{n-r}$ such that $i_{r+1}+\cdots+i_n\geq 2$.*

Then there exist coordinates $x_1,\dots,x_n$ in a neighbourhood of a general fixed point of f such that $f^{(i)}(x)=\lambda_i(x_1,\dots,x_r)x_i$ for all $1\leq i\leq n$, where $\lambda_1=\cdots=\lambda_r=1$ (and in the second case $\lambda_i(x_1,\dots,x_r)$ are constant for all $1\leq i\leq n$).

The desired coordinates are constructed, using Newton's methods, as limits in appropriate spaces of certain sequence of approximations $h_0,h_1,h_2,\dots,$ which are diffeomorphisms of the ρ_i-neighbourhood of a general fixed point of f for a decreasing sequence of radii $\rho_1>\rho_2>\rho_3>\cdots>\rho_\infty>0$.

After a preliminary step (Corollaries 4.3, 4.4), the procedure is the same as in [HY], but instead of working in the spaces $A^2_\rho(k^n)$ and $B^2_\rho(k^n)$ of loc. cit. we work in smaller (when $r>0$) spaces $A^{(r)}_\rho(k^n)$ and $B^{(r)}_\rho(k^n)$, cf. below.

Lemma 4.2 *Let f be an analytic diffeomorphism of an n-dimensional domain, F be the fixed set of f, and q be a general point of F. Assume that F is r-dimensional and that only r eigenvalues of the tangent map of f at q are equal to 1. Then there exist coordinates $x_1,\dots,x_n$ in a neighbourhood of q such that $x_{r+1},\dots,x_n$ generate the (f-invariant) ideal I_F of F and $f^{(i)}(x)\equiv x_i \pmod{I_F^2}$ for all $1\leq i\leq r$.*

Proof. We split the coordinates into two groups: $x':=(x_1,\dots,x_r)$ and $x'':=(x_{r+1},\dots,x_n)$. Then $f(x)=x+\sum_{I\in\mathbb{Z}^{n-r}_{\geq 0}:\ |I|\geq 1}a_I(x')(x'')^I$ for some analytic $a_I(x')$, so $f(x)\equiv(x'+\sum_{I\in\{0,1\}^{n-r}:\ |I|=1}a'_I(x')(x'')^I;x''+\sum_{I\in\{0,1\}^{n-r}:\ |I|=1}a''_I(x')(x'')^I \pmod{I_F^2}$, so $f\equiv(x'+a'x'';x''+a''x'') \pmod{I_F^2}$, where $a'=a'(x')$ is the matrix with columns $a'_I(x')$ for $|I|=1$ and $a''=a''(x')$ is the matrix with columns $a''_I(x')$ for $|I|=1$. Let $h(x)=(x'+a'(a'')^{-1}x'';x'')$. Then $h^{-1}(x)\equiv(x'-a'(a'')^{-1}x'';x'') \pmod{I_F^2}$, $f(h(x))\equiv(x'+a'(a'')^{-1}x''+a'x'';x''+a''x'')=(x'+a'(1+(a'')^{-1})x'';x''+a''x'')$, and finally, $h^{-1}(f(h(x)))\equiv(x';x''+a''x'') \pmod{I_F^2}$.

Corollary 4.3 *In the setting of Lemma 4.2, assume that* (i) *the tangent maps of f are semisimple at all points of F and* (ii) *their eigenvalues distinct from 1 are equal. Then there exist coordinates $x_1,\dots,x_n$ in a neighbourhood of a general point of F such that $x_{r+1},\dots,x_n$ generate the ideal I_F of F, $f^{(i)}(x)\equiv x_i \pmod{I_F^2}$ for all $1\leq i\leq r$ and $f^{(i)}(x)\equiv\lambda(x')x_i \pmod{I_F^2}$ for all $r<i\leq n$.*

Proof. In the setting of the proof of Lemma 4.2, the matrix a'' is diagonalizable with the same eigenvalues, so it is already diagonal.

Corollary 4.4 *In the setting of Lemma 4.2, assume that* (i) *the tangent maps of f are semisimple at all points of F and* (ii) *their eigenvalues $\lambda_1,\dots,\lambda_n$ do not vary.*

Then there exist coordinates $x_1, \dots, x_n$ in a neighbourhood of a general point of F such that $x_{r+1}, \dots, x_n$ generate the ideal I_F of F and $f^{(i)}(x) \equiv \lambda_i x_i \pmod{I_F^2}$ for all $1 \le i \le n$.

Proof. In the setting of the proof of Lemma 4.2, there is transform of the coordinates $x_1, \dots, x_n$, identical on $x_1, \dots, x_r$, which is linear on $x_{r+1}, \dots, x_n$ with coefficients in functions of $x_1, \dots, x_r$ making the matrix diagonal.

Namely, after a k-linear change of variables we can assume that a'' is diagonal at q. Let $p_i := \prod_{j:\ \lambda_j \neq \lambda_i} (\lambda_i - \lambda_j)^{-1}(a'' - \lambda_j)$ be a projector onto the λ_i-eigenspace of a''. If $\{e_{r+1}, \dots, e_n\}$ is an eigenbasis of $a''(q)$, considered as sections of the restriction of the tangent bundle to F, then $\{E_i := p_i e_i\}_{r<i\le n}$ is a system of eigenvectors of a'' and its reduction modulo the maximal ideal in $k[[x_1, \dots, x_r]]$ is the eigenbasis $\{e_{r+1}, \dots, e_n\}$ of $a''(q)$. This means that $E_1, \dots, E_n$ generate the tangent bundle, cf. [AtM, Proposition 2.8]. Then the dual basis of the cotangent bundle gives the desired linear transformation of the coordinates $x_{r+1}, \dots, x_n$ with coefficients in functions of $x_1, \dots, x_r$.

Now we modify [HY] a little bit to allow some "resonances". Let Λ be an $n \times n$ diagonal matrix with entries $\lambda_1, \dots, \lambda_n$, the first r being equal to 1. We assume that $\lambda_{r+1}, \dots, \lambda_n$ satisfy the following bad diophantine approximation property: $|\lambda_{r+1}^{i_{r+1}} \cdots \lambda_n^{i_n} - \lambda_j|_k \ge C(i_{r+1} + \dots + i_n)^{-\beta}$ for some $C > 0$ and $\beta \ge 0$, any $r < j \le n$ and $i = (i_{r+1}, \dots, i_n) \in \mathbb{Z}^{n-r}$ such that $i_{r+1} + \dots + i_n > 1$.

For real $\rho > 0$ define the spaces

- $A_\rho(k^n) := \{\phi = \sum_I a_I x^I \in k[[x_1, \dots, x_n]] \mid \sup |a_I| \rho^{|I|} =: \|\phi\|_\rho < \infty\}$ (this is a non-archimedian Banach algebra, denoted $A_\rho^2(k^n)$ in [HY]);
- $A_\rho^r(k^n)$ consists of all $\phi \in A_\rho(k^n)$ such that $\phi(x_1, \dots, x_r, 0, \dots, 0) = 0$ and $\frac{\partial \phi}{\partial x_i}(x_1, \dots, x_r, 0, \dots, 0) = 0$ for all $r < i \le n$ (this is an ideal in $A_\rho(k^n)$);
- $B_\rho^r(k^n) := \{\phi \in A_\rho^r(k^n) \mid \phi \circ \Lambda \in A_\rho^r(k^n)\}$. In particular, $B_\rho^r(k^n) = A_\rho^r(k^n)$ if $\|\lambda_1\| = \dots = \|\lambda_n\| = 1$.

The set $x + (A_\rho^r(k^n))^n$ is a group (with respect to the composition). This group acts on $A_\rho^r(k^n)$.

We are interested in inverting the linear operator $L : (B_\rho^r(k^n))^n \to (A_\rho^r(k^n))^n$, defined by $Lw = w \circ \Lambda - \Lambda \circ w$. Its injectivity is evident: $\phi = \sum_I a_I x^I \mapsto (\sum_I (\lambda^I - \lambda_i) a_I^{(i)} x^I)_{1 \le i \le n}$, where $\lambda^I = \lambda_1^{i_1} \cdots \lambda_n^{i_n}$. Moreover, it is also evident that for any $g \in (A_\rho^r(k^n))^n$ there exists a unique n-tuple of formal series w such that $Lw = g$, $w(x_1, \dots, x_r, 0, \dots, 0) = 0$ and $\frac{\partial w}{\partial x_i}(x_1, \dots, x_r, 0, \dots, 0) = 0$ for all $r < i \le n$.

Put on $(B_\rho^r(k^n))^n$ the max norm: $\|\phi\|_r = \max(\|\phi\|_r, \|\phi \circ \Lambda\|_r)$.

Lemma 4.5 *For any $\delta > 0$ the solution w belongs to $(B_{\rho-\delta}^r(k^n))^n$ and satisfies $\|w\|_{r-\delta} \le C_1 \frac{\|g\|_r}{\delta^\beta} r^\beta$, $\|Dw\|_{r-\delta} \le C_1 \frac{\|g\|_r}{\delta^\beta} \frac{r^\beta}{r-\delta}$, $\|Dw \circ \Lambda\|_{r-\delta} \le C_1 \frac{\|g\|_r}{\delta^\beta} \frac{r^\beta}{r-\delta}$, where C_1 is a constant depending only on C, β, $\|\Lambda\|$.*

Let $0 < \delta < s \leq 1$ such that $s - \delta \geq 1/2$. Assume as before that $f \in (A_1(k^n))^n$. Take $\hat{h} \in (B_s^r(k^n))^n$ such that $\|\hat{h}\|_s < 1/2$ and $\|D\hat{h} \circ \Lambda\|_s < 1/2$. Set $F_f(h) := f \circ h - h \circ \Lambda$, where $h(x) = x + \hat{h}(x)$. It belongs to $(A_s(k^n))^n$.

Set $E := (Dh)^{-1} \cdot \Delta h$. Then $F_f(h+\Delta h) = f \circ (h+\Delta h) - f \circ h + F_f(h) - \Delta h \circ \Lambda = F_f(h) + [f \circ (h+\Delta h) - f \circ h + DF_f(h) \cdot E - Df \circ h \cdot \Delta h] + (Dh \circ \Lambda)(\Lambda \cdot E - E \circ \Lambda)$.

By Cauchy's formula [HY, Lemma 10], $\|DF_f(h) \cdot E\|_{s-\delta} \leq 2\|F_f(h)\|_{s-\delta}\|E\|_{s-\delta}$. By Taylor's formula [HY, Prop. 7], $\|f \circ (h + \Delta h) - f \circ h - Df \circ h \cdot \Delta h\|_{s-\delta} \leq 4\|f\|_1\|\Delta h\|^2_{s-\delta}$.

According to Lemma 4.5=[HY, Lemma 15], there exists $E \in (B^r_{s-\delta}(k^n))^n$ such that $F_f(h) + (Dh \circ \Lambda)(\Lambda \cdot E - E \circ \Lambda) = 0$ and $\|E\|_{s-\delta} \leq C_2\frac{\|F_f(h)\|_s}{\delta^\beta}\|(Dh \circ \Lambda)^{-1}\|_s \leq C_2\frac{\|F_f(h)\|_s}{\delta^\beta}$, $\|DE\|_{s-\delta} \leq C_2\frac{\|F_f(h)\|_s}{\delta^\beta}$, $\|DE \circ \Lambda\|_{s-\delta} \leq C_2\frac{\|F_f(h)\|_s}{\delta^\beta}$, where we have used the estimate $\|Dh \circ \Lambda - id\| < 1/2$ and the following

Lemma 4.6 ([HY], Lemma 13) *If $\varphi \in A_\rho(k^n, \mathrm{End}(k^n))$ and $\|\varphi\|_\rho < 1$ then $1+\varphi$ is invertible in $A_\rho(k^n, \mathrm{End}(k^n))$ and $\|(1 + \varphi)^{-1} - 1\|_\rho = \|\varphi\|_\rho$.*

(This is evident from the identity $(1 - x)^{-1} = \sum_{i\geq 0} x^i$ for $\|x\| < 1$.)

It follows from the estimates $\|Dh\|_s \leq 1$, $\|Dh^{-1}\|_s \leq 1$, $\|Dh \circ \Lambda\|_s \leq 1$, $\|(Dh \circ \Lambda)^{-1}\|_s \leq 1$ that $\Delta h \in (B^r_{s-\delta}(k^n))^n$ and satisfies the same estimates as E.

To obtain the initial assumption $\|\Delta h\|_{s-\delta} < 1/2$, we assume that $C_2\frac{\|F_f(h)\|_s}{\delta^\beta} < 1/2$. Then we can choose Δh so that $\|F_f(h+\Delta h)\|_{s-\delta} \leq \frac{K}{\delta^\beta}\|F_f(h)\|^2_s$, where $K > C_2^2$ depends only on C, β, $\|A\|$, $\|f\|_1$. Moreover, $h + \Delta h$ satisfies, with respect to $s - \delta$, the same hypothesis that h with respect to s.

Let $\rho_i = 1/2 + 2^{-i-1}$, $\rho_\infty = 1/2$ and $\delta_i = s_i - s_{i+1} = 2^{-i-2}$ for all integer $i \geq 0$. Replacing if necessary f by $uf(u^{-1}x)$ ($u \in k$, $|u|_k \gg 1$), we can suppose that $\|f - \lambda\|_1 = \epsilon$ is as small as we want.

The rest of the iteration process goes exactly the same way as described in [HY, §4.4].

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