SINGULARITIES WITH CRITICAL LOCUS AN COMPLETE INTERSECTION AND TRANSVERSAL TYPE $A_1$

MAMUKA SHUBLADZE

Abstract. In this paper we study germs of holomorphic functions $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ with the following two properties:

(i) the critical locus $\Sigma$ of $f$ is an isolated complete intersection singularity (icis);
(ii) the transversal singularity of $f$ in points of $\Sigma \setminus \{0\}$ is of type $A_1$

we first compute the homology of the Milnor fibre and then show that the homotopy type of the Milnor fibre $F$ of $f$ is a bouquet of spheres.

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1. Introduction

Let $O$ be the ring of holomorphic germs $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$. Let $I \subset O$ be a reduced ideal defining an icis $\Sigma$ of arbitrary dimension $k$. As usual $J(f)$ denote the jacobian ideal of $f$, namely:

$$J(f) = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m} \right).$$

We consider, as in [Pe-1, Pe-2], the group $D_I$ of local analytic isomorphisms $\varphi : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ such that $\varphi^*(I) = I$.

Let $f \in O$ be a germ whose critical set contains $\Sigma$. Then by [Pe-1, Pe-2], $f \in I^2$. The group $D_I$ acts an $I^2$, and the extended codimension of the orbit of $f$ with respect to this action is

$$c_e(f) = \dim \frac{I^2}{I^2 \cap J(f)}$$

we shall focus our attention on germs $f \in I^2$ with $c_e(f) < \infty$. We are interested in the topology of Milnor fibre of $f$. We known if $k$ dimension of singular locus $\Sigma$ is 1 then Milnor fibre $F$ is homotopy equivalent of bouquet of some dimensional sphere [Si-1, Si-2].
If \( k = m - 1 \) i.e. \( \text{codim } \Sigma = 1 \), then again \( F \) is homotopy equivalent of bouquet of some dimensional sphere \([Sh-1, Sh-2, Ne-1]\). If \( k = 2 \) bouquet theorem also are valid the Milnor fibre \( F \) is homotopy equivalent bouquet of sphere \([Za, Ne-2]\).

We consider case when \( k \geq 3 \) and give the properties in which case we can prove the

**Theorem.** The Milnor fibre \( F \) of \( f = (\mathbb{C}^m, 0) \to (\mathbb{C}, 0) \) is homotopy equivalent of bouquet of spheres \( F \simeq S^n \vee S^{m-1} \vee S^{m-1} \vee \ldots \vee S^{m-1} \), where \( n = m - k \).

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2. **Non-Isolated Singularities with Transversal Type \( A_1 \)**

Let as above \( I \subset O \) be a reduced ideal defining an icis (isolated complete intersection singularity) \( \Sigma \) of dimension \( k \) and suppose that \( I = (g_1, \ldots, g_n) \) with \( n = m - k \). We shall assume that \( n \geq 2 \) and \( k \geq 3 \); the cases \( k = 1, k = 2 \) and \( n = 1 \) are situated in \([Si-2]\), \([Ne-2, Za]\) and respectively \([Sh-1]\) and \([Ne-1]\). Let \( f \in O \) be a germ whose critical set contains \( \Sigma \). It follows that \( f \in I^2 \) and we have decomposition

\[
f = \sum_{i,j=1}^{n} h_{ij} g_i g_j
\]

with \( h_{ij} = h_{ji} \) \([Pe-1, Pe-2]\). Moreover, the class of \( h_{ij} \) in \( O/I \) is uniquely determined by \( f \) \([Za]\).

In \([Pe-1]\) and \([Pe-2]\) were introduced \( D(k, p) \) singularity. Their local equations, in a suitable coordinate system \( x_{ij} (1 \leq i \leq j \leq p) \), \( z_1, \ldots, z_q, y_1, \ldots, y_n \), is

\[
f(x, y, z) = \sum_{1 \leq i \leq j \leq p} x_{ij} y_i y_j + \sum_{l=p+1}^{n} y_l^2.
\]

Note also the singular locus of a \( D(k, p) \) singularity is smooth and of dimension \( k = \frac{1}{2} p(p+1) + q \), while \( m = k + n \). \( D(k, 0) \) singularity in \([Pe-1, Pe-2]\) is also called \( A(k) \)

\[
A(k) := D(k, 0) : \sum_{l=1}^{n} y_l^2.
\]

We note also:

\[
D(k, 1) : xy_1^2 + \sum_{l=2}^{n} y_l^2.
\]
Remark 2.1. As in [Sh-3], see also [Za], it is easy to see that following are valid

(1) A singular point \( z \in \Sigma \) is a singular point of type \( D(k, 0) \) if the matrix \( (h_{ij}(z)) \) has rank \( n \).

(2) A singular point \( z \in \Sigma \) is a singular point of type \( D(k, 1) \) if the matrix rank \( (h_{ij}(z)) = n - 1 \) and \( \text{grad}_z (\det(h_{ij}(z))) |_{\Sigma} \neq 0 \).

Let \( D \) be defined as in [Za] by \( D(z) = \det(h_{ij}(z)) \) then if \( D(0) = 0 \) then the ideal \( I + D = (g_1, \ldots, g_n, D) \) defines a complete intersection in \( \mathbb{C}^m, 0 ) \), which depends only of \( f \) [Za]. Let us denote by \( \Delta \) the zero set of \( I + (D) \).

The following result is similar to [Si-1, Sh-1] criterion of finite codimension.

**Theorem 2.2.** Let \( f \in I^2, f = \sum_{ij=1}^n h_{ij} g_i g_j \) and \( I \), and \( I + (D) \) is isolated complete intersection and \( \Delta \) is an isolated singularity. Then

(a) the critical locus of \( f \) is \( \Sigma \) and the germ of \( f \) in every points of \( \Sigma \setminus \{0\} \) outside \( \Delta \) is equivalent to a \( D(k, 0) \) singularity and point an \( \Delta \) is equivalent to a \( D(k, 1) \).

(b) \( c_e(f) < +\infty \).

**Proof.** (a) If \( z \in \Delta \) and \( z \neq 0 \) then rank\((h_{ij}(z)) = n - 1 \) since \( \Sigma \) is icis of dimension \( k = m - n \). Since \( \Delta = \det(h_{ij}(z)) \) is isolated singularity on \( \Sigma \) so \( \text{grad} \Delta |_{\Sigma} \neq 0 \) at the point of \( \Delta \setminus \{0\} \), which means that \( f \) at \( z \) is of type \( D(k, 1) \) by the remark of 2.2. Let us now \( z\in\Delta \) and \( z \neq 0 \). Then we have \( \det(h_{ij}(z)) \neq 0 \) at this point \( z \), so rank\((h_{ij}(z)) = n \) and using Remark 2.2 \( f \) at this points \( z \) has \( D(k, 0) \) singularity.

(b) Let \( f \) be some representative of the germ of given singularity. In the domain where it is given we define a sheaf of \( O \) modules as follows

\[
\mathcal{F}(u) = I^2 / \tau_e(f),
\]

where \( I^2 \) and \( \tau_e(f) \) are considered as modules over the holomorphic functions on \( u \). It is clear that \( \mathcal{F} \) is coherent. We will use the fact: \( \mathcal{F} \) is concentrated in a point \( \iff \dim \Gamma(\mathcal{F}) < \infty \). For \( z \in \mathbb{C}^m \setminus \Sigma \), \( f \) is regular at \( z \) and we have \( \dim \mathcal{F}_z = 0 \) since \( I^2_z \cong O_z \) and \( (\tau_e(f))_z \cong O_z \). If \( z \in \Sigma \setminus \{0\} \) then as we proved above \( f \) is of type \( D(k, p) \), \( p \leq 1 \) at \( z \) and we have \( \dim \mathcal{F}_z = 0 \) since \( c_e(D(k, p)) = 0 \). So \( \mathcal{F} \) is concentrated at 0, hence \( c_e(f) < \infty \). \( \square \)

3. **The Deformation of Nonisolated Singularities**

First consider the case when singular locus \( \Sigma \) of \( f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0) \) is smooth \( k \)-dimensional submanifolds. Consider coordinates
$(x_1, \ldots, x_{m-n}, y_1, \ldots, y_n)$ in $\mathbb{C}^m$. Then $f = \sum_{i,j=1}^{n} h_{ij} y_i y_j$. Let us $\det(h_{ij})_{ij} = D(z)$ and $D(z)|_{\Sigma}$ is isolated singularity at $0 \in \Sigma$.

In case of an ordinary isolated singularity it is useful to consider a generic approximation $g$ of with only ordinary Morse point [Br]. At every Morse point one can study its local Milnor fibration, with Milnor fibre homotopy equivalent to one $n$-sphere $S^n$ (‘the vanishing cycle’). The Milnor fibre of the original $f$ then has the homotopy type of the wedge of those spheres.

We like to mimic the constructions in our case.

Let us $\Sigma$ is $k$-dimensional complete intersection defining by the ideal $I = (g_1, \ldots, g_n)$. Then $f = \sum_{i,j=1}^{n} h_{ij} g_i g_j$. Assume that $D(z) = \det((h_{ij})_{ij})$ is an isolated singularity and $I + (D)$ is complete intersection.

Let $G : (\mathbb{C}^m \times \mathbb{C}^r, 0) \to (\mathbb{C}^m \times \mathbb{C}^r, 0)$ be a versal deformation of $(\Sigma, 0)$ with $G(z, v) = (G_1(z, v), \ldots, G_n(z, v), v)$ and $G_i(z, 0) = g_i(z)$ [Loo]. Consider the deformation

$$f_s : (\mathbb{C}^m \times S, 0) = \mathbb{C}^m \times \mathbb{C}^r \times \mathbb{C}^n \times \mathbb{C}^{(n+1)/2} \times \mathbb{C}^{m-n}, 0) \to \mathbb{C}$$

given by

$$f_s(z) = f(z, v, u, a, b) = \sum_{i,j=1}^{n} \left( h_{ij}(z) + a_{ij} + \sum_{t=1}^{m} b_{ij} x_t \delta_{ij} \right) \cdot (G_i(z, v) - u_i)(G_j(z, v) - u_j),$$

where $a_{ij} = a_{ji}$, and $S$ is the space of parameters $(a, b, u, v)$. In case $k = \dim \Sigma = 1, 2$ or $m - 1$, there exists a dense subset $U$ in $S$ such that for every $s \in U$, the germ of $f_s$ at the points of $\Sigma_s$ is of type $D(k, 0)$ or $D(k, 1)$. Moreover, the set of points of $\Sigma_s$ where $f_s$ is of type $D(k, 1)$ is exactly $\Delta_s$ and this set is a Milnor fibre of the icis $\Delta$.

[Si-2, Sh-1, Za] For an arbitrary $k$, we know at least two cases when such deformation exists: i) the germ $f$ at any point $z \in \Sigma \setminus \{0\}$ is of type $D(k, 0)$, ii) the matrix $(h_{ij}(0))_{ij}$ has rank $n - 1$. From this page assume the existence of such deformation for the arbitrary $k$. The following are valid [Za-Bo-Ne-2].

**Lemma 3.1.** There exist an $\varepsilon$-ball $B_\varepsilon$ with center $D \in \mathbb{C}^m$, a proper analytic set $(A, 0) \subset (S, 0)$, and a neighborhood $U$ of $0 \in S$, such that for any $s \in U \setminus A$ has the following:

(a) $\Sigma_s = \{ z \in B_\varepsilon : G_i(x, v) = u_i, i = 1, \ldots, n \}$ is the Milnor fibre of $\Sigma$. 


(b) The zero set \( D_s(z) = \det(h_{ij}(z) + a_{ij} + \sum b_{ij} z_t \delta_{ij}) \) intersects \( \Sigma_s \) transversally; hence \( \Delta_s = D_s^{-1}(0) \cap \Sigma_s \) is smooth. In particular \( \Delta_s \) is (diffeomorphic to) the Milnor fibre of \( \Delta \).

(c) The singularities of \( f_s \) in \( B_\varepsilon \setminus \Sigma_s \) are of type \( A_1 \).

(d) The germ of \( f_s \) at any point of \( \Sigma_s \setminus \Delta_s \) is of type \( D(k,0) \) and at any point of \( \Delta_s \) is of type \( D(k,1) \).

(e) Fix \( \varepsilon \) sufficiently small and \( \delta \) sufficiently small with respect to \( \varepsilon \). If \( U \) is sufficiently small with respect to \( \varepsilon \) and \( \delta \), then \( f_s^{-1}(t) \) (as a stratified set) intersects \( \partial B_\varepsilon \) transversally for any \( s \in U \) and \( t \in \Lambda = \{|t| \leq \delta\} \). In particular, the topological type of the smooth fibres of the maps

\[
f_s : X_s = f_s^{-1}(\Lambda) \cap B_\varepsilon \to \Lambda \quad (s \in U)
\]

is independent of the parameter \( s \in U \). (In fact, even the vibrations \( f_s : f_s^{-1}(\partial \Lambda) \cap B_\varepsilon \to \partial \Lambda \) are equivalent to the fibration \( f : f^{-1}(\partial \Lambda) \cap B_\varepsilon \to \partial \Lambda \). In particular, the corresponding fibres are homotopically equivalent).

(f) The spaces \( X_s \) (\( s \in U \)) are contractible.

### 4. The Topology of Milnor Fibre

Let \( f_s \) be a deformation of \( f \) obtained by Lemma 3.1 and let us suppose that the number of \( A_1 \) (Morse) points is \( \sigma \). The critical set of \( f \) consists of

(a) A manifold \( \Sigma_0 \) with is the Milnor fibre \( \Sigma_s \) of \( k \)-dimensional isolated complete intersection singularity \( \Sigma \). The local singularities of \( f \) on \( \Sigma_0 \) are \( D(k,0) \) and \( D(k,1) \).

(b) \( \Sigma_1 = \{b_1\}, \ldots, \Sigma_\sigma = \{b_\sigma\} \), where the local singularity of \( f \) is isolated of type \( A_1 \).

Define \( B_1, B_2, \ldots, B_\sigma \) as disjoint \( 2m \) dimensional balls in the space \( \mathbb{C}^m \) with centered of the points \( b_1, \ldots, b_\sigma \) and \( D_1, D_2, \ldots, D_\sigma \) a disjoint two dimensional disks at the points \( f_s(b_1), \ldots, f_s(b_\sigma) \). Choose them such that \( \tilde{f} : B_i \cap \tilde{f}^{-1}(D_i) \to D_i \) define a locally trivial Milnor fibration, the following transversality condition holds: \( f_s^{-1}(t) \cap \partial B_i, \forall t \in D_i, \ i = 1, \ldots, \sigma \).

The situation at the points of \( b_1, \ldots, b_\sigma \) is well known we consider the situation along \( \Sigma_s \).

Firstly we define \( B^0 \) a tabular neighborhood

\[
B^0 = \left\{ z \in B_\varepsilon : \sum_{i=1}^{n} |G_i(z, v) - u|^2 \leq \rho \right\} \quad \text{of} \quad \Sigma_s \cap B_\varepsilon,
\]
which is diffeomorphic for sufficiently small $\rho$ to the product $(\Sigma_s \cap B_z) \times Q^n$, where $Q^n$ is a closed $n$-ball in $\mathbb{C}^n$ with center at the origin \([Si-1]\).

Let us denote $X_{t,s} = f^{-1}_s(t) \cap B_z$ and $F^0 = B^0 \cap X_{t,s}$ then for the sufficiently small $t$ we have

$$H_{s-1}(X_{s,t}) = H_{s}(X_s, X_{s,t}) = \begin{cases} H_{s}(B^0, F^0) & \text{if } * \neq m, \\ H_{m}(B^0, F^0) \oplus \mathbb{Z}^* & \text{if } * = m, \end{cases}$$

\([Si-2]\).

First compute the homology of the point $(B^0, F^0)$. Following \([Si-2, Z]\) we shall consider in $B^0$ coordinates $(w_1, \ldots, w_n, w_{m-k+1}, \ldots, w_m)$ such that $(w_1, \ldots, w_n) \in Q^n$ are the functions defined by $w_i(z) = G_i(x, v) - u_i$ and $w_{m-k+1}, \ldots, w_m \in \Sigma_s$ (recall that $\dim \Sigma = k$ and $m = n + k$). Then $(w_1, \ldots, w_n)$ are holomorphic functions on $B^0$ and $(w_{m-k+1}, \ldots, w_m)$ are real differentiable \([Si-2]\). Now consider the projection $\pi : (w_{m-k+1}, \ldots, w_m) : (B^0, F^0) \to \Sigma_s$. Then similarly \([Si-2, Z, Sh, Sh-3]\) we can prove

**Lemma 4.1.** If $\rho$ and tubular neighborhoods $U_1 \subset U_2 \subset \Sigma_s$ of $\Delta_s \subset \Sigma_s$ are sufficiently small then

(a) $\pi : (B^0 \setminus \pi^{-1}(U_1), F^0 \setminus \pi^{-1}(U_1)) \to \Sigma_s \setminus U_1$ is locally trivial fibration with fibre equal to the pair $((\mathbb{C}^{m-k}, \text{Milnor fibre of } x_1^2 + \cdots + x_n^2)$,

(b) the map given by the superposition $\pi^{-1}(U_2) \to U_2 \to \Delta_s$ is a fibration of the pair $(\pi^{-1}(U_2), F^0 \cap \pi^{-1}(U_2))$ with fibre equal to the pair $((\mathbb{C}^{n-k}, \text{Milnor fibre of } x_1^{n+1}x_2 + x_2^2 + \cdots + x_n^2)$.

For a subset $W \subset \Sigma_s$ we shall denote by $F_W$ the following set: $F_W = \pi^1(W) \cap F^0$.

The following statements holds

**Lemma 4.2.** $H_q(F_{\Sigma_s \setminus U_1}) = 0$ for $q = n - 2$ and $q = n$. Moreover $H_{n-1}(F_{\Sigma_s \setminus U_1}) = \mathbb{Z}_2$, $H_{m-1}(F_{\Sigma_s \setminus U_1}) = \mathbb{Z}^{\mu_s + \mu_{\Delta}}$, $H_1(F_{\Sigma_s \setminus U_1}) = 0$, $q \geq n - 2$.

**Proof.** If $n = 1$ this case was studied in \([Sh-1, Ne-1]\). $n = 2$ in \([Ne-2]\), so $n \geq 3$. We may assume that $n > k + 2$ because of if $w$ is a new variable, then the Milnor fiber $F_w$ of $f(z) + w^2$ is the suspension of the Milnor fibre $F$ of $f$, in particular $H_* = H_{*+1}(F_w)$.

Consider the fibration $\pi : F_{\Sigma_s \setminus U_1} \to \Sigma_s \setminus U_1$.

The base space $\Sigma_s \setminus U_1$ is homotopy equivalent $\Sigma_s \setminus U_1 \simeq \Sigma_s \setminus U_1 \times S^1 \simeq S^1 \vee S^k \vee \cdots \vee S^k$ bouquet of circle $S^1$ and $k$-dimensional spheres. The number of $k$-dimensional spheres $\mu$ is equal same of $\mu_S + \mu_{\Delta}$ \([Za, Sh-1]\). The homotopy type of fibre of $\pi$ is $S^{n-1}$ but unfortunately we cold’t use Gysin exact sequence for this fibration $\pi$ because $\pi$ is
not orientable. But the total space $F^0_s \setminus F_{U_1}$ is homotopy equivalent to $E' \cup E''$, where $E' \to S^1$ and $E'' \to \vee S^k_i$ are fibre bundles with fibre $S^{n-1}$ and in $E' \cup E''$ a fibre of $E'$ is identified with a fiber of $E''$.

For the fibration $E' \to S^1$ which is nonorientable and its monodromy is equal $-1$ [Ne-2] we may use Wang exact sequence. We obtain

$$
\rightarrow H_q(S^{n-1}) \rightarrow H_q(E') \rightarrow H_{q-1}(S^{n-1}) \rightarrow H_{q-1}(S^{n-1}) \rightarrow \cdots
$$

Finally, we receive short exact sequence

$$0 \longrightarrow H_n(E') \longrightarrow \mathbb{Z} \overset{\alpha}{\longrightarrow} \mathbb{Z} \longrightarrow 0,$$

$\alpha$ is multiplication by 2. Therefore $H_{n-1}(E') = \mathbb{Z}_2$, $H_q(E') = 0$, $q \neq 0$, $q \neq n - 1$.

On the other hands, we have orientable fibration $E'' \to \bigvee_{i=1}^k S^1_i$ because of $k \geq 3$. Hence we may use Gysin exact sequence we obtain

$$
\rightarrow H_q(E'') \rightarrow H_q(\bigvee_{i=1}^k S^1_i) \rightarrow H_{q-n}(\bigvee_{i=1}^k S^1_i) \rightarrow H_{q-1}(E'') \rightarrow \cdots
$$

Since $n \geq k + 2$ and $k \geq 3$ we receive $H_{n-2}(E'') = H_n(E'') = 0$ and $H_{n-1}(E'') = \mathbb{Z}$, $H_{m-2}(E'') \simeq \mathbb{Z}^{2+\mu q}$.

The total space $F_{\Sigma_2 \setminus U_1} = E' \cup E''$, where $E' \cap E'' \simeq S^{n-1}$. Using Mayer-Vietoris theorem we obtain

$$
\rightarrow H_q(E' \cap E'') \rightarrow H_q(E') \oplus H_q(E'') \rightarrow H_q(E' \cup E'') \rightarrow H_{q-1}(E' \cup E'') \rightarrow \cdots
$$

After short computations we receive short exact sequence

$$
0 \rightarrow H_n(E) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow H_{n-1}(E) \rightarrow 0.
$$

Therefore $H_q(E) = 0$, $H_{n-1}(E) = \mathbb{Z}_2$ and $H_{n-2}(E) = 0$.

Similarly we receive $H_{m-1}(E) = \mathbb{Z}^{2+\mu q}$ and $H_q(E) = 0$, $q \geq n - 2$.

**Lemma 4.3.** $H_{n-2}(F_{U_2 \setminus U_1}) = 0$, $H_{n-1}(F_{U_2 \setminus U_1}) = \mathbb{Z}_2$, $H_{m-2}(F_{U_2 \setminus U_1}) = \mathbb{Z}^{2\Delta}$ and $H_q(F_{U_2 \setminus U_1}) = 0$ if $q \geq n - 2$ and $q \neq n - 1, m - 2$.

**Proof.** We have fibration $\pi : F_{U_2 \setminus U_1} \to U_2 \setminus U_1$ with fibre $S^{n-1}$. Since $U_2 \setminus U_1$ is homotopy equivalent to $S^1 \times \Delta_s$ using homotopy exact sequence of fibration $\pi$ we receive $H_{n-2}(F_{U_2 \setminus U_1}) = 0$. Because of $\pi$ is not orientable $H_{n-1}(F_{U_2 \setminus U_1}) = \mathbb{Z}_2$. As in [Ne-2], since the base space has a product structure, one can write $F_{U_2 \setminus U_1}$ as a fibre bundle over $\Delta_s$ with fibre $\mathbb{Z}$ is the total space of a fibre bundle with base $S^1$ and fibre $S^{n-1}$. Using Wang exact sequence we obtain $H_{n-1}(Z) = \mathbb{Z}_2$, $H_q(Z) = 0$, $q \neq 0, n - 1$. Because $\Delta_s$ is simply connected, it follows from the Serre spectral sequence [Me] $H_s(\Delta_s; H_s(Z)) \Rightarrow H_s(F_{U_2 \setminus U_1})$.
that $H_{m-2}(F_{U_2\setminus U_1}) = \mathbb{Z}_2^{\mu_\Delta}$ and $H_q(F_{U_2\setminus U_1}) = 0$ if $q \geq n - 2$, $q \neq n - 1, m - 2$.

**Lemma 4.4.** $H_{n-1}(F_{U_2}) = 0$, $H_n(F_{U_2}) = \mathbb{Z}$ and $H_{m-1}(F_{U_2}) = \mathbb{Z}_2^{\mu_\Delta}$.

**Proof.** This follows from the fibration $F_{U_2} \to \Delta_s$ (cf. Lemma 3.1 (b)), whose fibre has the homotopy type of $S^n$.

**Corollary 4.5.**

$$H_q(F^0, F_{U_2}) = \begin{cases} 
\mathbb{Z}, & \text{if } q = 0, \\
\mathbb{Z}_2^{\mu_\Delta + \mu_\Sigma}, & \text{if } q = m - 1, \\
0, & \text{otherwise}.
\end{cases}$$

**Proof.** Using the long exact sequence for the pair $(F_{\Sigma_1 \setminus U_1}, F_{U_2 \setminus U_1})$ we receive

$$0 \to H_q(F_{U_2 \setminus U_1}) \to H_q(F_{\Sigma_1 \setminus U_1}) \to H_q(F_{\Sigma_1 \setminus U_1}, F_{U_2 \setminus U_1}) \to$$

$$\to H_{q-1}(F_{U_2 \setminus U_1}) \to \cdots$$

Since $F_{U_2 \setminus U_1} \hookrightarrow F_{\Sigma_1 \setminus U_1}$ is inclusion using excision $H_q(F_{\Sigma_1 \setminus U_1}, F_{U_2 \setminus U_1}) = H_q(F^0, F_{U_2})$, and Lemma 4.2, 4.3 we obtain $H_q(F^0, F_{U_2}) = 0$ if $q \neq 0, n, m - 1$. For $n$-dimensional homology group we have exact sequence

$$0 \to H_n(F^0, F_{U_2}) \to \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \to H_{n-1}(F^0, F_{U_2}) \to 0.$$ So $H_n(F^0, F_{U_2}) = 0$ and $H_{n-1}(F^0, F_{U_2}) = 0$. For $m - 1$ dimensional homology group we have following exact sequence

$$0 \to \mathbb{Z}_2^{\mu_\Delta + \mu_\Sigma} \to H_{m-1}(F^0, F_{U_2}) \to \mathbb{Z}_2^{\mu_\Delta} \to 0.$$

As we known we have fibrations

$$F_{\Sigma_1 \setminus U_1} \xrightarrow{i_1} F_{U_2 \setminus U_1} \xrightarrow{i_2} S^{1} \times \Delta_s$$

Let $b_1, \ldots, b_{\mu_\Sigma}$ generators of $H_{m-1}(F_{\Sigma_1 \setminus U_1})$. Take into account $\Delta_s \simeq \bigvee_{i=1}^{\mu_\Sigma} S^{k-1}_i$. Let $f_{i, \pm}$ be the map

$$D_{i, \pm}^k = \left[0, \frac{1}{2}\right] \times S^{k-1}_i/\{1\} \times S^{k-1}_i \to S^{1} \times \Delta_s/\{1\} \times \Delta_s \to$$

$$\to \Sigma_s \sqcup (S^{1} \times \Delta_s), \quad i = 1, \ldots, \mu_0.$$
The pullback of the fibration $F_{\Sigma_i \setminus U_1} \to \Sigma_s \sqcup (S^1 \times \Delta_s)$ along $f_{i,+}$ is trivial. Therefore we have following diagram

\[
\begin{array}{c}
(D_i^k \times S^{n-1}, S_i^{k-1} \times S^{n-1}) \xrightarrow{\bar{f}_{i,+}} (F_{\Sigma_i \setminus U_1}, F_{U_2 \setminus U_1}) \\
(D_i^k, S_i^{k-1}) \xrightarrow{f_{i,+}} (\Sigma_s \sqcup (S^1 \times \Delta_s), S^1 \times \Delta_s)
\end{array}
\]

Let $a_i \in H_{m-1}(F_{\Sigma_i \setminus U_1}, F_{U_2 \setminus U_1})$ be image of a generator of $H_{m-1}(D_i^k \times S^{n-1}, S_i^{k-1} \times S^{n-1}) \cong \mathbb{Z}$ under $(\bar{f}_{i,+})$. There is a homotopy between $f_{i,+}$ and $f_{i,-}$ (as a map of pairs), namely

\[
D_i^k \times [0, 1] = ([0, 1] \times S_i^{k-1}/\{0\} \times S_i^{k-1}) \times [0, 1] \to \Sigma_s \sqcup (S^1 \times \Delta_s).
\]

\[
([t, x], S) \mapsto \begin{cases} f_{i,+}([1 - 2s]t, x), & 0 \leq s \leq \frac{1}{2}, \\ f_{i,-}([2s - 1]t, x), & \frac{1}{2} \leq s \leq 1. \end{cases}
\]

Therefore $(\bar{f}_{i,+})_*$ and $(\bar{f}_{i,-})_*$ define the same element $a_i$. Hence $2a_i$ as an element $H_{m-1}(F_{\Sigma_i \setminus U_1})$ is represented by $\bar{f}_{i,+} \cup \bar{f}_{i,-}$, which means that

\[
H_{m-1}(F_{\Sigma_i \setminus U_1}, F_{U_2 \setminus U_1}) = H_{m-1}(F^0, F^0) = \mathbb{Z}^{\mu_\Delta + \mu_\Sigma}.
\]

\[\square\]

Corollary 4.6.

\[
H_q(F^0) = \begin{cases} \mathbb{Z}, & \text{if } q = q = n, \ q = 0, \\ \mathbb{Z}^{2\mu_\Delta + \mu_\Sigma}, & \text{if } q = m - 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Proof. Use the long exact sequence of the pair $(F^0, F^0)$ and above lemmas.

Now we consider the pair $(B^0, F^0)$ and the corresponding exact sequence in homology we obtain $H_\ast(B^0, F^0) = H_{\ast-1}(F^0)$. As we mentioned in the beginning of this section for the Milnor fibre $F = X_{i,s}$ the homology group is equal

\[
H_{\ast-1}(F) = \begin{cases} H_{\ast-1}(F^0) & \text{if } \ast \neq m, \\ H_{m-1}(F^0) \oplus \mathbb{Z}^\sigma & \text{if } \ast = m. \end{cases}
\]

Therefore finally we receive

\[
H_\ast(F) = \begin{cases} \mathbb{Z} & \text{if } \ast = 0, n, \\ \mathbb{Z}^{2\mu_\Delta + \mu_\Sigma + \sigma} & \text{if } \ast = m - 1, \\ 0 & \text{otherwise}. \end{cases}
\]
Now we will show that our Milnor fibre is homotopy equivalent to a wedge of spheres $S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}$ following are valid.

**Lemma 4.7.** Let $X$ be a $(n-2)$-connected CW complex of dimension $n \geq 3$ with given homology $H_n(X, \mathbb{Z}) = \mathbb{Z}$, $H_{m-1}(X, \mathbb{Z}) = \mathbb{Z}^\mu$, $\tilde{H}_k(X, \mathbb{Z}) = 0$ if $k \neq n, m - 1$. Then we have a homotopy equivalence

$$X \simeq S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}.$$ 

**Proof.** For $n \geq 3$ we have that $X$ is simple connected. According to Herewicz theorem $\pi_n(X) \simeq H_n(X) = \mathbb{Z}$. We may attach an $n$-cell $e_n$ corresponding to a generator $\varphi$ of $\pi_{n-1}(X)$. Let $\tilde{X} = X \cup e_n$. So we have $\pi_{n-1}(\tilde{X}) = 0$ and $\pi_k(\tilde{X}) = \pi_k(X) = 0$, $k \leq n - 2$.

Moreover we can prove that $\tilde{X}$ is homotopy equivalent bouquet of $n - 1$ dimensional $\mu$ copies of sphere (see [Si-2], Prop. 6.1).

Consider the following Hurewicz diagram

\[
\begin{array}{ccc}
\mathbb{Z}^\mu = H_{m-1}(X) & \xleftarrow{\alpha_1} & \pi_{m-1}(X) \\
\downarrow & & \downarrow \\
H_{m-1}(\tilde{X}) & \xleftarrow{\cong} & \pi_{m-1}(\tilde{X}) \\
\downarrow & & \downarrow \\
H_{m-1}(\tilde{X}, X) & \xleftarrow{\beta_1} & \pi_{m-1}(\tilde{X}, X) \\
\downarrow & & \downarrow \\
H_{m-2}(X) & \xleftarrow{\gamma_2} & \pi_{m-2}(X)
\end{array}
\]

This implies $\beta_2 = 0$ so $\alpha_2$ is surjective. \qed

Let now $Y = S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}$, and $\tilde{Y} = D^{n+1} \vee S^{m-1} \vee \cdots \vee S^{m-1}$, where $\partial D^{n+1} = S^n$. Define $h : Y \to X$ and $\tilde{h} : \tilde{Y} \to \tilde{X}$ as follows

- $h|_{S^n} = \text{generator of } \pi_n(X)$,
- $h|_{S^{m-1}} = \text{lifted generator of } \pi_{m-1}(\tilde{X})$,
- $h|_{D} = e_n$. 

\[\square\]
It is obvious that $H_q(X) = H_q(Y)$, if $q \neq m - 1$. For $q = m - 1$ consider

\[\begin{array}{c}
H_{m-1}(X) \\
\cong
\end{array} \begin{array}{c}
\cong
\end{array} \begin{array}{c}
\pi_{m-1}(X)
\end{array}\]

\[\begin{array}{c}
H_{m-1}(Y) \\
\cong
\end{array} \begin{array}{c}
\cong
\end{array} \begin{array}{c}
\pi_{m-1}(Y)
\end{array}\]

\[\begin{array}{c}
\cong
\end{array} \begin{array}{c}
\pi_{m-1}(	ilde{X})
\end{array}\]

\[\begin{array}{c}
\cong
\end{array} \begin{array}{c}
\pi_{m-1}(	ilde{Y})
\end{array}\]

The following maps are isomorphisms

\[\tilde{h} : \pi_{m-1}(\tilde{Y}) \to \pi_{m-1}(\tilde{X}) \quad \text{by construction,}\]
\[\pi_{m-1}(\tilde{X}) \to H_{m-1}(\tilde{X}) \quad \text{by Hyrevicz-theorem,}\]
\[\pi_{m-1}(\tilde{Y}) \to H_{m-1}(\tilde{Y}) \quad \text{by Hyrevicz-theorem,}\]
\[H_{m-1}(Y) \to H_{m-1}(\tilde{Y}) \quad \text{by exactness,}\]
\[H_{m-1}(X) \to H_{m-1}(\tilde{X}) \quad \text{by exactness.}\]

It follows that $h$ is homotopy equivalence, because of $H_\ast(Y) \cong H_\ast(X)$, $X$ and $Y$ are simple connected, as a consequence of whiteheads theorem.

**Main Theorem 4.8.** Let $\Sigma = \{g_1 = \cdots = g_n = 0\}$ be a isolated complete intersection and $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ i.e. $f = \sum_{i,j=1}^{n} h_{ij} g_i g_j$, with $D = \det((h_{ij}))$ isolated singularity at the origin and $(g_1, \ldots, g_n, D)$ icis and deformation $f_s$ described above exists. Then the Milnor fibre of $f$ is homotopy equivalent to a bouquet of $\mu_{m-1}(f) = 2\mu_{\Delta} + \mu_{\Sigma} + \sigma$ copies of $S^{m-1}$ and one copy of $S^n$, where $\mu_{\Sigma}$ (respectively $\mu_{\Delta}$) is the Milnor number of $\Sigma$ (respectively of $\Delta$), and $\sigma$ is the number of Morse points which occur in a special deformation of $f$.

**Proof.** We know that Milnor fibre $F$ is $n - 2$ connected (see [Ka-Ma]). As we mansion above $n \geq 3$ so $F$ is simple connected and we can apply Lemma 4.6 and find $F \simeq S^n \vee S^{m-1} \vee \cdots \vee S^{m-1}$. This finishes the proof of the main theorem.
References


