

# ON REGULAR COURANT ALGEBROIDS

ZHUO CHEN, MATHIEU STIÉNON, AND PING XU

ABSTRACT. For any regular Courant algebroid, we construct a characteristic class *à la* Chern-Weil. This intrinsic invariant of the Courant algebroid is a degree-3 class in its naive cohomology. When the Courant algebroid is exact, it reduces to the Ševera class (in  $H_{DR}^3(M)$ ). On the other hand, when the Courant algebroid is a quadratic Lie algebra  $\mathfrak{g}$ , it coincides with the class of the Cartan 3-form (in  $H^3(\mathfrak{g})$ ). We also give a complete classification of regular Courant algebroids and discuss its relation to the characteristic class.

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## INTRODUCTION

Courant algebroids were introduced in [8] as a way to merge the concept of Lie bialgebra and the bracket on  $\mathfrak{X}(M) \oplus \Omega^1(M)$  first discovered by Courant [5] — here  $M$  is a smooth manifold. Roytenberg gave an equivalent definition phrased in terms of the Dorfman bracket [9], which highlighted the relation of Courant algebroids to  $L_\infty$ -algebras [13] observed by Roytenberg & Weinstein [11].

Despite its importance, the subject suffered from the lack of examples for a long time. One important class of Courant algebroids, called exact Courant algebroids, was discovered by Ševera [12]. A Courant algebroid  $E$  is said to be exact if its underlying vector bundle fits into an exact sequence  $0 \rightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \rightarrow 0$ , where  $\rho$  is the anchor map. Ševera proved that the isomorphism classes of exact Courant algebroids are classified by a degree-3 class in the de Rham cohomology of  $M$ : the Ševera class. Furthermore, the structure of transitive Courant algebroids — Courant algebroids with surjective anchors — was described independently by Vaisman [15], Ševera (in a private correspondence with Weinstein [12]) and Bressler [2]. Ševera and Bressler also classified transitive Courant algebroids as extensions of transitive Lie algebroids [2, 12]. Indeed Ševera also outlined some nice ideas of classification of transitive Courant algebroids in [12].

In [14], the second and third authors introduced the modular class of Courant algebroids, a degree 1 characteristic class in the naive cohomology  $H_{naive}^1(E)$  of the Courant algebroid  $E$ . It is natural to ask whether there is a degree 3 characteristic class for a general Courant algebroid resembling the Ševera class, and if so, what role is played by such a class in the classification of Courant algebroids. The main purpose of this paper is to answer these questions for regular Courant algebroids, that is, Courant algebroids with a constant rank anchor. Such Courant algebroids  $E$  are particularly easy to handle, for their associated characteristic distribution  $\rho(E)$  does not have any singularity. Note that characteristic classes of Lie algebroids were studied by Evens-Lu-Weinstein [3], Crainic and Fernandes [4, 6]. It would be interesting to explore if there is any intrinsic connection between these constructions.

For any regular Courant algebroid, we will construct a degree-3 class, called the characteristic class, in the naive cohomology  $H_{naive}^3(E)$  of the Courant algebroid, an analogue of Ševera's class. Note that when  $E$  is an exact Courant algebroid,  $H_{naive}^3(E)$  is isomorphic to the de Rham cohomology  $H_{DR}^3(M)$ . However, unlike the exact case,  $H_{naive}^3(E)$  does not classify regular Courant algebroids. The classification problem is much subtler in this case.

Given a regular Courant algebroid  $E$  with anchor  $\rho$ ,  $\mathcal{A} := E/(\ker \rho)^\perp$  is clearly a regular Lie algebroid, i.e. a Lie algebroid with a constant rank anchor. It is called the ample Lie algebroid of  $E$ . The kernel  $\mathcal{G}$  of the anchor of  $\mathcal{A}$  is a bundle of quadratic Lie algebras, which satisfies certain compatibility conditions. Therefore, the Lie algebroid  $\mathcal{A}$  is quadratic (see Definition 1.8).

The first natural question is whether every quadratic Lie algebroid  $\mathcal{A}$  arises in this way. It turns out that there is an obstruction: the first Pontryagin class, an element in  $H^4(F)$  naturally associated to any quadratic Lie algebroid. Here  $F$  is the image of the anchor (an integrable subbundle of  $TM$ ) and  $H^4(F)$  stands for the leafwise de Rham cohomology of  $F$ . This obstruction is similar to the one described by Ševera and Bressler in the transitive case [2, 12].

To recover a Courant algebroid from a given a quadratic Lie algebroid  $\mathcal{A}$  with vanishing first Pontryagin class, one needs an extra piece of data: a coherent form  $C$ . This is a closed 3-form on the Lie algebroid  $\mathcal{A}$  satisfying certain compatibility conditions. In this case  $(\mathcal{A}, C)$  is called a characteristic pair.

An equivalence is introduced on characteristic pairs. Let  $H_{\leftarrow}^{\bullet}(\mathcal{A})$  denote the cohomology groups of the subcomplex  $\mathbf{C}_{\leftarrow}^{\bullet}(\mathcal{A})$  of  $(\Gamma(\wedge^{\bullet}\mathcal{A}^*), d)$ , where  $\mathbf{C}_{\leftarrow}^k(\mathcal{A})$  consists of the sections of  $\wedge^k\mathcal{A}^*$  which are annihilated by all sections of  $\wedge^k\mathcal{G}$ . Roughly speaking, two coherent forms  $C_1$  and  $C_2$  on  $\mathcal{A}$  are equivalent if and only if the class of  $C_1 - C_2$  in  $H_{\leftarrow}^3(\mathcal{A})$  is zero. We prove that there is a one-to-one correspondence between regular Courant algebroids up to isomorphisms and equivalence classes of characteristic pairs.

The inclusion  $i : \mathcal{G} \rightarrow \mathcal{A}$  is a Lie algebroid morphism. Therefore, it induces a morphism  $i^* : H^3(\mathcal{A}) \rightarrow H^3(\mathcal{G})$ . We denote the class of the well-known Cartan 3-form of Lie theory in  $H^3(\mathcal{G})$  by  $\alpha$ .

The main result of this paper can be summarized as the following

- Theorem.** (a) *There is a natural map from regular Courant algebroids to quadratic Lie algebroids with vanishing first Pontryagin class.*
- (b) *For any quadratic Lie algebroid  $\mathcal{A}$  with vanishing first Pontryagin class, the isomorphism classes of Courant algebroids whose ample Lie algebroids are isomorphic to  $\mathcal{A}$  are parametrized by  $\frac{a^*H^3(F)}{\mathbb{I}} \cong \frac{H^3(F)}{(a^*)^{-1}(\mathbb{I})}$ . Here  $\mathbb{I}$  is a certain abelian subgroup of  $a^*H^3(F) \subset H_{\leftarrow}^3(\mathcal{A})$  and  $a$  denotes the anchor of  $\mathcal{A}$ .*
- (c) *For any regular Courant algebroid  $E$ , there is a degree-3 characteristic class, which is a cohomology class in  $(i^*)^{-1}(\alpha) \subset H^3(\mathcal{A})$ . Here  $\mathcal{A}$  is the ample Lie algebroid of  $E$ , and  $\alpha \in H^3(\mathcal{G})$  is the Cartan 3-form.*

Hence, we have the “exact sequence”

$$\boxed{\begin{array}{l} \text{isomorphism classes of reg-} \\ \text{ular Courant algebroids} \\ \text{with characteristic distri-} \\ \text{bution } F \end{array}} \xrightarrow{\Phi} \boxed{\begin{array}{l} \text{isomorphism classes of} \\ \text{quadratic Lie algebroids} \\ \text{with characteristic distri-} \\ \text{bution } F \end{array}} \xrightarrow{\text{FPC}} \boxed{H^4(F)} .$$

To this day, little is known about Courant algebroid cohomology [10]: Roytenberg computed it for  $TM \oplus T^*M$  and Ginot & Grützmann for transitive and some very special regular Courant algebroids [7]. Our result should be useful for computing the Courant algebroid cohomology of arbitrary regular Courant algebroids.

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## 1. PRELIMINARIES

**1.1. Regular Courant algebroids.** A Courant algebroid consists of a vector bundle  $E \rightarrow M$ , a fiberwise nondegenerate pseudo-metric  $(-, -)$ , a bundle map  $\rho : E \rightarrow TM$

called anchor and an  $\mathbb{R}$ -bilinear operation  $\circ$  on  $\Gamma(E)$  called the Dorfman bracket, which, for all  $f \in C^\infty(M)$  and  $e_1, e_2, e_3 \in \Gamma(E)$  satisfy the following relations:

$$e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3); \quad (1)$$

$$\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]; \quad (2)$$

$$e_1 \circ (fe_2) = (\rho(e_1)f)e_2 + f(e_1 \circ e_2); \quad (3)$$

$$\frac{1}{2}(e_1 \circ e_2 + e_2 \circ e_1) = \mathcal{D}(e_1, e_2); \quad (4)$$

$$\mathcal{D}f \circ e_1 = 0; \quad (5)$$

$$\rho(e_1)(e_2, e_3) = (e_1 \circ e_2, e_3) + (e_2, e_1 \circ e_3), \quad (6)$$

where  $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(E)$  is the  $\mathbb{R}$ -linear map defined by

$$(\mathcal{D}f, e) = \frac{1}{2}\rho(e)f. \quad (7)$$

The symmetric part of the Dorfman bracket is given by Eq. (4). The Courant bracket is defined as the skew-symmetric part  $\llbracket e_1, e_2 \rrbracket = \frac{1}{2}(e_1 \circ e_2 - e_2 \circ e_1)$  of the Dorfman bracket. Thus we have the relation  $e_1 \circ e_2 = \llbracket e_1, e_2 \rrbracket + \mathcal{D}(e_1, e_2)$ .

Using the identification  $\Xi : E \rightarrow E^*$  induced by the pseudo-metric  $(-, -)$ :

$$\langle \Xi(e_1) | e_2 \rangle := (e_1, e_2), \quad \forall e_1, e_2 \in E,$$

we can rewrite Eq. (7) as

$$\mathcal{D}f = \frac{1}{2}\Xi^{-1}\rho^*df.$$

It is easy to see that  $(\ker \rho)^\perp$ , the subbundle of  $E$  orthogonal to  $\ker \rho$  with respect to the pseudo-metric, coincides with  $\Xi^{-1}\rho^*(T^*M)$ , the subbundle of  $E$  generated by the image of  $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(E)$ . From Eq. (6), it follows that  $\rho(\mathcal{D}f) = 0$  for any  $f \in C^\infty(M)$ . Therefore, the kernel of the anchor is coisotropic:

$$(\ker \rho)^\perp \subset \ker \rho.$$

The spaces of sections of  $\ker \rho$  and  $(\ker \rho)^\perp$  are two-sided ideals of  $\Gamma(E)$  with respect to the Dorfman bracket.

A Courant algebroid  $E$  is said to be **regular** if  $F := \rho(E)$  has constant rank, in which case  $F$  is an integrable distribution on the base manifold  $M$ . Moreover, if  $E$  is regular, then  $\ker \rho$  and  $(\ker \rho)^\perp$  are smooth (constant rank) subbundles of  $E$  and the quotients  $E/\ker \rho$  and  $E/(\ker \rho)^\perp$  are Lie algebroids. Obviously,  $E/\ker \rho$  and  $F$  are canonically isomorphic. We call  $E/(\ker \rho)^\perp$  the **ample** Lie algebroid associated to  $E$ . It will be denoted by the symbol  $\mathcal{A}_E$ .

The inclusions

$$(\ker \rho)^\perp \subset \ker \rho \subset E$$

yield four exact sequences:

$$\begin{array}{ccccccc}
 & & & & \rho & & \\
 & \curvearrowright & & \curvearrowright & & \curvearrowright & \\
 0 & & (\ker \rho)^\perp & & E & & \frac{E}{\ker \rho} & & 0 \\
 & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\
 & 0 & & \ker \rho & & \frac{E}{(\ker \rho)^\perp} & & 0 & \\
 & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \\
 0 & & 0 & & \mathcal{G} & & 0 & & 0 \\
 & \curvearrowleft & & \curvearrowleft & & \curvearrowleft & & \curvearrowleft & \\
 & & & & & & & & 
 \end{array}$$

Here  $\mathcal{G} := \ker \rho / (\ker \rho)^\perp$ . We use the symbol  $\pi$  (resp.  $q$ ) to denote the projection  $\ker \rho \rightarrow \mathcal{G}$  (resp.  $E \rightarrow \mathcal{A}_E = E / (\ker \rho)^\perp$ ).

**1.2. Bundle of quadratic Lie algebras.** A  $C^\infty(M)$ -bilinear and skew-symmetric bracket on  $\Gamma(\mathcal{G})$  determined by the Dorfman bracket of  $E$  through the relation

$$[\pi(r), \pi(s)]^{\mathcal{G}} = \pi(r \circ s), \quad \forall r, s \in \Gamma(\ker \rho),$$

turns  $\mathcal{G}$  into a bundle of Lie algebras. Moreover, the map

$$\pi(r) \otimes \pi(s) \mapsto (r, s)$$

is a well-defined nondegenerate symmetric and ad-invariant pseudo-metric on  $\mathcal{G}$ , which we will denote by the symbol  $(-, -)^{\mathcal{G}}$ . Hence  $\mathcal{G}$  is a bundle of quadratic Lie algebras<sup>1</sup>.

Note that  $\mathcal{G}$  is also a module over  $\mathcal{A}_E$ ; the representation is given by

$$x \cdot \pi(r) = [x, q(r)], \quad \forall x \in \Gamma(\mathcal{A}_E), r \in \Gamma(\ker \rho).$$

This representation is compatible with the pseudo-metric on  $\mathcal{G}$ :

$$a(x)(\mathbf{r}, \mathbf{s})^{\mathcal{G}} = (x \cdot \mathbf{r}, \mathbf{s})^{\mathcal{G}} + (\mathbf{r}, x \cdot \mathbf{s})^{\mathcal{G}}, \quad \forall x \in \Gamma(\mathcal{A}_E), \mathbf{r}, \mathbf{s} \in \Gamma(\mathcal{G}). \quad (8)$$

**1.3. Dissections of regular Courant algebroids.** Let  $E$  be a regular Courant algebroid with characteristic distribution  $F$  and bundle of quadratic Lie algebras  $\mathcal{G}$ . A **dissection** of  $E$  is an isomorphism of vector bundles

$$\Psi : F^* \oplus \mathcal{G} \oplus F \rightarrow E$$

such that

$$(\Psi(\xi + \mathbf{r} + x), \Psi(\eta + \mathbf{s} + y)) = \frac{1}{2} \langle \xi | y \rangle + \frac{1}{2} \langle \eta | x \rangle + (\mathbf{r}, \mathbf{s})^{\mathcal{G}},$$

for all  $\xi, \eta \in \Gamma(F^*)$ ,  $\mathbf{r}, \mathbf{s} \in \Gamma(\mathcal{G})$  and  $x, y \in \Gamma(F)$ . Such an isomorphism transports the Courant algebroid structure of  $E$  to  $F^* \oplus \mathcal{G} \oplus F$ .

Each dissection of  $E$  induces three canonical maps:

(a)  $\nabla : \Gamma(F) \otimes \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G})$ :

$$\nabla_x \mathbf{r} = \text{Pr}_{\mathcal{G}}(x \circ \mathbf{r}), \quad \forall x \in \Gamma(F), \mathbf{r} \in \Gamma(\mathcal{G}); \quad (9)$$

(b)  $R : \Gamma(F) \otimes \Gamma(F) \rightarrow \Gamma(\mathcal{G})$ :

$$R(x, y) = \text{Pr}_{\mathcal{G}}(x \circ y), \quad \forall x, y \in \Gamma(F); \quad (10)$$

(c)  $\mathcal{H} : \Gamma(F) \otimes \Gamma(F) \otimes \Gamma(F) \rightarrow C^\infty(M)$ :

$$\mathcal{H}(x, y, z) = \langle \text{Pr}_{F^*}(x \circ y) | z \rangle, \quad \forall x, y, z \in \Gamma(F). \quad (11)$$

**Proposition 1.1.** (a) *The map  $\nabla$  satisfies*

$$\nabla_{fx} \mathbf{r} = f \nabla_x \mathbf{r} \quad \text{and} \quad \nabla_x (f \mathbf{r}) = f \nabla_x \mathbf{r} + (x(f)) \mathbf{r},$$

for all  $x \in \Gamma(F)$ ,  $\mathbf{r} \in \Gamma(\mathcal{G})$  and  $f \in C^\infty(M)$ .

(b) *The map  $R$  is skew-symmetric and  $C^\infty(M)$ -bilinear. It can thus be regarded as a bundle map  $\wedge^2 F \rightarrow \mathcal{G}$ .*

(c) *The map  $\mathcal{H}$  is skew-symmetric and  $C^\infty(M)$ -bilinear. It can thus be regarded as a section of  $\wedge^3 F^*$ .*

The following lemma shows that dissections always exist.

**Lemma 1.2.** *Let  $(E, (-, -), \circ, \rho)$  be a regular Courant algebroid. And set  $F = \rho(E)$ .*

<sup>1</sup>A Lie algebra is said to be quadratic if there exists a nondegenerate, ad-invariant inner product on its underlying vector space.

- (a)
- There exists a splitting  $\lambda : F \rightarrow E$  of the short exact sequence*

$$0 \rightarrow \ker \rho \rightarrow E \xrightarrow{\rho} F \rightarrow 0 \quad (12)$$

*whose image  $\lambda(F)$  is isotropic in  $E$ .*

- (b)
- Given such a splitting  $\lambda$ , there exists a unique splitting  $\sigma_\lambda : \mathcal{G} \rightarrow \ker \rho$  of the short exact sequence*

$$0 \rightarrow (\ker \rho)^\perp \rightarrow \ker \rho \xrightarrow{\pi} \mathcal{G} \rightarrow 0 \quad (13)$$

*with image  $\sigma_\lambda(\mathcal{G})$  orthogonal to  $\lambda(F)$  in  $E$ .*

- (c)
- Given a pair of splittings  $\lambda$  and  $\sigma_\lambda$  as above, the map  $\Psi_\lambda : F^* \oplus \mathcal{G} \oplus F \rightarrow E$  defined by*

$$\Psi_\lambda(\xi + \mathbf{r} + x) = \frac{1}{2}\Xi^{-1}\rho^*(\xi) + \sigma_\lambda(\mathbf{r}) + \lambda(x)$$

*is a dissection of  $E$ .*

*Proof.* (a) Take any section  $\lambda_0$  of  $\rho : E \rightarrow F$ . Consider the bundle map  $\varphi : F \rightarrow F^*$  defined by

$$\langle \varphi(x)|y \rangle = (\lambda_0(x), \lambda_0(y)),$$

where  $x, y \in \Gamma(F)$ . Then  $\lambda = \lambda_0 - \frac{1}{2}\Xi^{-1}\rho^*\varphi$  is a section of  $\rho : E \rightarrow F$  such that

$$(\lambda x, \lambda y) = 0, \quad \forall x, y \in \Gamma(F).$$

- (b) Take any section
- $\sigma_0$
- of
- $\pi : \ker \rho \rightarrow \mathcal{G}$
- . Consider the bundle map
- $\psi : \mathcal{G} \rightarrow F^*$
- defined by

$$\langle \psi(\mathbf{r})|x \rangle = (\sigma_0(\mathbf{r}), \lambda(x)),$$

where  $\mathbf{r} \in \mathcal{G}$  and  $x \in F$ . Then  $\sigma_\lambda = \sigma_0 - \Xi^{-1}\rho^*\psi$  is the only section of  $\pi : \ker \rho \rightarrow \mathcal{G}$  such that

$$(\sigma_\lambda \mathbf{r}, \lambda x) = 0, \quad \forall \mathbf{r} \in \mathcal{G}, x \in \Gamma(F).$$

- (c) This is obvious.
- 

**1.4. Covariant derivatives on regular Courant algebroids.** Let us first recall the notion of  $E$ -connections [1]. Vaisman's metric connections are a related notion [15].

**Definition 1.3.** *An  $E$ -connection (or  $E$ -covariant derivative) on  $E$  is an  $\mathbb{R}$ -bilinear map*

$$\nabla^E : \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E) : (e, s) \mapsto \nabla_e^E s$$

*such that*

$$\begin{aligned} \nabla_{fe}^E s &= f \nabla_e^E s, \\ \nabla_e^E (fs) &= f \nabla_e^E s + (\rho(e)f)s, \end{aligned}$$

*for all  $f \in C^\infty(M)$  and  $e, s \in \Gamma(E)$ .*

The next lemma shows that each regular Courant algebroid  $E$  admits  $E$ -connections.

If  $E$  is a regular Courant algebroid, there always exists a torsion-free connection  $\nabla^F$  on the integrable distribution  $F = \rho(E) \subset TM$ . Indeed, it suffices to consider the restriction to  $F$  of the Levi-Civita connection of some Riemannian metric on  $M$ . In the sequel, the symbol  $\nabla^F$  will be used to denote a chosen torsion-free connection on  $F$  and the dual connection on  $F^*$  as well.

**Lemma 1.4.** (a) *Given a dissection  $\Psi$  of a regular Courant algebroid  $E$  (identifying  $E$  with  $F^* \oplus \mathcal{G} \oplus F$ ) and the data  $\nabla$ ,  $R$  and  $\mathcal{H}$  it induces, each torsion-free connection  $\nabla^F$  on  $F$  determines an  $E$ -connection  $\nabla^E$  on  $E$  through the defining relation*

$$\nabla_{\xi+\mathbf{r}+x}^E(\eta + \mathbf{s} + y) = (\nabla_x^F \eta - \frac{1}{3}\mathcal{H}(x, y, -) + (\nabla_x \mathbf{s} + \frac{2}{3}[\mathbf{r}, \mathbf{s}]^{\mathcal{G}}) + \nabla_x^F y, \quad (14)$$

where  $x, y \in \Gamma(F)$ ,  $\xi, \eta \in \Gamma(F^*)$  and  $\mathbf{r}, \mathbf{s} \in \Gamma(\mathcal{G})$ .

(b) *This covariant derivative  $\nabla^E$  preserves the pseudo-metric:*

$$\rho(e_1)(e_2, e_3) = (\nabla_{e_1}^E e_2, e_3) + (e_2, \nabla_{e_1}^E e_3), \quad \forall e_1, e_2, e_3 \in \Gamma(E).$$

(c) *Moreover, we have*

$$\rho(\nabla_{e_1}^E e_2 - \nabla_{e_2}^E e_1) = [\rho(e_1), \rho(e_2)].$$

We omit the proof as it is straightforward.

**1.5. Naive cohomology.** Let  $E$  be a Courant algebroid. The nondegenerate pseudo-metric on  $E$  induces a nondegenerate pseudo-metric on  $\wedge^k E$ :

$$(e_1 \wedge \cdots \wedge e_k, f_1 \wedge \cdots \wedge f_k) = \begin{vmatrix} (e_1, f_1) & (e_1, f_2) & \cdots & (e_1, f_k) \\ (e_2, f_1) & (e_2, f_2) & \cdots & (e_2, f_k) \\ \vdots & \vdots & \ddots & \vdots \\ (e_k, f_1) & (e_k, f_2) & \cdots & (e_k, f_k) \end{vmatrix}.$$

The isomorphism of vector bundles  $\wedge^k E \rightarrow \wedge^k E^*$  coming from this nondegenerate pseudo-metric will be denoted by the same symbol  $\Xi$ . In the sequel,  $\wedge^k E$  and  $\wedge^k E^*$  are sometimes identified with each other by means of  $\Xi$  when it is clear from the context.

The sections of  $\wedge^k \ker \rho$  are called naive  $k$ -cochains. The operator

$$\check{d} : \Gamma(\wedge^k \ker \rho) \rightarrow \Gamma(\wedge^{k+1} \ker \rho)$$

defined by the relation

$$\begin{aligned} (\check{d}s, e_0 \wedge e_1 \wedge \cdots \wedge e_k) &= \sum_{i=0}^k (-1)^i \rho(e_i)(s, e_0 \wedge \cdots \wedge \widehat{e}_i \wedge \cdots \wedge e_k) \\ &+ \sum_{i < j} (-1)^{i+j} (s, [[e_i, e_j]] \wedge e_0 \wedge \cdots \wedge \widehat{e}_i \wedge \cdots \wedge \widehat{e}_j \wedge \cdots, e_k), \end{aligned} \quad (15)$$

where  $s \in \Gamma(\wedge^k \ker \rho)$  and  $e_0, \dots, e_k \in \Gamma(E)$ , makes  $(\Gamma(\wedge^\bullet \ker \rho), \check{d})$  a cochain complex. Its cohomology  $H_{\text{naive}}^\bullet(E)$  is called the **naive cohomology** of the Courant algebroid  $E$  [14].

**Remark 1.5.** *If  $E$  is regular, then  $\mathcal{A}_E = E/(\ker \rho)^\perp$  is a regular Lie algebroid. And it is easy to see that the cochain complexes  $(\Gamma(\wedge^\bullet \ker \rho), \check{d})$  and  $(\Gamma(\wedge^\bullet \mathcal{A}_E^*), d)$  are isomorphic. Indeed, we have*

$$\Xi(\ker \rho) = ((\ker \rho)^\perp)^0 = q^*(\mathcal{A}_E^*)$$

and

$$(\Xi^{-1} \circ q^*) \circ d = \check{d} \circ (\Xi^{-1} \circ q^*).$$

Hence the naive cohomology of  $E$  is isomorphic to the Lie algebroid cohomology of  $\mathcal{A}_E$ .

**1.6. A degree-3 characteristic class.** The map  $K : \Gamma(\wedge^3 E) \rightarrow \mathbb{R}$  defined by

$$K(e_1, e_2, e_3) = (\llbracket e_1, e_2 \rrbracket, e_3) + \text{c.p.},$$

where  $e_1, e_2, e_3 \in \Gamma(E)$ , is *not* a 3-form on  $E$  as it is not  $C^\infty(M)$ -linear.

However, it can be modified using an  $E$ -connection so that the result is  $C^\infty(M)$ -linear.

**Lemma 1.6** ([1]). *If  $\nabla^E$  is a covariant derivative on a Courant algebroid  $E$ , then*

$$C_{\nabla^E}(e_1, e_2, e_3) = \frac{1}{3}(\llbracket e_1, e_2 \rrbracket, e_3) - \frac{1}{2}(\nabla_{e_1}^E e_2 - \nabla_{e_2}^E e_1, e_3) + \text{c.p.}, \quad (16)$$

where  $e_1, e_2, e_3 \in \Gamma(E)$ , defines a 3-form on  $E$ .

The following is an analogue of the Chern-Weil construction.

**Theorem 1.7.** *Let  $E$  be a regular Courant algebroid.*

- (a) *If  $\nabla^E$  is the  $E$ -connection on  $E$  given by (14), then the 3-form  $C_{\nabla^E}$  does not depend on the chosen torsion-free connection  $\nabla^F$ , but only on the chosen dissection  $\Psi$ . It will henceforth be denoted by the symbol  $C_\Psi$ .*
- (b) *The 3-form  $C_\Psi$  is a naive 3-cocycle.*
- (c) *The cohomology class of  $C_\Psi$  does not depend on the chosen dissection  $\Psi$ .*

The class of  $C_\Psi$  is called the **characteristic class** of the Courant algebroid  $E$ . The proof of this theorem is deferred to Section 3.

**1.7. First Pontryagin class of a quadratic Lie algebroid.** We fix a regular Lie algebroid  $(\mathcal{A}, [-, -]_{\mathcal{A}}, a)$  over the base manifold  $M$ , with cohomology differential operator  $d : \Gamma(\wedge^\bullet \mathcal{A}^*) \rightarrow \Gamma(\wedge^{\bullet+1} \mathcal{A}^*)$ .

We have the short exact sequence of vector bundles

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{A} \xrightarrow{a} F \rightarrow 0,$$

where  $F = a(\mathcal{A})$  and  $\mathcal{G} = \ker a$ .

It is clear that  $\mathcal{G}$  is a bundle of Lie algebras, called the Lie-algebra bundle of  $\mathcal{A}$ . The fiberwise bracket is denoted by  $[-, -]^{\mathcal{G}}$ .

**Definition 1.8.** (a) *A regular Lie algebroid  $\mathcal{A}$  is said to be a **quadratic Lie algebroid** if the kernel  $\mathcal{G}$  of its anchor  $a$  is equipped with a fiberwise nondegenerate ad-invariant symmetric bilinear form  $(-, -)^{\mathcal{G}}$  satisfying:*

$$a(X)(\mathbf{r}, \mathbf{s})^{\mathcal{G}} = ([X, \mathbf{r}]_{\mathcal{A}}, \mathbf{s})^{\mathcal{G}} + (\mathbf{r}, [X, \mathbf{s}]_{\mathcal{A}})^{\mathcal{G}}, \quad \forall X \in \Gamma(\mathcal{A}), \mathbf{r}, \mathbf{s} \in \Gamma(\mathcal{G}).$$

- (b) *Two regular quadratic Lie algebroids  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are said to be isomorphic if there is a Lie algebroid isomorphism  $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ , whose restriction  $\mathcal{G}_1 \rightarrow \mathcal{G}_2$  to the kernels of the anchors is an isomorphism of quadratic Lie algebras.*

For example, the ample Lie algebroid  $\mathcal{A}_E = E/(\ker \rho)^\perp$  of any regular Courant algebroid  $(E, (-, -), \circ, \rho)$  is a quadratic Lie algebroid.

Now a natural question arises as to whether every quadratic Lie algebroid can be realized as the ample Lie algebroid of a Courant algebroid. It turns out that there is an obstruction. To see this, let us introduce some notations.

A **hoist** of the Lie algebroid  $\mathcal{A}$  is a section  $\kappa : F \rightarrow \mathcal{A}$  of the anchor  $a$ . The bundle map  $R^\kappa : \wedge^2 F \rightarrow \mathcal{G}$  given by

$$R^\kappa(x, y) = [\kappa(x), \kappa(y)]_{\mathcal{A}} - \kappa([x, y]), \quad \forall x, y \in \Gamma(F) \quad (17)$$



is its **curvature**.

Given a quadratic Lie algebroid endowed with a hoist, we can consider the 4-form  $\langle R^\kappa \wedge R^\kappa \rangle \in \Gamma(\wedge^4 F^*)$  given by

$$\langle R^\kappa \wedge R^\kappa \rangle(x_1, x_2, x_3, x_4) = \frac{1}{4} \sum_{\sigma \in S_4} \text{sgn}(\sigma) (R^\kappa(x_{\sigma(1)}, x_{\sigma(2)}), R^\kappa(x_{\sigma(3)}, x_{\sigma(4)}))^{\mathcal{G}},$$

for all  $x_1, x_2, x_3, x_4 \in \Gamma(F)$ .

**Lemma 1.9.** *The 4-form  $\langle R^\kappa \wedge R^\kappa \rangle$  is closed and its cohomology class in  $H^4(F)$  does not depend on the choice of  $\kappa$ .*

Following [2, 12], this cohomology class is called the **first Pontryagin class** of the quadratic Lie algebroid  $\mathcal{A}$ .

**Theorem 1.10.** *A regular quadratic Lie algebroid is isomorphic to the ample Lie algebroid associated to some regular Courant algebroid if and only if its first Pontryagin class vanishes.*

The proof of this theorem is deferred to Section 2.2.

### 1.8. Characteristic pairs and regular Courant algebroids.

**Definition 1.11.** (a) *Let  $\mathcal{A}$  be a regular quadratic Lie algebroid and let  $\kappa$  be a hoist of  $\mathcal{A}$ . A closed 3-form  $C \in \Gamma(\wedge^3 \mathcal{A}^*)$  is said to be  $\kappa$ -coherent if*

$$C(\mathbf{r}, \mathbf{s}, \mathbf{t}) = -([\mathbf{r}, \mathbf{s}]^{\mathcal{G}}, \mathbf{t})^{\mathcal{G}}; \quad (18)$$

$$C(\mathbf{r}, \mathbf{s}, \kappa(x)) = 0; \quad (19)$$

$$C(\mathbf{r}, \kappa(x), \kappa(y)) = (\mathbf{r}, R^\kappa(x, y))^{\mathcal{G}}, \quad (20)$$

for all  $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \mathcal{G}$  and  $x, y \in F$ . We also say that  $(C, \kappa)$  is a coherent pair.

- (b) *A closed 3-form  $C \in \Gamma(\wedge^3 \mathcal{A}^*)$  on a regular quadratic Lie algebroid is called **coherent** if there exists a hoist  $\kappa$  such that  $C$  is  $\kappa$ -coherent.*
- (c) *A **characteristic pair** is a couple  $(\mathcal{A}, C)$  made of a regular quadratic Lie algebroid  $\mathcal{A}$  and a coherent 3-form  $C$  on it.*

**Example 1.12.** *If  $\mathcal{A} = \mathcal{G}$  is a quadratic Lie algebra, then the only coherent 3-form is the Cartan 3-form defined by Eq. (18). If  $\mathcal{A} = \mathcal{G} \times TM$  is the product Lie algebroid with the obvious quadratic Lie algebroid structure and  $\mathcal{H}$  is a closed 3-form on  $M$ , then the 3-form  $C \in \Gamma(\wedge^3 \mathcal{A}^*)$  defined by*

$$C(\mathbf{r} + x, \mathbf{s} + y, \mathbf{t} + z) = -([\mathbf{r}, \mathbf{s}]^{\mathcal{G}}, \mathbf{t})^{\mathcal{G}} + \mathcal{H}(x, y, z),$$

for all  $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \Gamma(\mathcal{G})$  and  $x, y, z \in \Gamma(TM)$ , is coherent.

**Proposition 1.13.** *Let  $\mathcal{A}$  be a regular quadratic Lie algebroid. The following statements are equivalent.*

- (a) *The first Pontryagin class of  $\mathcal{A}$  vanishes.*
- (b) *There exists a coherent 3-form on  $\mathcal{A}$ .*
- (c) *There exists a Courant algebroid whose ample Lie algebroid is  $\mathcal{A}$ .*

Consider the subcomplex  $\mathbf{C}_{\leftarrow}^\bullet(\mathcal{A})$  of  $(\Gamma(\wedge^\bullet \mathcal{A}^*), d)$ , where  $\mathbf{C}_{\leftarrow}^k(\mathcal{A})$  consists of the sections of  $\wedge^k \mathcal{A}^*$  which are annihilated by all sections of  $\wedge^k \mathcal{G}$ . Its cohomology groups are denoted  $H_{\leftarrow}^\bullet(\mathcal{A})$ . Given two coherent 3-forms  $C_1$  and  $C_2$  on  $\mathcal{A}$ , we have  $C_2 - C_1 \in \mathbf{C}_{\leftarrow}^3(\mathcal{A})$ .

**Definition 1.14.** *Two characteristic pairs  $(\mathcal{A}_1, C_1)$  and  $(\mathcal{A}_2, C_2)$  are said to be **equivalent** if there exists an isomorphism of quadratic Lie algebroids  $\sigma : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  such that  $[C_1 - \sigma^*C_2] = 0 \in H^3_{\leftrightarrow}(\mathcal{A}_1)$ . The equivalence class of a characteristic pair  $(\mathcal{A}, C)$  will be denoted  $[(\mathcal{A}, C)]$ .*

Our main result is the following:

**Theorem 1.15.** (a) *There is a one-to-one correspondence between regular Courant algebroids up to isomorphisms and equivalence classes of characteristic pairs.*  
 (b) *If  $E$  is a Courant algebroid corresponding to the characteristic pair  $(\mathcal{A}, C)$ , then the characteristic class of  $E$  is equal to  $[C] \in H^3(\mathcal{A})$ .*

The proof is postponed to Section 3.

## 2. STANDARD STRUCTURES

**2.1. Standard Courant algebroid structures on  $F^* \oplus \mathcal{G} \oplus F$ .** Let  $F$  be an integrable subbundle of  $TM$  and  $\mathcal{G}$  be a bundle of quadratic Lie algebras over  $M$ .

In this section, we are only interested in those Courant algebroid structures on  $F^* \oplus \mathcal{G} \oplus F$  whose anchor map is

$$\rho(\xi_1 + \mathbf{r}_1 + x_1) = x_1, \quad (21)$$

whose pseudo-metric is

$$\langle \xi_1 + \mathbf{r}_1 + x_1, \xi_2 + \mathbf{r}_2 + x_2 \rangle = \frac{1}{2} \langle \xi_1 | x_2 \rangle + \frac{1}{2} \langle \xi_2 | x_1 \rangle + (\mathbf{r}_1, \mathbf{r}_2)^{\mathcal{G}}, \quad (22)$$

and whose Dorfman bracket satisfies

$$\text{Pr}_{\mathcal{G}}(\mathbf{r}_1 \circ \mathbf{r}_2) = [\mathbf{r}_1, \mathbf{r}_2]^{\mathcal{G}}, \quad (23)$$

where  $\xi_1, \xi_2 \in F^*$ ,  $\mathbf{r}_1, \mathbf{r}_2 \in \mathcal{G}$ , and  $x_1, x_2 \in F$ . We call them **standard** Courant algebroid structures on  $F^* \oplus \mathcal{G} \oplus F$ .

Given such a standard Courant algebroid structure on  $F^* \oplus \mathcal{G} \oplus F$ , we define  $\nabla$ ,  $R$ , and  $\mathcal{H}$  as in Eqs. (9)-(11). The following Lemma shows that the Dorfman bracket on  $\Gamma(F^* \oplus \mathcal{G} \oplus F)$  can be recovered from  $\nabla$ ,  $R$ , and  $\mathcal{H}$ .

**Lemma 2.1.** *Let  $\mathcal{P} : \Gamma(\mathcal{G}) \otimes \Gamma(\mathcal{G}) \rightarrow \Gamma(F^*)$  and  $\mathcal{Q} : \Gamma(F) \otimes \Gamma(\mathcal{G}) \rightarrow \Gamma(F^*)$  be the maps defined by*

$$\langle \mathcal{P}(\mathbf{r}_1, \mathbf{r}_2) | y \rangle = 2(\mathbf{r}_2, \nabla_y \mathbf{r}_1)^{\mathcal{G}} \quad (24)$$

and

$$\langle \mathcal{Q}(x, \mathbf{r}) | y \rangle = (\mathbf{r}, R(x, y))^{\mathcal{G}}. \quad (25)$$

Then we have

$$x_1 \circ x_2 = \mathcal{H}(x_1, x_2, -) + R(x_1, x_2) + [x_1, x_2], \quad (26)$$

$$\mathbf{r}_1 \circ \mathbf{r}_2 = \mathcal{P}(\mathbf{r}_1, \mathbf{r}_2) + [\mathbf{r}_1, \mathbf{r}_2]^{\mathcal{G}}, \quad (27)$$

$$\xi_1 \circ \mathbf{r}_2 = \mathbf{r}_1 \circ \xi_2 = \xi_1 \circ \xi_2 = 0, \quad (28)$$

$$x_1 \circ \xi_2 = L_{x_1} \xi_2, \quad (29)$$

$$\xi_1 \circ x_2 = -L_{x_2} \xi_1 + d^F \langle \xi_1 | x_2 \rangle, \quad (30)$$

$$x_1 \circ \mathbf{r}_2 = -\mathbf{r}_2 \circ x_1 = -2\mathcal{Q}(x_1, \mathbf{r}_2) + \nabla_{x_1} \mathbf{r}_2, \quad (31)$$

for all  $\xi_1, \xi_2 \in \Gamma(F^*)$ ,  $\mathbf{r}_1, \mathbf{r}_2 \in \Gamma(\mathcal{G})$ ,  $x_1, x_2 \in \Gamma(F)$ . Here  $d^F : C^\infty(M) \rightarrow \Gamma(F^*)$  denotes the leafwise de Rham differential.

*Proof.* Eqs. (26) and (28) are quite obvious. To get Eq. (27), we observe that, for all  $x \in \Gamma(F)$ ,

$$\begin{aligned} \langle \text{Pr}_{F^*}(\mathbf{r}_1 \circ \mathbf{r}_2) | x \rangle &= 2(\text{Pr}_{F^*}(\mathbf{r}_1 \circ \mathbf{r}_2), x) = 2(\mathbf{r}_1 \circ \mathbf{r}_2, x) = -2(\mathbf{r}_2, \mathbf{r}_1 \circ x) \\ &= 2(\mathbf{r}_2, x \circ \mathbf{r}_1) = 2(\mathbf{r}_2, \text{Pr}_{\mathcal{G}}(x \circ \mathbf{r}_1)) = 2(\mathbf{r}_2, \nabla_x \mathbf{r}_1). \end{aligned}$$

Eq. (31) can be proved similarly. To prove Eq. (29), we only need to show that  $\text{Pr}_{\mathcal{G}}(x_1 \circ \xi_2) = 0$ . In fact, for all  $\mathbf{r} \in \Gamma(\mathcal{G})$ ,

$$(\text{Pr}_{\mathcal{G}}(x_1 \circ \xi_2), \mathbf{r})^{\mathcal{G}} = (x_1 \circ \xi_2, \mathbf{r}) = L_{x_1}(\xi_2, \mathbf{r}) - (\xi_2, x_1 \circ \mathbf{r}) = -\frac{1}{2} \langle \xi_2 | \text{Pr}_F(x_1 \circ \mathbf{r}) \rangle = 0.$$

Eq. (30) follows directly from Eq. (29).  $\square$

**Proposition 2.2.** *The following identities hold:*

$$L_x(\mathbf{r}, \mathbf{s})^{\mathcal{G}} = (\nabla_x \mathbf{r}, \mathbf{s})^{\mathcal{G}} + (\mathbf{r}, \nabla_x \mathbf{s})^{\mathcal{G}}, \quad (32)$$

$$\nabla_x[\mathbf{r}, \mathbf{s}]^{\mathcal{G}} = [\nabla_x \mathbf{r}, \mathbf{s}]^{\mathcal{G}} + [\mathbf{r}, \nabla_x \mathbf{s}]^{\mathcal{G}}, \quad (33)$$

$$(\nabla_x R(y, z) - R([x, y], z)) + c.p. = 0, \quad (34)$$

$$\nabla_x \nabla_y \mathbf{r} - \nabla_y \nabla_x \mathbf{r} - \nabla_{[x, y]} \mathbf{r} = [R(x, y), \mathbf{r}]^{\mathcal{G}}, \quad (35)$$

for all  $x, y, z \in \Gamma(F)$  and  $\mathbf{r}, \mathbf{s} \in \Gamma(\mathcal{G})$ .

Moreover, we have

$$d^F \mathcal{H} = \langle R \wedge R \rangle, \quad (36)$$

where  $\langle R \wedge R \rangle$  denotes the 4-form on  $F$  given by

$$\langle R \wedge R \rangle(x_1, x_2, x_3, x_4) = \frac{1}{4} \sum_{\sigma \in S_4} \text{sgn}(\sigma) (R(x_{\sigma(1)}, x_{\sigma(2)}), R(x_{\sigma(3)}, x_{\sigma(4)}))^{\mathcal{G}},$$

where  $x_1, x_2, x_3, x_4 \in F$ .

*Proof.* Applying the expressions found for the Dorfman bracket in Lemma 2.1, Eq. (32) follows from

$$L_x(\mathbf{r}, \mathbf{s})^{\mathcal{G}} = (x \circ \mathbf{r}, \mathbf{s})^{\mathcal{G}} + (\mathbf{r}, x \circ \mathbf{s})^{\mathcal{G}},$$

Eq. (33) from

$$x \circ (\mathbf{r} \circ \mathbf{s}) = (x \circ \mathbf{r}) \circ \mathbf{s} + \mathbf{r} \circ (x \circ \mathbf{s}),$$

Eqs. (34) and (36) from

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z),$$

and Eq. (35) from

$$x \circ (y \circ \mathbf{r}) = (x \circ y) \circ \mathbf{r} + y \circ (x \circ \mathbf{r}). \quad \square$$

In summary, we have proved the following

**Theorem 2.3.** *A Courant algebroid structure on  $F^* \oplus \mathcal{G} \oplus F$ , with pseudo-metric (22) and anchor map (21), and satisfying Eq. (23), is completely determined by an  $F$ -connection  $\nabla$  on  $\mathcal{G}$ , a bundle map  $R : \wedge^2 F \rightarrow \mathcal{G}$ , and a 3-form  $\mathcal{H} \in \Gamma(\wedge^3 F^*)$  satisfying the compatibility conditions (32)-(36). The Dorfman bracket on  $F^* \oplus \mathcal{G} \oplus F$  is then given by Eqs. (26)-(31).*

This Courant algebroid structure will be called the standard Courant algebroid determined by the quintuple  $(F, \mathcal{G}; \nabla, R, \mathcal{H})$ .

**2.2. Standard quadratic Lie algebroids.** We assume again that  $F$  is an integrable subbundle of  $TM$  and  $\mathcal{G}$  is a bundle of quadratic Lie algebras over  $M$ .

**Proposition 2.4.** *Let  $\mathcal{A}$  be a quadratic Lie algebroid with anchor  $a : \mathcal{A} \rightarrow TM$  such that  $a(\mathcal{A}) = F$ , and  $\ker a$  is isomorphic, as a bundle of quadratic Lie algebras, to  $\mathcal{G}$ . Then any hoist of  $\mathcal{A}$  determines an isomorphism  $\mathcal{A} \cong \mathcal{A}^s = \mathcal{G} \oplus F$ . The transported Lie bracket on  $\Gamma(\mathcal{A}^s)$  is completely determined by an  $F$ -connection  $\nabla$  on  $\mathcal{G}$  and a bundle map  $R : \wedge^2 F \rightarrow \mathcal{G}$  satisfying the conditions (32)-(35), through the following relations:*

$$[\mathbf{r}, \mathbf{s}]_{\mathcal{A}} = [\mathbf{r}, \mathbf{s}]^{\mathcal{G}}, \quad [x, y]_{\mathcal{A}} = R(x, y) + [x, y], \quad [x, \mathbf{r}]_{\mathcal{A}} = \nabla_x \mathbf{r},$$

for all  $\mathbf{r}, \mathbf{s} \in \Gamma(\mathcal{G})$ ,  $x, y \in \Gamma(F)$ .

This quadratic Lie algebroid  $\mathcal{A}^s$  will be called the standard quadratic Lie algebroid determined by the quadruple  $(F, \mathcal{G}; \nabla, R)$ .

Although it is similar to the proof of Corollary 3.2. in [2], we now sketch a proof of Theorem 1.10 for completeness.

*Proof of Theorem 1.10.* Given a regular Courant algebroid  $E$ , choose a dissection to identify it to  $F^* \oplus \mathcal{G} \oplus F$ . Then, according to Eq. (36), the Pontryagin 4-cocycle of the ample Lie algebroid  $\mathcal{A}_E$  associated to  $E$  is a coboundary. Conversely, given a quadratic Lie algebroid  $\mathcal{A}$  with anchor  $a : \mathcal{A} \rightarrow TM$  such that  $a\mathcal{A} = F$  and  $\ker a$  is isomorphic to  $\mathcal{G}$  as a bundle of quadratic Lie algebras, assume there is a hoist  $\kappa$  of the Lie algebroid  $\mathcal{A}$  such that the Pontryagin 4-cocycle  $\langle R^\kappa \wedge R^\kappa \rangle$  is a coboundary, say  $d^F \mathcal{H}$  for some  $\mathcal{H} \in \Gamma(\wedge^3 F^*)$ . We define an  $F$ -connection  $\nabla^\kappa$  on  $\mathcal{G}$  by setting

$$\nabla_x^\kappa \mathbf{r} = [\kappa(x), \mathbf{r}]_{\mathcal{A}}, \quad \forall x \in \Gamma(F), \mathbf{r} \in \Gamma(\mathcal{G}). \quad (37)$$

The quintuple  $(F, \mathcal{G}; \nabla^\kappa, R^\kappa, \mathcal{H})$  satisfies all conditions in Theorem 2.3 and it is clear that  $\mathcal{A}$  is the ample Lie algebroid associated to the Courant algebroid determined by this quintuple.  $\square$

**2.3. Standard 3-forms on  $\mathcal{A}^s$  and  $E^s$ .** Let  $\mathcal{A}^s$  denote the standard quadratic Lie algebroid determined by the quadruple  $(F, \mathcal{G}; \nabla, R)$ . Given any  $\mathcal{H} \in \Gamma(\wedge^3 F^*)$ , we define a 3-form  $C^s \in \Gamma(\wedge^3 (\mathcal{A}^s)^*)$  by

$$C^s(\mathbf{r}+x, \mathbf{s}+y, \mathbf{t}+z) = \mathcal{H}(x, y, z) - ([\mathbf{r}, \mathbf{s}]^{\mathcal{G}}, \mathbf{t})^{\mathcal{G}} + (R(x, y), \mathbf{t})^{\mathcal{G}} + (R(y, z), \mathbf{r})^{\mathcal{G}} + (R(z, x), \mathbf{s})^{\mathcal{G}}, \quad (38)$$

for all  $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \Gamma(\mathcal{G})$  and  $x, y, z \in \Gamma(F)$ .

**Proposition 2.5.** *The following statements are equivalent:*

- (a)  $\mathcal{H}$  satisfies Eq. (36);
- (b)  $C^s$  is a closed 3-form;
- (c)  $C^s$  is a coherent 3-form.

*Proof.* Straightforward calculations lead to

$$(\check{d}C^s)(x, y, z, w) = (d^F \mathcal{H})(x, y, z, w) - \langle R \wedge R \rangle(x, y, z, w), \quad (39)$$

$$(\check{d}C^s)(x, y, z, \mathbf{r}) = ((\nabla_x R)(y, z) - R([x, y], z)) + \text{c.p.}, \mathbf{r})^{\mathcal{G}}, \quad (40)$$

$$(\check{d}C^s)(x, y, \mathbf{r}, \mathbf{s}) = 0, \quad (41)$$

$$(\check{d}C^s)(x, \mathbf{r}, \mathbf{s}, \mathbf{t}) = -L_x([\mathbf{r}, \mathbf{s}]^{\mathcal{G}}, \mathbf{t})^{\mathcal{G}} + ([\nabla_x \mathbf{r}, \mathbf{s}]^{\mathcal{G}} + [\mathbf{r}, \nabla_x \mathbf{s}]^{\mathcal{G}}, \mathbf{t})^{\mathcal{G}} + ([\mathbf{r}, \mathbf{s}]^{\mathcal{G}}, \nabla_x \mathbf{t})^{\mathcal{G}}, \quad (42)$$

$$(\check{d}C^s)(\mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}) = 0, \quad (43)$$

for all  $x, y, z, w \in \Gamma(F)$  and  $\mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t} \in \Gamma(\mathcal{G})$ .

The right hand side of Eqs. (40) and (42) vanish due to Eqs. (32)-(35). Thus  $\check{d}C^s = 0$  if and only if the right hand side of Eq. (39) vanishes. The latter is equivalent to Eq. (36). This proves the equivalence between the first two statements. The equivalence between the last two statements follows from Definition 1.11 and Eq. (38).  $\square$

Let  $E^s$  be the standard Courant algebroid determined by the quintuple  $(F, \mathcal{G}; \nabla, R, \mathcal{H})$ . Clearly its associated ample quadratic Lie algebroid is the standard quadratic Lie algebroid  $\mathcal{A}^s$  determined by the quadruple  $(F, \mathcal{G}; \nabla, R)$ .

It follows from Proposition 2.5 and the compatibility conditions satisfied by  $\nabla, R, \mathcal{H}$  that the form  $C^s$  defined by formula (38) is a closed — and thus coherent — 3-form on  $\mathcal{A}^s$ . Since the cocycles of the Lie algebroid  $\mathcal{A}^s$  can be identified to the naive cocycles on  $E^s$ , we obtain a naive 3-cocycle on  $E^s$  called the **standard 3-form** on  $E^s$ .

At this point, Proposition 1.13 and the following result are quite obvious.

**Proposition 2.6.** *Let  $E^s$  be the standard Courant algebroid determined by the quintuple  $(F, \mathcal{G}; \nabla, R, \mathcal{H})$ , and  $C^s$  the 3-form on  $\mathcal{A}^s$  defined by (38). Then the characteristic class of  $E^s$  is equal to  $[C^s] \in H^3(\mathcal{A}^s)$ .*

#### 2.4. Isomorphic standard Courant algebroid structures on $F^* \oplus \mathcal{G} \oplus F$ .

**Proposition 2.7.** *Given two different standard Courant algebroid structures  $E_1^s$  and  $E_2^s$  on  $F^* \oplus \mathcal{G} \oplus F$ , any isomorphism of Courant algebroids  $\Theta : E_1^s \rightarrow E_2^s$  is of the form*

$$\Theta(\xi + \mathbf{r} + x) = (\xi + \beta(x) - 2\varphi^*\tau(\mathbf{r})) + (\tau(\mathbf{r}) + \varphi(x)) + x, \quad (44)$$

for  $\xi \in F^*$ ,  $\mathbf{r} \in \mathcal{G}$  and  $x \in F$ . Here  $\tau$  is an automorphism of the bundle of quadratic Lie algebras  $\mathcal{G}$  and  $\varphi : F \rightarrow \mathcal{G}$  and  $\beta : F \rightarrow F^*$  are bundle maps satisfying the following compatibility conditions:

$$\frac{1}{2}\langle \beta(x)|y \rangle + \frac{1}{2}\langle x|\beta(y) \rangle + (\varphi(x), \varphi(y))^{\mathcal{G}} = 0, \quad (45)$$

$$\nabla_x^2 \tau(\mathbf{r}) - \tau \nabla_x^1(\mathbf{r}) = [\tau(\mathbf{r}), \varphi(x)]^{\mathcal{G}}, \quad (46)$$

$$R^2(x, y) - \tau R^1(x, y) = \tau(\nabla_y^1 \tau^{-1} \varphi(x) - \nabla_x^1 \tau^{-1} \varphi(y)) + \varphi[x, y] + [\varphi(x), \varphi(y)]^{\mathcal{G}}, \quad (47)$$

$$\begin{aligned} & \mathcal{H}^2(x, y, z) - \mathcal{H}^1(x, y, z) + 2(\varphi(x), [\varphi(y), \varphi(z)]^{\mathcal{G}})^{\mathcal{G}} \\ & = (2(\varphi(x), \tau \nabla_y^1 \tau^{-1} \varphi(z) - \tau R^1(y, z))^{\mathcal{G}} + L_x \langle y|\beta(z) \rangle + \langle x|\beta[y, z] \rangle) + c.p., \end{aligned} \quad (48)$$

for all  $x, y, z \in \Gamma(F)$  and  $\mathbf{r} \in \Gamma(\mathcal{G})$ .

*Proof.* Consider the bundle maps

$$\begin{aligned} \tau : \quad \mathcal{G} &\hookrightarrow E_1^s \xrightarrow{\Theta} E_2^s \twoheadrightarrow \mathcal{G}; \\ \varphi : \quad F &\hookrightarrow E_1^s \xrightarrow{\Theta} E_2^s \twoheadrightarrow \mathcal{G}; \\ \beta : \quad F &\hookrightarrow E_1^s \xrightarrow{\Theta} E_2^s \twoheadrightarrow F^*; \\ \gamma : \quad \mathcal{G} &\hookrightarrow E_1^s \xrightarrow{\Theta} E_2^s \twoheadrightarrow F^*, \end{aligned}$$

where  $E_i^s$  stands for  $F^* \oplus \mathcal{G} \oplus F$ . It is clear that  $\Theta(x) = \beta(x) + \varphi(x) + x$ , for all  $x \in F$ . Since  $\Theta$  respects the pseudo-metric, we have  $(\Theta(x), \Theta(y)) = 0$ , for all  $x, y \in F$  and Eq. (45) follows. Since  $\Theta(\mathbf{r}) = \gamma(\mathbf{r}) + \tau(\mathbf{r})$  for all  $\mathbf{r} \in \mathcal{G}$  and  $(\Theta(\mathbf{r}), \Theta(x)) = (\mathbf{r}, x) = 0$ , we have  $\frac{1}{2}\langle \gamma(\mathbf{r})|x \rangle + (\tau(\mathbf{r}), \varphi(x))^{\mathcal{G}} = 0$ . Hence  $\gamma = -2\varphi^*\tau$ . From Eq. (4), it is easy to get that  $\Theta(\xi) = \xi$ , for all  $\xi \in F^*$ . Thus  $\Theta$  must be of the form (44). Since  $\Theta$  takes the Dorfman bracket of  $E_1^s$  to  $E_2^s$ , Eqs. (46)-(48) follow from Lemma 2.1.  $\square$

The following two technical lemmas follow from direct verifications.

**Lemma 2.8.** *Given a bundle map  $J : F \rightarrow \mathcal{G}$ , define  $\Phi_J \in \mathbf{C}_{\rightarrow}^2(\mathcal{A}^s)$  by*

$$\Phi_J(\mathbf{r} + x, \mathbf{s} + y) = (\mathbf{r}, J(y))^{\mathcal{G}} - (\mathbf{s}, J(x))^{\mathcal{G}}, \quad (49)$$

for all  $x, y \in F$  and  $\mathbf{r}, \mathbf{s} \in \mathcal{G}$ . Then the differential of  $\Phi_J$  is given by:

$$\begin{aligned} (d\Phi_J)(\mathbf{r}_1 + x_1, \mathbf{r}_2 + x_2, \mathbf{r}_3 + x_3) &= (\mathbf{r}_1, \nabla_{x_3} J(x_2) - \nabla_{x_2} J(x_3) + J[x_2, x_3])^{\mathcal{G}} \\ &\quad - (J(x_1), R(x_2, x_3) + [\mathbf{r}_2, \mathbf{r}_3]^{\mathcal{G}})^{\mathcal{G}} + c.p., \end{aligned}$$

for all  $x_1, x_2, x_3 \in \Gamma(F)$  and  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \Gamma(\mathcal{G})$ .

**Lemma 2.9.** *Given a bundle map  $K : F \rightarrow F^*$ , define  $\Psi_K \in \mathbf{C}_{\rightarrow}^2(\mathcal{A}^s)$  by*

$$\Psi_K(\mathbf{r} + x, \mathbf{s} + y) = \langle x | K(y) \rangle - \langle y | K(x) \rangle, \quad (50)$$

for all  $x, y \in F$  and  $\mathbf{r}, \mathbf{s} \in \mathcal{G}$ . Then the differential of  $\Psi_K$  is given by:

$$d\Psi_K(\mathbf{r} + x, \mathbf{s} + y, \mathbf{t} + z) = (\langle x | L_z K(y) - L_y K(z) + K([y, z]) \rangle + \langle [x, y] | K(z) \rangle) + c.p.,$$

for all  $x_1, x_2, x_3 \in \Gamma(F)$  and  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \Gamma(\mathcal{G})$ .

**Proposition 2.10.** *The isomorphism  $\Theta : E_1^s \rightarrow E_2^s$  of Proposition 2.7 yields an isomorphism  $\tilde{\Theta} : \mathcal{A}_1^s \rightarrow \mathcal{A}_2^s$  between the associated ample Lie algebroid structures on  $\mathcal{G} \oplus F$ :*

$$\tilde{\Theta}(\mathbf{r} + x) = (\tau(\mathbf{r}) + \varphi(x)) + x, \quad \forall \mathbf{r} \in \mathcal{G}, x \in F.$$

If  $C_1^s$  and  $C_2^s$  denote the standard coherent 3-forms (defined in Section 2.3) on  $\mathcal{A}_1^s$  and  $\mathcal{A}_2^s$ , respectively, then  $\tilde{\Theta}^* C_2^s - C_1^s = d(\frac{1}{2}\Psi_{\beta} + \Phi_{\tau^{-1}\varphi})$ .

*Proof.* The proof is a direct but cumbersome calculation using Proposition 2.7, Lemmas 2.8 and 2.9, and Eq. (45).  $\square$

### 3. PROOFS OF THE MAIN THEOREMS

**3.1. Proof of Theorem 1.7.** For  $C_{\nabla E}$  defined in (16), it is easy to see the following equivalent formula

$$\begin{aligned} C_{\nabla E}(e_1, e_2, e_3) &= (e_1 \circ e_2, e_3) - \frac{1}{2}\rho(e_1)(e_2, e_3) + \frac{1}{2}\rho(e_2)(e_3, e_1) - \frac{1}{2}\rho(e_3)(e_1, e_2) \\ &\quad - \frac{1}{2}(\nabla_{e_1}^E e_2 - \nabla_{e_2}^E e_1, e_3) + c.p.. \end{aligned}$$

Using the expressions of the Dorfman brackets given by Proposition 2.1, one obtains

$$\begin{aligned} C_{\nabla E}(\xi + \mathbf{r} + x, \eta + \mathbf{s} + y, \zeta + \mathbf{t} + z) &= (R(x, y), \mathbf{t})^{\mathcal{G}} + (R(y, z), \mathbf{r})^{\mathcal{G}} + (R(z, x), \mathbf{s})^{\mathcal{G}} \\ &\quad - ([\mathbf{r}, \mathbf{s}]^{\mathcal{G}}, \mathbf{t})^{\mathcal{G}} + \mathcal{H}(x, y, z), \quad (51) \end{aligned}$$

for all  $\xi, \eta, \zeta \in \Gamma(F^*)$ ,  $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \Gamma(\mathcal{G})$ ,  $x, y, z \in \Gamma(F)$ . This shows that  $C_{\nabla E}$  coincides with the standard 3-form  $C^s$  on  $E^s$  (defined by Eq. (38)). Hence it is naive, closed, does not depend on the choice of  $\nabla^F$  and we get Statements (a) and (b).

Finally, we prove Statement (c). Any two dissections  $\Psi_1$  and  $\Psi_2$  of  $E$  induce two different Courant algebroid structures on  $E_i^s = F^* \oplus \mathcal{G} \oplus F$ , but  $E_1^s$  and  $E_2^s$  are isomorphic. On the other hand, the 3-form  $C_{\Psi_i}$  on  $E$  is the pull-back of a standard naive 3-form  $(C^s)_i$  on  $E_i^s$ . Therefore, by Proposition 2.10 and Remark 1.5,  $C_{\Psi_1} - C_{\Psi_2}$  must be a coboundary  $\check{d}\varphi$ , for some naive 2-form  $\varphi$  on  $E$ .

**3.2. Proof of Theorem 1.15.** We first show how to construct a Courant algebroid out of a characteristic pair. Let  $\mathcal{A}$  be a regular quadratic Lie algebroid and let  $(C, \kappa)$  be a coherent pair. Set  $F = a(\mathcal{A})$  and  $\mathcal{G} = \ker a$ . Together,  $\mathcal{A}$  and  $(C, \kappa)$  induce the  $F$ -connection  $\nabla^\kappa$  on  $\mathcal{G}$ ; the bundle map  $R^\kappa : \wedge^2 F \rightarrow \mathcal{G}$ ; and the 3-form  $\mathcal{H}^{(C, \kappa)} \in \Gamma(\wedge^3 F^*)$  defined by  $\mathcal{H}^{(C, \kappa)}(x, y, z) = C(\kappa(x), \kappa(y), \kappa(z))$ , for  $x, y, z \in \Gamma(F)$ . By Proposition 2.5,  $C$  being a  $\kappa$ -coherent 3-form implies that  $\mathcal{H}^{(C, \kappa)}$  satisfies Eq. (36) and thus the quintuple  $(F, \mathcal{G}; \nabla^\kappa, R^\kappa, \mathcal{H}^{(C, \kappa)})$  satisfies the requirements to construct a standard Courant algebroid structure on the bundle  $E^s = F^* \oplus \mathcal{G} \oplus F$  described by Theorem 2.3. We shall denote such a Courant algebroid by  $E^s(\mathcal{A}; C, \kappa)$ . If  $\mathcal{A}$  is understood, we just write  $E^s(C, \kappa)$ .

In the sequel, the isomorphism class of a regular Courant algebroid  $E$  will be denoted by  $[E]$ . Similarly, the equivalence class of a characteristic pair  $(\mathcal{A}, C)$  is denoted by  $[(\mathcal{A}, C)]$ .

Define a map  $\mathfrak{i}$  from the set of equivalence classes of characteristic pairs to the set of isomorphism classes of regular Courant algebroids sending  $[(\mathcal{A}, C)]$  to  $[E^s(\mathcal{A}; C, \kappa)]$ . Here  $\kappa$  is any hoist of  $\mathcal{A}$  such that  $(C, \kappa)$  is a coherent pair on  $\mathcal{A}$ .

**Proposition 3.1.** *The map  $\mathfrak{i}([(A, C)]) = [E^s(\mathcal{A}; C, \kappa)]$  is well-defined.*

Before we can prove it, let us establish some useful facts. First, we study the change of hoists. It is obvious that for any two hoists of  $\mathcal{A}$ :  $\kappa$  and  $\kappa'$ , there exists a unique map  $J : F \rightarrow \mathcal{G}$  such that  $\kappa'(x) = \kappa(x) - J(x)$ ,  $\forall x \in F$ . Recall the notation  $\nabla^\kappa$  defined by formula (37). It is easy to check that

$$\begin{aligned} \nabla_x^{\kappa'} \mathbf{r} - \nabla_x^\kappa \mathbf{r} &= [\mathbf{r}, J(x)]^{\mathcal{G}}, \\ R^{\kappa'}(x, y) - R^\kappa(x, y) &= [J(x), J(y)] - \nabla_x^\kappa J(y) + \nabla_y^\kappa J(x) + J[x, y], \end{aligned}$$

for all  $x, y \in \Gamma(F)$ ,  $\mathbf{r} \in \Gamma(\mathcal{G})$ .

By Lemma 2.8 and the above two equalities, one can easily prove the following lemma.

**Lemma 3.2.** *If  $(C, \kappa)$  is a coherent pair, then so is the pair  $(C + d\Phi_J, \kappa - J)$ . Here  $\Phi_J$  is a 2-form on  $\mathcal{A}$  defined by (49).*

An important fact is the following

**Lemma 3.3.** *The Courant algebroids  $E^s(C, \kappa)$  and  $E^s(C + d\Phi_J, \kappa - J)$  are isomorphic.*

*Proof.* Write  $C^1 = C$ ,  $C^2 = C + d\Phi_J$ ,  $\kappa^1 = \kappa$ ,  $\kappa^2 = \kappa - J$  and let  $E_i = E^s(C^i, \kappa^i)$ . Clearly  $E_i$  is determined by the quintuple  $(F, \mathcal{G}; \nabla^i, R^i, \mathcal{H}^i)$ , where  $\nabla^i = \nabla^{\kappa^i}$ ,  $R^i = R^{\kappa^i}$  and  $\mathcal{H}^i = \mathcal{H}^{(C^i, \kappa^i)}$ . We have the following equalities:

$$\begin{aligned} \nabla_x^2 \mathbf{r} - \nabla_x^1 \mathbf{r} &= [\mathbf{r}, J(x)]^{\mathcal{G}}, \\ R^2(x, y) - R^1(x, y) &= [J(x), J(y)]^{\mathcal{G}} - \nabla_x^1 J(y) + \nabla_y^1 J(x) + J[x, y], \\ \mathcal{H}^2(x, y, z) - \mathcal{H}^1(x, y, z) &= (J(x), \nabla_y^1 J(z) - \nabla_z^1 J(y) - J[y, z] - 2R^1(y, z))^{\mathcal{G}} \\ &\quad - 2([J(x), J(y)]^{\mathcal{G}}, J(z))^{\mathcal{G}} + \text{c.p.}, \end{aligned}$$

for all  $x, y, z \in \Gamma(F)$ ,  $\mathbf{r} \in \Gamma(\mathcal{G})$ . Define a map  $I : E_1 \rightarrow E_2$  by

$$I(\xi + \mathbf{r} + x) = (\xi + \beta(x) - 2J^*(\mathbf{r})) + (\mathbf{r} + J(x)) + x,$$

where  $\xi \in F^*$ ,  $\mathbf{r} \in \mathcal{G}$ ,  $x \in F$ , and  $\beta : F \rightarrow F^*$  denotes the map  $-J^* \circ J$ . According to Proposition 2.7 and the above equalities,  $I$  is a Courant algebroid isomorphism.  $\square$

**Lemma 3.4.** *Assume that  $(C, \kappa)$  and  $(C + d\Phi_J, \kappa)$  are both coherent pairs, for some bundle map  $J : F \rightarrow \mathcal{G}$ . Then the Courant algebroids  $E^s(C, \kappa)$  and  $E^s(C + d\Phi_J, \kappa)$  are isomorphic.*

*Proof.* Let  $E_1 = E^s(C, \kappa)$ ,  $E_2 = E^s(C + d\Phi_J, \kappa)$ . Clearly,  $E_i$  is the standard Courant algebroid determined by the data  $\nabla = \nabla^\kappa$ ,  $R = R^\kappa$  and  $\mathcal{H}^i$ , where  $\mathcal{H}^1 = \mathcal{H}^{(C, \kappa)}$ ,  $\mathcal{H}^2 = \mathcal{H}^{(C + d\Phi_J, \kappa)}$ . Since  $(C, \kappa)$  and  $(C + d\Phi_J, \kappa)$  are both coherent pairs, we have  $\iota_{\mathbf{r}}(d\Phi_J) = 0$ , for all  $\mathbf{r} \in \mathcal{G}$ . By Lemma 2.8, one can check that this is equivalent to two conditions:

$$J(F) \subset Z(\mathcal{G}) \quad \text{and} \quad \nabla_x J(y) - \nabla_y J(x) - J[x, y] = 0, \quad \forall x, y \in \Gamma(F).$$

Using these, one has

$$\mathcal{H}^2(x, y, z) - \mathcal{H}^1(x, y, z) = -(J(x), R(y, z))^{\mathcal{G}} + \text{c.p.},$$

for all  $x, y, z \in \Gamma(F)$ . Then by Proposition 2.7, the map  $I : E_1 \rightarrow E_2$  defined by

$$I(\xi + \mathbf{r} + x) = (\xi + \beta(x) - 2\varphi^*(\mathbf{r})) + (\mathbf{r} + \varphi(x)) + x,$$

where  $\xi \in F^*$ ;  $\mathbf{r} \in \mathcal{G}$ ;  $x \in F$ ;  $\varphi = \frac{1}{2}J$ ; and  $\beta = -\frac{1}{4}J^* \circ J$ , is an isomorphism of Courant algebroids.  $\square$

The following Corollary implies that  $[E^s(\mathcal{A}; C, \kappa)]$  actually does not depend on the choice of the hoist  $\kappa$  of  $\mathcal{A}$ .

**Corollary 3.5.** *If  $(C, \kappa)$  and  $(C, \kappa')$  are both coherent pairs on  $\mathcal{A}$ , then the Courant algebroids  $E^s(C, \kappa)$  and  $E^s(C, \kappa')$  are isomorphic.*

*Proof.* Suppose that  $\kappa' = \kappa + J$ , for some bundle map  $J : F \rightarrow \mathcal{G}$ , then according to Lemma 3.2, the pair  $(C + d\Phi_J, \kappa)$  is also coherent. By Lemma 3.3 and 3.4, we have isomorphisms  $E^s(C, \kappa') \cong E^s(C + d\Phi_J, \kappa) \cong E^s(C, \kappa)$ .  $\square$

**Lemma 3.6.** *If  $(C, \kappa)$  is a coherent pair, then so is the pair  $(C + a^*(d^F\omega), \kappa)$ , for any 2-form  $\omega \in \Gamma(\wedge^2 F^*)$ . Moreover, the Courant algebroids  $E^s(C + a^*(d^F\omega), \kappa)$  and  $E^s(C, \kappa)$  are isomorphic.*

*Proof.* It is clear that  $\iota_{\mathbf{r}}(a^*(d^F\omega)) = 0$ ,  $\forall \mathbf{r} \in \mathcal{G}$ . Thus one can easily check that  $(C + a^*(d^F\omega), \kappa)$  is a coherent pair. Let  $E_i$  be the standard Courant algebroid determined by the quintuple  $(F, \mathcal{G}; \nabla, R, \mathcal{H}^i)$ , where  $\mathcal{H}^1 = \mathcal{H}^{(C, \kappa)}$ ,  $\mathcal{H}^2 = \mathcal{H}^{(C + a^*(d^F\omega), \kappa)}$ . Obviously,  $\mathcal{H}^2 - \mathcal{H}^1 = d^F\omega$ . By Proposition 2.7, we can construct an isomorphism  $I : E_1 \rightarrow E_2$  by setting  $I(\xi + \mathbf{r} + x) = (\xi - \omega^\sharp(x)) + \mathbf{r} + x$ , for  $\xi \in F^*$ ,  $\mathbf{r} \in \mathcal{G}$ , and  $x \in F$ . Here  $\omega^\sharp : F \rightarrow F^*$  is defined by  $\langle \omega^\sharp(x) | y \rangle = \omega(x, y)$ ,  $\forall x, y \in \Gamma(F)$ .  $\square$

**Lemma 3.7.** *Let  $(C_1, \kappa_1)$  and  $(C_2, \kappa_2)$  be two coherent pairs. Assume that  $C_2 = C_1 + d\varphi$ , for some  $\varphi \in \mathbf{C}_{\rightarrow}^2(\mathcal{A})$ . Then the Courant algebroids  $E^s(C_1, \kappa_1)$  and  $E^s(C_2, \kappa_2)$  are isomorphic.*

*Proof.* Choose  $J : F \rightarrow \mathcal{G}$  so that  $\kappa_2 = \kappa_1 + J$ . Lemma 3.3 implies that

$$E^s(C_2, \kappa_2) = E^s(C_2, \kappa_1 + J) \cong E^s(C_2 + d\Phi_J, \kappa_1) = E^s(C_1 + d(\varphi + \Phi_J), \kappa_1).$$

The hoist  $\kappa_1$  produces a decomposition  $\mathcal{A} \cong \mathcal{G} \oplus F$ . Hence, there exist  $J' : F \rightarrow \mathcal{G}$  and  $\omega \in \Gamma(\wedge^2 F^*)$  such that  $\varphi + \Phi_J = \Phi_{J'} + a^*\omega$ . By Lemmas 3.6 and 3.4, we have

$$E^s(C_1 + d(\varphi + \Phi_J), \kappa_1) = E^s(C_1 + d\Phi_{J'} + a^*(d^F\omega), \kappa_1) \cong E^s(C_1 + d\Phi_{J'}, \kappa_1) \cong E^s(C_1, \kappa_1). \quad \square$$

*Proof of Proposition 3.1.* First we show that  $\mathfrak{i}$  is well defined. Suppose that two characteristic pairs  $(\mathcal{A}, C)$  and  $(\overline{\mathcal{A}}, \overline{C})$  are equivalent, i.e. there is an isomorphism  $\sigma : \mathcal{A} \rightarrow \overline{\mathcal{A}}$  such that  $C - \sigma^*\overline{C} = d\varphi$ , for some  $\varphi \in \mathbf{C}_{\rightarrow}^2(\mathcal{A})$ . We need to show that there is an isomorphism



of Courant algebroids  $E^s(\mathcal{A}; C, \kappa) \cong E^s(\overline{\mathcal{A}}; \overline{C}, \overline{\kappa})$ . It is quite obvious that  $(\sigma^*\overline{C}, \sigma^{-1} \circ \overline{\kappa})$  is also a coherent pair on  $\mathcal{A}$  and

$$E^s(\mathcal{A}; \sigma^*\overline{C}, \sigma^{-1} \circ \overline{\kappa}) \cong E^s(\overline{\mathcal{A}}; \overline{C}, \overline{\kappa}).$$

Since  $C - \sigma^*\overline{C} = d\varphi$ , Lemma 3.7 gives the isomorphism

$$E^s(\mathcal{A}; C, \kappa) \cong E^s(\mathcal{A}; \sigma^*\overline{C}, \sigma^{-1} \circ \overline{\kappa}).$$

Thus we conclude that  $E^s(\mathcal{A}; C, \kappa) \cong E^s(\overline{\mathcal{A}}; \overline{C}, \overline{\kappa})$  and  $\mathfrak{i}$  is well defined.  $\square$

With these preparations, we are able to prove the second main theorem in this paper.

*Proof of Theorem 1.15.* (a) Let us denote by  $\mathfrak{R}$  the set of isomorphism classes of regular Courant algebroids, and by  $\mathfrak{C}$  the set of equivalence classes of characteristic pairs. We need to establish a one-to-one correspondence between  $\mathfrak{R}$  and  $\mathfrak{C}$ .

For any regular Courant algebroid  $E$ , one may choose a dissection so as to identify  $E$  with  $E^s = F^* \oplus \mathcal{G} \oplus F$ . In Section 2.2, we showed that there is a standard naive 3-cocycle  $C^s$  on  $E^s$  and it corresponds to a coherent 3-form on  $\mathcal{A}^s = \mathcal{G} \oplus F$ , also denoted by  $C^s$ .

So we can define a map  $\mathfrak{b} : \mathfrak{R} \rightarrow \mathfrak{C}$ ,  $[E] \mapsto [(\mathcal{A}^s, C^s)]$ . By Proposition 2.10, this map is well defined and does not depend on the choice of the dissection.

On the other hand, Proposition 3.1 indicates that there is a well defined map  $\mathfrak{i} : \mathfrak{C} \rightarrow \mathfrak{R}$ . An easy verification shows that it is indeed the inverse of  $\mathfrak{b}$ . This completes the proof.

(b) This follows from Proposition 2.6.  $\square$

### 3.3. Classification.

**Lemma 3.8.** *Let  $C \in \Gamma(\wedge^3 \mathcal{A}^*)$  be a coherent 3-form. Then any other coherent 3-form is of the form  $C + a^*(\varpi) + d\Phi_J$ , where  $\varpi \in \Gamma(\wedge^3 F^*)$  is a closed 3-form on  $F$  and  $\Phi_J$  is defined by (49).*

*Proof.* Assume that  $C$  (resp.  $C'$ ) is  $\kappa$  (resp.  $\kappa'$ )-coherent, for some hoist  $\kappa$  (resp.  $\kappa'$ ) of  $\mathcal{A}$ . The difference between  $\kappa$  and  $\kappa'$  determines a bundle map  $J : F \rightarrow \mathcal{G}$  such that  $\kappa' = \kappa - J$ . A direct calculation shows that the 3-form  $C' - C - d\Phi_J$  must be the pull back of some 3-form  $\varpi$  by the anchor map  $a$ . Conversely, it is easy to verify that  $C + a^*(\varpi) + d\Phi_J$  is coherent with respect to  $\kappa' = \kappa - J$ .  $\square$

**Definition 3.9.** *A 3-form  $\vartheta \in \mathbf{C}_{\rightarrow}^3(\mathcal{A})$  of a quadratic Lie algebroid  $\mathcal{A}$  is said to be **intrinsic**, if there exists a coherent 3-form  $C$  and an automorphism  $\sigma : \mathcal{A} \rightarrow \mathcal{A}$  over the identity such that  $\vartheta = \sigma^*C - C$ .*

It is simple to see that every intrinsic 3-form is closed, and therefore defines a class in  $H_{\rightarrow}^3(\mathcal{A})$ . It turns out that this is the obstruction class to lift the automorphism  $\sigma$  to a Courant algebroid automorphism.

**Lemma 3.10.** *Let  $\mathcal{A}_E$  be the ample Lie algebroid of a regular Courant algebroid  $E$  and  $\vartheta = \sigma^*C - C$  an intrinsic form of  $\mathcal{A}_E$ . Then  $[\vartheta] = 0$  if and only if  $\sigma$  can be lifted to an automorphism  $\tilde{\sigma}$  of the Courant algebroid  $E$ .*

Let  $\mathbb{I}$  denote the subset in  $H_{\rightarrow}^3(\mathcal{A})$  consisting of the cohomology classes of all intrinsic 3-forms. The following result is an easy consequence of Lemma 3.8.

**Proposition 3.11.**

- $\mathbb{I}$  consists of the cohomology classes  $[\sigma^*C_0 - C_0]$  for a fixed coherent 3-form  $C_0$  and all automorphisms  $\sigma : \mathcal{A} \rightarrow \mathcal{A}$  over the identity;
- $\mathbb{I}$  is an abelian subgroup of  $a^*H^3(F)$ .

Hence the abelian group  $\mathbb{I}$  measures how many automorphisms of  $\mathcal{A}$  that are essentially nontrivial, i.e., cannot be lifted to the Courant algebroid level.

Let  $\Phi$  be the natural map from the isomorphism classes of regular Courant algebroids to the isomorphism classes of quadratic Lie algebroids with vanishing first Pontryagin class. We are now ready to prove the last main theorem of this paper.

**Theorem 3.12.** *For any quadratic Lie algebroid  $\mathcal{A}$  with vanishing first Pontryagin class,  $\Phi^{-1}([\mathcal{A}])$  is isomorphic to  $\frac{a^*H^3(F)}{\mathbb{I}} \cong \frac{H^3(F)}{(a^*)^{-1}(\mathbb{I})}$ .*

*Proof.* By Theorem 1.15, we understand that every isomorphism class of Courant algebroid in  $\Phi^{-1}([\mathcal{A}])$  can be characterized by an equivalence class of characteristic pairs  $(\mathcal{A}, C)$ , where  $C$  is a coherent 3-form. Lemma 3.8 implies that  $a^*(H^3(F))$  acts transitively on  $\Phi^{-1}([\mathcal{A}])$ . Now we determine the isotropy group of this action. Assume that  $\varpi \in \Gamma(\wedge^3 F^*)$  is a closed 3-form such that the characteristic pairs  $(\mathcal{A}, C)$  and  $(\mathcal{A}, C + a^*(\varpi))$  are equivalent. By definition, there exists an automorphism  $\sigma$  of  $\mathcal{A}$  such that  $[\sigma^*(C) - (C + a^*(\varpi))] = 0$  in  $H_{\leftarrow}^3(\mathcal{A})$ . Hence it follows that  $[a^*(\varpi)] \in \mathbb{I}$ . Conversely, it is simple to see that  $\mathbb{I}$  acts trivially on  $\Phi^{-1}([\mathcal{A}])$ . This concludes the proof of the theorem.  $\square$

$(a^*)^{-1}(\mathbb{I})$  is called the intrinsic group of  $\mathcal{A}$ . Note that the classification problem investigated here differs from that studied by Ševera and Bressler even in the transitive case. Ševera and Bressler classified the isomorphism classes of transitive Courant algebroid extensions of a given Lie algebroid [2, 12], while we are interested in isomorphism classes of Courant algebroids themselves.

**Remark 3.13.** Note that  $a^* : H^3(F) \rightarrow H_{\leftarrow}^3(\mathcal{A})$  is not necessarily injective.

Let  $M$  be a contact manifold with  $\theta \in \Omega^1(M)$  being the contact 1-form. Consider the central extension  $\mathcal{A} = (\mathbb{R} \times M) \oplus TM$ , of the Lie algebroid  $TM$  by the exact 2-form  $d\theta$  considered as a 2-cocycle of the Lie algebroid  $TM$ . It is a transitive Lie algebroid. Endow  $\mathbb{R} \times M \rightarrow M$  with its standard fiberwise metric. It is clear that  $\mathcal{A}$  is a quadratic Lie algebroid. It is easy to check that  $a^*[\theta d\theta] = 0 \in H_{\leftarrow}^3(\mathcal{A})$ . However  $[\theta d\theta]$  is a nontrivial class in  $H_{DR}^3(M)$ .

For instance, if  $M = S^3$  with the canonical contact structure,  $H_{DR}^3(M) = \mathbb{R}$  and  $a^*H_{DR}^3(M) = 0$ . As a consequence, there is a unique Courant algebroid (up to isomorphisms) whose induced quadratic Lie algebroid is isomorphic to  $\mathcal{A} = (\mathbb{R} \times M) \oplus TM$ . On the other hand, according to [2, 12], Courant algebroid extensions of  $\mathcal{A} = (\mathbb{R} \times M) \oplus TM$  are parametrized by  $\mathbb{R}$ .

In general, it is hard to describe explicitly the intrinsic group  $(a^*)^{-1}(\mathbb{I})$ . The following example, due to Ševera [12], gives a nice description of the intrinsic group in a special case. We are grateful to Ševera, who pointed out this example to us, which led us to find an error in an earlier version of the draft.

**Example 3.14.** *Let  $\mathfrak{g}$  be a compact Lie algebra such that both  $\text{Out}(\mathfrak{g})$  and  $Z(\mathfrak{g})$  are trivial. Assume that  $M$  is a connected and simply connected manifold. Choose an ad-invariant*

non-degenerate bilinear form on  $\mathfrak{g}$ . Consider the canonical quadratic Lie algebroid  $\mathcal{A} = (M \times \mathfrak{g}) \oplus TM$ . The Cartan 3-form induces a coherent 3-form  $C \in \Gamma(\wedge^3 \mathcal{A}^*)$ :

$$C(\mathbf{r} + x, \mathbf{s} + y, \mathbf{t} + z) = -([\mathbf{r}, \mathbf{s}]^{\mathfrak{g}}, \mathbf{t}), \quad (52)$$

$\forall \mathbf{r}, \mathbf{s}, \mathbf{t} \in C^\infty(M, \mathfrak{g})$  and  $x, y, z \in \Gamma(TM)$ .

Let  $G$  be a connected and simply connected Lie group whose Lie algebra is  $\mathfrak{g}$ . Then any automorphism  $\sigma$  of  $\mathcal{A}$  is determined by a smooth map  $\tau : M \rightarrow G$ , where  $\sigma$  can be written as

$$\sigma(\mathbf{r} + v) = \text{Ad}_{\tau(x)}(\mathbf{r} - J(v)) + v, \quad \forall \mathbf{r} \in \mathfrak{g}, v \in T_x M.$$

Here  $J : TM \rightarrow \mathfrak{g}$  is the bundle map defined by:

$$J(v) = L_{\tau(x)^{-1}}(\tau_*(v)), \quad \forall v \in T_x M. \quad (53)$$

To see this, one can consider  $\mathcal{A}$  as the Atiyah Lie algebroid of the trivial principal bundle  $M \times G \rightarrow M$ . Then  $\sigma$  is induced by the principal bundle automorphism defined by the gauge group element  $\tau : M \rightarrow G$ .

Using Lemma 2.8, one obtains

$$(\sigma^* C - C)(\mathbf{r} + x, \mathbf{s} + y, \mathbf{t} + z) = -(d\Phi_J)(\mathbf{r} + x, \mathbf{s} + y, \mathbf{t} + z) + ([J(x), J(y)]^{\mathfrak{g}}, J(z)).$$

It thus follows that

$$[\sigma^* C - C] = -(J \circ a)^*[\alpha] = -a^*(J^*[\alpha]),$$

where  $\alpha = ([\cdot, \cdot], \cdot) \in \wedge^3 \mathfrak{g}^*$ . It is simple to see that  $J^*(\alpha) = \tau^*(\tilde{\alpha})$ , where  $\tilde{\alpha}$  is the corresponding Cartan 3-form on  $G$ . As a consequence, the intrinsic group  $(a^*)^{-1}(\mathbb{I})$  is generated by elements of the form  $\tau^*[\tilde{\alpha}]$ , where  $\tau : M \rightarrow G$  is any smooth map. Note that  $(a^*)^{-1}(\mathbb{I})$  is discrete, because when  $\tau_1$  and  $\tau_2$  are homotopic, then  $\tau_1^*[\tilde{\alpha}] = \tau_2^*[\tilde{\alpha}]$ .

Here is one example. Let  $M = G = SU(2)$ . Then  $H_{DR}^3(M) \cong \mathbb{R}$  and the intrinsic group  $(a^*)^{-1}(\mathbb{I})$  is isomorphic to  $\mathbb{Z}$  with the generator being  $[\tilde{\alpha}]$ . Hence by Theorem 3.12, the isomorphism classes of Courant algebroids with ample Lie algebroid isomorphic to the quadratic Lie algebroid  $(S^3 \times \mathfrak{su}(2)) \oplus TS^3$  is parametrized by  $\mathbb{R}/\mathbb{Z} \cong S^1$ .

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TSINGHUA UNIVERSITY, DEPARTMENT OF MATHEMATICS

*E-mail address:* `zchen@math.tsinghua.edu.cn`

UNIVERSITÉ PARIS DIDEROT<sup>2</sup>, INSTITUT DE MATHÉMATIQUES DE JUSSIEU<sup>3</sup>

*E-mail address:* `stienon@math.jussieu.fr`

PENNSYLVANIA STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS

*E-mail address:* `ping@math.psu.edu`

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<sup>2</sup>(Paris 7)

<sup>3</sup>(CNRS, UMR 7586)