A lift into Siegel modular forms over the theta group in degree two and the chiral superstring measure

by

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A LIFT INTO SIEGEL MODULAR FORMS OVER THE
THETA GROUP IN DEGREE TWO AND THE CHIRAL
SUPERSTRING MEASURE

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ABSTRACT. We prove in degree two, that the Siegel modular form
of D’Hoker and Phong that gives the chiral superstring measure is
a lift. This gives a fast algorithm for computing its Fourier coef-
ficients. We prove a general lifting from Jacobi cusp forms of half
integral index $t/2$ over the theta group $\Gamma_1(1, 2)$ to Siegel modu-
lar cusp forms over certain subgroups $\Gamma_{\text{para}}(t; 1, 2)$ of paramodular
groups. The theta group lift given here is a modification of the
Gritsenko lift.

1. Introduction

We construct a lifting $L$ from Jacobi cusp forms of index $t/2$ for the
theta group $\Gamma_1(1, 2)$ to Siegel modular forms on subgroups $\Gamma_{\text{para}}(t; 1, 2)$
of the paramodular groups $\Gamma_{\text{para}}(t)$:

$$L : J_{k,t/2}^{\text{cusp}}(\Gamma_1(1, 2)) \rightarrow M_k(\Gamma_{\text{para}}(t; 1, 2)).$$

Our construction imitates the construction of the lift due to Gritsenko,
Grit : $J_{k,m}(\text{SL}_2(\mathbb{Z})) \rightarrow M_k(\Gamma_{\text{para}}(m))$, which sends Jacobi forms of
index $m$ on $\text{SL}_2(\mathbb{Z})$ to Siegel modular forms on the paramodular group
$\Gamma_{\text{para}}(m)$, see [8]. Although we proceed in greater generality, our main
interest is the case where $t$ is odd. In order to properly call $L$ a lift, we
should really discuss the $L$-series of the lifted Siegel modular form but
here we content ourselves with giving the Fourier coefficients.

Theorem 1. Let $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. There is a monomorphism

$$L : J_{k,t/2}^{\text{cusp}}(\Gamma_1(1, 2)) \rightarrow M_k(\Gamma_{\text{para}}(t; 1, 2))$$

such that if $\phi \in J_{k,t/2}^{\text{cusp}}(\Gamma_1(1, 2))$ has the Fourier expansion

$$\phi(\tau, z) = \sum_{n, r \in \mathbb{Z}: tn-r^2>0, n>0} c(n, r)e\left(\frac{1}{2}n\tau + rz\right),$$

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for \( \tau \in \mathcal{H}_1 \) and \( z \in \mathbb{C} \), then \( L(\phi) \in M_k (\Gamma_{\text{para}}(t; 1, 2)) \) has the Fourier expansion

\[
L(\phi)(\Omega) = \sum_{T = \begin{pmatrix} n & r \\ r & m \end{pmatrix} : t|m, \quad mn - r^2 > 0, n > 0, m > 0.} a(T) e\left(\frac{1}{2} \text{tr}(T\Omega)\right),
\]

for \( \Omega \in \mathcal{H}_2 \), where

\[
a\left(\begin{pmatrix} n & r \\ r & m \end{pmatrix}\right) = (-1)^{(m/t+1)(n+1)} \sum_{a\mid(n,r,m/t) \ a \text{ odd}} a^{k-1} c\left(\frac{mn}{ta^2}, \frac{r}{a}\right).
\]

If \( t \not\equiv 0 \mod 4 \) then \( L(\phi) \) is a cusp form.

Although the lifting \( L \) is adequately described as an imitation of the Gritsenko lift, the choice of Hecke operators used to construct \( L \) was not obvious to us. The special case \( t = 1 \) is a lifting from Jacobi cusp forms of index \( 1/2 \) for the Jacobi theta group \( \Gamma_1(1, 2)^J \) to Siegel modular cusp forms for the theta group \( \Gamma_2(1, 2) \),

\[
L : J_{k,1/2}^{\text{cusp}}(\Gamma_1(1, 2)^J) \to S_k(\Gamma_2(1, 2))
\]

The groups that arise in this construction have natural geometric interpretations. The moduli space \( \Gamma_{\text{para}}(t)\backslash\mathcal{H}_2 \) is the equivalence classes of polarized abelian surfaces whose polarization has type \( E_t = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \) with \( T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \). This implies that there exists a divisor on the abelian surface with Chern class \( E_t \). To each equivalence class of type \( E_t \) polarized abelian surface in \( \Gamma_{\text{para}}(t; 1, 2)\backslash\mathcal{H}_2 \), one may associate a distinguished rank \( t \) vector space of sections of a divisor of Chern class \( E_t \), compare the transformation of theta functions under the paramodular group in [12], page 175. For many purposes, the theta group \( \Gamma_g(1, 2) \) is just as natural, or even more natural, than the full modular group \( \Gamma_g \). For example, the theta series of an integral unimodular lattice of even rank is always automorphic with respect to the theta group for a character, whereas the theta series is only automorphic with respect to the full modular group when the lattice happens to be even. We can also connect the lift \( L \) with elliptic modular forms on the theta group \( \Gamma_1(1, 2) \) if we make use of multiplication by the theta function \( \theta[0] \in J_{1/2,1/2}(\Gamma_1(1, 2)^J, v_0) \). Here, \( v_0 \) is the multiplier of the theta function and takes values in the eighth roots of unity.

**Corollary 2.** For \( k \in \mathbb{N} \), with \( 4\mid k \), there are monomorphisms

\[
S_{k-\frac{1}{2}}(\Gamma_1(1, 2), v_0^{2k-1}) \xrightarrow{\cdot \theta[0](z,r)} J_{k,1/2}^{\text{cusp}}(\Gamma_1(1, 2)^J) \xrightarrow{L} S_k(\Gamma_2(1, 2)).
\]
The point is that $L$, which is defined with respect to the theta group, is just as fundamental as any member of the Saito-Kurokawa family of lifts, to which $L$ belongs. For a general context and for an extended family of lifts, see the thesis of F. Clery [3].

An Application.

D’Hoker and Phong [5] have computed the chiral superstring measure $d\nu(g)[e] = \Xi_g[e] d\mu(g)$ in $g=2$ and it is determined by $\Xi_2[0] \in S_8(\Gamma_2(1,2))$, which can be defined, for example, as a polynomial of degree 16 in the thetanullwerte, see [10]:

$$\Xi_2[0] = \frac{1}{1024} \left( 2 \theta(0 0 0)^{16} - \theta(0 0 0)^8 \sum_{\zeta \text{ even}} \theta[\zeta]^8 + 2 \theta(0 0 0)^4 F \right),$$

with $F =
\theta(0 0 1)^4 \theta(0 1 0)^4 \theta(0 0 0)^4 + \theta(0 0 0)^4 \theta(0 1 0)^4 \theta(0 1 0)^4 + \theta(0 0 0)^4 \theta(0 0 1)^4 \theta(0 1 0)^4
+ \theta(0 1 0)^4 \theta(0 0 0)^4 \theta(1 0 0)^4 + \theta(0 0 1)^4 \theta(0 0 0)^4 \theta(0 1 0)^4.$

The solution $\Xi_2[0]$ may be variously viewed as a Siegel modular form, a Teichmuller modular form or as a binary invariant depending upon whether it is viewed as a section over the moduli space of abelian varieties, curves or hyperelliptic curves. In the first setting, the Ansatz of D’Hoker and Phong [5][2][10][11] asks for a family of Siegel modular forms satisfying: 1) $\Xi_{g_1+g_2}[0](\Omega_1 \Omega_2) = \Xi_{g_1}[0](\Omega_1) \Xi_{g_2}[0](\Omega_2)$ for $\Omega_1, \Omega_2$ in the Jacobian loci. 2) $\text{tr}(\Xi_g[0])$ vanishes on the Jacobian locus. 3) The family $\{\Xi_g[0]\}$ is uniquely determined on the Jacobian loci by the genus one solution $\Xi_1[0] = \theta_{000}^4$. This Ansatz can be satisfied through $g \leq 5$ but is thought unlikely to extend further [14]. Over the hyperelliptic locus, however, the corresponding conditions are solved for all $g$ by a family of binary invariants, see [15]. As of this writing it remains an open question whether the corresponding conditions can be satisfied by a Teichmuller modular form beyond $g = 5$. See [13] for an entry to the physics literature. We write $T = \begin{pmatrix} n & r \\ r & m \end{pmatrix} = [n, r, m]$ and in Table 1 give some Fourier coefficients for $a(T; \Xi_2[0])$ using the above polynomial in the thetanullwerte (1).

**Table 1.** Fourier coefficients for $\Xi_2[0]$.

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A rapid method exists for computing these Fourier coefficients because $\Xi_2[0]$ is a lift. Consider $\Phi = \theta^{11} F_2 - 16 \theta^7 F_2^2 \in S_{15/2} (\Gamma_0(4)^*, \tilde{v}_d^{15})$, where $\tilde{v}_d : \Gamma_0(4)^* \to \mathbb{C}^*$ is conjugate to $v_d$; note $\Gamma_1(1, 2)$ is conjugate to $\Gamma_0(4)^*$ via $\Gamma_1(1, 2) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right)^{-1} \Gamma_0(4)^* \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right)$. We also note that $\Phi$ is not in the Kohnen plus space and that its Fourier expansion begins:

$$\Phi(\tau) = q + 6q^2 - 64q^4 - 84q^5 + 252q^6 + 512q^7 - 384q^8 - 1107q^9 + 28q^{10} + 3724q^{13} + 792q^{14} - 4608q^{15} + 4096q^{16} - 168q^{17} - 15390q^{18} + 5376q^{20} + 1944q^{21} + 27676q^{22} + 10752q^{23} - 16128q^{24} - 11635q^{25} - 20748q^{26} - 32768q^{28} - 31836q^{29} + 79704q^{30} + 21504q^{31} + 24576q^{32} + 60984q^{33} - 107464q^{34} + 70848q^{36} - 41492q^{37} - 20748q^{38} - 124416q^{39} - 1792q^{40} + 63504q^{41} - 68616q^{42} + 215460q^{45} + 175640q^{46} + 64512q^{47} - 315783q^{49} + O(q^{50}).$$

Use Corollary 2 to define a Jacobi form $\phi(\tau, z) = \theta_{00}(z, \tau) \Phi(\tau/2) \in J_{8,1/2}(\Gamma_1(1, 2))$. The lift $L(\phi)$ is then in the one dimensional space $S_8(\Gamma_2(1, 2))$. By checking agreement on one Fourier coefficient we conclude $\Xi_2[0] = L(\phi)$ and obtain the formula

$$a \left( \binom{n}{r} \binom{r}{m}; \Xi_2[0] \right) = (-1)^{(m+1)(n+1)} \sum_{a \text{ odd}} \sum_{a | (m, r, m)} \sum_{a | (n, r, m)} a^7 c \left( \frac{mn - r^2}{a^2}; \Phi \right).$$

Thus, the entries in Table 1 can be easily verified from the $q$-expansion of the elliptic modular form $\Phi$.

We thank M. Oura and R. Salvati Manni for introducing us to the subject of superstring measures. We thank V. Gritsenko and T. Ibukiyama for their comments on this work. We thank the Max Planck Institute for Mathematics in Bonn for its hospitality during March, 2010 when parts of this article were written.
2. Groups

The justifications for working out a variant of the Saito-Kurokawa lift are the precise specification of the group of automorphy and the cuspidality of the lift. For index $1/2$, the lift $L(\phi)$ is automorphic with respect to the theta group $\Gamma_2(1, 2)$. For index $t/2$, this role is played by $\Gamma_{\text{para}}(t; 1, 2)$, a subgroup of the paramodular group $\Gamma_{\text{para}}(t)$. In order to determine the group of automorphy for the lift we will need to know generators of $\Gamma_{\text{para}}(t; 1, 2)$. The thesis of Delzeith [4] shows that $\Gamma_{\text{para}}(t)$ is generated by its translations and by $J(t) = \left( \begin{array}{cc} 0 & T \\ -T & 0 \end{array} \right)$ for $T = \left( \begin{array}{cc} 1 & 0 \\ 0 & t \end{array} \right)$. In order to show the cuspidality of the lift for $t \not\equiv 0 \pmod{4}$, we require coset decompositions of $\text{Sp}_2(\mathbb{Q})$ with respect to $\Gamma_{\text{para}}(t)$ and $\Gamma_{\text{para}}(t; 1, 2)$. Let $J = \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right) \in \text{GL}_2(\mathbb{Z})$.

Definition 3. For $F = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$, define groups of matrices:

- $\text{Sp}_g(F) = \{ \gamma \in M_{2g \times 2g}(F) : \gamma J \gamma' = J \}$,
- $\text{GSp}_g^+(F) = \{ \gamma \in M_{2g \times 2g}(F) : \exists \mu(\gamma) \in F^+ : \gamma J \gamma' = \mu(\gamma) J \}$.

The theta group of genus $g$ is $\Gamma_g(1, 2) = \{(A \ B \\
C \ D) \in \text{Sp}_g(\mathbb{Z}) : A'C, B'D \text{ have even diagonal entries} \}$.

The real symplectic group $\text{Sp}_g(\mathbb{R})$ has a natural action on the Siegel upper half space $H_g$. For a domain $D \subseteq \mathbb{C}$, let $V_g(D)$ be the $g$-by-$g$ symmetric matrices with coefficients in $D$. For $D \subseteq \mathbb{R}$, let $\mathcal{P}_g(D)^{\text{semi}} \subseteq V_g(D)$ be the semidefinite elements and let $\mathcal{P}_g(D)$ be the definite elements. Let $\mathcal{H}_g$ be the Siegel upper half space of degree $g$, the subset of $V_g(\mathbb{C})$ with positive definite imaginary part. The symplectic group $\text{Sp}_g(\mathbb{R})$ acts on $\Omega \in \mathcal{H}_g$ via

$$(A \ B) \circ \Omega := (A\Omega + B)(C\Omega + D)^{-1}.$$  

Here we think of elements of $\text{Sp}_g(\mathbb{R})$ as consisting of four $g \times g$ blocks. The group of symplectic similitudes $\text{GSp}_g^+(\mathbb{Q})$ is useful in the construction of Hecke algebras. The theta function $\theta[0]^8$ is automorphic with respect to theta group $\Gamma_g(1, 2)$. Because $\Gamma_g(1, 2)$ is closed under transposition, we may also use the conditions that $AB'$ and $CD'$ are even matrices.

Definition 4. The parabolic subgroup of the symplectic group is

$$\Gamma_\infty(F) = \{ \begin{pmatrix} * & 0 & * & * \\
* & * & * & * \\
* & 0 & * & * \\
0 & 0 & 0 & * \end{pmatrix} \in \text{Sp}_2(F) \}.$$
Also define

\[ \Gamma_\infty(F) = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \text{GSp}_2^+(F) \right\}. \]

Denote \( \Gamma_2(1, 2)_\infty = \Gamma_2(1, 2) \cap \Gamma_\infty(\mathbb{Z}) \).

For an element \( \gamma \in \text{GSp}_2^+(F) \) to be in \( \Gamma_\infty(F) \), it suffices that the second column be of the correct form. Introduce the notation \((\gamma)_2\) to mean the second column of \(\gamma\) written as a row 4-tuple for typesetting convenience.

**Lemma 5.** \( \Gamma_\infty(F) = \{ \gamma \in \text{GSp}_2^+(F) : (\gamma)_2 = (0, *, 0, 0) \text{ for } * \in F \} \).

**Proof.** Writing \( \gamma = (A B C D) \), we assume that \( A_{12} = C_{12} = C_{22} = 0 \) and need to show that \( C_{21}D_{21} = 0 \). The defining conditions for \( \text{GSp}_2^+(F) \) are \( AB', CD' \) symmetric and \( AD' - BC' = \mu I \) and we deduce:

\[
C_{21}D_{11} = C_{11}D_{21}, \quad A_{11}D_{11} - B_{11}C_{11} = \mu, \quad A_{11}D_{21} = B_{11}C_{21}.
\]

The conclusion then follows since \( \mu > 0 \) and

\[
\mu C_{21} = A_{11}D_{11}C_{21} - B_{11}C_{11}C_{21} = A_{11}C_{11}D_{21} - B_{11}C_{11}C_{21} = C_{11} (A_{11}D_{21} - B_{11}C_{21}) = 0;
\]

\[
\mu D_{21} = A_{11}D_{11}D_{21} - B_{11}C_{11}D_{21} = D_{11}B_{11}C_{21} - B_{11}C_{11}D_{21} = B_{11} (D_{11}C_{21} - C_{11}D_{21}) = 0.
\]

The parabolic group \( \Gamma_\infty(\mathbb{R}) \) is used in the construction of Fourier Jacobi expansions. The intersection of this parabolic subgroup with the theta group may be constructed in terms of more elementary groups as follows: Consider the Heisenberg group \( H(\mathbb{F}) = \mathbb{F}^3 = \{ (\lambda, v, k) \in \mathbb{F}^3 \} \) with the multiplication \((\lambda_1, v_1, k_1)(\lambda_2, v_2, k_2) = (\lambda_1 + \lambda_2, v_1 + v_2, k_1 + k_2 + \frac{v_1 v_2}{\lambda_2})\). The theta group \( \Gamma_1(1, 2) \) produces two orbits in \( H(\mathbb{Z}) \) under the action that sends \((\lambda, v, k)\) to \((\lambda, v, k)(\sigma + 1)\) for \( \sigma \in \Gamma_1(1, 2) \):

\[
H_e(\mathbb{Z}) = \{ (\lambda, v, k) \in \mathbb{Z}^3 : \lambda v + k \text{ is even } \},
\]

\[
H_o(\mathbb{Z}) = \{ (\lambda, v, k) \in \mathbb{Z}^3 : \lambda v + k \text{ is odd } \}.
\]

We denote by \( \Gamma_1(1, 2)^J \) the semidirect product \( \Gamma_1(1, 2) \ltimes H_e(\mathbb{Z}) \). By choosing the orbit which is a subgroup, this notation is consistent with that for the level one Jacobi group \( \text{SL}_2(\mathbb{Z})^J = \text{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z}) \).
Lemma 6. The following multiplicative homomorphisms are injective: 
\[ i : \text{GL}_2(\mathbb{R}) \to \Gamma_\infty(\mathbb{R}), \] 
\[ w : H(\mathbb{R}) \to \Gamma_\infty(\mathbb{R}), \]

\[ i \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & ad - bc & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

and \[ w(\lambda, v, k) = \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

We have a group isomorphism \( \Gamma_1(1, 2) \rtimes H_e(\mathbb{Z}) \to \Gamma_2(1, 2)_\infty / \{ \pm I \} \), sending \((g, h) \mapsto \pm i(g)w(h)\). Let \( \epsilon = \text{diag}(1, -1, 1, -1) \). We have an exact sequence

\[ \{ I \} \to \langle w(H_e(\mathbb{Z})), \epsilon \rangle \to \Gamma_2(1, 2)_\infty \to \Gamma_1(1, 2) \to \{ I \} \]
given by sending \( \gamma \in \Gamma_2(1, 2)_\infty \) to \( \left( \begin{smallmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{31} & \gamma_{33} \end{smallmatrix} \right) \in \Gamma_1(1, 2) \).

Proof. Left to reader. \( \square \)

Definition 7. For \( t \in \mathbb{N} \), define the paramodular group to be

\[ \Gamma_{\text{para}}(t) = \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & /t \\ t* & t* & t* & * \\ * & * & * & * \end{pmatrix} : * \in \mathbb{Z} \right\} \cap \text{Sp}_2(\mathbb{Q}). \]

Define the paramodular theta group, \( \Gamma_{\text{para}}(t; 1, 2) = \)

\[ \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A'C' = \begin{pmatrix} a & * \\ * & d/t \end{pmatrix}, B'D = \begin{pmatrix} c & * \\ * & d/t \end{pmatrix}, a, b, c, d \in 2\mathbb{Z} \right\}. \]

The moduli space interpretation of these groups was mentioned in the Introduction. The groups \( \Gamma_{\text{para}}(t) \) and \( \Gamma_{\text{para}}(t; 1, 2) \) have a common normalizer \( \mu_t \in \text{Sp}_2(\mathbb{R}) \) with the property that \( \mu_t^2 = -I \); we have

\[ \mu_t = \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 \\ \sqrt{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t} \\ 0 & 0 & -\sqrt{t} & 0 \end{pmatrix}. \]

We now determine the parabolic part of the paramodular groups. For \( t \in \mathbb{N} \), define \( \gamma_t \) as below and set \( \Gamma_2(1, 2)_\infty[t] = \langle \Gamma_2(1, 2)_\infty, \gamma_t \rangle \), the
Lemma 8. We have $\Gamma_\infty(Q) \cap \Gamma_{\text{para}}(t; 1, 2) = \Gamma_2(1, 2)_\infty[t]$.

Proof. The “$\supset$” inclusions is easy. Take $\delta \in \Gamma_\infty(Q) \cap \Gamma_{\text{para}}(t; 1, 2)$, and write

$$\delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & c \\ d & \epsilon_1 & e & f/t \\ g & 0 & h & i \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}$$

for some $a, b, c, d, e, f, g, h, i \in \mathbb{Z}$ and $\epsilon_1, \epsilon_2 \in \{-1, +1\}$. The diagonal of $A'C$ and the upper left entry of $B'D$ are even integers because $\delta \in \Gamma_{\text{para}}(t; 1, 2)$; the lower right entry of $B'D$ is an even multiple of $1/t$. So $ci + \epsilon_2 f/t = 2z/t$ for some $z \in \mathbb{Z}$. In particular, $tci + \epsilon_2 f$ is even. We proceed by cases.

If $t$ is odd, then let $\beta = \gamma_t^{2f(t-1)/2} \delta$ and we see that its $(2, 4)$ entry is $\epsilon_2^2 f = f$ and so $\beta \in \Gamma_{\text{para}}(t; 1, 2) \cap \Gamma_\infty(2)$, $\Gamma_{\text{para}}(t; 1, 2) \cap \Gamma_\infty(z)$. We now show that the lower right entry of its “$B'D$” is even; it is $ci + \epsilon_2 f \equiv tci + \epsilon_2 f \equiv 0 \mod 2$ because $t$ is odd. Thus $\beta \in \Gamma_2(1, 2)_\infty$. Then $\delta = \gamma_t^{2f(t-1)/2 - 2ci/2} \beta \in \Gamma_2(1, 2)_\infty[t]$.

If $t$ is even, then the condition that $tci + \epsilon_2 f$ is an even integer forces $f$ to be even. Then let $\beta = \gamma_t^{-2f/2 - 4ci} \delta$ to see that its $(2, 4)$ entry is $-\epsilon_2^2 ci$ and so $\beta \in \Gamma_{\text{para}}(t; 1, 2) \cap \Gamma_\infty(2)$, $\Gamma_{\text{para}}(t; 1, 2) \cap \Gamma_\infty(z)$. We now show that the lower right entry of its “$B'D$” is an even integer; it is $ci - \epsilon_1^2 ci = 0$. Thus $\beta \in \Gamma_2(1, 2)_\infty$ and $\delta = \gamma_t^{-2f/2 + 4ci} \beta \in \Gamma_2(1, 2)_\infty[t]$.

Proofs about generators are best done inside an integral version of the group. To this end, denote $T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ and $I_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$, and $E_t = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$. For any group $G$, denote $G^I = I_t^{-1} G I_t$. Then

$$(2) \quad \Gamma_{\text{para}}(t; 1, 2)^I = \{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}_4(\mathbb{Z}) : g'E_t g = E_t \\
and (T^{-1} \alpha' T \gamma)_{0} \equiv 0 \mod 2, \quad (T^{-1} \beta' T \delta)_{0} \equiv 0 \mod 2 \}.$$ 

The presentation (2) makes it clear that the integral version of the paramodular theta group $\Gamma_{\text{para}}(t; 1, 2)^I$ is a natural generalization of the theta group to nonprincipal polarizations and that when $t = 1$, we have the equalities $\Gamma_{\text{para}}(t; 1, 2) = \Gamma_{\text{para}}(t; 1, 2)^I = \Gamma_2(1, 2)$. 

A group generated by $\Gamma_2(1, 2)_\infty$ and $\gamma_t$ inside $\text{Sp}_2(Q)$, where

$$\gamma_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2/t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Definition 9. For \( t \in \mathbb{N} \), define the group
\[
G_t = (\Gamma_2(1,2)_{\infty}[t], \mu_t \Gamma_2(1,2)_{\infty}[t] \mu_t).
\]

We will in due course show \( G_t = \Gamma_{\text{para}}(t;1,2) \). Compare this with the generators given by Gritsenko for \( \Gamma_{\text{para}}(t) \), see [9].

Lemma 10. The following matrices are elements of \( G_t^1 \): \( J = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \),
\[
E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad g(\lambda, v, k, \ell) = \begin{pmatrix} 1 & 0 & 0 & tv \\ \lambda & 1 & v & kt + 2\ell \\ 0 & 0 & 1 & -\lambda t \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
\[
Jg(\lambda, v, k, \ell)J^{-1} = \begin{pmatrix} 1 & -\lambda t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -vt & 1 & 0 \\ -v & -kt - 2\ell & \lambda & 1 \end{pmatrix},
\]
whenever \( k + \lambda v \in 2\mathbb{Z} \) and \( k, \lambda, v, \ell \in \mathbb{Z} \).

Also, \( i(A) \) and \( j(A) \) for \( A \in \Gamma_1(1,2) \), where
\[
j((a \ b) \ (c \ d)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}.
\]

Proof. We have \( J = I_t^{-1} \mu_t i((0 \ 1) \mu_t^{-1} i((0 \ 1)) I_t \in G_t^1 \) and \( E_1 = I_t^{-1} \mu_t i((\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}) \mu_t^{-1} I_t \in G_t^1 \). The element \( g(k, \lambda, v, \ell) \) is in the Heisenberg part \( w(H_\varepsilon(\mathbb{Z}))^I \subseteq \Gamma_2(1,2)_{\infty}^I \) and the conjugate \( Jg(k, \lambda, v, \ell)J^{-1} \) is therefore in \( G_t^1 \). Since \( (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) and \( (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \) generate \( \Gamma_1(1,2) \), we have \( \forall A \in \Gamma_1(1,2), j(A) \in G_t^1 \). We already know \( \forall A \in \Gamma_1(1,2), i(A) \in G_t^1 \). \( \square \)

Proposition 11. For \( t \in \mathbb{N} \), \( \Gamma_{\text{para}}(t;1,2) = G_t \).

Proof. Since \( G_t \subseteq \Gamma_{\text{para}}(t;1,2) \) is easily checked, we only prove the inclusion \( \Gamma_{\text{para}}(t;1,2)^I \subseteq G_t^I \). Take any \( \gamma \in \Gamma_{\text{para}}(t;1,2)^I \). Recall the notation \( \gamma_2 = (u_1, u_2, u_3, u_4) \) to mean the second column of \( \gamma \) written as a row 4-tuple. Since \( \gamma \) is integral of determinant one, we know that \( \gcd(u_1, u_2, u_3, u_4) = 1 \).

STEP 1: \( \exists \beta_0 \in G_t^I : (\beta_0 \gamma)_2 = (x_1, x_2, x_3, x_4) \) and \( x_4 \neq 0 \).

Note at least one of the \( u_i \) must be nonzero. If \( u_4 \neq 0 \), then \( \beta_0 = I \) works. If \( u_4 = 0 \) and \( u_3 \neq 0 \), then \( \beta_0 = Jg(1,0,0,0)J^{-1} \) works. If \( u_4 = 0 \) and \( u_1 \neq 0 \), then \( \beta_0 = Jg(1,0,0,0)J^{-1}i((0 \ 1)) \) works. Finally, if \( u_4 = 0 \) and \( u_2 \neq 0 \), then \( \beta_0 = j((0 \ 1)) \) works. Note that we always have \( \gcd(x_1, x_2, x_3, x_4) = 1 \).
STEP 2: \( \exists \beta_1 \in G_t^I : (\beta_1 \beta_0 \gamma)_2 = (y_1, y_2, y_3, y_4) \) and \( \gcd(y_2, y_4) = 1 \). Set \( w = \gcd(x_2, x_4) \) and \( w_2 = \gcd(x_1, x_3) \) and \( w_3 = \gcd(x_4/w, w(x_4)) \). We make the following comments: \( w \neq 0 \) because \( x_4 \neq 0 \). There are \( a, b \in \mathbb{Z} \) such that \( w_2 = ax_1 + bx_3 \). Finally, \( \gcd(x_4/(ww_3), w) = 1 \) and for any prime \( p, p|w_3 \) implies \( p|w \).

Let \( \beta_1 = g(\lambda, v, k, \ell) \) with \( \lambda = ax_4/(ww_3), v = bx_4/(ww_3), k = -\lambda v \) and \( \ell = 0 \) so that

\[
\beta_1 = g(\lambda, v, k, \ell) = \begin{pmatrix} 1 & 0 & 0 & \frac{bx_4}{ww_3} \\ \frac{ax_4}{ww_3} & 1 & \frac{bx_4}{ww_3} & k \\ 0 & 0 & 1 & -\frac{ax_4}{ww_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Then \( (\beta_1 \beta_0 \gamma)_2 = (y_1, y_2, y_3, y_4) \) where

\[
y_4 = x_4 \quad \text{and} \quad y_2 = x_2 + \gcd(x_1, x_3)(\frac{x_4}{ww_3} + ktx_4).
\]

It is already the case that \( \gcd(y_2, y_4) = 1 \). Consider any prime \( p|y_4 \).

Case \( p|w \): If \( p|w \) then \( p|x_2 \), and \( p \nmid \gcd(x_1, x_3) \) since \( \gcd(x_1, x_2, x_3, x_4) = 1 \). But also \( p \nmid \frac{x_4}{ww_3} \) since \( \gcd(x_4/(ww_3), w) = 1 \) and so \( p \nmid y_3 \).

Case \( p \nmid w \): If \( p \nmid w \), then \( p \nmid x_2 \) and \( p \nmid \frac{x_4}{ww_3} \) since \( p|x_4 \) and \( p \nmid w \). Furthermore \( p \nmid w_3 \) (else \( p|w \)) so that \( p \nmid \frac{x_4}{ww_3} \). Then \( p \nmid y_3 \). In either case \( p \nmid \gcd(y_2, y_4) \) and thus \( \gcd(y_2, y_4) = 1 \).

STEP 3: \( \exists \beta_2 \in G_t^I \) such that \( (\beta_2 \beta_1 \beta_0 \gamma)_2 = (z_1, 1, z_3, 0) \).
Note that if one of \( y_2, y_4 \) is even (and hence the other is odd) then we can find \( A \in \Gamma_1(1, 2) \) such that \( (j(A)\beta_1 \beta_0 \gamma)_2 = (z_1, 1, z_3, 0) \). In this case we may take \( \beta_2 = j(A) \). If both \( y_2, y_4 \) are odd, then \( (y_1, y_2, y_3, y_4) \) being the second column of a \( \Gamma_{\text{para}}(t; 1, 2)^I \) matrix forces \( t|y_1, t|y_3 \) and \( y_1y_3/t + y_2y_4 \equiv 0 \mod 2 \) which forces \( y_1, y_3 \) to be odd as well. Then \( g(1, 0, 0, 0)\beta_1 \beta_0 \gamma \) satisfies \( (g(1, 0, 0, 0)\beta_1 \beta_0 \gamma)_2 = (y_1, y_1 + y_2, y_3 - t y_4, y_4) \).

Then \( y_1 + y_2 \) is even and \( y_4 \) is still odd, so that \( \beta_2 = j(A)g(1, 0, 0, 0) \) suffices by the first argument.

STEP 4: \( \exists \beta_3 \in G_t^I \) such that \( (\beta_3 \beta_2 \beta_1 \beta_0 \gamma)_2 = (0, 1, z_3, z_1z_3/t) \).
From \( \beta_2 \beta_1 \beta_0 \gamma \in \Gamma_{\text{para}}(t; 1, 2)^I \) we see that \( z_1z_3/t \equiv 0 \mod 2 \), and that \( t|z_1 \) and \( t|z_3 \). Then \( \beta_3 = g(z_1/t, 0, 0, 0) \) gives us \( (\beta_3 \beta_2 \beta_1 \beta_0 \gamma)_2 = (0, 1, z_3, z_1z_3/t) \).

STEP 5: \( \exists \beta_4 \in G_t^I \) such that \( (\beta_4 \beta_3 \beta_2 \beta_1 \beta_0 \gamma)_2 = (z_3, 1, 0, 0) \).
Note that \( z_1z_3/t \) is even and \( \beta_4 = i((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})j((\begin{smallmatrix} 1 \\ -z_1z_3/t \end{smallmatrix})) \in G_t^I \) works.

STEP 6: \( \exists \beta_5 \in G_t^I \) such that \( (\beta_5 \beta_4 \beta_3 \beta_2 \beta_1 \beta_0 \gamma)_2 = (0, 1, 0, 0) \).
Use \( \beta_5 = g(z_3/t, 0, 0, 0) \in G_t^I \).

By Lemma 5, we have \( \beta_5 \beta_4 \beta_3 \beta_2 \beta_1 \beta_0 \gamma \in GT_{\infty}(\mathbb{Z}) \cap \Gamma_{\text{para}}(t; 1, 2)^I \).
Now, \( GT_{\infty}(\mathbb{Z}) \cap \Gamma_{\text{para}}(t; 1, 2)^I \subseteq (GT_{\infty}(\mathbb{Q}) \cap \Gamma_{\text{para}}(t; 1, 2))^I \) and we have
G\Gamma_\infty(\mathbb{Q}) \cap \Gamma^\text{para}(t; 1, 2) = \Gamma_\infty(\mathbb{Q}) \cap \Gamma^\text{para}(t; 1, 2) = \Gamma_2(1, 2)_\infty[t], where the last equality is by Lemma 8. Thus \beta_5\beta_4\beta_3\beta_2\beta_1\beta_0\gamma \in \Gamma_2(1, 2)_\infty[t]^I \subseteq G^I_t and \gamma \in G^I_t. 

Lemma 12. We have Sp_2(\mathbb{Z}) \subseteq \Gamma^\text{para}(t) U \Gamma_\infty(\mathbb{Q}), where

\[ U = \{ \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c \end{pmatrix} : c \in \mathbb{Z} \}. \]

Proof. Take any \alpha \in Sp_2(\mathbb{Z}). Since the second column of \alpha must have relatively prime entries, by a similar argument to Steps 1 and 2 of the proof to Proposition 11 we can find a \beta_1 = \begin{pmatrix} 1 & 0 & 0 & \eta \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^\text{para}(t) \cap Sp_2(\mathbb{Z}) such that (\beta_1\alpha)_2 = (y_1, y_2, y_3, y_4) where gcd(y_2, y_4) = 1. Let \( g = \gcd(ty_2, y_4) = aty_2 + by_4 \) for some \( a, b \in \mathbb{Z} \). Note \( g|t \). Then let \( \beta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b/t \\ 0 & 0 & 1 & 0 \\ -ty_4/g & 0 & 1 & g \end{pmatrix} \in \Gamma^\text{para}(t) \) so that (\beta_2\beta_1\alpha)_2 = (y_1, g/t, y_3, 0). Next let \( (\begin{smallmatrix} a \\ b \\ c \\ d \end{smallmatrix}) \in SL_2(\mathbb{Z}) \) such that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix} \) where \( z = \gcd(y_1, y_3) \).

Then let \( \beta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^\text{para}(t) \) so that (\beta_3\beta_2\beta_1\alpha)_2 = (z, g/t, 0, 0).

Finally, let \( u = \begin{pmatrix} 1 & zt/g & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -zt/g & 1 \end{pmatrix} \) so that (\( u^{-1}\beta_3\beta_2\beta_1\alpha\))_2 = (0, g/t, 0, 0). The \( \frac{zt}{g} \) are the integers \( c \) in the statement of the Lemma. Calling \( \gamma = u^{-1}\beta_3\beta_2\beta_1\alpha \), then \( \gamma \in Sp_2(\mathbb{Q}) \) and \( (\gamma)_2 = (0, g/t, 0, 0) \) forces \( \gamma \in \Gamma_\infty(\mathbb{Q}) \) by Lemma 5. Then \( \alpha = \beta_1^{-1}\beta_2^{-1}\beta_3^{-1}u\gamma \) says that \( \alpha \in \Gamma^\text{para}(t) U \Gamma_\infty(\mathbb{Q}) \).

Lemma 13. For \( \Gamma^\text{para}(t; 1, 2) \backslash \Gamma^\text{para}(t) \), a complete list of right coset representatives can be taken to be

\[
C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
C_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_6 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
C_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\]

for \( t \) odd, and we can omit \( C_{10} \) for \( t \) even.

Proof. It is a straightforward calculation to check that the set of cosets \{\Gamma^\text{para}(t; 1, 2)C_i\}_{i=1}^{10} is stable under right multiplication by the following set of generators for \( \Gamma^\text{para}(t) \):

\[
\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

for \( t \) odd, and we can omit \( C_{10} \) for \( t \) even.
which shows that $\Gamma_{\text{para}}(t) = \bigcup_{i=1}^{10} \Gamma_{\text{para}}(t; 1, 2) G_i$. It is another simple calculation to see that $C_i G_j \not\in \Gamma_{\text{para}}(t; 1, 2)$ when $i \neq j$ except in the case when $t$ is even and $\{i, j\} = \{9, 10\}$; this shows that the coset representatives are nonredundant except that we can omit $C_{10}$ when $t$ is even. One can check that the permutations of cosets induced by the right multiplication of these generators as cycles in $S_{10}$ (or in $S_9$ for even $t$) are: $\alpha = (12)(34)(78)$ and $\beta = (13)(24)(56)$ and $\delta = (25)(37)(49)(68)$ in either case, whereas $\gamma$ is the identity for even $t$ and $(56)(78)(910)$ for odd $t$. 

Let $\Delta_2(\mathbb{F}) = \{(\begin{smallmatrix} A & B \\ 0 & \beta \end{smallmatrix}) \in \text{Sp}_2(\mathbb{F})\}$.

**Proposition 14.** Let $t \in \mathbb{N}$.

1. For $t$ odd, we have
   $$\text{Sp}_2(\mathbb{Z}) \subseteq \Gamma_{\text{para}}(t; 1, 2) \Gamma_{\infty}(\mathbb{Z}) \Delta_2(\mathbb{Q}) \Gamma_{\infty}(\mathbb{Q}).$$

2. For $t$ even but with $t/2$ odd, we have
   $$\text{Sp}_2(\mathbb{Z}) \subseteq \Gamma_{\text{para}}(t; 1, 2) \Gamma_{\infty}(\mathbb{Z}) \Delta_2(\mathbb{Q}) \Gamma_{\infty}(\mathbb{Q})$$
   $$\cup \Gamma_{\text{para}}(t; 1, 2) \mu_t^{-1} \Gamma_{\infty}(\mathbb{Z}) \mu_t \Delta_2(\mathbb{Q}) \Gamma_{\infty}(\mathbb{Q}).$$

**Proof.** From Lemma 12 and Lemma 13, we have that
$$\text{Sp}_2(\mathbb{Z}) \subseteq \bigcup_{i=1}^{10} \Gamma_{\text{para}}(t; 1, 2) C_i \ U \Gamma_{\infty}(\mathbb{Q}),$$
where $U$ is as defined in Lemma 12. It is clear that $C_i \subseteq \Gamma_{\infty}(\mathbb{Z}) \Delta_2(\mathbb{Q})$ for $i = 1, \ldots, 6$, and so we have the inclusion $\Gamma_{\text{para}}(t; 1, 2) C_i \ U \Gamma_{\infty}(\mathbb{Q}) \subseteq \Gamma_{\text{para}}(t; 1, 2) \Gamma_{\infty}(\mathbb{Z}) \Delta_2(\mathbb{Q}) \Gamma_{\infty}(\mathbb{Q})$ for these $i$.

For the case where $t$ is odd, we have
$$C_7 = \left(\begin{array}{cccc}
1 & -t & 0 & 0 \\
-1 & t + 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-2t & t(t+1) & 1 & 1
\end{array}\right) \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \left(\begin{array}{cccc}
t + 1 & t & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -t & t + 1
\end{array}\right)$$
and $C_8 = C_7 \left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{array}\right)$ so that $C_7, C_8 \in \Gamma_{\text{para}}(t; 1, 2) \Gamma_{\infty}(\mathbb{Z}) \Delta_2(\mathbb{Q})$. And we have
$$C_9 = \left(\begin{array}{cccc}
1 & -t & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-2t & t & 1 & 0
\end{array}\right) \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \left(\begin{array}{cccc}
1 & t & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)$$
and $C_{10} = C_9 \left(\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{array}\right)$ so that $C_9, C_{10} \in \Gamma_{\text{para}}(t; 1, 2) \Gamma_{\infty}(\mathbb{Z}) \Delta_2(\mathbb{Q})$.

Then $\Gamma_{\text{para}}(t; 1, 2) C_i \ U \Gamma_{\infty}(\mathbb{Q}) \subseteq \Gamma_{\text{para}}(t; 1, 2) \Gamma_{\infty}(\mathbb{Z}) \Delta_2(\mathbb{Q}) \Gamma_{\infty}(\mathbb{Q})$ for $i = 7, 8, 9, 10$ as well, which proves item (1).

For $t$ even (and $t/2$ odd), we use that
$$C_7 = \mu_t^{-1} C_5 \mu_t$$
so that \( C_7, C_8 \in \mu_t^{-1} \Gamma_\infty(\mathbb{Z}) \mu_t \Delta_2(\mathbb{Q}) \). The final case is \( C_9 \) when \( t \) is even. We will manipulate \( C_9 u \) for any \( u \in U \). Any \( u \in U \) is of the form \( u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) with \( c \in \mathbb{Z} \). So \( C_9 u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \). For the case where \( c \) is odd, let \( g = \gcd(c - t, c) = a(c - t) + bc \) for some \( a, b \in \mathbb{Z} \). We can verify that

\[
C_9 u = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{t}{g} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{a+b}{g} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{a+b}{g} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

and this proves \( C_9 u \in \Gamma_{\text{para}}(t; 1, 2) \Gamma_\infty(\mathbb{Z}) \Delta_2(\mathbb{Q}) \Gamma_\infty(\mathbb{Q}) \) when \( c \) is odd. For the case where \( c \) is even, we can verify that

\[
C_9 u = \begin{pmatrix} 1 & t/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

One important note is that we are assuming that \( t/2 \) is odd so that \( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) \( \in \Gamma_{\text{para}}(t; 1, 2) \), and so the above proves

\[
C_9 u \in \Gamma_{\text{para}}(t; 1, 2) \mu_t^{-1} \Gamma_\infty(\mathbb{Z}) \mu_t \Delta_2(\mathbb{Q}) \Gamma_\infty(\mathbb{Q}) \] when \( c \) is even. Thus

\[
\Gamma_{\text{para}}(t; 1, 2) C_9 U \Gamma_\infty(\mathbb{Q}) \subseteq \Gamma_{\text{para}}(t; 1, 2) \Gamma_\infty(\mathbb{Z}) \Delta_2(\mathbb{Q}) \Gamma_\infty(\mathbb{Q})
\]

\[
\cup \Gamma_{\text{para}}(t; 1, 2) \mu_t^{-1} \Gamma_\infty(\mathbb{Z}) \mu_t \Delta_2(\mathbb{Q}) \Gamma_\infty(\mathbb{Q})
\]

The proof of item (2) is now complete. \( \square \)

3. Hecke Algebras

We recall the abstract Hecke algebra \( \mathcal{H}_R(U, S) \). Let \( U \subseteq S \) be a group contained in a semigroup inside of some larger group. For a ring \( R \), let \( \mathcal{L}_R(U, S) \) be the free \( R \)-module of finite linear combinations of the basis \( U \setminus S \). A right action of \( U \) on \( \mathcal{L}_R(U, S) \) is defined by \( (Us)u \mapsto U(su) \), extended \( R \)-linearly. The invariant \( R \)-module is denoted

\[
\mathcal{H}_R(U, S) = \{ T \in \mathcal{L}_R(U, S) : \forall u \in U, T u = T \}.
\]

The right invariance of \( \mathcal{H}_R(U, S) \) under \( U \) allows us to define a product \( \mathcal{H}_R(U, S) \times \mathcal{L}_R(U, S) \to \mathcal{L}_R(U, S) \) by \( (\sum c_\alpha U s_\alpha)Us = \sum c_\alpha Us_\alpha s \) for \( c_\alpha \in R \) and \( s_\alpha \in S \). The restriction of this product to \( \mathcal{H}_R(U, S) \times \mathcal{H}_R(U, S) \to \mathcal{H}_R(U, S) \) makes \( \mathcal{H}_R(U, S) \) an associative \( R \)-algebra and \( \mathcal{H}_R(U, S) \) also acts on \( \mathcal{L}_R(U, S) \) from the left.
Lemma 15. Let $U_0 \subseteq S_0$ and $U \subseteq S$ be groups contained in semigroups inside of some larger groups. Let $i : (U_0, S_0) \to (U, S)$ be a relative homomorphism. Let $R$ be a ring. If

(L1) There exists a subgroup $H \subseteq U$ such that $U = i(U_0)H$,
(L2) For all $s \in i(S_0)$, we have $sHs^{-1} \subseteq U$,

then there is an $R$-algebra homomorphism $j : H_R(U_0, S_0) \to H_R(U, S)$ such that $j(\sum c_ax) = \sum c_au_i(x)$ for $c_a \in R$ and $x \in S_0$. Furthermore, if $i$ is injective and

(L3) $i(S_0)i(S_0)^{-1} \cap U \subseteq i(U_0)$,

then $j$ is injective.

Proof. Since $i(U_0) \subseteq U$ we may define a $R$-linear map $j : \mathcal{L}_R(U_0, S_0) \to \mathcal{L}_R(U, S)$ by $\sum c_au_\alpha \mapsto \sum c_au_i(\alpha)$. To show that $j$ restricts to an $R$-linear map on the Hecke algebras, select $T = \sum c_au_\alpha \mathcal{H}_R(U_0, S_0)$. The right invariance of $T$ under $U_0$ implies that $j(T)$ is right invariant under $i(U_0)$: $j(T)u_\alpha = j(Tu_\alpha) = j(T)$. The right invariance of $j(T)$ under $h \in H$ follows from (L2): $j(T)h = \sum c_au_i(\alpha)h = \sum c_au_i(\alpha)h i(x) = \sum c_au_i(\alpha)$. Since $U = i(U_0)H$ by (L1), we have $j(T) \in H_R(U, S)$.

To show that $j : H_R(U_0, S_0) \to H_R(U, S)$ is a homomorphism it suffices to prove the commutativity of the following diagram:

\[
\begin{array}{ccc}
H_R(U_0, S_0) \times \mathcal{L}_R(U_0, S_0) & \overset{x}{\longrightarrow} & \mathcal{L}_R(U_0, S_0) \\
\downarrow j & & \downarrow j \\
H_R(U, S) \times \mathcal{L}_R(U, S) & \overset{x}{\longrightarrow} & \mathcal{L}_R(U, S).
\end{array}
\]

We have

\[
j(T(U_0x)) = \sum c_au_i(\alpha) x = \left(\sum c_au_i(\alpha)\right) (U_i(\alpha)) = j(T)j(U_0x).
\]

To show the injectivity of $j$ given the injectivity of $i$ and (L3), write $T = \sum c_au_\alpha \in H_R(U_0, S_0)$ with distinct cosets $U_0\alpha$. It suffices to show that the cosets $j(U_0\alpha) = U_i(\alpha)$ are distinct. If $U_i(x_1) = U_i(x_2)$ then $i(x_1)i(x_2)^{-1} \in i(S_0)i(S_0)^{-1} \cap U \subseteq i(U_0)$ by (L3) and we conclude $U_0x_1 = U_0x_2$ by the injectivity of $i$.  

We will apply Lemma 15 with the following choices:
Lemma 16. Consider the Hecke pairs \((U_0, S_0)\) and \((U, S)\):

\[ U_0 = \Gamma_1(1, 2), \]
\[ S_0 = \{(a b \ b c) \in G \text{Sp}_1^+(\mathbb{Z}) : ac, bd \text{ even}, ad - bc \in \mathbb{N}\}; \]
\[ U = \Gamma_2(1, 2)_{\infty}, \]
\[ S = \{(A B \ C D) \in G \text{Sp}_2^+(\mathbb{Z}) : A'C, B'D \text{ even matrices}, AD' - BC' \in \mathbb{N}\} \]

and the relative injection \(i : (U_0, S_0) \rightarrow (U, S)\) given in Lemma 6. We have \(i(S_0)i(S_0)^{-1} \cap U = i(U_0)\). Let \(H \subseteq U\) be the subgroup given by \(H = \pm w(H_{e}(\mathbb{Z})).\) \(H\) is a normal subgroup with \(U = i(U_0)H\). In fact, for all \(s \in i(S_0)\), we have \(sHs^{-1} \subseteq H\). Therefore, \((U_0, S_0)\) and \((U, S)\) satisfy (L1), (L2) and (L3) of Lemma 15.

Proof. That \(H\) is a normal subgroup of \(U = \Gamma_2(1, 2)_{\infty}\) with \(U = i(U_0)H\) follows from Lemma 6. We have

\[ H = \left\{ \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} : v, k, \lambda \in \mathbb{Z}, \rho = \pm 1, k + v\lambda \text{ even} \right\} \tag{3} \]

For condition (L2), take any \(s = i((a b \ b c)) \in i(S_0)\). So \(ad - bc \in \mathbb{N}\) and \(ac, bd\) are even. Take a general \(h \in H\) as in (3). Then

\[ sHs^{-1} = \rho \begin{pmatrix} 1 & 0 & 0 & -b\lambda + av \\ d\lambda - cv & 1 & -b\lambda + av & (ad - bc)k \\ 0 & 0 & 1 & -d\lambda + cv \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

For this to be in \(H\), we need that the following is even:

\[ (ad - bc)k + (d\lambda - cv)(-b\lambda + av). \]

But this can be rearranged to

\[ (ad - bc)(k + v\lambda) + 2bcv\lambda - acv^2 - bd\lambda^2 \]

which is even because \((k + v\lambda), ac, bd\) are all even. Thus \(sHs^{-1} \subseteq H\), and condition (L2) of Lemma 15 is also satisfied.

We now show \(i(S_0)i(S_0)^{-1} \cap U = i(U_0)\) even though it is easy. The general element \(u \in U\) may be written

\[ u = \begin{pmatrix} a & 0 & b & g \\ e & f & h & j \\ c & 0 & d & r \\ 0 & 0 & 0 & n \end{pmatrix} \in \Gamma_2(1, 2) \]

where the conditions are \(ad - bc = 1, fn = 1, de - ch + fr = 0, be + fg - ah = 0\) and \(ab, cd, eh + fj\) are even. The alternate parity
conditions are $ac$, $bd$ and $gr + jn$ even. If this $u$ is in $i(S_0)i(S_0)^{-1} \subseteq i(\text{GL}_2^+(\mathbb{Q}))$ then $e, g, h, j$, and $r$ vanish while $n = 1$ and $f > 0$. From the equation $1 = \det(u) = (ad - bc)f$ and from $a, b, c, d, f \in \mathbb{Z}$ we get $(ad - bc) = f = 1$. Therefore $u = i((\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix})) \in i(U_0)$.

**Definition 17.** For each $m \in \mathbb{N}$, consider the operator

$$T_m^{(1)} = \sum_{a,b,d} U_0(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}) \in L \mathcal{T}(U_0, S_0),$$

where the sum is over $a, b, d \in \mathbb{N}$ with $ad = m$, $0 \leq b < 2d$, and $a, (b + d)$ both odd.

**Lemma 18.** For each $m \in \mathbb{N}$, $T_m^{(1)} \in \mathcal{H}_Z(U_0, S_0)$.

**Proof.** First note that the left cosets in the above sum are disjoint because

$$(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix})(\begin{smallmatrix} a_2 & b_2 \\ 0 & d_2 \end{smallmatrix})^{-1} = \left(\begin{smallmatrix} a & b - b_2d_2 \\ 0 & d \end{smallmatrix}\right),$$

and the only way that this could be in $U_0$ is if $a = a_2$, hence $d = d_2$, and $\frac{b - b_2}{d}$ is even, which means that $b - b_2$ would have to be a multiple of $2d$. Next, we will show that $T_m^{(1)}$ is right invariant by elements from $U_0$. Since $U_0 = \Gamma_1(1, 2)$ is generated as a group by the two elements $(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$, we only need to show right invariance by these two elements. In fact, because the above cosets are disjoint, we only need to show that a coset representative multiplied on the right by these generators always land in another of the cosets above.

First, we can easily calculate that

$$(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix})(\begin{smallmatrix} 1 & 0 \\ 2\ell & 1 \end{smallmatrix}) = \left(\begin{smallmatrix} a & b + 2a - 2\ell d \\ 0 & d \end{smallmatrix}\right).$$

By picking $\ell \in \mathbb{Z}$ such that $0 \leq b + 2a - 2\ell d < 2d$ and noting that $(b + 2a - 2\ell d)$ has the same parity as $b$, then $(\begin{smallmatrix} a & b + 2a - 2\ell d \\ 0 & d \end{smallmatrix})$ is one of the coset representatives.

Second, let $u = \gcd(b, d)$, so that $u$ is odd. Let $x, y \in \mathbb{Z}$ such that $bx + dy = u$. Since $u$ is odd, we can choose $x, y$ such that $b, x$ have the same parity and $d, y$ have the same parity. (Just replace by $b(x + d) + d(y - b) = u$ if necessary.) Let $A = \left(\begin{smallmatrix} -b/u & y \\ -d/u & -x \end{smallmatrix}\right)$. Note $A \in U_0$.

One can easily verify that

$$(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix})(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) = A(\begin{smallmatrix} u & -ax \\ 0 & ad/u \end{smallmatrix}) = A(\begin{smallmatrix} 1 & 2\ell \\ 0 & 1 \end{smallmatrix})(\begin{smallmatrix} u & -ax - 2\ell d/u \\ 0 & ad/u \end{smallmatrix})$$

where we choose $\ell \in \mathbb{Z}$ so that $0 \leq -ax - 2\ell d/u < 2ad/u$. Since $ax$ has the same parity as $b$, and since $\frac{ad}{u}$ has the same parity as $d$, then $(\begin{smallmatrix} u & -ax - 2\ell d/u \\ 0 & ad/u \end{smallmatrix})$ is one of the coset representatives, and we have shown that $T_m^{(1)}$ is right invariant by $U_0$. \qed
So we may apply Lemma 15 to conclude the operator \( j(T_m^{(1)}) \) is in \( \mathcal{H}_Z(U, S) \). We denote \( T_m = j(T_m^{(1)}) \); namely, we have proven the following Corollary:

**Corollary 19.** We may define

\[
T_m = j(T_m^{(1)}) = \sum \Gamma(1, 2) \in \mathcal{H}_Z(U, S)
\]

where the sum is over \( a, b, d \in \mathbb{N} \) with \( ad = m, 0 \leq b < 2d \), and \( a, (b + d) \) both odd.

4. Jacobi forms and Siegel forms and the lift

For \( r \in \mathbb{Q} \) and \( \gamma = (A B C D) \in \text{Sp}_g(\mathbb{R}) \) and \( \Omega \in \mathcal{H}_g \), we set

\[
(f|_r \gamma)(\Omega) = \det(C\Omega + D)^{-r} f(\gamma \circ \Omega),
\]

for the choice of holomorphic root on \( \mathcal{H}_g \) determined by the condition that \( \det(\Omega/\imath)^r > 0 \) for \( \Omega = \imath Y \) with \( Y \in \mathcal{P}_g(\mathbb{R}) \). Let \( \Gamma \) be a subgroup commensurable with \( \Gamma_g \). A holomorphic function \( f: \mathcal{H}_g \to \mathbb{C} \) is a modular form of weight \( r \) with respect to \( \Gamma \) and a map \( v: \Gamma \to \mathbb{C}^* \) if

\[
\forall \gamma \in \Gamma, \forall \Omega \in \mathcal{H}_g, \quad (f|_r \gamma)(\Omega) = v(\gamma) f(\Omega),
\]

and if additionally, for all \( \gamma \in \Gamma_g \) and for all \( Y_0 \in \mathcal{P}_g(\mathbb{R}) \), \( f|_r \gamma \) is bounded on domains of type \( \{ \Omega \in \mathcal{H}_g : \im \Omega > Y_0 \} \). By a result of Koecher, this boundedness condition is redundant for \( g \geq 2 \). We denote by \( M_r(\Gamma, v) \) the vector space of such functions and use the notation \( M_r(\Gamma) \) when the map \( v \) is identically 1. The space \( M_r(\Gamma_g) \) is trivial unless \( \mu(\gamma, \Omega) = \det(C\Omega + D)^r v(\gamma) \) is a factor of automorphy; that is, \( \mu: \Gamma \times \mathcal{H}_g \to \mathbb{C}^* \) satisfies the cocycle condition:

\[
\mu(\gamma_1 \gamma_2, \Omega) = \mu(\gamma_1, \gamma_2 \circ \Omega) \mu(\gamma_2, \Omega).
\]

For integral weights \( k \), \( \det(C\Omega + D)^k \) is already a factor of automorphy and hence \( v: \Gamma \to \mathbb{C}^* \) is a character.

The transformation formula for the theta function, see pages 176 and 182 of [12],

\[
\exists v_g^{(g)} : \Gamma_g(1, 2) \to e(1/8) : \forall \gamma \in \Gamma_g(1, 2), \quad \theta[0]|_{1/2} \gamma = v_g^{(g)}(\gamma) \theta[0],
\]

gives an example of a Siegel modular form of weight 1/2; the standard thenull \( \theta[0](0, \Omega) \) gives an element of \( M_{1/2}(\Gamma_g(1, 2), v_g^{(g)}) \). We write \( v_g = v_g^{(g)} \) when the degree \( g \) is clear from the context.

For holomorphic \( f: \mathcal{H}_g \to \mathbb{C} \) we define

\[
\Phi(f)(\Omega_1) = \lim_{\lambda \to +\infty} f \left( \begin{array}{cc} \Omega_1 & 0 \\ 0 & i\lambda \end{array} \right)
\]

when this limit exists for all \( \Omega_1 \in \mathcal{H}_{g-1} \). In particular, this operator maps \( M_r(\Gamma_g) \) to \( M_r(\Gamma_{g-1}) \) and \( M_r(\Gamma_g(1, 2)) \) to \( M_r(\Gamma_{g-1}(1, 2)) \), see [7].
for details. A modular form is a cusp form if ∀γ ∈ Γ, Φ(f|γ) = 0. We shall denote by Sr(Γ, v) the subspace of cusp forms and use the notation Sγ(Γ) when v is identically 1. We let e(z) = e^{2πiz} for z ∈ C.

**Definition 20.** Let k, m ∈ Q. Let Γ ⊆ Γ∞(Z) and fix a map v : Γ → C*. The Jacobi forms with respect to Γ and v, denoted J_{k,m}(Γ, v), are the vector space of holomorphic φ : H × C → C such that for all γ ∈ Γ, we have

\[\tilde{\phi}|_{kγ} = v(γ) \tilde{\phi},\]

where we define \(\tilde{\phi}(\tau z \omega) = \phi(\tau, z)e(mw),\) and for all γ ∈ Γ∞(Z), we have that the Fourier expansion for \(\tilde{\phi}|_{kγ}\) is supported on semidefinite index matrices, namely

\[\sum_{s ≥ 0} c(s)e(\text{tr}(s(\tau z \omega))),\]

where s ≥ 0 indicates s is summed over only semidefinite 2 × 2 matrices. Furthermore, we say φ is a Jacobi cusp form and write φ ∈ J_{cusp}k,m(Γ, v) if for all γ ∈ Γ∞(Z), we have that the Fourier expansion for \(\tilde{\phi}|_{kγ}\) has no nonzero coefficients at indefinite index matrices, namely

\[\sum_{s > 0} c(s)e(\text{tr}(s(\tau z \omega))),\]

where s > 0 indicates s is summed over only positive definite 2 × 2 matrices. When v is identically 1, we write J_{k,m}(Γ) = J_{k,m}(Γ, v) and similarly for cusp forms.

We study J_{k,t/2}(Γ2(1, 2)∞) in this article. Note that Γ2(1, 2)∞ contains translation matrices of the form \(\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}\) where S is symmetric integral with even diagonal entries. This implies that φ ∈ J_{cusp}k,m(Γ2(1, 2)∞) has a Fourier expansion of the form

\[\phi(\tau, z) = \sum_{n, r ∈ Z: \quad tn − r^2 > 0, n > 0} c(n, r)e(\frac{1}{2}nτ + rz).\]

For g = 2, the Fourier Jacobi expansion of θ[0]

\[\theta[0](\tau z \omega) = \theta[0](0, τ) + 2θ[0](z, τ)e(\omega/2) + \ldots\]

shows that θ[0](z, τ) is automorphic with respect to Γ2(1, 2) ∩ Γ∞(Z) = Γ2(1, 2)∞ of weight 1/2 and index 1/2. Thus θ[0](z, τ) gives an element of J_{\frac{1}{4}, \frac{1}{4}}(Γ2(1, 2)∞, vθ).

The definition of Jacobi form above is equivalent to the usual one. The group Γ1(1, 2)F = Γ1(1, 2) × He(Z) is isomorphic to Γ2(1, 2)∞/±I by Lemma 6, and this shows the equivalence to the usual definition by
taking generators of $\Gamma(1, 2)$ and $H_c(\mathbb{Z})$. These transformations are
\[
\forall (a b \overline{c} d) \in \Gamma(1, 2),
\phi \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = v \left( \left( \frac{a b}{c d} \right) \right) e \left( \frac{cmz^2}{c \tau + d} \right) \phi(\tau, z),
\forall (\lambda, v, \kappa) \in H_c(\mathbb{Z}),
\phi(\tau, z + \lambda \tau + v) = v(w(\lambda, v, \kappa)) e \left( (m(\lambda^2 \tau + 2 \lambda z + (\lambda v + \kappa))) \right) \phi(\tau, z).
\]
The first equation shows that if $\phi \in J_{k,0}(\Gamma, v)$ then $\phi(\tau, 0)$ gives an element of $J_{k,0}(\Gamma, v)$. Using the isomorphism $M_\kappa(i^{-1}(\Gamma), i^*v) = J_{k,0}(\Gamma, v)$ we have $M_\kappa(i^{-1}(\Gamma), i^*v_1)J_{k_2,m}(\Gamma, v_2) \subseteq J_{k_1+k_2,m}(\Gamma, v_1v_2)$. We use this containment in the statement of Corollary 2 to write
\[
S_{k-\frac{1}{2}}(\Gamma(1, 2), (v_\theta^{(1)})^{2k-1})(\theta[0](z, \tau)) \subseteq J_{\frac{1}{2}, k}(\Gamma(1, 2))^J, (v_\theta^{(2)})^{2k}.
\]
Here one needs to check that $i^*(v_\theta^{(2)}) = v_\theta^{(1)}$ on $\Gamma(1, 2)$. This can be done by restricting the theta function to diagonal $(\theta_0 \omega) \in \mathcal{H}_2$.

**Definition 21.** Fix $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. For $\phi \in J_{\frac{1}{2}, \frac{k}{2}}(\Gamma_2(1, 2)_\infty)$, define
\[
\tilde{\phi}(\left( \frac{z}{w} \right)) = \phi(\tau, z)e^{(\frac{1}{2}tw)}.
\]
Define a formal series $F_\phi$ by
\[
F_\phi = \sum_{m=1}^{\infty} m^{2-k}(-1)^m+1 \phi |_k \tau \sum_{m=1}^{\infty} m^{2-k}(-1)^m+1 \phi |_k \tau \left( \left( \frac{a b}{c d} \right) \right),
\]
where the inner sum is over $a, b, d \in \mathbb{N}$ with $ad = m$, $0 \leq b < 2d$, and $a, (b + d)$ both odd.

**Proposition 22.** Let $\phi \in J_{\frac{1}{2}, \frac{k}{2}}^c(\Gamma_2(1, 2)_\infty)$ have expansion
\[
\phi(\tau, z) = \sum_{n, r \in \mathbb{Z};mn-r^2>0, n>0} c(n, r) e \left( \frac{1}{2}n \tau + rz \right).
\]
Then the formal series $F_\phi(\Omega)$ may be rearranged to
\[
F_\phi(\Omega) = \sum_{t/m} a(T)e^{(\frac{1}{2}tr(T\Omega))}
\]
where the coefficients $a(T)$ are given by
\[
a(T) = \left( \frac{n}{r} \right) \sum_{a \mid (n, r, m/m)} a^{k-1} c(\frac{mn}{ta^2}, \frac{r}{a}).
\]
Proof. Applying the action of \( T_m \) to \( \tilde{\phi} \), we get

\[
F_{\phi}(\left( \frac{r}{z}, \frac{z}{w} \right)) = \sum_{m=1}^{\infty} m^{2-k} (-1)^{m+1} \sum_{0 \leq b < 2d \atop a, b \text{ odd}} \sum_{ad=m} m^{2k-3} d^{-k} \phi\left( \frac{ar+b}{d}, az \right) e\left( \frac{1}{2} tmw \right)
\]

\[
= \sum_{m=1}^{\infty} m^{k-1} (-1)^{m+1} \sum_{0 \leq b < 2d \atop a, b \text{ odd}} d^{-k} \sum_{tn-r^2 > 0} c(n, r) e\left( \frac{1}{2} n \frac{ar+b}{d} + raz + \frac{1}{2} tmw \right)
\]

\[
= \sum_{m=1}^{\infty} \sum_{tn-r^2 > 0} \sum_{0 \leq b < 2d \atop b, d \text{ odd}} m^{k-1} (-1)^{m+1} c(n, r) d^{-k} \sum_{0 \leq b < 2d} e\left( \frac{nb}{2d} \right).
\]

If \( m \) is odd, then \( d \) is odd, and \( b \) must be even and we would have

\[
\sum_{0 \leq b < 2d} e\left( \frac{nb}{2d} \right) = \sum_{j=0}^{d-1} e\left( \frac{n(2j+1)}{d} \right) = \begin{cases} 
\frac{d}{2} & \text{if } d|n \\
0 & \text{otherwise.}
\end{cases}
\]

If \( m \) is even, then \( d \) is even and \( b \) must be odd and we would have

\[
\sum_{0 \leq b < 2d} e\left( \frac{nb}{2d} \right) = \sum_{j=0}^{d-1} e\left( \frac{n(2j+1)}{2d} \right) = e\left( \frac{1}{2} n \frac{a^2}{d} \right) \sum_{j=0}^{d-1} e\left( \frac{nj}{d} \right) = \begin{cases} 
\frac{d(-1)^{\frac{n}{d}}}{2} & \text{if } d|n \\
0 & \text{otherwise.}
\end{cases}
\]

We can unify these two cases of \( m \) even or odd by

\[
\sum_{0 \leq b < 2d} e\left( \frac{nb}{2d} \right) = \begin{cases} 
d(-1)^{(m+1)\frac{n}{d}} & \text{if } d|n \\
0 & \text{otherwise.}
\end{cases}
\]
Plugging this into the formula, we can make the substitution \( n = dn_1 \) where \( n_1 \in \mathbb{Z} \) to get

\[
F_\phi((\tau \ z \ w)) = \sum_{m=1}^{\infty} \sum_{a \text{ odd}} \sum_{t \text{ even}} m^{k-1}(-1)^{(m+1)}(-1)^{(m+1)n_1} c(dn_1, r)d^{-k+1}e(\frac{1}{2}n_1a\tau + raz + \frac{1}{2}tmw) = \sum_{m=1}^{\infty} \sum_{a \text{ odd}} \sum_{t \text{ even}} a^{k-1}(-1)^{(m+1)(n_1+1)}c(\frac{m}{a}n_1, r) e(\frac{1}{2}n_1a\tau + raz + \frac{1}{2}tmw).
\]

Making another substitution \( R = ar \) and \( N = an_1 \), where we sum over \( R, N \) which must be multiples of \( a \) (or equivalently, we must only use \( a \) which divide all of \( m, R, N \), we get

\[
(4) \quad F_\phi((\tau \ z \ w)) = \sum_{m=1}^{\infty} \sum_{tmN - R^2 > 0} \sum_{a \text{ odd}} a^{k-1}(-1)^{(m+1)(N+1)} c(\frac{mN}{a^2}, \frac{R}{a}) e(\frac{1}{2}N\tau + Rz + \frac{1}{2}tmw),
\]

where we used the fact that \( N \) has the same parity as \( n_1 \) because \( a \) is odd when we replaced \( n_1 \) by \( N \) in the exponent of \((-1)^{(m+1)(N+1)}\). A final substitution \( M = mt \) where \( M \) ranges over \( \mathbb{N} \) with \( t | M \) gives

\[
F_\phi((\tau \ z \ w)) = \sum_{M,N \in \mathbb{N}, R \in \mathbb{Z}} \sum_{M \text{ odd}, R^2 > 0} \sum_{a \text{ odd}} a^{k-1}(-1)^{(M/2+1)(N+1)}c(\frac{MN}{ta^2}, \frac{R}{a}) e(\frac{1}{2}N\tau + Rz + \frac{1}{2}Mw),
\]

and this proves the proposition. \( \square \)

**Proposition 23.** Fix \( t \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Let \( \phi \in \mathcal{J}^\text{cusp}_{k,\frac{1}{2}t}(\Gamma_2(1,2)) \). The series \( F_\phi(\Omega) \) converges absolutely for all \( \Omega \in \mathcal{H}_2 \) and \( F_\phi : \mathcal{H}_2 \rightarrow \mathbb{C} \) defines a holomorphic function. Also, for \((\tau \ z \ w) \in \mathcal{H}_2 \) we have

\[
F_\phi((tw \ z \ \frac{1}{1}t\tau)) = F_\phi((\tau \ z \ w)).
\]

**Proof.** Since \( \phi \) has its Fourier coefficients \( c(n, r; \phi) \) bounded by polynomial growth, so does \( F_\phi \) have its Fourier coefficients \( a((\frac{n}{t} \ \frac{r}{t})) \) bounded by polynomial growth. In more detail, the cusp form \( \phi \) has a bound \( |\phi(\tau, z)| \leq M_\phi v^{-k/2}e^{\pi vy^2/\varphi} \), where we write \( z = x + iy \) and \( \tau = u + iv \) for real \( x, y, u, v \). This implies that the Fourier coefficients of \( \phi \) have a polynomial bound \( |c(n, r; \phi)| \leq A_\phi (2tn - r^2)^{k/2} \) where \( A_\phi = (2\pi e/kt)^{k/2}M_\phi \).
A crude estimate shows \( |a \left( \left( \begin{smallmatrix} n \\ m \end{smallmatrix} \right) \right) | \leq A_\delta m^k(2tn - r^2)^{k/2} \). This suffices to show the absolute convergence of \( F_\phi \) on compact subsets of \( \mathcal{H}_2 \). Note that in the above proof of Proposition 22, in equation 4, the expression is nearly symmetric in \( m \) and \( N \). Thus switching \( m \) and \( N \), we see that

\[
F_\phi\left( \left( \begin{smallmatrix} tw \\ z \\ \frac{1}{t} \tau \end{smallmatrix} \right) \right) = F_\phi\left( \left( \begin{smallmatrix} \tau \ z \\ w \end{smallmatrix} \right) \right).
\]

\( \square \)

**Proposition 24.** Fix \( t \in \mathbb{N} \) and let \( \phi \) and \( F_\phi \) be as in Proposition 23. Then \( F_\phi|_{k \mu_t} = (-1)^k F_\phi \).

**Proof.** Note that \( \tilde{\phi} \mid_k \left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) implies \( \phi(\tau, -z) = (-1)^k \phi(\tau, z) \) and so \( c(n, -r; \phi) = (-1)^k c(n, r; \phi) \). We have

\[
(F_\phi|_{k \mu})(\left( \begin{smallmatrix} tw z \\ \tau \end{smallmatrix} \right)) = F_\phi\left( \left( \begin{smallmatrix} tw z \\ \frac{1}{t} \tau \end{smallmatrix} \right) \right) = (-1)^k F_\phi\left( \left( \begin{smallmatrix} z \gamma \phi \end{smallmatrix} \right) \right) = (-1)^k F_\phi\left( \left( \begin{smallmatrix} z \gamma \phi \end{smallmatrix} \right) \right)
\]

\( \square \)

The following Theorem completes the proof of Theorem 1 from the Introduction. The form of the Fourier coefficients has already been given in Proposition 22.

**Theorem 25.** Let \( t \in \mathbb{N} \) and \( k \in \mathbb{Z} \). For \( \phi \in j_{k, \frac{1}{2}}^c(\Gamma_2(1, 2)_\infty) \), we have \( F_\phi \in M_k(\Gamma_{\text{para}}(t; 1, 2)) \) and \( F_\phi|_{\mu_t} = (-1)^k F_\phi \). If \( t \not\equiv 0 \mod 4 \), then we have \( F_\phi \in S_k(\Gamma_{\text{para}}(t; 1, 2)) \).

**Proof.** We know that \( F_\phi \) is holomorphic from Proposition 23. From Definition 21, we know that \( F_\phi \) is invariant under \( \Gamma_2(1, 2)_\infty \) because the series defining it is term by term invariant. From the form of \( F_\phi \) in Proposition 22, it is clear that \( F_\phi \) is invariant under \( \gamma_t \) and so \( F_\phi \) is invariant under \( \Gamma_2(1, 2)_{\infty}[t] \). From Proposition 24, we know \( F_\phi|_{k \mu_t} = (-1)^k F_\phi \) and therefore \( F_\phi \) is invariant under \( G_t = \langle \Gamma_2(1, 2)_\infty[t], \mu_t \Gamma_2(1, 2)_\infty[t] \mu_t \rangle = \Gamma_{\text{para}}(t; 1, 2) \) by Proposition 11.

We only need to prove that \( F_\phi \) is a cusp form when \( t \not\equiv 0 \mod 4 \). Take any \( \beta \in \text{Sp}(2, \mathbb{Z}) \). Since \( t \not\equiv 0 \mod 4 \), by Proposition 14, we have that \( \beta = \alpha \gamma_1 \delta \gamma_2 \), or \( \beta = \alpha \mu_t^{-1} \gamma_1 \mu_t \delta \gamma_2 \), where \( \alpha \in \Gamma_{\text{para}}(t; 1, 2) \), \( \delta \in \Delta_2(\mathbb{Q}) \) and \( \gamma_1 \in \Gamma_\infty(\mathbb{Z}) \) and \( \gamma_2 \in \Gamma_\infty(\mathbb{Q}) \). Then \( F_\phi \mid \beta = F_\phi \mid \gamma_1 \delta \gamma_2 \) or \( F_\phi \mid \beta = (-1)^k F_\phi \mid \gamma_1 \mu_t \delta \gamma_2 \).

Since \( F_\phi \) has no nonzero indefinite coefficients in its Fourier expansion, and since \( \gamma_1 \in \Gamma_\infty(\mathbb{Z}) \), we have that \( F_\phi \mid \gamma_1 \) has no nonzero indefinite coefficients. Since \( \delta \) and \( \mu_t \delta \) are upper triangular, then \( F_\phi \mid \gamma_1 \delta \) and \( F_\phi \mid \gamma_1 \mu_t \delta \) have no nonzero indefinite coefficients either; these two cases can be unified together by saying that \( F_\phi \mid \beta \gamma_2^{-1} \) has no nonzero indefinite coefficients.
Consider the Siegel operator \((\Phi_2 f)(\tau) = \lim_{s \to \infty} f((z^s \tau^s)w))\) for a modular form \(f\). Since \(\gamma_2 \in \Gamma_{\infty}(\mathbb{Q})\) and \((f|\gamma_2)((z^s \tau^s)w) = (*)f((z^s \tau^s))\) where the \(*\) depend only on \(\tau, z\) and not on \(\omega\), then \(\Phi_2 f = 0\) would imply \(\Phi_2 (f|\gamma_2) = 0\). Thus

\[
\Phi_2(F_{\phi}|\beta) = \Phi_2((F_{\phi}|\beta\gamma_2^{-1})|\gamma_2) = 0.
\]

Since this is true for all \(\beta \in \text{Sp}_2(\mathbb{Z})\), \(F_{\phi}\) is a cusp form. \(\square\)

When \(t = 1\) and \(k\) is even, we get the following corollary which we state as a theorem because it is of particular interest for the degree two chiral superstring measure.

**Theorem 26. Lifting to Degree Two Theta Group for even \(k\).** Let \(k \in \mathbb{N}\) be even and \(\phi \in J_{k, \frac{1}{2}}^\text{cusp}(\Gamma_2(1, 2)_\infty)\). Then \(F_{\phi} \in S_k(\Gamma_2(1, 2))\).

**Corollary 27.** For \(t = 1\), if \(k \in \mathbb{N}\) is odd and \(\phi \in J_{k, \frac{1}{2}}^\text{cusp}(\Gamma_2(1, 2)_\infty)\) then \(F_{\phi} = 0\).

**Proof.** Since \(k\) is odd, then by Proposition 24, \(F_{\phi}|\mu_t = -F_{\phi}\). Let

\[
g = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \epsilon = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Note that both \(g, \epsilon \in \Gamma_2(1, 2)_\infty\) (see Lemma 6) and so \(F_{\phi}|g = F_{\phi}\) and \(F_{\phi}|\epsilon = F_{\phi}\). But it is straightforward to check that

\[
\mu_{1}g\epsilon\mu_{1}g^{-1}\epsilon\mu_{1}g = I
\]

is the identity matrix. But \(F_{\phi}|(\mu_{1}g\epsilon\mu_{1}g^{-1}\epsilon\mu_{1}g) = (-1)^3F_{\phi} = -F_{\phi}\) and \(F_{\phi}|I = F_{\phi}\). This forces \(F_{\phi} = 0\). \(\square\)

5. **The Chiral String Modular Form in Genus 2**

Now we discuss the weight \(15/2\) form that gives \(\Xi_2[0]\). We define here the variety of theta functions that we use. For \(\Omega \in \mathcal{H}_g\), \(z \in \mathbb{C}^g\) and \(a, b \in \mathbb{R}^g\), define the first order theta function with characteristics \(a\) and \(b\) as a holomorphic function on \(\mathbb{C}^g \times \mathcal{H}_g\) given by the series

\[
\theta[a/b](z, \Omega) = \sum_{n \in \mathbb{Z}^g} e\left(\frac{1}{2}(n + a)'\Omega(n + a) + (n + a)'(n + z + b)\right).
\]

For \(r \in \mathbb{N}\), the \(r^{th}\) order theta functions \(\theta_{r}[\nu] : \mathcal{H}_g \to \mathbb{C}\) are given by

\[
\theta_{r}[\nu](\Omega) = \theta\left[\frac{\nu}{r}\right](0, r\Omega).
\]
In $g = 1$, we use the standard abbreviations $\theta_{ab}(\tau) = \theta \left[ \frac{a/2}{b/2} \right] (0, \tau)$ for $a, b \in \{0, 1\}$. In $g = 2$, we use $\theta(\frac{a_1}{b_1}, \frac{a_2}{b_2})(\Omega) = \theta \left[ \frac{a_1/2}{b_1/2}, \frac{a_2/2}{b_2/2} \right] (0, \Omega)$.

The Dedekind eta function, mentioned only in connection with the $g = 1$ chiral superstring measure in the Introduction, is the standard one. In the introduction, we have given $\Phi = \theta^{11} F_2 - 16 \theta^7 F_2^2 \in S_{15/2} (\Gamma_0(4)^*; \tilde{\nu}_\theta)$ in terms of the generators of $\bigoplus_{\ell=0}^\infty M_{\ell/2}(\Gamma_0(4), \tilde{\nu}_\theta^\ell)$:

$$\theta(\tau) = \theta_2[0](\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \ldots$$

$$F_2(\tau) = \left( \frac{\theta_2[1](\tau)}{2} \right)^4 = \sum_{n \in \mathbb{N}; n \text{ odd}} \sigma_1(n) q^n = q + 4q^3 + 6q^5 + 8q^7 + \ldots$$

Here $\tilde{\nu}_\theta : \Gamma_0(4)^* \rightarrow \mathbb{C}^*$ is given explicitly by $\tilde{\nu}_\theta(\frac{a}{c}, \frac{b}{d}) = \nu_\theta(\frac{a}{c}, 2\frac{b}{d})$. In these terms we can show directly that, for $W_4 = \left( \begin{array}{rr} 0 & \frac{1}{2} \\ -2 & 0 \end{array} \right)$, we have

$$F_2|W_4 = F_2 - \frac{1}{2^4} \theta^4.$$  

However, the following alternate expression immediately shows modularity with respect to the theta group,

$$\Phi(\tau/2) = \theta_{00}(\tau)^3 \left( \frac{\theta_{00}(\tau)\theta_{01}(\tau)\theta_{10}(\tau)}{2} \right)^4 \in S_{15/2} (\Gamma_1(1, 2), \nu_{\theta}^{15}) .$$

Consider a form $g \in S_{k-1}(\Gamma_1(1, 2), \nu_\theta^{2k-1})$ whose Fourier expansion is $g(\tau) = \sum_{n \in \mathbb{N}} c(n; g) q^{n/2}$. Multiplication by $\theta[0] \in J_{1/2, 1/2}(\Gamma_1(1, 2)^J, \nu_\theta)$ whose Fourier expansion is $\theta[0](z, \tau) = \sum_{n \in \mathbb{N}} q^{n^2/2} \zeta^n$ produces a Jacobi form $\phi(\tau, z) = g(\tau) \theta[0](z, \tau) \in J_{k, 1/2}(\Gamma_1(1, 2)^J, \nu_\theta^{2k})$ whose Fourier expansion is $\phi(\tau, z) = \sum_{n \in \mathbb{N}, r \in \mathbb{Z}} c(n-r^2; g) q^{n/2} \zeta^r$. Note that when $4|k$, $\phi$ has trivial character. In this case we have $c(n, r; \phi) = c(n-r^2; g)$ and the formula for the Fourier coefficients of the lift $L(\phi)$ is

$$a \left( \begin{array}{rr} n & r \\ r & m \end{array} ; L(\phi) \right) = (-1)^{(m+1)(n+1)} \sum_{a | (n, r, m)} a^{k-1} c \frac{mn - r^2}{a^2} ; g .$$

This proves the formula for the Fourier coefficients of the chiral superstring form $\Xi_2[0]$ that was given at the conclusion of the Introduction.
6. Final Remarks

A final remark is that when \( t \equiv 0 \mod 4 \), we can prove that

\[
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
0 & t & -2 & 1
\end{pmatrix}
\notin \Gamma_{\text{para}}(t; 1, 2)\Gamma_\infty(Z)\Delta_2(Q)\Gamma_\infty(Q)
\cup \Gamma_{\text{para}}(t; 1, 2)\mu_t^{-1}\Gamma_\infty(Z)\mu_t\Delta_2(Q)\Gamma_\infty(Q)
\]

by showing that any matrix in the coset \( \Gamma_{\text{para}}(t; 1, 2) \) cannot have a 0 in the (3, 2) or (4, 2) entry but a matrix in \( \Gamma_\infty(Z)\Delta_2(Q)\Gamma_\infty(Q) \) must have a 0 in the (4, 2) entry and any matrix that happens to be in \( \Gamma_{\text{para}}(t; 1, 2)\mu_t^{-1}\Gamma_\infty(Z)\mu_t\Delta_2(Q)\Gamma_\infty(Q) \) must have a 0 in the (3, 2) entry.

Thus the above method of proof in Theorem 25 that the lift is a cuspidal form does not work when \( t \equiv 0 \mod 4 \). It is conceivable that the lift of a Jacobi cusp form might not be a cusp form in general when \( t \equiv 0 \mod 4 \) but we don’t know any examples of this. The intended case where \( t/2 \) is strictly half integral has been fully treated, as well as the slightly more general case when \( t \not\equiv 0 \mod 4 \).

References


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