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Hypergroups

by

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Preface

Let S be a set, and let μ be a map from $S \times S$ to the power set of S. For any two elements p and q of S, we write pq instead of $\mu(p, q)$ and assume that pq is not empty.

For any two nonempty subsets P and Q of S, we define the *complex product* PQ to be the union of the sets pq with $p \in P$ and $q \in Q$. If one of the two factors in a complex product consists of a single element, say s, we write s instead of $\{s\}$ in that product.

Following (and generalizing) Frédéric Marty's terminology in [3] we call S a *hypergroup* (with respect to μ) if the following three conditions hold.

H1 For any three elements p, q, and r in S, we have $p(qr) = (pq)r$.

H2 The set S possesses an element e such that $se = \{s\}$ for each element s in S.

H3 For each element s in S, there exists an element s^* in S such that $p \in rq^*$ and $q \in p^*r$ for any three elements p, q, and r in S satisfying $r \in pq$.

The present text is a first attempt to see how far a theory of hypergroups can be developed.

The interest in hypergroups is motivated by the observation that each association scheme satisfies the above three axioms H1, H2, and H3; cf. [9; Lemma 1.3.1], [9; Lemma 1.3.3(ii)], and [9; Lemma 1.3.3(i)]. Thus, hypergroups generalize association schemes. In particular, they generalize groups.

The above observation would not be so exciting if not a significant part of scheme theory relies just on the above three axioms. A closer look at scheme theory reveals that most (if not all) of the reasoning in scheme theory consists of a part that refers only to the complex multiplication in schemes (and, therefore, just relies on the above three axioms) and a part that requires the analysis of the underlying set of the scheme under investigation and its arithmetic. In fact, quite a few results in scheme theory do not make use at all of the underlying set, so that, in reality, these results are results on hypergroups. It is natural that one wishes to conceptually isolate the first (algebraic) part from the latter (geometric) part of the theory.

The analysis of scheme theory that was mentioned in the previous paragraph implies of course that many themes of the first three chapters of these notes (not all) are just formal generalizations of results in [9]. However, putting them here in the right place of the theory might have its own value.

An additional motive for investigating hypergroups is the fact that (via association schemes) hypergroups provide a natural conceptual framework for buildings and twin buildings. This topic will be discussed in the final (sixth) chapter of these notes. Chapter 3 and 4 show how natural buildings are embedded in the theory of hypergroups when viewed as closed subsets generated by Coxeter sets.

The contents of the individual chapters and a motivation for the choice of the topics will be previewed in the introduction of the corresponding chapters.

There are two topics which have been left out in these notes and which would make the notes more complete. Firstly, we have not considered quotient hypergroups, although the definition and first results should be straightforward. Secondly, it would be interesting to have sufficient and necessary conditions for a hypergroup to be a scheme. In [10; Theorem A], there was given a sufficient and necessary condition for a scheme to be schurian, and it would be nice to have a similar criterion for a hypergroup to be a scheme.

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Bonn, May 2010 Paul-Hermann Zieschang

Contents

1

Basic Facts

In this chapter, we compile some basic facts about hypergroups. Most of the results were proven earlier for schemes; cf. [9].

The letter S will always stand for a hypergroup.

1.1 First observations

All results of this section will be used throughout these notes, often without further mentioning.

Lemma 1.1.1 *Let s be an element in S. Then the following hold.*

(i) *For each neutral element e of S, we have $e \in s^*s$.*

(ii) *We have $s^{**} = s$.*

(iii) *For each neutral element e of S, we have $es = \{s\}$.*

Proof. (i) Let e be a neutral element of S. Then, by H2, $s \in se$. Thus, by H3, $e \in s^*s$.

(ii) From (i) we know that $e \in s^*s$. Thus, by H3, $s \in s^{**}e$. On the other hand, by H2, $s^{**}e = \{s^{**}\}$. Thus, $s \in \{s^{**}\}$, and that means that $s^{**} = s$.

(iii) Let r be an element in es. Then, by H3, $e \in rs^*$. A second application of H3 yields $s^* \in r^*e$. On the other hand, by H2, $r^*e = \{r^*\}$. Thus, $s^* \in \{r^*\}$, and that means that $r^* = s^*$. Thus, by (ii), $r = r^{**} = s^{**} = s$. □

Lemma 1.1.2 *The hypergroup S possesses exactly one neutral element.*

Proof. Let c and d be neutral elements of S. Then, by Lemma 1.1.1(iii), $\{c\} = cd = \{d\}$. It follows that $c = d$. □

Until further notice the uniquely determined neutral element of S will be denoted by e.

Lemma 1.1.3 *We have $e^* = e$.*

Proof. From H2 we know that $e \in ee$. Thus, by H3, $e \in e^*e$. On the other hand, by H2, $e^*e = \{e^*\}$. Thus, $e \in \{e^*\}$, and that means $e^* = e$. □

Lemma 1.1.4 *Let p and q be elements in S. Then the following hold.*

(i) *We have $e \in pq$ if and only if $p = q^*$.*

(ii) *Let r be and element in S satisfying $r \in pq$. Then $r^* \in q^*p^*$.*

(iii) *Let o and r be elements of S, and assume that $op \cap qr$ is not empty. Then $o^*q \cap pr^*$ is not empty.*

Proof. (i) Assume that $e \in pq$. Then, by H3, $p \in eq^*$. On the other hand, by Lemma 1.1.1(iii), $eq^* = \{q^*\}$. Thus, $p \in \{q^*\}$, and that means that $p = q^*$.

Conversely, assume that $p = q^*$. Then, by Lemma 1.1.1(i), $e \in q^*q = pq$.

(ii) Applying H3 three times we obtain from $r \in pq$ first $p \in rq^*$, then $q^* \in r^*p$, and, finally, $r^* \in q^*p^*$.

(iii) Let s be an element in $op \cap qr$. From $s \in op$ we obtain $o \in sp^*$; cf. H3. Thus, as $s \in qr$, $o \in qrp^*$. Thus, rp^* possesses an element t such that $o \in qt$; cf. H1. From $o \in qt$ we obtain $t \in q^*o$; cf. H3. It follows that $t \in rp^* \cap q^*o$. Thus, by (ii) (together with Lemma 1.1.1(ii)), $t^* \in o^*q \cap pr^*$. □

Lemma 1.1.5 *For any three nonempty subsets P, Q, and R of S, the following hold.*

(i) *Assume that $P \subseteq Q$, Then $PR \subseteq QR$ and $RP \subseteq RQ$.*

(ii) *We have $P(QR) = (PQ)R$.*

Proof. (i) This follows immediately from the definition of the complex product.

(ii) Let s be an element in $P(QR)$. Then, by definition, there exist elements p in P and t in QR such that $s \in pt$. Since $t \in QR$, there exist elements q in Q and r in R such that $t \in qr$. It follows that $s \in p(qr)$. Thus, by definition, $s \in (pq)r \subseteq (PQ)R$.

Since s was chosen arbitrarily in $P(QR)$, we have shown that $P(QR) \subseteq (PQ)R$. That $(PQ)R \subseteq P(QR)$ is shown similarly. □

Lemma 1.1.5(ii) says that the set of all nonempty subsets of S is a monoid with respect to complex multiplication and with neutral element $\{e\}$. This fact will be crucial in Chapter 6 where we shall deal with buildings.

1.2 Closed subsets

For each subset R of S, we set

$$R^* := \{s \in S \mid s^* \in R\}.$$

A nonempty subset R of S is called *closed* if $R^*R \subseteq R$.

From H2 (together with Lemma 1.1.3) we obtain that $\{e\}$ is closed. Note also that S is closed and that intersections of closed subsets are closed.

Lemma 1.2.1 *Let T be a closed subset of S. Then the following hold.*

(i) *We have $e \in T$.*

(ii) *We have $T^* = T$.*

(iii) *We have $TT = T$.*

Proof. (i) Since T is assumed to be a closed subset of S, T is not empty. Let t be an element in T. Then, by definition, $t^*t \subseteq T^*T \subseteq T$. On the other hand, by Lemma 1.1.1(i), $e \in t^*t$. Thus, $e \in T$.

(ii) From (i) we know that $e \in T$. Thus, by H2,

$$T^* = T^*e \subseteq T^*T \subseteq T.$$

It follows that $T^* \subseteq T$, and from this we obtain $T^{**} \subseteq T^*$. Now recall from Lemma 1.1.1(ii) that $T^{**} = T$. Therefore, $T^* = T$.

(iii) From (i) we know that $e \in T$. Thus, $T = Te \subseteq TT$. From (ii) we obtain $TT = T^*T \subseteq T$. Thus, $TT = T$. $\square$

Lemma 1.2.2 *Let T and U be closed subsets of S. Then $\{TsU \mid s \in S\}$ is a partition of S.*

Proof. Let p and q be elements in S such that $p \in TqU$. From $p \in TqU$ we obtain $TpU \subseteq TqU$; cf. Lemma 1.2.1(iii). Thus, it suffices to show that $q \in TpU$.

Since $p \in TqU$, there exist elements t in T and u in U such that $p \in tqu$. From $p \in tqu$ we obtain an element r in qu such that $p \in tr$.

From $r \in qu$ we obtain $r^* \in u^*q^*$; cf. Lemma 1.1.4(ii). From $p \in tr$ we obtain $t \in pr^*$; cf. H3. Thus, $t \in pu^*q^*$, and from this, we obtain similarly $q^* \in up^*t$. Thus, by Lemma 1.1.4(ii),

$$q \in t^*pu^* \subseteq TpU;$$

cf. Lemma 1.2.1(ii). $\square$

For each nonempty subset R of S, we set

$$S/R := \{sR \mid s \in S\}.$$

The following lemma gives a sufficient and necessary condition for a subset of S to be closed.

Lemma 1.2.3 *Let R be a subset of S with $e \in R$. Then R is closed if and only if S/R is a partition of S.*

Proof. If R is closed, S/R is a partition by Lemma 1.2.2. (Set $T = \{e\}$ and $U = R$ in that lemma.)

Conversely, assume that S/R is a partition of S, and let r be an element in R. Then, by Lemma 1.1.1(i), $e \in r^*r \subseteq r^*R$. On the other hand, we are assuming that $e \in R$. Thus, as S/R is assumed to be a partition of S, $r^*R = R$. Thus, as r has been chosen arbitrarily in R, we have shown that R is closed. □

Lemma 1.2.4 *Let T and U be closed subsets of S. Then TU is closed if and only if $TU = UT$.*

Proof. Assume first that TU is closed. Then, by Lemma 1.2.1(ii), $(TU)^* = TU$. Since T and U are assumed to be closed, we also have $T^* = T$ and $U^* = U$. Thus, by Lemma 1.1.4(ii),

$$TU = (TU)^* = U^*T^* = UT.$$

Conversely, assume that $TU = UT$. Then, referring once more to Lemma 1.1.4(ii),

$$(TU)^*TU = U^*T^*TU \subseteq U^*TU = U^*UT \subseteq UT = TU.$$

Therefore, TU is closed. □

Lemma 1.2.5 *Let T and U be closed subsets of S. Then we have $T \cap U = \{e\}$ if and only if, for each element s in TU, there exist uniquely determined elements t in T and u in U such that $s \in tu$.*

Proof. Assume first that $T \cap U = \{e\}$, and let s be an element in TU. Then, by definition, there exist elements t in T and u in U such that $s \in tu$.

Let t' be an element in T and u' an element in U such that $s \in t'u'$. We have to show that $t' = t$ and $u' = u$.

Since $s \in tu \cap t'u'$, $t^*t' \cap uu'^*$ is not empty; cf. Lemma 1.1.4(iii). Since T is assumed to be closed, $t^*t' \subseteq T$. Similarly, as U is assumed to be closed, $uu'^* \subseteq U$. It follows that

$$t^*t' \cap uu'^* \subseteq T \cap U = \{e\}.$$

Therefore, $e \in t^*t'$ and $e \in uu'^*$. Thus, by Lemma 1.1.4(i), $t' = t$ and $u' = u$.

Conversely, let s be an element in $T \cap U$. Then $s \in TU$ and, by Lemma 1.1.1(iii), $es = \{s\} = se$. Thus, as $e \in T \cap U$, $s = e$. □

The following lemma is a very specific case of a general observation due to Richard Dedekind; cf. [2; Theorem VIII].

Lemma 1.2.6 *Let P and Q be nonempty subsets of S, and let T be a closed subset of S. Then we have the following.*

(i) *If $P \subseteq T$, $T \cap PQ = P(T \cap Q)$.*

(ii) *If $Q \subseteq T$, $T \cap PQ = (T \cap P)Q$.*

Proof. (i) We first show that $T \cap PQ \subseteq P(T \cap Q)$. Let t be an element in $T \cap PQ$. From $t \in PQ$ we obtain elements p in P and q in Q with $t \in pq$. Thus, by H3, $p \in tq^*$ and then $q^* \in t^*p$. Now recall that T is assumed to be closed. Thus, as $t \in T$ and $p \in P \subseteq T$, $t^*p \subseteq T$. It follows that $q^* \in T$, and then $q \in T$. Thus, as $t \in pq$, $t \in P(T \cap Q)$.

Conversely, as T is assumed to be closed, $P \subseteq T$ yields $P(T \cap Q) \subseteq T$; cf. Lemma 1.2.1(iii). Thus, as $P(T \cap Q) \subseteq PQ$, $P(T \cap Q) \subseteq T \cap PQ$.

(ii) Setting $P = Q^*$ and $Q = P^*$ in (i) we obtain $T \cap Q^*P^* = Q^*(T \cap P^*)$. Thus, the claim follows from Lemma 1.1.4(ii). □

Corollary 1.2.7 *Let P and Q be nonempty subsets of S, let T be a closed subset of S, and assume that PT and QT are closed. Then*

$$(P \cap QT)(Q \cap PT) = PQ \cap QT \cap PT.$$

Proof. Since T is assumed to be closed, we have $e \in T$; cf. Lemma 1.2.1(i). Thus, $P \subseteq PT$. Thus, as PT is assumed to be closed, Lemma 1.2.6(i) yields

$$PT \cap (P \cap QT)Q = (P \cap QT)(PT \cap Q).$$

From $e \in T$ we also obtain $Q \subseteq QT$. Thus, as QT is assumed to be closed, Lemma 1.2.6(ii) yields

$$QT \cap PQ = (QT \cap P)Q.$$

The desired equation follows easily from the last two equations. □

Corollary 1.2.8 *Let P and Q be nonempty subsets of S, let T be a closed subset of S, and assume that $Q \subseteq T$. Then we have the following.*

(i) *If $T \subseteq QPQ$, $Q(T \cap P)Q = T$.*

(ii) *If $P \cup Q = PQ \cap QP$, $(T \cap P) \cup Q = (T \cap P)Q \cap Q(T \cap P)$.*

Proof. (i) Since we are assuming that $Q \subseteq T$, we obtain from Lemma 1.2.6 that

$$Q(T \cap P)Q = (T \cap QP)Q = T \cap QPQ.$$

Thus, if $T \subseteq QPQ$, $Q(T \cap P)Q = T$.

(ii) We are assuming that $Q \subseteq T$. Thus, $(T \cap P) \cup Q = T \cap (P \cup Q)$. On the other hand, we know from Lemma 1.2.6 that

$$PQ \cap T \cap QP = (T \cap P)Q \cap Q(T \cap P).$$

Thus, if $P \cup Q = PQ \cap QP$, $(T \cap P) \cup Q = (T \cap P)Q \cap Q(T \cap P)$. □

1.3 Normalizer, strong normalizer, and centralizer

Let P and Q be subsets of S, and assume that Q is not empty. We set

$$N_P(Q) := \{p \in P \mid Qp \subseteq pQ\}$$

and call this set the *normalizer* of Q *in* P.

Lemma 1.3.1 *Let T be a closed subset of S. Then the following hold.*

(i) *We have $e \in N_S(T)$.*

(ii) *We have $TN_S(T) \subseteq N_S(T)$.*

(iii) *Let s be an element in $N_S(T)$ with $s^* \in N_S(T)$. Then $Ts = sT$.*

Proof. (i) This follows immediately from the definition of $N_S(T)$.

(ii) Let s be an element in $TN_S(T)$. Then there exists an element r in $N_S(T)$ such that $s \in Tr$. Thus, by Lemma 1.2.2, $Tr = Ts$.

Since $r \in N_S(T)$, $Tr \subseteq rT$. Thus, as $s \in Tr$, $s \in rT$. Thus, by Lemma 1.2.3, $rT = sT$. From $Tr = Ts$ and $Tr \subseteq rT$ we now obtain $Ts \subseteq sT$. Thus, by definition, $s \in N_S(T)$.

(iii) We are assuming that $s^* \in N_S(T)$. Thus, by definition, $Ts^* \subseteq s^*T$. Thus, by Lemma 1.1.4(ii), $sT \subseteq Ts$. Since s has been chosen from $N_S(T)$, we have $Ts \subseteq sT$, too. □

In general, $N_S(T)$ is not closed if T is a closed subset of S.

Lemma 1.3.2 *Let T and U be closed subsets of S. Then the following hold.*

(i) *Assume that $T \subseteq N_S(U)$. Then TU is closed.*

(ii) *Assume that $T \subseteq N_S(U)$. Then $TU \subseteq N_S(U)$.*

(iii) *We have $N_S(T) \cap N_S(U) \subseteq N_S(TU)$.*

(iv) *We have $N_T(U) \subseteq N_T(T \cap U)$.*

Proof. (i) Considering Lemma 1.2.4 this is a consequence of Lemma 1.3.1(iii).

(ii) From (i) we know that TU is closed. Thus, by Lemma 1.2.4, $TU = UT$. From Lemma 1.3.1(ii) we obtain $UT \subseteq UN_S(U) \subseteq N_S(U)$. Thus, $TU \subseteq N_S(U)$.

(iii) For each element s in $N_S(T) \cap N_S(U)$, we have

$$TUs \subseteq TsU \subseteq sTU.$$

Thus, $s \in N_S(TU)$.

(iv) Let t be an element in $N_T(U)$. Then, by definition, $Ut \subseteq tU$. Thus, by Lemma 1.2.6(i),

$$(T \cap U)t \subseteq T \cap Ut \subseteq T \cap tU = t(T \cap U).$$

Thus, by definition, $t \in N_T(T \cap U)$. □

Let P and Q be subsets of S, and assume Q to be not empty. We set

$$K_P(Q) := \{p \in P \mid p^*Qp \subseteq Q\}$$

and call this set the *strong normalizer* of Q *in* P.

For each nonempty subset R of S, we obviously have $e \in K_S(R)$. From Lemma 1.2.1(iii) we also obtain that $T \subseteq K_S(T)$ for each closed subset T of S.

Lemma 1.3.3 *For each nonempty subset R of S, we have $K_S(R) \subseteq N_S(R)$.*

PROOF. Let s be an element in $K_S(R)$. Then, by definition, $s^*Rs \subseteq R$. Thus, as $e \in ss^*$, $Rs \subseteq ss^*Rs \subseteq sR$. It follows that $Rs \subseteq sR$, and that means that $s \in N_S(R)$. □

Lemma 1.3.4 *Let T and U be closed subsets of S. Then the following hold.*

(i) *We have $K_S(T) \cap K_S(U) \subseteq K_S(T \cap U)$.*

(ii) *Assume that $T \subseteq N_S(U)$. Then $K_S(T) \cap N_S(U) \subseteq K_S(TU)$.*

(iii) *Assume that $T \subseteq U$. Then $K_S(T) \cap N_S(U) \subseteq K_S(U)$.*

PROOF. (i) Let s be an element in $K_S(T) \cap K_S(U)$. Then

$$s^*(T \cap U)s \subseteq s^*Ts \cap s^*Us \subseteq T \cap U.$$

Thus, $s \in K_S(T \cap U)$.

(ii) Let s be an element in $K_S(T) \cap N_S(U)$. Then

$$s^*TUs \subseteq s^*TsU \subseteq TU.$$

Thus, $s \in K_S(TU)$.

(iii) From Lemma 1.3.1(i), (ii) we obtain $U \subseteq N_S(U)$. Thus, the claim follows from (ii). □

Let T and U be closed subsets of S, and assume that $T \subseteq U$. The closed subset T is said to be a *normal* closed subset *of* U if $U \subseteq N_S(T)$. In this case, we say that T is *normal in* U. The closed subset T is said to be a *strongly normal* closed subset *of* U if $U \subseteq K_S(T)$. In this case, the set T is called *strongly normal in* U.

Lemma 1.3.3 implies that, if a closed subset T of S is strongly normal in a closed subset U of S, then T is normal in U.

Lemma 1.3.5 *Let T and U be closed subsets of S, and assume that T is strongly normal in U. Let V be a closed subset of S, and assume that $U \subseteq N_S(V)$. Then TV is strongly normal in UV.*

Proof. We are assuming that $U \subseteq N_S(V)$. Thus, by Lemma 1.3.2(i), UV is closed. Similarly, we obtain from $T \subseteq U \subseteq N_S(V)$ that TV is closed.

In order to show that TV is strongly normal in UV, we pick an element s in UV. We shall see that $s^*TVs \subseteq TV$.

Since $s \in UV$, there exist elements u in U and v in V such that $s \in uv$. From $u \in U$ and $U \subseteq N_S(V)$ we obtain $u \in N_S(V)$, and that means $Vu \subseteq uV$. From $u \in U$ and $U \subseteq K_S(T)$ we obtain $u \in K_S(T)$, and that means $u^*Tu \subseteq T$. Thus,

$$s^*TVs \subseteq v^*u^*TVuv \subseteq v^*u^*TuVv \subseteq v^*TVv \subseteq TV.$$

(Recall that, by Lemma 1.1.4(ii), $s^* \in v^*u^*$. Note also that $v \in TV$.) □

Lemma 1.3.6 *Let R be a nonempty subset of S, and let p and q be elements in $K_S(R)$. Then $pq \subseteq K_S(R)$.*

Proof. Let s be an element in pq. Then,

$$s^*Rs \subseteq q^*p^*Rpq \subseteq q^*Rq \subseteq R.$$

Thus, $s \in K_S(R)$. □

From Lemma 1.1.1(i) we know that $e \in s^*s$ for each element s in S. An element s in S is called *thin* if $s^*s = \{e\}$.

For each subset R of S, we define $O_\vartheta(R)$ to be the set of all thin elements in R. The set $O_\vartheta(R)$ is called the *thin radical* of R.

Lemma 1.3.7 *The following statements hold.*

(i) *We have $O_\vartheta(S) = K_S(\{e\})$.*

(ii) *For any two elements p and q in $O_\vartheta(S)$, we have $pq \subseteq O_\vartheta(S)$.*

Proof. (i) Let s be an element in S. Then $s \in O_\vartheta(S)$ if and only if $s^*s = \{e\}$ if and only if $s \in K_S(\{e\})$.

(ii) From (i) we know that $O_\vartheta(S) = K_S(\{e\})$. Thus, our claim is a consequence of Lemma 1.3.6. □

Lemma 1.3.8 *Let s be an element in S and t be an element in $O_\vartheta(S)$. Then $|st| = 1$.*

Proof. Let r be an element in st. Then, by H3, $s \in rt^*$. Thus,

$$st \subseteq rt^*t = re = \{r\}.$$

(Recall that t is assumed to be thin.) □

Let R be a subset of S, and let s be an element in S. We set

$$C_R(s) := \{r \in R \mid sr = rs\}$$

and call this set the *centralizer* of s *in* R.

Let P and Q be subsets of S, and assume that Q is not empty. We define $C_P(Q)$ to be the intersection of the sets $C_P(q)$ with $q \in Q$. The set $C_P(Q)$ is called the *centralizer* of Q *in* P.

Note that we have $C_P(Q) \subseteq N_P(Q)$ for any two nonempty subsets P and Q of S.

Lemma 1.3.9 *Let T and U be closed subsets of S, and assume that $T \subseteq N_S(U)$, $U \subseteq N_S(T)$, and $T \cap U = \{e\}$. Then $T \subseteq C_S(U)$.*

Proof. Let t be an element in T, let u be an element in U, and let s be an element in tu. Since the hypotheses of the lemma are symmetric in T and U, we shall be done if we succeed in showing that $s \in ut$.

From $s \in tu$ (together with the hypothesis that $U \subseteq N_S(T)$) we obtain $s \in uT$. Thus, T possesses an element p such that $s \in up$.

From $s \in up$ (together with the hypothesis that $T \subseteq N_S(U)$) we obtain $s \in pU$. Thus, U possesses an element q such that $s \in pq$. Thus, as $s \in tu$, Lemma 1.2.5 yields $t = p$. Thus, as $s \in up$, $s \in ut$. □

1.4 Hypergroups of order 6

It is the purpose of this section to show by an example how similar hypergroup theory can be to group theory; cf. Corollary 1.4.6. The letter T will stand for a closed subset of S.

Lemma 1.4.1 *Let p be an element in S with $pT \subseteq Tp$, and let q be an element in Tp such that $q^*T = qT$. Then $q \in pT$.*

Proof. From $q \in Tp$ we obtain $q^* \in p^*T$; cf. Lemma 1.1.4(ii) and Lemma 1.2.1(ii). Thus, by Lemma 1.2.3, $p^* \in q^*T$. Thus, as $q^*T = qT$, $q \in Tp$, and $pT \subseteq Tp$,

$$p^* \in qT \subseteq TpT = Tp = Tq.$$

It follows that $p \in q^*T = qT$, so that $q \in pT$. □

Recall that $N_S(T)$ is defined to be the set of all elements s in S such that $Ts \subseteq sT$.

Lemma 1.4.2 *Let s be an element in S such that $sT = \{s\}$ and $s^*T = \{s^*\}$. Then $s \in N_S(T)$.*

Proof. From our hypothesis $s^*T = \{s^*\}$ we obtain $Ts = \{s\}$; cf. Lemma 1.1.4(ii). On the other hand, we are assuming that $sT = \{s\}$. Thus, $Ts = sT$, and that implies that $s \in N_S(T)$. □

Lemma 1.4.3 *Let s be an element in S such that $(sT)^* \in S/T$. Then $s \in N_S(T)$.*

Proof. We are assuming that $(sT)^* \in S/T$. Thus, as $s^* \in (sT)^*$, $(sT)^* = s^*T$; cf. Lemma 1.2.3. Thus, by Lemma 1.1.4(ii), $Ts = sT$. □

Recall that the closed subset T is called normal in S if $N_S(T) = S$. The following two results give sufficient conditions for T to be normal in S.

An element s in S is called *symmetric* if $s^* = s$.

Lemma 1.4.4 *Assume that $|S \setminus T| = |S/T|$ and that $S \setminus T$ possesses exactly $|S \setminus T| - 2$ symmetric elements. Then T is normal in S.*

Proof. We are assuming that $|S \setminus T| = |S/T|$. Thus, $S/T \setminus \{T\}$ possesses exactly one element of cardinality 2, all other elements of $S/T \setminus \{T\}$ consist of a single element. Let p and q be the two elements in the uniquely determined element of $S/T \setminus \{T\}$ of cardinality 2.

If $\{p, q\}^* = \{p, q\}$, we are done by Lemma 1.4.3. Thus, we shall be done if we succeed in showing that $\{p, q\}^* \neq \{p, q\}$ leads to a contradiction.

If $\{p, q\}^* \neq \{p, q\}$, we must have $p^* \notin \{p, q\}$ or $q^* \notin \{p, q\}$. Without loss of generality, we assume that $p^* \notin \{p, q\}$.

Since $\{p, q\}$ is the only element in $S/T \setminus \{T\}$ that has more than one element, $p^* \notin \{p, q\}$ forces $p^*T = \{p^*\}$. Thus, $p^*T \subseteq Tp^*$.

We are assuming that $S \setminus T$ has exactly $|S \setminus T| - 2$ symmetric elements. Thus, as $p^* \neq p \neq q$, we must have $q^* = q$. Thus, $q^*T = qT$ and, since $q \in pT$, $q \in Tp^*$; cf. Lemma 1.1.4(ii).

From $p^*T \subseteq Tp^*$, $q \in Tp^*$, and $q^*T = qT$ we obtain $q \in p^*T$; cf. Lemma 1.4.1. Thus, as $p^*T = \{p^*\}$, $q = p^*$, contradiction. □

Theorem 1.4.5 *Assume that $|S \setminus T| \leq 3$. Then T is normal in S.*

Proof. If $|S/T| = 1$, $T = S$, and we are done. If $|S/T| = 2$, we have $Ts = S \setminus T = sT$ for each element s in $S \setminus T$. Thus, T is normal also in this case.

Assume that $3 \leq |S/T|$. Then as $|S/T| \leq |S \setminus T| + 1 \leq 4$, we must have $|S/T| = |S \setminus T| = 3$ or $|S/T| = |S \setminus T| + 1$. In the former case, we are done by Lemma 1.4.3 and by Lemma 1.4.4. In the latter case, we are done by Lemma 1.4.2. □

Corollary 1.4.6 *Assume that $|S| = 6$ and that T is not normal in S. Then $|T| = 2$.*

Proof. This follows immediately from Theorem 1.4.5. □

1.5 Conjugates of closed subsets

Let R be a nonempty subset of S, and let s be an element of S. We define

$$R^s := \{r \in S \mid sr \subseteq Rs\}.$$

Note that $sR^s \subseteq Rs$.

Lemma 1.5.1 *Let s be an element in S, and let R be a nonempty subset of S. Then the following hold.*

(i) *We have $R^s \subseteq s^*Rs$.*

(ii) *If $e \in R$, $e \in R^s$.*

(iii) *Let P and Q be nonempty subsets of S such that $PQ \subseteq R$. Then we have $P^sQ^s \subseteq R^s$.*

PROOF. (i) From $e \in s^*s$ and $sR^s \subseteq Rs$ we obtain

$$R^s \subseteq s^*sR^s \subseteq s^*Rs.$$

(ii) Assume that $e \in R$. Then, by Lemma 1.1.1(iii), $se = \{s\} = es \subseteq Rs$. It follows that $e \in R^s$.

(iii) Let r be an element in P^sQ^s. Then there exist elements t in P^s and u in Q^s such that $r \in tu$. Since $t \in P^s$, $st \subseteq Ps$. Since $u \in Q^s$, $su \subseteq Qs$. Thus, as we are assuming that $PQ \subseteq R$,

$$sr \subseteq stu \subseteq Psu \subseteq PQs \subseteq Rs,$$

so that $sr \subseteq Rs$. Thus, by definition, $r \in R^s$. □

Lemma 1.5.2 *Let p and q be elements in S, and let R be a subset of S such that $e \in R$, $RR \subseteq R$, and $Rp = Rq$. Then $R^p = R^q$.*

PROOF. Let s be an element in R^p. Then, by definition, $ps \subseteq Rp$.

From $e \in R$ we obtain $q \in Rq$. Thus, as $Rp = Rq$, $q \in Rp$. It follows that $qs \subseteq Rps$.

From $qs \subseteq Rps$ and $ps \subseteq Rp$ we obtain $qs \subseteq RRp$. Thus, as we are assuming $RR \subseteq R$, we obtain $qs \subseteq Rp$. Since we are assuming that $Rp = Rq$, this implies $qs \subseteq Rq$. Thus, $s \in R^q$.

So far, we have seen that $R^p \subseteq R^q$. The proof for $R^q \subseteq R^p$ is similar. □

Lemma 1.5.3 *Let s be an element in S, and let T be a closed subset of S. Then $s^* \in N_S(T)$ if and only if $T \subseteq T^s$.*

PROOF. By definition, we have $s^* \in N_S(T)$ if and only if $Ts^* \subseteq s^*T$. According to Lemma 1.1.4(ii), $Ts^* \subseteq s^*T$ is equivalent to $sT \subseteq Ts$, and this means that $T \subseteq T^s$. □

Lemma 1.5.4 *Let s be an element in S, and let T be a closed subset of S such that $ss^* \subseteq T$. Then the following hold.*

(i) *We have $T^s = s^*Ts$.*

(ii) *The set s^*Ts is closed.*

(iii) *We have $K_S(s^*Ts) = s^*K_S(T)s$.*

PROOF. (i) From Lemma 1.5.1(i) we know that $T^s \subseteq s^*Ts$. Conversely, assuming $ss^* \subseteq T$ we also have $ss^*Ts \subseteq Ts$, so that, by definition, $s^*Ts \subseteq T^s$.

(ii) Assuming $ss^* \subseteq T$ we have

$$(s^*Ts)^*s^*Ts = s^*Tss^*Ts = s^*Ts;$$

cf. Lemma 1.1.4(ii) and Lemma 1.2.1(iii). Thus, s^*Ts is closed.

(iii) Let q be an element in $s^*K_S(T)s$. Then $K_S(T)$ possesses an element p such that $q \in s^*ps$. Thus, as $ss^* \subseteq T$ is assumed, we obtain from $p \in K_S(T)$ that

$$q^*s^*Tsq \subseteq s^*p^*ss^*Tss^*ps \subseteq s^*p^*Tps \subseteq s^*Ts.$$

It follows that $q \in K_S(s^*Ts)$.

Since q has been chosen arbitrarily in $s^*K_S(T)s$, we, thus, have shown that

$$s^*K_S(T)s \subseteq K_S(s^*Ts).$$

Let q be an element in $K_S(s^*Ts)$. Then, as $ss^* \subseteq T$ is assumed,

$$sq^*s^*Tsqs^* \subseteq ss^*Tss^* = T.$$

Thus proves that $sqs^* \subseteq K_S(T)$, whence $q \in s^*sqs^*s \subseteq s^*K_S(T)s$.

Since q has been chosen arbitrarily in $K_S(s^*Ts)$, we, thus, have shown that

$$K_S(s^*Ts) \subseteq s^*K_S(T)s.$$

This finishes the proof. □

Lemma 1.5.5 *Let T and V be closed subsets of S, and set $U := K_V(T)$. Assume that $T \subseteq V$ and that U is closed. Then we have the following.*

(i) *Let p and q be elements in V such that $Up = Uq$. Then $pp^* \subseteq T$ if and only if $qq^* \subseteq T$.*

(ii) *The number of sets s^*Ts with $s \in V$ and $ss^* \subseteq T$ is equal to the number of sets Us with $s \in V$ and $ss^* \subseteq T$.*

Proof. (i) Since we are assuming that $Up = Uq$, there exists an element u in U such that $p \in uq$. Thus, $q \in u^*p$. Thus, assuming that $pp^* \subseteq T$, we obtain

$$qq^* \subseteq u^*pp^*u \subseteq u^*Tu \subseteq T.$$

Similarly, one obtains $pp^* \subseteq T$ from $qq^* \subseteq T$.

(ii) Let p and q be elements in V, and assume first that $p^*Tp \subseteq q^*Tq$ and $qq^* \subseteq T$. Then, $qp^*Tpq^* \subseteq qq^*Tqq^* \subseteq T$. It follows that $pq^* \subseteq U$. Thus, $p \in Uq$. Thus, by Lemma 1.2.2, $Up = Uq$.

Assume now that $Up = Uq$. Then U possesses an element u such that $p \in uq$. It follows that

$$p^*Tp \subseteq q^*u^*Tuq \subseteq q^*Tq.$$

Similarly, one shows that $q^*Tq \subseteq p^*Tp$, so that $p^*Tp = q^*Tq$. □

Lemma 1.5.6 *Let T and U be closed subsets of S such that $T \cap U = \{e\}$. Let t be an element in T, and let u be an element in U. Then the following hold.*

(i) *Assume that $|tu| = 1$. Then $|u^*tu \cap T| \leq 1$.*

(ii) *If $U \subseteq N_S(T)$, $1 \leq |u^*tu \cap T|$.*

Proof. (i) Let p and q be elements in $u^*tu \cap T$. We have to show that $p = q$.

We are assuming that tu contains exactly one element. Let us call this element s. From $p \in u^*tu$ and $tu = \{s\}$ we obtain $p \in u^*s$. Thus, $s \in up$. Similarly, we obtain $s \in uq$, so that $s \in up \cap uq$.

Since $s \in up \cap uq$, $u^*u \cap pq^*$ is not empty; cf. Lemma 1.1.4(iii). Thus, as $u^*u \cap pq^* \subseteq T \cap U = \{e\}$, $u^*u \cap pq^* = \{e\}$. It follows that $e \in pq^*$, so that, according to Lemma 1.1.4(i), $p = q$.

(ii) Let s be an element in tu, and assume that $U \subseteq N_S(T)$. Then $s \in uT$. Thus, T possesses an element r such that $s \in ur$. It follows that $r \in u^*s \subseteq u^*tu$. Thus, as $r \in T$, $r \in u^*tu \cap T$. □

Lemma 1.5.7 *Let p and q be elements in S such that $|pq| = 1$, and let T be a closed subset of S such that $q \in p^*Tp$. Then $q \in T^p$.*

Proof. We are assuming that $q \in p^*Tp$. Thus, p^*T possesses an element r such that $q \in rp$. It follows that $p \in r^*q \subseteq Tpq$; cf. Lemma 1.1.4(ii). Thus, $p \in Tpq$. Now recall that we are assuming that $|pq| = 1$. Thus, $pq \subseteq Tp$. Thus, by definition, $q \in T^p$. □

Corollary 1.5.8 *Let s be an element in S, and let T and U be closed subsets of S. Assume that, for each element u in $s^*Ts \cap U$, $|su| = 1$. Then $T^s \cap U = s^*Ts \cap U$.*

Proof. We are assuming $|su| = 1$ for each element u in $s^*Ts \cap U$. Thus, by Lemma 1.5.7, $s^*Ts \cap U \subseteq T^s$, so that the desired equation follows from Lemma 1.5.1(i). □

Corollary 1.5.9 *Let s be an element in S, and assume that $|sr| = 1$ for each element r in s^*s. Then the following hold.*

(i) *We have $\{e\}^s = s^*s$.*

(ii) *The set s^*s is closed.*

Proof. (i) Apply Corollary 1.5.8 to $\{e\}$ and S in place of T and U.

(ii) From (i) we obtain $ss^*s = \{s\}$. Thus, $s^*ss^*s = s^*s$, and that means that s^*s is closed. □

Let s be an element in S. For each subset R of S, we define $D_R(s)$ to be the set of all elements r in R such that $r^*r \subseteq s^*s$.

Lemma 1.5.10 *Let s be an element in S, and assume that $|sr| = 1$ for each element r in s^*s. Let T be a closed subset of S with $T \subseteq D_S(s^*)$. Assume that $T \subseteq T^s$ and that $|st| = 1$ for each element t in T. Then $T \subseteq K_S(s^*s)$.*

Proof. Let t be an element in T. We have to show that $t \in K_S(s^*s)$.

Since $t \in T$, $|st| = 1$. On the other hand, we are assuming that $T \subseteq T^s$. Thus, as $t \in T$, $t \in T^s$, and that means that $st \subseteq Ts$.

Since $|st| = 1$ and $st \subseteq Ts$, there exists an element r in T such that $st \subseteq rs$. From $st \subseteq rs$ we obtain

$$t^*s^*st \subseteq s^*r^*rs.$$

From $r \in T$ and $T \subseteq D_S(s^*)$ we obtain $r \in D_S(s^*)$. Thus, $r^*r \subseteq ss^*$. From Corollary 1.5.9(ii) we also know that s^*s is closed. Thus,

$$t^*s^*st \subseteq s^*r^*rs \subseteq s^*ss^*s \subseteq s^*s.$$

This means that $t \in K_S(s^*s)$. □

1.6 Generating sets

Throughout this section, the letter R stands for a subset of S.

We define $\langle R\rangle$ to be the intersection of the closed subsets of S which contain R as a subset.

Note that $\langle R\rangle$ is closed. The set $\langle R\rangle$ is said to be *generated* by R. It is also called the *span* of R.

If R possesses an element r with $R = \{r\}$, we shall write $\langle r\rangle$ instead of $\langle R\rangle$. Similarly, we write $\langle p, q\rangle$ instead of $\langle R\rangle$ if p and q are elements in R with $R = \{p, q\}$.

For the remainder of this section, we assume R to be not empty.

We set $R^0 := \{e\}$ and define inductively $R^n := R^{n-1}R$ for each positive integer n.

Lemma 1.6.1 *The set $\langle R\rangle$ is equal to the union of the sets $(R^* \cup R)^n$ with n a nonnegative integer.*

Proof. Set $P := R^* \cup R$ and define Q to be the union of the sets P^n with n a nonnegative integer. We have to show that $\langle R\rangle = Q$.

Since $P^* = P$, $(P^n)^* = P^n$ for each nonnegative integer n; cf. Lemma 1.1.4(ii). Thus, for any two nonnegative integers l and m,

$$(P^l)^* P^m = P^l P^m = P^{l+m} \subseteq Q.$$

It follows that Q is closed. Thus, as $R \subseteq Q$, $\langle R\rangle \subseteq Q$.

Conversely, for each non-negative integer n, we have

$$P^n \subseteq \langle P\rangle^n \subseteq \langle P\rangle = \langle R\rangle.$$

Therefore, $Q \subseteq \langle R\rangle$. □

From Lemma 1.6.1 we obtain that, for each element s in $\langle R\rangle$, there exists a nonnegative integer n such that $s \in (R^* \cup R)^n$. The smallest of these integers is called the *R-length* of s or simply the *length* of s and will be denoted by $\ell_R(s)$.

Since the subset R is fixed within this section, we shall write ℓ instead of ℓ_R for the remainder of this section.

Lemma 1.6.2 *Let s be an element in $\langle R\rangle \setminus \{e\}$. Then there exist elements q in $\langle R\rangle$ and r in $R^* \cup R$ such that $s \in qr$ and $\ell(s) = \ell(q) + 1$.*

Proof. We set $n := \ell(s)$. Then, by definition, $s \in (R^* \cup R)^n$. On the other hand, we are assuming that $s \neq e$. Therefore, $1 \leq n$. From $s \in (R^* \cup R)^n$ and $1 \leq n$ we obtain elements q in $(R^* \cup R)^{n-1}$ and r in $R^* \cup R$ such that $s \in qr$.

From $q \in (R^* \cup R)^{n-1}$ we obtain $\ell(q) \leq n - 1$. From $n = \ell(s)$ and $s \in qr$ we obtain $n \leq \ell(q) + 1$. □

Lemma 1.6.3 *Let p, q, and r be elements in $\langle R\rangle$ satisfying $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$. Let t and u be elements in $\langle R\rangle$ satisfying $q \in tu$ and $\ell(q) = \ell(t) + \ell(u)$. Then, pt possesses an element s such that $r \in su$, $\ell(s) = \ell(p) + \ell(t)$, and $\ell(r) = \ell(s) + \ell(u)$.*

Proof. Since $r \in pq$ and $q \in tu$, $r \in ptu$. Thus, pt possesses an element s such that $r \in su$.

Since $s \in pt$, $\ell(s) \leq \ell(p) + \ell(t)$. Since $r \in su$, $\ell(r) \leq \ell(s) + \ell(u)$. Thus, as we are assuming that $\ell(r) = \ell(p) + \ell(q)$ and that $\ell(q) = \ell(t) + \ell(u)$, we conclude

that
$$\ell(r) \le \ell(s) + \ell(u) \le \ell(p) + \ell(t) + \ell(u) = \ell(r).$$
It follows that $\ell(s) = \ell(p) + \ell(t)$ and $\ell(r) = \ell(s) + \ell(u)$. □

Lemma 1.6.4 *Let s be an element in S, and assume that $R^* = R$. Then the following hold.*

(i) *Assume that $Rs \subseteq s\langle R\rangle$. Then $s \in N_S(\langle R\rangle)$.*

(ii) *We have $\langle R^s\rangle \subseteq \langle R\rangle^s$.*

Proof. (i) Define Q to be the set of all elements q in $\langle R\rangle$ with $qs \not\subseteq s\langle R\rangle$. By way of contradiction, we assume that Q is not empty. We fix an element q in Q such that $\ell(q)$ is as small as possible.

Since $s \in s\langle R\rangle$, $q \neq e$. Thus, by Lemma 1.6.2, there exist elements p in $\langle R\rangle$ and r in R such that $q \in pr$ and $\ell(q) = \ell(p) + 1$. Since $\ell(p) = \ell(q) - 1$, $p \notin Q$. Thus, as $p \in \langle R\rangle$, $ps \subseteq s\langle R\rangle$. Thus, as $rs \subseteq s\langle R\rangle$ (by hypothesis),
$$qs \subseteq prs \subseteq ps\langle R\rangle \subseteq s\langle R\rangle,$$
contradiction.

(ii) Define Q to be the set of all elements q in $\langle R^s\rangle$ with $sq \not\subseteq \langle R\rangle s$. By way of contradiction, we assume that Q is not empty. We fix an element q in Q such that $\ell(q)$ is as small as possible.

Since $s \in \langle R\rangle s$, $q \neq e$. Thus, by Lemma 1.6.2, there exist elements p in $\langle R^s\rangle$ and r in R^s such that $q \in pr$ and $\ell(q) = \ell(p) + 1$. Since $\ell(p) = \ell(q) - 1$, $p \notin Q$. Thus, as $p \in \langle R^s\rangle$, $sp \subseteq \langle R\rangle s$.

On the other hand, as $r \in R^s$, $sr \subseteq Rs$. Thus,
$$sq \subseteq spr \subseteq \langle R\rangle sr \subseteq \langle R\rangle Rs = \langle R\rangle s,$$
contradiction. □

Lemma 1.6.5 *Let s be an element in S, and assume that $s^*Rs \subseteq \langle R\rangle$. Then $s \in K_S(\langle R\rangle)$.*

Proof. Define Q to be the set of all elements q in $\langle R\rangle$ with $s^*qs \not\subseteq \langle R\rangle$. By way of contradiction, we assume that Q is not empty. We fix an element q in Q such that $\ell(q)$ is as small as possible.

We are assuming that $s^*Rs \subseteq \langle R\rangle$. Thus,
$$s^*s \subseteq s^*Rss^*R^*s \subseteq \langle R\rangle.$$

This shows that $e \notin Q$. Thus, as $q \in Q$, $q \neq e$. Thus, by Lemma 1.6.2, there exist elements p in $\langle R\rangle$ and r in $R^* \cup R$ such that $q \in pr$ and $\ell(q) = \ell(p) + 1$. Since $\ell(p) = \ell(q) - 1$, $p \notin Q$. Thus, as $p \in \langle R\rangle$, $s^*ps \subseteq \langle R\rangle$.

We are assuming that $s^*Rs \subseteq \langle R\rangle$. Thus, as $\langle R\rangle$ is closed, $s^*R^*s = (s^*Rs)^* \subseteq \langle R\rangle$; cf. Lemma 1.1.4(ii). Thus, no matter whether $r \in R$ or $r \in R^*$, we have $s^*rs \subseteq \langle R\rangle$.

From $s^*ps \subseteq \langle R\rangle$ and $s^*rs \subseteq \langle R\rangle$ we now obtain

$$s^*qs \subseteq s^*prs \subseteq s^*pss^*rs \subseteq \langle R\rangle,$$

contradiction. □

Subsets of S are called *thin* if all of their elements are thin.

Lemma 1.6.6 *The following hold.*

(i) *The set $R^* \cup R$ is thin if and only $\langle R\rangle$ is thin.*

(ii) *If $R^* = R$, $\langle O_\vartheta(R)\rangle \subseteq O_\vartheta(\langle R\rangle)$.*

Proof. (i) Since $R^* \cup R \subseteq \langle R\rangle$, $R^* \cup R$ is thin if $\langle R\rangle$ is thin.

Assume now that $R^* \cup R$ is thin and that $\langle R\rangle$ is not thin. Assuming $\langle R\rangle$ not to be thin we find an element s in $\langle R\rangle$ such that s is not thin. Among the non-thin elements of $\langle R\rangle$ we fix s in such a way that $\ell(s)$ is as small as possible.

Since s is not thin, $s \neq e$. Thus, by Lemma 1.6.2, there exist elements q in $\langle R\rangle$ and r in $R^* \cup R$ such that $s \in qr$ and $\ell(s) = \ell(q) + 1$. Since $\ell(q) = \ell(s) - 1$, the minimal choice of s forces q to be thin. Since $r \in R^* \cup R$ and $R^* \cup R$ is assumed to be thin, r is thin. Thus, as $s \in qr$, Lemma 1.3.7(ii) forces s to be thin, contradiction.

(ii) Set $Q := O_\vartheta(R)$. Then $Q \subseteq O_\vartheta(\langle R\rangle)$. Thus, for each non-negative integer n, $Q^n \subseteq O_\vartheta(\langle R\rangle)$; cf. Lemma 1.3.7(ii). It follows that $\langle Q\rangle \subseteq O_\vartheta(\langle R\rangle)$; cf. Lemma 1.6.1. □

1.7 The thin residue

In this section, the letter T stands for a closed subset of S.

We define $O^\vartheta(T)$ to be the intersection of all strongly normal closed subsets of T and call it the *thin residue* of T.

Note that $O^\vartheta(T)$ is closed.

Theorem 1.7.1 *The following statements hold.*

(i) *The set $O^\vartheta(T)$ is strongly normal in T.*

(ii) *Let R denote the union of the sets t^*t with $t \in T$. Then $O^\vartheta(T) = \langle R\rangle$.*

(iii) *For each closed subset U of S with $T \subseteq U$, we have $O^\vartheta(T) \subseteq O^\vartheta(U)$.*

Proof. (i) This follows from Lemma 1.3.4(i).

(ii) Let p and q be elements in T. We first prove $q^*p^*pq \subseteq \langle R\rangle$.

Let t be an element in pq. Then $p \in tq^*$, whence

$$t^*pq \subseteq t^*tq^*q \subseteq RR \subseteq \langle R\rangle.$$

Since t has been chosen arbitrarily in pq, this yields $q^*p^*pq \subseteq \langle R\rangle$.

Since p and q have been chosen arbitrarily in T, we have shown $t^*Rt \subseteq \langle R\rangle$ for each element t in T. Thus, by Lemma 1.6.5, $\langle R\rangle$ is strongly normal in T. Thus, by definition, $O^\vartheta(T) \subseteq \langle R\rangle$.

In order to show that $\langle R\rangle \subseteq O^\vartheta(T)$ it suffices to show that $R \subseteq O^\vartheta(T)$. (This is because $O^\vartheta(T)$ is closed.)

Let t be an element in T. Then, as $e \in O^\vartheta(T)$, $t^*t \subseteq t^*O^\vartheta(T)t$. On the other hand, we know from (i) that $O^\vartheta(T)$ is strongly normal in T, so that $t^*O^\vartheta(T)t \subseteq O^\vartheta(T)$. Thus, $t^*t \subseteq O^\vartheta(T)$. Since t has been chosen arbitrarily in T, we have shown that $R \subseteq O^\vartheta(T)$.

(iii) This is an immediate consequence of (ii). □

Lemma 1.7.2 *Let U be a closed subset of S such that $T \subseteq N_S(U)$. Then we have $O^\vartheta(T)U = O^\vartheta(TU)U$.*

Proof. From Theorem 1.7.1(iii) we know that $O^\vartheta(T) \subseteq O^\vartheta(TU)$, and from this we obtain $O^\vartheta(T)U \subseteq O^\vartheta(TU)U$.

By Theorem 1.7.1(i), $O^\vartheta(T)$ is strongly normal in T. Moreover, we are assuming that $T \subseteq N_S(U)$. Thus, by Lemma 1.3.5, $O^\vartheta(T)U$ is strongly normal in TU. Thus, by definition of $O^\vartheta(TU)$, $O^\vartheta(TU) \subseteq O^\vartheta(T)U$. Thus, as U is closed, $O^\vartheta(TU)U \subseteq O^\vartheta(T)U$. □

Lemma 1.7.3 *Let U be a closed subset of S. Assume that TU is closed and that $O^\vartheta(U) \subseteq T$. Then $O^\vartheta(TU)$ is the intersection of all strongly normal closed subsets of T which contain $O^\vartheta(U)$.*

Proof. Define $\mathcal{V}$ to be the set of all strongly normal closed subsets of T which contain $O^\vartheta(U)$ and W to be the intersection of all elements of $\mathcal{V}$. We have to show that $O^\vartheta(TU) = W$.

We first show that $O^\vartheta(TU) \subseteq W$. In order to do so we pick an element s in TU and an element V in $\mathcal{V}$. Since TU is assumed to be closed, $TU = UT$; cf. Lemma 1.2.4. Thus, as $s \in TU$, $s \in UT$. Thus, there exist elements t in T and u in U such that $s \in ut$. Since $u \in U$, $u^*u \subseteq O^\vartheta(U)$; cf. Theorem 1.7.1(ii). Since $V \in \mathcal{V}$, $t^*Vt \subseteq V$ and $O^\vartheta(U) \subseteq V$. Thus,

$$s^*s \subseteq t^*u^*ut \subseteq t^*O^\vartheta(U)t \subseteq V.$$

Now, as s has been chosen arbitrarily in TU, we conclude that $O^\vartheta(TU) \subseteq V$; cf. Theorem 1.7.1(ii). But also V has been chosen arbitrarily in $\mathcal{V}$. Therefore, we have shown that $O^\vartheta(TU) \subseteq W$.

Let us now prove that, conversely, $W \subseteq O^\vartheta(TU)$.

From $O^\vartheta(TU) \subseteq W$ and $W \subseteq T$ we obtain $O^\vartheta(TU) \subseteq T$. In particular, $O^\vartheta(TU)$ is strongly normal in T. On the other hand, we know from Theorem 1.7.1(iii) that $O^\vartheta(U) \subseteq O^\vartheta(TU)$. Thus, $O^\vartheta(TU) \in \mathcal{V}$, so that, by definition, $W \subseteq O^\vartheta(TU)$. □

Recall that subsets of S are called thin if all of their elements are thin.

Corollary 1.7.4 *Let U be a thin closed subset of S, and assume that TU is closed. Then we have $O^\vartheta(T) = O^\vartheta(TU)$.*

Proof. Since U is assumed to be thin, $O^\vartheta(U) = \{e\}$; cf. Lemma 1.7.1(ii). Thus, the claim follows from Lemma 1.7.3. □

We set $(O^\vartheta)^0(T) := T$. For each positive integer n, we inductively define

$$(O^\vartheta)^n(T) := O^\vartheta((O^\vartheta)^{n-1}(T)).$$

Note that $(O^\vartheta)^n(T)$ is closed for each non-negative integer n. Note also that, for each positive integer n, $(O^\vartheta)^n(T)$ is a strongly normal closed subset of $(O^\vartheta)^{n-1}(T)$.

Here is a generalization of Theorem 1.7.1(iii).

Lemma 1.7.5 *Let n be a non-negative integer, and let U be a closed subset of S such that $T \subseteq U$. Then $(O^\vartheta)^n(T) \subseteq (O^\vartheta)^n(U)$.*

Proof. Our lemma is certainly true if $n = 0$. By induction, we may assume that $(O^\vartheta)^{n-1}(T) \subseteq (O^\vartheta)^{n-1}(U)$. Then, by Theorem 1.7.1(iii),

$$(O^\vartheta)^n(T) = O^\vartheta((O^\vartheta)^{n-1}(T)) \subseteq O^\vartheta((O^\vartheta)^{n-1}(U)) = (O^\vartheta)^n(U),$$

and that proves the lemma. □

The second part of the following lemma generalizes Lemma 1.7.2.

Lemma 1.7.6 *Let n be a non-negative integer, and let U be a closed subset of S such that $T \subseteq N_S(U)$. Then the following hold.*

(i) *We have $(O^\vartheta)^n(TU)U \subseteq N_S(U)$.*

(ii) *We have $(O^\vartheta)^n(T)U = (O^\vartheta)^n(TU)U$.*

(iii) *If $O^\vartheta(TU)U = TU$, $(O^\vartheta)^n(TU)U = TU$.*

Proof. (i) We are assuming that $T \subseteq N_S(U)$. Thus, by Lemma 1.3.2(ii), $TU \subseteq N_S(U)$. In particular, $(O^\vartheta)^n(TU) \subseteq N_S(U)$.

Now recall that $(O^\vartheta)^n(TU)$ is closed. Thus, a second application of Lemma 1.3.2(ii) yields $(O^\vartheta)^n(TU)U \subseteq N_S(U)$.

(ii) There is nothing to show if $n = 0$. Therefore, we assume that $1 \leq n$.
By induction, we may assume that

$$(O^\vartheta)^{n-1}(T)U = (O^\vartheta)^{n-1}(TU)U.$$

Thus, by Theorem 1.7.1(iii),

$$O^\vartheta((O^\vartheta)^{n-1}(T)U)U = O^\vartheta((O^\vartheta)^{n-1}(TU)U)U.$$

On the other hand, applying Lemma 1.7.2 to $(O^\vartheta)^{n-1}(T)$ in place of T, we obtain

$$(O^\vartheta)^n(T)U = O^\vartheta((O^\vartheta)^{n-1}(T))U = O^\vartheta((O^\vartheta)^{n-1}(T)U)U.$$

Moreover, by (i), $(O^\vartheta)^{n-1}(TU) \subseteq N_S(U)$. Thus, applying Lemma 1.7.2 to $(O^\vartheta)^{n-1}(TU)$ in place of T, we obtain

$$(O^\vartheta)^n(TU)U = O^\vartheta((O^\vartheta)^{n-1}(TU))U = O^\vartheta((O^\vartheta)^{n-1}(TU)U)U.$$

Thus, we have $(O^\vartheta)^n(T)U = (O^\vartheta)^n(TU)U$.
(iii) By (i), $(O^\vartheta)^{n-1}(TU) \subseteq N_S(U)$. Thus, by Lemma 1.7.2,

$$O^\vartheta((O^\vartheta)^{n-1}(TU))U = O^\vartheta((O^\vartheta)^{n-1}(TU)U)U.$$

On the other hand, we are assuming that $O^\vartheta(TU)U = TU$, and, by induction, we may assume $(O^\vartheta)^{n-1}(TU)U = TU$. Thus,

$$O^\vartheta((O^\vartheta)^{n-1}(TU)U)U = TU.$$

Thus, as $O^\vartheta((O^\vartheta)^{n-1}(TU)) = (O^\vartheta)^n(TU)$, $(O^\vartheta)^n(TU)U = TU$. $\square$

2

Involutions

In this chapter, we shall look at involutions of hypergroups. We start with two basic observations on involutions. After that, our focus will be on closed subsets generated by involutions.

There are three conditions on closed subsets generated by involutions which we shall investigate, the condition of being constrained, the exchange condition, and dichotomy. We shall first see that the exchange condition implies dichotomy. After that, our focus will be on constrained sets of involutions that satisfy the exchange condition.

Constrained sets of involutions that satisfy the exchange condition will be called Coxeter sets. These sets will be the subject of Chapter 3.

As before, the letter S stands for a hypergroup in this chapter.

2.1 Basic Facts

An element s in $S \setminus \{e\}$ is called an *involution* if $\{e, s\}$ is closed.

Note that the element in $T \setminus \{e\}$ in Corollary 1.4.6 is an involution.

Lemma 2.1.1 *Let l be an involution of S. Then the following hold.*

(i) *We have $l^* = l$.*

(ii) *If l is not thin, $ll = \{e, l\}$.*

Proof. (i) Since l is assumed to be an involution, $l \neq e$ and $\{e, l\}$ is closed. Since $\{e, l\}$ is closed, $l \in \{e, l\}$ implies $l^* \in \{e, l\}$; cf. Lemma 1.2.1(ii). Thus, as $l^* \neq e$, $l^* = l$.

(ii) From Lemma 1.1.1(i) we know that $e \in l^*l$. Now assume that l is not thin. Then, by definition, $l^*l \neq \{e\}$. On the other hand, as $\{e, l\}$ is closed, $l^*l \subseteq \{e, l\}$. Thus, $l^*l = \{e, l\}$. Thus, by (i), $ll = \{e, l\}$. □

At several instances we shall refer to Lemma 2.1.1(i) without further mention.

Lemma 2.1.2 *Let p and q be elements in S such that $p \neq q$, and let l be an involution of S such that $pl = \{q\}$. Then the following hold.*

(i) *If l is thin, $ql = \{p\}$.*

(ii) *If l is not thin, $ql = \{p, q\}$.*

Proof. (i) Assume that l is thin. Then $|ql| = 1$; cf. Lemma 1.3.8. On the other hand, we are assuming that $q \in pl$. Thus, by H3, $p \in ql$.

(ii) We are assuming that $pl = \{q\}$. Thus,

$$ql = pll = p\{e, l\} = \{p\} \cup pl = \{p, q\};$$

cf. Lemma 2.1.1(ii) for the second equation. □

2.2 Sets of involutions

In this section, the letter L stands for a nonempty set of involutions of S.

We set $P_0(L) := \{e\}$. For each positive integer n, we define $P_n(L)$ to be the union of all sets $l_1 \cdots l_n$ where $l_1, \ldots, l_n$ are elements in L satisfying $l_{i-1} \neq l_i$ for each element i in $\{2, \ldots, n\}$.

Note that $P_1(L) = L$.

We set $U_0(L) := \{e\}$. For each positive integer n, we define $U_n(L)$ to be the union of the sets $P_i(L)$ with $i \in \{0, \ldots, n\}$.

It is clear that $U_{n-1}(L) \subseteq U_n(L)$ for each positive integer n.

Lemma 2.2.1 *Let L be a set of involutions of S, and let n be a nonnegative integer. Then $L^n \subseteq U_n(L)$.*

Proof. The statement is obviously true if $n = 0$ and for $n = 1$. Thus, we assume that $2 \leq n$.

Let $l_1, \ldots, l_n$ be elements in L. We have to show that $l_1 \cdots l_n \subseteq U_n(L)$.

If $l_{i-1} \neq l_i$ for each element i in $\{2, \ldots, n\}$, we have

$$l_1 \cdots l_n \subseteq P_n(L) \subseteq U_n(L)$$

and we are done. If $\{2, \ldots, n\}$ possesses an element i with $l_{i-1} = l_i$, we have

$$l_1 \cdots l_n \subseteq l_1 \cdots l_{i-2} l_i \cdots l_n \cup l_1 \cdots l_{i-1} l_i \cdots l_n \subseteq L^{n-2} \cup L^{n-1};$$

cf. Lemma 2.1.1(ii). On the other hand, by induction,

$$L^{n-2} \subseteq U_{n-2}(L) \quad \text{and} \quad L^{n-1} \subseteq U_{n-1}(L).$$

Thus, the claim follows from $U_{n-2}(L) \subseteq U_n(L)$ and $U_{n-1}(L) \subseteq U_n(L)$. □

For the remainder of this section, we write ℓ instead of ℓ_L.

Recall from Lemma 2.1.1(i) that $l^* = l$ for each involution of S. In particular, we have $\ell(s^*) = \ell(s)$ for each element s in $\langle L \rangle$.

Lemma 2.2.2 *Let s be an element in $\langle L \rangle$, and set $n := \ell(s)$. Then $s \in P_n(L)$.*

PROOF. Since $P_0(L) = \{e\}$, the statement is true if $n = 0$. Thus, we assume that $1 \leq n$.

Since $\ell(s) = n$, $s \in L^n$. Thus, by Lemma 2.2.1, $s \in U_n(L)$, so that $s \in P_i(L)$ for some element i in $\{1, \ldots, n\}$. From $s \in P_i(L)$ we obtain $n = \ell(s) \leq i$.

From $i \leq n$ and $n \leq i$ we obtain $i = n$. Thus, as $s \in P_i(L)$, $s \in P_n(L)$. □

It is obvious that

$$\ell(r) \leq \ell(p) + \ell(q)$$

for any three elements p, q, and r in $\langle L \rangle$ with $r \in pq$. We now shall look at elements p, q, and r satisfying $\ell(r) \leq \ell(p) + \ell(q)$.

For each element q in $\langle L \rangle$, we define $L_1(q)$ to be the set of all elements p in $\langle L \rangle$ such that pq contains an element r satisfying $\ell(r) = \ell(p) + \ell(q)$.

For each element q in $\langle L \rangle$, we define $L_{-1}(q)$ to be the set of all elements r in $\langle L \rangle$ such that there exists an element p in $\langle L \rangle$ satisfying $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$.

Lemma 2.2.3 *For any two elements p and q in $\langle L \rangle$, we have the following.*

(i) *If $p \in L_1(q)$, $q^* \in L_1(p^*)$.*

(ii) *If $L_{-1}(p) \cap L_1(q)$ is not empty, $p \in L_1(q)$.*

(iii) *If $q \in L_{-1}(p)$, $L_{-1}(q) \subseteq L_{-1}(p)$.*

(iv) *If $q \in L_{-1}(p)$, $L_1(q^*) \subseteq L_1(p^*)$.*

PROOF. (i) This follows from $\ell(p^*) = \ell(p)$ and $\ell(q^*) = \ell(q)$.

(ii) Let r be an element in $L_{-1}(p) \cap L_1(q)$. From $r \in L_{-1}(p)$ we obtain an element t in $\langle L \rangle$ such that $r \in tp$ and $\ell(r) = \ell(t) + \ell(p)$. Since $r \in L_1(q)$, rq possesses an element s such that $\ell(s) = \ell(r) + \ell(q)$. Thus, by Lemma 1.6.3, pq possesses an element u such that $s \in tu$, $\ell(u) = \ell(p) + \ell(q)$, and $\ell(s) = \ell(t) + \ell(u)$.

From $u \in pq$ and $\ell(u) = \ell(p) + \ell(q)$ we obtain $p \in L_1(q)$.

(iii) Assume that $q \in L_{-1}(p)$ and fix an element s in $L_{-1}(q)$. We shall show that $s \in L_{-1}(p)$.

Since $s \in L_{-1}(q)$, $\langle L \rangle$ possesses an element u such that $s \in uq$ and $\ell(s) = \ell(u) + \ell(q)$. Since we are assuming that $q \in L_{-1}(p)$, $\langle L \rangle$ possesses an element

t such that $q \in tp$ and $\ell(q) = \ell(t) + \ell(p)$. Now, by Lemma 1.6.3, ut possesses an element r such that $s \in rp$, $\ell(r) = \ell(u) + \ell(t)$, and $\ell(s) = \ell(r) + \ell(p)$.

From $s \in rp$ and $\ell(s) = \ell(r) + \ell(p)$ we obtain $s \in L_{-1}(p)$.

(iv) Assume that $q \in L_{-1}(p)$ and fix an element s in $L_1(q^*)$. We shall show that $s \in L_1(p^*)$.

Since $s \in L_1(q^*)$, $q \in L_1(s^*)$; cf. (i). Thus, as we are assuming that $q \in L_{-1}(p)$, $p \in L_1(s^*)$; cf. (ii). Thus, by (i), $s \in L_1(p^*)$. □

For each subset R of $\langle L \rangle$, we define $L_1(R)$ to be the intersection of the sets $L_1(r)$ with $r \in R$.[1]

Lemma 2.2.4 *Let q be an element in $\langle L \rangle$, let p be an element in $L_1(q)$, and let r be an element in $L_1(pq)$. Then $rp \cap L_1(q)$ is not empty.*

Proof. Since $p \in L_1(q)$, pq possesses an element u such that $\ell(u) = \ell(p)+\ell(q)$.

Since $r \in L_1(pq)$ and $u \in pq$, $r \in L_1(u)$. Thus, ru possesses an element t such that $\ell(t) = \ell(r) + \ell(u)$. Thus, by Lemma 1.6.3, rp possesses an element s such that $t \in sq$ and $\ell(t) = \ell(s) + \ell(q)$. From $t \in sq$ and $\ell(t) = \ell(s) + \ell(q)$ we obtain $s \in L_1(q)$. □

Lemma 2.2.5 *Let K be a subset of L, and assume that $\langle L \rangle = L_{-1}(k) \cup L_1(k)$ for each element k in K. Then $\langle L \rangle = L_1(K)\langle K \rangle$.*

Proof. Assume, by way of contradiction, that $\langle L \rangle \neq L_1(K)\langle K \rangle$. Among the elements in $\langle L \rangle \setminus L_1(K)\langle K \rangle$ we fix s such that $\ell(s)$ is as small as possible.

Since $s \notin L_1(K)\langle K \rangle$, $s \notin L_1(K)$. Thus, by definition, K possesses an element k such that $s \notin L_1(k)$. Thus, as $s \in \langle L \rangle$, our hypothesis implies $s \in L_{-1}(k)$. Thus, $\langle L \rangle$ possesses an element r such that $s \in rk$ and $\ell(s) = \ell(r) + 1$.

Since $\ell(s) = \ell(r) + 1$, the (minimal) choice of s yields $r \in L_1(K)\langle K \rangle$. Thus, as $s \in rk$ and $k \in K$, $s \in L_1(K)\langle K \rangle$, contradiction. □

2.3 Pairs of involutions

The results in this section will be needed in Section 6.4 in order to show how Coxeter sets are related to free monoids. This fact in turn is needed in order to show that Coxeter schemes and buildings (in the sense of Tits) are the same thing.

In this section, the letters h and k will stand for involutions of S.

Let n be a positive integer, and assume (for a moment) that $h \neq k$.

We define $R_n(h, k)$ to be the set $l_1 \cdots l_n$ where $l_i = h$ if i is an odd and $l_i = k$ if i is an even element in $\{1, \ldots, n\}$.

[1] Our convention is that $L_1(R) = \langle L \rangle$ if R is empty.

Note that $P_n(\{h,k\}) = R_n(h,k) \cup R_n(k,h)$.

Lemma 2.3.1 *Assume that $h \neq k$, and let i and j be positive integers such that $i \leq j-1$ and $R_i(h,k) \cap R_j(h,k)$ is not empty. Then there exists an element n in $\{j-i, \ldots, j+i-1\}$ such that $e \in R_n(h,k)$ or $e \in R_n(k,h)$.*

PROOF. Let s be an element in $R_i(h,k) \cap R_j(h,k)$. Then, as $e \in s^*s$,

$$e \in R_j(h,k)^* R_i(h,k) = R_{j-1}(k,h)^* R_{i-1}(k,h) \cup R_{j-1}(k,h)^* h R_{i-1}(k,h);$$

cf. Lemma 2.1.1(ii).

Note that

$$R_{j-1}(k,h)^* h R_{i-1}(k,h) = R_{i+j-1}(h,k)$$

or

$$R_{j-1}(k,h)^* h R_{i-1}(k,h) = R_{i+j-1}(k,h).$$

Thus, we are done if $e \in R_{j-1}(k,h)^* h R_{i-1}(k,h)$.

If $e \in R_{j-1}(k,h)^* R_{i-1}(k,h)$, then there exist elements p in $R_{i-1}(k,h)$ and q in $R_{j-1}(k,h)$ such that $e \in q^*p$. From $e \in q^*p$ we obtain $p = q$; cf. Lemma 1.1.4(i). Thus, $R_{i-1}(k,h) \cap R_{j-1}(k,h)$ is not empty. Thus, by induction, there exists an element n in $\{j-i, \ldots, j+i-3\}$ such that $e \in R_n(h,k)$ or $e \in R_n(k,h)$. $\square$

We define $C(h,k)$ to be the set of all positive integers n such that $e \in (hk)^n$ and set

$$c(h,k) := \begin{cases} \min(C(h,k)) & \text{if } C(h,k) \neq \emptyset, \\ \infty & \text{if } C(h,k) = \emptyset. \end{cases}$$

We call $c(h,k)$ the *Coxeter number* of h and k.

From Lemma 2.1.1(i) one obtains that $c(h,k) = 1$ if $h = k$. Note also that $c(h,k) = c(k,h)$.

For the remainder of this section, we assume that $h \neq k$.

Lemma 2.3.2 *Assume that $C(h,k)$ is not empty. Then $2c(h,k)$ is the smallest positive integer n which satisfies $e \in R_n(h,k)$.*

PROOF. The definition of $c(h,k)$ yields $e \in (hk)^{c(h,k)} = R_{2c(h,k)}(h,k)$.

Among the positive integers n with $e \in R_n(h,k)$ we chose n as small as possible. We shall be done if we succeed in showing that $2c(h,k) \leq n$.

Assume first that n is odd. We shall see that this leads to a contradiction.

If n is odd, there exists a positive integer i such that $n = 2i+1$. From $e \in R_n(h,k)$ and $n = 2i+1$ we obtain $e \in (hk)^i h$. Thus, $h \in (kh)^i$. It follows that $h \in (kh)^{i-1} kh \cap eh$. Thus, by Lemma 1.1.4(iii), $(kh)^{i-1} k \cap hh$ is not empty. Thus, as $hh \subseteq \{e,h\}$,

$$e \in (kh)^{i-1}k \quad \text{or} \quad h \in (kh)^{i-1}k.$$

In the second case, we obtain $e \in (hk)^i = R_{2i}(h,k) = R_{n-1}(h,k)$, contrary to the choice of n.

Assume that $e \in (kh)^{i-1}k$. Then, as before,

$$e \in (hk)^{i-2}h \quad \text{or} \quad k \in (hk)^{i-2}h.$$

In the first case, we obtain $e \in R_{2i-3}(h,k) = R_{n-4}(h,k)$, contradiction. In the second case, we obtain $e \in (hk)^{i-1} = R_{2i-2}(h,k) = R_{n-3}(h,k)$, contradiction.

Assume now that n is even. Then there exists a positive integer i such that $n = 2i$. From $n = 2i$ we obtain $R_n(h,k) = (hk)^i$. Thus, as $e \in R_n(h,k)$, $e \in (hk)^i$. Thus, the definition of $c(h,k)$ yields, $c(h,k) \leq i$. It follows that $2c(h,k) \leq 2i = n$. □

Lemma 2.3.3 *Assume that $C(h,k)$ is not empty. Set $L := \{h,k\}$, let n be an element in $\{1,\ldots,c(h,k)\}$, and let s be an element in $R_n(h,k)$. Then $\ell_L(s) = n$.*

Proof. Set $\ell := \ell_L$ and $i := \ell(s)$. We have to show that $i = n$.

From $\ell(s) = i$ we obtain

$$s \in P_i(\{h,k\}) = R_i(h,k) \cup R_i(k,h);$$

cf. Lemma 2.2.2. On the other hand, we are assuming $s \in R_n(h,k)$. Thus, $i \leq n$ and

$$s \in R_i(h,k) \cap R_n(h,k) \quad \text{or} \quad s \in R_i(k,h) \cap R_n(h,k).$$

Assume first that $s \in R_i(h,k) \cap R_n(h,k)$. If $i \leq n-1$, there exists an element m in $\{n-i,\ldots,n+i-1\}$ such that $e \in R_m(h,k)$ or $e \in R_m(k,h)$; cf. Lemma 2.3.1. Thus, as $i \leq n \leq c(h,k)$,

$$1 \leq m \leq n+i-1 \leq 2c(h,k)-1,$$

contrary to Lemma 2.3.2. Thus, we must have $i = n$ in this case.

Assume now that $s \in R_i(k,h) \cap R_n(h,k)$. Then $e \in R_{n+i}(h,k)$ or $e \in R_{n+i}(k,h)$. Thus, by Lemma 2.3.2, $2c(h,k) \leq n+i$. Thus, as $i \leq n \leq c(h,k)$, $i = n$. □

Lemma 2.3.4 *Assume that $C(h,k)$ is not empty, and set $n := c(h,k)$. Then $R_n(h,k) \cap R_n(k,h)$ is not empty.*

Proof. We are assuming that $C(h,k)$ is not empty. Thus, as $n = c(h,k)$,

$$e \in (hk)^n = R_n(h,k)R_n(k,h)^*.$$

Thus, there exist elements p in $R_n(h,k)$ and q in $R_n(k,h)$ such that $e \in pq^*$. Thus, by Lemma 1.1.4(i), $p = q$. It follows that $p \in R_n(h,k) \cap R_n(k,h)$. $\square$

For the remainder of this section, we set $L := \{h,k\}$ and $\ell := \ell_L$.

Lemma 2.3.5 *The set $C(h,k)$ is empty if and only if $L_{-1}(L)$ is empty.*

PROOF. Assume first that $C(h,k)$ is not empty, and set $n := c(h,k)$. Then, by Lemma 2.3.4, $R_n(h,k) \cap R_n(k,h)$ is not empty. Let s be an element in $R_n(h,k) \cap R_n(k,h)$.

Since $s \in R_n(h,k)$, $\ell(s) = n$; cf. Lemma 2.3.3.

From $s \in R_n(h,k)$ we obtain an element p in $R_{n-1}(h,k)$ such that $s \in ph$ if n is odd and $s \in pk$ if n is even. From $p \in R_{n-1}(h,k)$ we obtain $\ell(p) = n-1$; cf. Lemma 2.3.3.

Similarly, as $s \in R_n(k,h)$ we obtain an element q in $R_{n-1}(k,h)$ such that $\ell(q) = n-1$, $s \in qk$ if n is odd, and $s \in qh$ if n is even.

It follows that $s \in L_{-1}(L)$.

Assume now, conversely, that $L_{-1}(L)$ is not empty, fix an element s in $L_{-1}(L)$, and set $n := \ell(s)$.

Since $s \in L_{-1}(h)$, $\langle L\rangle$ possesses an element p such that $s \in ph$ and $\ell(s) = \ell(p)+1$. From $\ell(s) = \ell(p)+1$ and $\ell(s) = n$ we obtain $\ell(p) = n-1$. Thus, by Lemma 2.2.2, $p \in R_{n-1}(h,k)$ or $p \in R_{n-1}(k,h)$. Note that $p \in R_{n-1}(h,k)$ if n is odd and $p \in R_{n-1}(k,h)$ if n is even.

Similarly, $\langle L\rangle$ possesses an element q such that $s \in qk$, $\ell(q) = n-1$, $q \in R_{n-1}(k,h)$ if n is odd, and $q \in R_{n-1}(h,k)$ if n is even. Thus,

$$e \in s^*s \subseteq hp^*qk \subseteq R_{2n}(h,k) = (hk)^n.$$

It follows that $n \in C(h,k)$. $\square$

Proposition 2.3.6 *Assume that $L_{-1}(L)$ is not empty, and chose s in $L_{-1}(L)$ such that $\ell(s)$ is as small as possible. Then $\ell(s) = c(h,k)$.*

PROOF. We set $n := \ell(s)$ and shall be done if we succeed in showing that $n = c(h,k)$.

The definition of $c(h,k)$ yields $e \in (hk)^{c(h,k)}$. Thus, there exist elements p in $R_{c(h,k)}(h,k)$ and q in $R_{c(h,k)}(k,h)^*$ such that $e \in pq^*$. It follows that $p = q$. Thus,

$$p \in R_{c(h,k)}(h,k) \cap R_{c(h,k)}(k,h).$$

From this we obtain $\ell(p) = c(h,k)$ and $p \in L_{-1}(L)$. Thus, the choice of s forces $n \leq c(h,k)$.

Since $s \in L_{-1}(h)$, there exists an element p in $\langle L\rangle$ such that $s \in ph$ and $\ell(s) = \ell(p)+1$. It follows that $p \in L_1(h)$.

From $\ell(s) = \ell(p) + 1$ and $\ell(s) = n$ we obtain $\ell(p) = n - 1$. Thus, by Lemma 2.2.2, $p \in R_{n-1}(h, k)$ or $p \in R_{n-1}(k, h)$. Since $p \in L_1(h)$, we must have $p \in R_{n-1}(h, k)$ if n is odd and $p \in R_{n-1}(k, h)$ if n is even.

Similarly, $\langle L \rangle$ possesses an element q such that $s \in qk$, $\ell(s) = \ell(p) + 1$, $q \in R_{n-1}(k, h)$ if n is odd and $q \in R_{n-1}(h, k)$ if n is even. Thus, we have

$$e \in s^* s \subseteq hp^* qk \subseteq R_n(h, k)^* R_n(k, h) = R_{2n}(h, k) = (hk)^n$$

in both cases. It follows that $c(h, k) \leq n$. □

2.4 Constrained sets of involutions

A nonempty set L of involutions of S is said to be *constrained* if $|pq| = 1$ for any two elements q in $\langle L \rangle$ and p in $L_1(q)$.

For the remainder of this section, the letter L stands for a constrained set of involutions of S, and we write ℓ instead of ℓ_L.

Lemma 2.4.1 *For any two elements q and r in $\langle L \rangle$, there exists at most one element p in $\langle L \rangle$ such that $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$.*

Proof. We fix an element r in $\langle L \rangle$ and define Q to be the set of all elements q in $\langle L \rangle$ such that there exist elements p and p' in $\langle L \rangle$ with $r \in pq$, $r \in p'q$, $\ell(r) = \ell(p) + \ell(q)$, $\ell(p') = \ell(p)$, and $p' \neq p$.

By way of contradiction, we assume that Q is not empty. We pick an element q in Q, and we do this in such a way that $\ell(q)$ is as small as possible.

Note that $e \notin Q$. Thus, as $q \in Q$, $q \neq e$. Thus, by Lemma 1.6.2, there exist elements l in L and u in $\langle L \rangle$ such that $q \in lu$ and $\ell(q) = 1 + \ell(u)$. Thus, as $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$, pl possesses an element t such that $r \in tu$, $\ell(t) = \ell(p) + 1$, and $\ell(r) = \ell(t) + \ell(u)$; cf. Lemma 1.6.3.

Similarly, we find an element t' in $p'l$ such that $r \in t'u$, $\ell(t') = \ell(p') + 1$, and $\ell(r) = \ell(t') + \ell(u)$.

Since $\ell(q) = 1 + \ell(u)$, the (minimal) choice of q yields $u \notin Q$. Thus, as $r \in tu$, $r \in t'u$, $\ell(r) = \ell(t) + \ell(u)$, and $\ell(t') = \ell(t)$, we may conclude that $t' = t$.

Recall that L is assumed to be constrained. Thus, as $t \in pl$ and $\ell(t) = \ell(p)+1$, we have $pl = \{t\}$. Similarly, $p'l = \{t'\}$. Thus, as $t' = t$, $p'l = pl$. It follows that $p' \in \{p, t\}$.

Since $t' = t$ and $\ell(t') = \ell(p') + 1$, $p' \neq t$. Thus, $p' = p$, contradiction. □

In the remainder of this section, we investigate the set $O_\vartheta(L)$. (Recall that L is assumed to be constrained.)

Lemma 2.4.2 *Let R be a subset of $\langle L \rangle$. Assume that $O_\vartheta(L) \subseteq R$, $R^2 \subseteq R$, and $r^* r \subseteq R$ for each element r in R. Then $R \subseteq \langle L \cap R \rangle$.*

Proof. Set $K := L \cap R$. We have to show that $R \subseteq \langle K \rangle$.

Suppose, by way of contradiction, that $R \not\subseteq \langle K \rangle$. We pick an element r in $R \setminus \langle K \rangle$, and we do this in such a way that $\ell(r)$ is as small as possible.

Since $r \notin \langle K \rangle$, $r \neq e$. Thus, by Lemma 1.6.2, there exist elements q in $\langle L \rangle$ and l in L such that $r \in ql$ and $\ell(r) = \ell(q) + 1$. Since L is assumed to be constrained, we obtain from $r \in ql$ and $\ell(r) = \ell(q) + 1$ that $ql = \{r\}$.

Let us first prove that $l \in R$. If l is thin, this follows from our hypothesis that $O_\vartheta(L) \subseteq R$. Assume that l is not thin. Then, by Lemma 2.1.1, $l \in l^*l$. Thus, as $e \in q^*q$ and $ql = \{r\}$, $l \in l^*q^*ql = r^*r$. Thus, as we are assuming $r^*r \subseteq R$, $l \in R$.

Since $r \in ql$, $q \in rl$. Thus, as $r \in R$ and $l \in R$, $q \in R^2$. Thus, as we are assuming that $R^2 \subseteq R$, we obtain $q \in R$.

Recall that $\ell(r) = \ell(q) + 1$. Thus, the (minimal) choice of r yields $q \notin R \setminus \langle K \rangle$. Thus, as $q \in R$, $q \in \langle K \rangle$. Thus, as $r \in ql$ and $l \in K$, $r \in \langle K \rangle$. This contradiction finishes the proof. □

Corollary 2.4.3 *Let T be a closed subset of $\langle L \rangle$, and assume that $O_\vartheta(L) \subseteq T$. Then $\langle L \cap T \rangle = T$.*

Proof. Since T is assumed to be closed, we have $\langle L \cap T \rangle \subseteq T$. From Lemma 2.4.2 we obtain $T \subseteq \langle L \cap T \rangle$. Thus, $\langle L \cap T \rangle = T$. □

Corollary 2.4.4 *The following statements hold.*

(i) *We have $\langle O_\vartheta(L) \rangle = O_\vartheta(\langle L \rangle)$.*

(ii) *The set $\langle L \rangle$ is thin if and only if L is thin.*

(iii) *The set $O_\vartheta(L)$ is empty if and only if $O_\vartheta(\langle L \rangle) = \{e\}$.*

Proof. (i) From Lemma 1.3.7(ii) we know that $O_\vartheta(\langle L \rangle)^2 \subseteq O_\vartheta(\langle L \rangle)$. Moreover, for each element s in $O_\vartheta(\langle L \rangle)$, we have $s^*s = \{e\} \subseteq O_\vartheta(\langle L \rangle)$. Thus, by Lemma 2.4.2,

$$O_\vartheta(\langle L \rangle) \subseteq \langle O_\vartheta(L) \rangle.$$

Since $O_\vartheta(L) \subseteq O_\vartheta(\langle L \rangle)$, we obtain from $O_\vartheta(\langle L \rangle) \subseteq \langle O_\vartheta(L) \rangle$ that we shall be done if we succeed in showing that $O_\vartheta(\langle L \rangle)$ is closed.

In order to show that $O_\vartheta(\langle L \rangle)$ is closed we fix elements p and q in $O_\vartheta(\langle L \rangle)$. We have to show that $p^*q \in O_\vartheta(\langle L \rangle)$.

From $p \in O_\vartheta(\langle L \rangle)$ and $O_\vartheta(\langle L \rangle) \subseteq \langle O_\vartheta(L) \rangle$ we obtain $p \in \langle O_\vartheta(L) \rangle$. Thus, by Lemma 1.6.1, $p \in O_\vartheta(L)^n$ for some positive integer n. Thus, $O_\vartheta(L)$ possesses elements $l_1, \ldots, l_n$ such that $p \in l_1 \cdots l_n$. Thus, by Lemma 2.1.1(i), $p^* \in l_n \cdots l_1$. Thus, by Lemma 1.3.7(ii), p^* is thin. Thus, as $p^* \in \langle L \rangle$, $p^* \in O_\vartheta(\langle L \rangle)$.

From $p^* \in O_\vartheta(\langle L \rangle)$ and $q \in O_\vartheta(\langle L \rangle)$ we obtain $p^*q \subseteq O_\vartheta(\langle L \rangle)$; cf. Lemma 1.3.7(ii).

(ii) If $\langle L\rangle$ is thin, so is L. Conversely, assume that L is thin. Then $O_\vartheta(L) = L$. Thus, by (i), $\langle L\rangle = O_\vartheta(\langle L\rangle)$, and that means that $\langle L\rangle$ is thin.

(iii) This is an immediate consequence of (i). □

As a consequence of Corollary 2.4.4(i) one obtains that $O_\vartheta(\langle L\rangle)$ is a closed subset of S. (In general, $O_\vartheta(T)$ is not closed for closed subsets T of S.)

We finish this section with an observation on the case where $|L| = 2$.

Lemma 2.4.5 *Assume L possesses elements h and k such that $L = \{h, k\}$ and $C(h, k) \neq \emptyset$. Then $|R_n(h, k)| = 1$ for each element n in $\{1, \ldots, c(h, k)\}$.*

Proof. Let n be an element in $\{1, \ldots, c(h, k)\}$. The statement is obviously true if $n = 1$. Thus, we assume that $2 \leq n$.

Let s be an element in $R_n(h, k)$. We shall be done if we succeed in showing that $R_n(h, k) = \{s\}$.

Since $s \in R_n(h, k)$, $s \in hR_{n-1}(k, h)$. Thus, $R_{n-1}(k, h)$ possesses an element r such that $s \in hr$. By induction, $|R_{n-1}(k, h)| = 1$. Thus, as $r \in R_{n-1}(k, h)$, $R_{n-1}(k, h) = \{r\}$.

By Lemma 2.3.3, $\ell(r) = n-1$ and $\ell(s) = n$. Thus, $\ell(s) = \ell(h)+\ell(r)$. Thus, as $s \in hr$, $s \in L_1(r)$. Thus, as L is assumed to be constrained, $hr = \{s\}$. Thus, as $R_{n-1}(k, h) = \{r\}$, $R_n(h, k) = \{s\}$. □

2.5 Dichotomy and the exchange condition

Throughout this section, the letter L stands for a set of involutions of S. We assume L to be not empty and write ℓ instead of ℓ_L.

The set L is called *dichotomic* if $\langle L\rangle = L_{-1}(l) \cup L_1(l)$ for each element l in L.

For each subset R of $\langle L\rangle$, we define $L_{-1}(R)$ to be the intersection of the sets $L_{-1}(r)$ with $r \in R$.[2]

Lemma 2.5.1 *Assume L to be dichotomic, and let s be an element in $\langle L\rangle$ such that $\ell(r) \leq \ell(s)$ for each element r in $\langle L\rangle$. Then $s \in L_{-1}(L)$.*

Proof. Let l be an element in L. Then $s \in L_{-1}(l)$ or $s \in L_1(l)$. If $s \in L_1(l)$, sl possesses an element t such that $\ell(t) = \ell(s) + 1$, contrary to the choice of s. Thus, $s \in L_{-1}(l)$.

Since l has been chosen arbitrarily in L, we have proved $s \in L_{-1}(L)$. □

The set L is said to satisfy the *exchange condition* if

$$hs \subseteq sk \cup L_1(k)$$

[2] Again, our convention is that $L_{-1}(R) = \langle L\rangle$ if R is empty.

for any three elements k in L, s in $L_1(k)$, and h in L with $h \in L_1(s)$.[3]

Lemma 2.5.2 *Assume that L satisfies the exchange condition. Then L is dichotomic.*

Proof. Let l be an element in L, and let s be an element in $\langle L \rangle$. We have to show that $s \in L_{-1}(l) \cup L_1(l)$.

Since $e \in L_1(l)$, we may assume that $s \neq e$. Thus, by Lemma 1.6.2, there exist elements k in L and r in $\langle L \rangle$ such that $s \in kr$ and $\ell(s) = 1 + \ell(r)$. It follows that $k \in L_1(r)$ and $s \in L_{-1}(r)$.

Assume first that $r \in L_{-1}(l)$. Then $L_{-1}(r) \subseteq L_{-1}(l)$; cf. Lemma 2.2.3(iii). Thus, as $s \in L_{-1}(r)$, $s \in L_{-1}(l)$, and we are done.

Assume now $r \notin L_{-1}(l)$. Then, by induction, $r \in L_1(l)$. Thus, as $k \in L_1(r)$ and L is assumed to satisfy the exchange condition, $kr \subseteq rl \cup L_1(l)$. Thus, as $s \in kr$, $s \in rl \cup L_1(l)$. If $s \in rl$, we obtain from $\ell(s) = \ell(r) + 1$ that $s \in L_{-1}(l)$. Thus, $s \in L_{-1}(l) \cup L_1(l)$. $\square$

Assume that L satisfies the exchange condition. Then, by Lemma 2.2.5 and Lemma 2.5.2, $\langle L \rangle = L_1(K)\langle K \rangle$ for each subset K of L. We do not explicitly state this result here since we shall prove a stronger result in Corollary 3.1.7.

Lemma 2.5.3 *Assume that L satisfies the exchange condition. Let l be an element in L, let p be an element in $L_1(l)$, and let r be an element in $L_{-1}(p) \cap L_{-1}(l)$. Then pl possesses an element q with $\ell(q) = \ell(p) + 1$ and $r \in L_{-1}(q)$.*

Proof. Define R to be the set of all elements r in $L_{-1}(p) \cap L_{-1}(l)$ such that $r \notin L_{-1}(q)$ for each element q in pl with $\ell(q) = \ell(p) + 1$. By way of contradiction, we assume that R is not empty. We fix an element r in R, and we do this in such a way that $\ell(r)$ is as small as possible.

Since $r \in R$, $r \in L_{-1}(p)$ and $r \in L_{-1}(l)$.

Since $r \in L_{-1}(p)$, $\langle L \rangle$ possesses an element u with $r \in up$ and $\ell(r) = \ell(u) + \ell(p)$. Since $r \in L_{-1}(l)$ and $p \in L_1(l)$, $r \neq p$. From $r \in up$ and $r \neq p$ we obtain $u \neq e$. Thus, there exist elements h in L and t in $\langle L \rangle$ such that $u \in ht$ and $\ell(u) = 1 + \ell(t)$; cf. Lemma 1.6.2. Thus, by Lemma 1.6.3, tp possesses an element s such that $r \in hs$, $\ell(s) = \ell(t) + \ell(p)$, and $\ell(r) = 1 + \ell(s)$.

Since $\ell(r) = 1 + \ell(s)$, the minimal choice of r yields $s \notin R$. Moreover, as $s \in tp$ and $\ell(s) = \ell(t) + \ell(p)$, $s \in L_{-1}(p)$.

Suppose that $s \in L_{-1}(l)$. Then, as $s \in L_{-1}(p)$ and $s \notin R$, pl possesses an element q such that $\ell(q) = \ell(p) + 1$ and $s \in L_{-1}(q)$. From $s \in L_{-1}(q)$ we obtain $L_{-1}(s) \subseteq L_{-1}(q)$; cf. Lemma 2.2.3(iii). On the other hand, as $r \in hs$ and $\ell(r) = 1 + \ell(s)$, $r \in L_{-1}(s)$. Thus, $r \in L_{-1}(q)$, contrary to the choice of r in R.

[3] If $hs \subseteq sk$, we obtain immediately $hs = sk$, for symmetry reasons.

This contradiction yields $s \notin L_{-1}(l)$. Thus, by Lemma 2.5.2, $s \in L_1(l)$. Thus, as $h \in L_1(s)$ and L is assumed to satisfy the exchange condition, $hs \subseteq sl \cup L_1(l)$. Thus, as $r \in hs$, $r \in sl \cup L_1(l)$. Thus, as $r \in L_{-1}(l)$, $r \in sl \subseteq tpl$. Thus, pl possesses an element q such that $r \in tq$. It follows that

$$\ell(r) \leq \ell(t) + \ell(q) \leq \ell(t) + 1 + \ell(p) = \ell(u) + \ell(p) = \ell(r).$$

Thus, $\ell(q) = \ell(p)+1$ and $\ell(r) = \ell(t)+\ell(q)$. From $r \in tq$ and $\ell(r) = \ell(t)+\ell(q)$ we obtain $r \in L_{-1}(q)$, contrary to $r \in R$. □

A constrained set of involutions of S that satisfies the exchange condition is called *Coxeter set* of S.

Lemma 2.5.4 *Assume that L is a Coxeter set of S. Let h be an element in L, let p be an element in $\langle L\rangle$, and let t be an element in hp such that $\ell(t) = 1 + \ell(p)$. Let k be an element in L, and let q be an element in $\langle L\rangle$, and let u be an element in qk such that $\ell(u) = \ell(q) + 1$. Assume that $t \in L_1(q)$ and $p \in L_1(u)$. Then we have $tq = pu$ or $t \in L_1(u)$.*

Proof. We are assuming that $t \in L_1(q)$. Thus, tq possesses an element s with $\ell(s) = \ell(t)+\ell(q)$. Thus, as $t \in hp$ and $\ell(t) = 1+\ell(p)$, pq possesses an element r such that $s \in hr$, $\ell(r) = \ell(p) + \ell(q)$, and $\ell(s) = 1 + \ell(r)$; cf. Lemma 1.6.3.

From $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$ we obtain $pq = \{r\}$. (Recall that L is assumed to be constrained.) From $s \in hr$ and $\ell(s) = 1 + \ell(r)$ we obtain $h \in L_1(r)$.

Similarly, using $pq = \{r\}$, we conclude from $p \in L_1(u)$ that $r \in L_1(k)$. Thus, as L is assumed to satisfy the exchange condition, we have $hr = rk$ or $hr \subseteq L_1(k)$.

If $hr = rk$, we obtain from $hp = \{t\}$, $pq = \{r\}$, and $qk = \{u\}$ that $tq = pu$.

If $hr \subseteq L_1(k)$, we obtain from $s \in hr$ that $s \in L_1(k)$. Thus, by definition, sk possesses an element v such that $\ell(v) = \ell(s) + 1$. From $v \in sk$ and

$$sk \subseteq hrk = hpqk = tu$$

we obtain $v \in tu$. From $\ell(v) = \ell(s) + 1$ and

$$\ell(s) + 1 = 1 + \ell(r) + 1 = 1 + \ell(p) + \ell(q) + 1 = \ell(t) + \ell(u)$$

we obtain $\ell(v) = \ell(t) + \ell(u)$. Thus, by definition, $t \in L_1(u)$. □

3

Coxeter sets

In this chapter, we compile basic facts about Coxeter sets of hypergroups. Most of the results were proven earlier for schemes; cf. [9; Section 3.6], [9; Section 11.1], [9; Section 11.2], [9; Section 11.4], [9; Section 12.1].

It is easy to see that closed subsets generated by thin Coxeter sets are the same thing Coxeter groups. Our interest in a general investigation of Coxeter sets is based on the observation that closed subsets generated by Coxeter sets share many features with Coxeter groups; cf. Theorem 3.1.4, Theorem 3.1.5, Theorem 3.1.8, and Section 3.3.

In Chapter 6 we shall see that Coxeter sets provide a conceptional framework for the class of buildings.

Throughout this chapter, the letter S stands for a hypergroup, the letter L for a Coxeter set of S. Instead of ℓ_L we write ℓ.

3.1 Subsets of Coxeter sets I

In this section, the letter K stands for a nonempty subset of L.

Lemma 3.1.1 *Let s be an element in $\langle K\rangle$. Then $\ell_K(s) = \ell(s)$.*

Proof. Assume the claim to be false. Among the elements in $\langle K\rangle$ which do not satisfy the equation in question we choose s in such a way that $\ell_K(s)$ is as small as possible.

Since $\ell_K(s) \neq \ell(s)$, $s \neq e$. Thus, by Lemma 1.6.2, there exist elements h in K and r in $\langle K\rangle$ such that $s \in hr$ and $\ell_K(s) = 1 + \ell_K(r)$.

Since $\ell_K(s) \neq \ell(s)$ and $K \subseteq L$, $s \notin K$. Thus, as $s \in hr$ and $h \in K$, $r \neq e$. Thus, by Lemma 1.6.2, there exist elements p in $\langle K\rangle$ and k in K such that $r \in pk$ and $\ell_K(r) = \ell_K(p)+1$. Now, by Lemma 1.6.3, hp possesses an element q such that $s \in qk$, $\ell_K(q) = 1 + \ell_K(p)$, and $\ell_K(s) = \ell_K(q) + 1$.

Since $\ell_K(s) = \ell_K(q) + 1$, the (minimal) choice of s yields $\ell_K(q) = \ell(q)$. Similarly, as $\ell_K(s) = 1 + \ell_K(r)$ and $\ell_K(r) = \ell_K(p) + 1$, $\ell_K(p) = \ell(p)$. Thus, as $q \in hp$ and $\ell_K(q) = 1 + \ell_K(p)$, $h \in L_1(p)$.

Similarly, one obtains $p \in L_1(k)$. Thus, as L is assumed to satisfy the exchange condition, we must have $hp = pk$ or $hp \subseteq L_1(k)$.

Since $s \in hpk$, the first of these two cases yields $s \in pkk \subseteq \{p\} \cup pk$, contrary to $\ell_K(s) = \ell_K(p) + 2$. Since $q \in hp$, the second case yields $q \in L_1(k)$. Thus, as $s \in qk$, $\ell(s) = \ell(q) + 1$. (Here we use the hypothesis that L is constrained.) Thus, as $\ell_K(q) = \ell(q)$ and $\ell_K(s) = \ell_K(q) + 1$, $\ell_K(s) = \ell(s)$, contradiction. $\square$

Lemma 3.1.2 *Let s be an element in $\langle K \rangle$. Then $K_1(s) = \langle K \rangle \cap L_1(s)$.*

Proof. Let p be an element in $K_1(s)$. Then $p \in \langle K \rangle$ and, by definition, ps contains an element q such that $\ell_K(q) = \ell_K(p) + \ell_K(s)$. Thus, by Lemma 3.1.1, $\ell(q) = \ell(p) + \ell(s)$, and this means that $p \in L_1(s)$.

Let p be an element in $\langle K \rangle \cap L_1(s)$. Since $p \in L_1(s)$, there exists an element q in ps such that $\ell(q) = \ell(p) + \ell(s)$. From p, $s \in \langle K \rangle$ and $q \in ps$ we obtain $q \in \langle K \rangle$. From $\ell(q) = \ell(p) + \ell(s)$ we obtain $\ell_K(q) = \ell_K(p) + \ell_K(s)$; cf. Lemma 3.1.1. It follows that $p \in K_1(s)$. $\square$

Corollary 3.1.3 *We have the following.*

(i) *The set K is a Coxeter set of S.*

(ii) *We have $K = L \cap \langle K \rangle$.*

(iii) *Let H be a subset of L satisfying $\langle H \rangle = \langle K \rangle$. Then $H = K$.*

Proof. (i) This is an immediate consequence of Lemma 3.1.2.

(ii) Since $K \subseteq L$, $K \subseteq L \cap \langle K \rangle$. Thus, we just have to show that $L \cap \langle K \rangle \subseteq K$.

Let l be an element in $L \cap \langle K \rangle$. Since $l \in L$, $\ell(l) = 1$. Since $l \in \langle K \rangle$, $\ell_K(l) = \ell(l)$; cf. Lemma 3.1.1. It follows that $\ell_K(l) = 1$, and that means that $l \in K$.

(iii) From (ii) we obtain $H = L \cap \langle H \rangle = L \cap \langle K \rangle = K$. $\square$

Considering Corollary 3.1.3(i) it might be worth mentioning that subsets of constrained sets are not necessarily constrained.

Recall from Section 2.2 that, for each subset R of $\langle L \rangle$, $L_1(R)$ is our notation for the intersection of the sets $L_1(r)$ with $r \in R$.

Theorem 3.1.4 *We have $L_1(\langle K \rangle) = L_1(K)$.*

Proof. Assume that $L_1(\langle K \rangle) \neq L_1(K)$. Then, as $L_1(\langle K \rangle) \subseteq L_1(K)$, $L_1(K) \not\subseteq L_1(\langle K \rangle)$. Among the elements in $L_1(K)$ which are not in $L_1(\langle K \rangle)$ we choose t such that $\ell(t)$ is as small as possible.

Since $t \notin L_1(\langle K \rangle)$ and $e \in L_1(\langle K \rangle)$, $t \neq e$. Thus, by Lemma 1.6.2, there exist elements h in L and p in $\langle L \rangle$ such $t \in hp$ and $\ell(t) = 1 + \ell(p)$.

Since $t \in hp$ and $\ell(t) = 1 + \ell(p)$, $t \in L_{-1}(p)$. Thus, as $t \in L_1(K)$, $p \in L_1(K)$; cf. Lemma 2.2.3(ii). Thus, as $\ell(t) = 1 + \ell(p)$, the (minimal) choice of t yields $p \in L_1(\langle K \rangle)$.

Since $t \notin L_1(\langle K \rangle)$, $\langle K \rangle$ possesses an element u such that $t \notin L_1(u)$. Among the elements u in $\langle K \rangle$ satisfying $t \notin L_1(u)$ we choose u in such a way that $\ell(u)$ is as small as possible. Since $t \notin L_1(u)$, $u \neq e$. Thus, by Lemma 1.6.2 together with Lemma 3.1.1, there exist elements q in $\langle K \rangle$ and k in K such that $u \in qk$ and $\ell(u) = \ell(q) + 1$.

Since $\ell(u) = \ell(q) + 1$ and $q \in \langle K \rangle$, the (minimal) choice of u yields $t \in L_1(q)$. Since $p \in L_1(\langle K \rangle)$ and $u \in \langle K \rangle$, $p \in L_1(u)$. Thus, by Lemma 2.5.4, $tq = pu$ or $t \in L_1(u)$. Thus, as $t \notin L_1(u)$, $tq = pu$. Thus, as q, $u \in \langle K \rangle$, $t \in p\langle K \rangle$; cf. Lemma 1.2.2. Thus, $\langle K \rangle$ possesses an element s with $t \in ps$.

Since $p \in L_1(\langle K \rangle)$ and $s \in \langle K \rangle$, $p \in L_1(s)$. Thus, as $t \in ps$, $\ell(t) = \ell(p) + \ell(s)$. (Here we use the hypothesis that L is constrained.) Since $\ell(t) = 1 + \ell(p)$, this means that $\ell(s) = 1$. Thus, by Lemma 3.1.1, $s \in K$. On the other hand, as $t \in ps$ and $\ell(t) = \ell(p) + \ell(s)$, $t \in L_{-1}(s)$. Thus, $t \notin L_1(K)$, contrary to our choice of t. □

Theorem 3.1.5 *We have $L_{-1}(\langle K \rangle) = L_{-1}(K)$.*

Proof. Assume that $L_{-1}(\langle K \rangle) \neq L_{-1}(K)$. Then, as $L_{-1}(\langle K \rangle) \subseteq L_{-1}(K)$, $L_{-1}(K) \nsubseteq L_{-1}(\langle K \rangle)$. Thus, $L_{-1}(K)$ possesses an element s such that $s \notin L_{-1}(\langle K \rangle)$.

Since $s \notin L_{-1}(\langle K \rangle)$, there exists an element q in $\langle K \rangle$ such that $s \notin L_{-1}(q)$. We pick an element q in $\langle K \rangle$ with $s \notin L_{-1}(q)$, and we do this in such a way that $\ell_K(q)$ is as small as possible.

Since $s \notin L_{-1}(q)$, $q \neq e$. Thus, Lemma 1.6.2 gives us elements p in $\langle K \rangle$ and k in K such that $q \in pk$ and $\ell_K(q) = \ell_K(p) + 1$. Now, the minimal choice of q yields $s \in L_{-1}(p)$. Thus, as $s \in L_{-1}(K) \subseteq L_{-1}(k)$, $s \in L_{-1}(q)$; cf. Lemma 2.5.3. This contradiction finishes our proof. □

Lemma 3.1.6 *We have $\langle K \rangle \subseteq L_1(\langle L \setminus K \rangle)$.*

Proof. Let l be an element in $L \setminus K$. Then, as L is assumed to satisfy the exchange condition, $l \in L_1(K)$. Thus, by Theorem 3.1.4, $l \in L_1(\langle K \rangle)$. Thus, by Lemma 2.2.3(i), $\langle K \rangle \subseteq L_1(l)$.

Since l has been chosen arbitrarily in $L \setminus K$, we have shown that $\langle K \rangle \subseteq L_1(L \setminus K)$. Thus, by Theorem 3.1.4, $\langle K \rangle \subseteq L_1(\langle L \setminus K \rangle)$. □

The following lemma generalizes Lemma 2.2.5 for Coxeter sets.

Corollary 3.1.7 *For each subset H of L, $\langle H \rangle = (\langle H \rangle \cap L_1(K))\langle H \cap K \rangle$.*

Proof. From Lemma 3.1.6 we know that $\langle H \rangle \subseteq L_1(K \setminus H)$. Thus,

$$\langle H\rangle \cap L_1(H\cap K) = \langle H\rangle \cap L_1(K).$$

From Lemma 2.2.5 (together with Lemma 2.5.2) we obtain

$$\langle H\rangle = \langle H\rangle \cap L_1(H\cap K)\langle H\cap K\rangle,$$

and according to Lemma 1.2.6(ii), the right hand side of this equation is equal to $(\langle H\rangle \cap L_1(H\cap K))\langle H\cap K\rangle$. □

Recall that, for each element q in S, $C_S(q)$ is our notation for the set of all elements s in S such that $sq = qs$. Recall also that, for each nonempty subset Q in S, $C_S(Q)$ is our notation for the intersection of the sets $C_S(q)$ with $q \in Q$.

Theorem 3.1.8 *Assume that $K \neq L$. Then the following conditions are equivalent.*

(a) *We have $K \subseteq C_{\langle L\rangle}(L \setminus K)$.*
(b) *We have $\langle K\rangle \subseteq C_{\langle L\rangle}(\langle L \setminus K\rangle)$.*
(c) *The closed subset $\langle K\rangle$ is normal in $\langle L\rangle$.*

Proof. (a) ⇒ (b) Let p be an element in $\langle K\rangle$, and let q be an element in $\langle L \setminus K\rangle$. We have to show that $pq = qp$.

If $p = e$ or $q = e$, the claim is obvious. Thus, we may assume $p \neq e$ and $q \neq e$. From $p \neq e$ we obtain elements t in $\langle K\rangle$ and h in K such that $p \in th$ and $\ell(p) = \ell(t)+1$; cf. Lemma 1.6.2. Similarly, we obtain elements k in $L\setminus K$ and u in $\langle L \setminus K\rangle$ such that $q \in ku$ and $\ell(q) = 1+\ell(u)$.

From $p \in th$ and $\ell(p) = \ell(t) + 1$ we obtain $\{p\} = th$. From $q \in ku$ and $\ell(q) = 1 + \ell(u)$ we obtain $\{q\} = ku$. We are assuming that $hk = kh$, and, by induction, we may assume that $tk = kt$, that $hu = uh$, and that $tu = ut$. Thus,

$$pq = thku = tkhu = ktuh = kuth = qp.$$

(b) ⇒ (c) Let s be an element in $\langle L\rangle$. We shall be done if we succeed in showing that $\langle K\rangle s \subseteq s\langle K\rangle$.

There is nothing to show if $s = e$. Therefore, we assume that $s \neq e$. From $s \neq e$ we obtain elements r in $\langle L\rangle$ and l in L such that $s \in rl$ and $\ell(s) = \ell(r) + 1$; cf. Lemma 1.6.2. From $s \in rl$ and $\ell(s) = \ell(r) + 1$ we obtain $r \in L_1(l)$. Thus, as L is assumed to be constrained, we conclude that $rl = \{s\}$.

By induction, we have $\langle K\rangle r \subseteq r\langle K\rangle$, and from (b) we obtain $\langle K\rangle l \subseteq l\langle K\rangle$. Thus, as $rl = \{s\}$,

$$\langle K\rangle s = \langle K\rangle rl \subseteq r\langle K\rangle l \subseteq rl\langle K\rangle = s\langle K\rangle.$$

(c) ⇒ (a) Let k be an element in K and let l be an element in $L\setminus K$. We have to show that $kl = lk$.

Since $k \in K$ and $l \in L \setminus K$, $k \in L_1(l)$; cf. Lemma 3.1.6. Thus, kl possesses an element s such that $\ell(s) = \ell(k) + \ell(l) = 2$. Since L is assumed to be constrained, we obtain from $k \in L_1(l)$ and $s \in kl$ that $kl = \{s\}$.

Since $\langle K \rangle$ is assumed to be normal in $\langle L \rangle$, we have $kl \subseteq l\langle K \rangle$. Thus, $s \in l\langle K \rangle$. Thus, $\langle K \rangle$ possesses an element r such that $s \in lr$. From $l \in L\setminus K$ and $r \in \langle K \rangle$ we obtain $l \in L_1(r)$; cf. Lemma 3.1.6. From $l \in L_1(r)$ and $s \in lr$ we obtain $\ell(s) = 1 + \ell(r)$ and $lr = \{s\}$. Thus, as $kl = \{s\}$, $kl = lr$. Thus, as $k \neq l$, $r \neq l$.

From $\ell(s) = 2$ and $\ell(s) = 1 + \ell(r)$ we obtain $\ell(r) = 1$. Thus, $r \in L$. On the other hand, as $kl = lr$, $r \in \langle k, l \rangle$. Thus, by Corollary 3.1.3(ii), $r \in \{k, l\}$. Thus, as $r \neq l$, $r = k$. Thus, as $kl = lr$, $kl = lk$. □

3.2 Subsets of Coxeter sets II

In this section, the letter K stands for a nonempty subset of L.

Lemma 3.2.1 *Let q be an element in $L_{-1}(K)$, and let p be an element in $\langle K \rangle \setminus \{q\}$. Then $\ell(p) + 1 \leq \ell(q)$.*

Proof. We are assuming that $q \in L_{-1}(K)$. Thus, by Theorem 3.1.5, $q \in L_{-1}(\langle K \rangle)$. Thus, as p is assumed to be an element in $\langle K \rangle$, $q \in L_{-1}(p)$. Thus, by definition, there exists an element s in $\langle L \rangle$ such that $q \in sp$ and $\ell(q) = \ell(s) + \ell(p)$.

From $\ell(q) = \ell(s) + \ell(p)$ and $0 \leq \ell(s)$ we obtain $\ell(p) \leq \ell(q)$.

Assume that $\ell(q) = \ell(p)$. Then $\ell(s) = 0$, and that means $s = e$. Thus, as $q \in sp$, $p = q$. □

Corollary 3.2.2 *The set $L_{-1}(L)$ has at most one element.*

Proof. This follows from Lemma 3.2.1. (Set $K = L$.) □

Lemma 3.2.3 *Let p be an element in $L_1(K)$, and let q be an element in $p\langle K \rangle$. Then $q^* \in L_{-1}(p^*)$.*

Proof. We are assuming that $q \in p\langle K \rangle$. Thus, there exists an element s in $\langle K \rangle$ such that $q \in ps$.

We are assuming that $p \in L_1(K)$. Thus, by Theorem 3.1.4, $p \in L_1(\langle K \rangle)$. Thus, as $s \in \langle K \rangle$, $p \in L_1(s)$. Thus, as $q \in ps$ and L is assumed to be constrained, $\ell(q) = \ell(p) + \ell(s)$.

From $q \in ps$, we obtain $q^* \in s^* p^*$; cf. Lemma 1.1.4(ii). From $\ell(q) = \ell(p) + \ell(s)$ we obtain $\ell(q^*) = \ell(s^*) + \ell(p^*)$. Thus, $q^* \in L_{-1}(p^*)$. □

The following result will be useful in the proof of Lemma 3.3.2.

Lemma 3.2.4 *Let p and q be elements in $\langle L\rangle$, and let l be an element in $L \cap L_1(p) \cap L_1(q)$. Assume that $lp \subseteq lq\langle K\rangle$. Then $p \in q\langle K\rangle$.*

PROOF. We are assuming that $lp \subseteq lq\langle K\rangle$. Thus, we must have $p \in q\langle K\rangle$ or $p \in lq\langle K\rangle$. In the first case, we are done, so we assume $p \in lq\langle K\rangle$.

From Corollary 3.1.7 we know that $\langle L\rangle = L_1(K)\langle K\rangle$. Thus, as $q \in \langle L\rangle$, there exists an element u in $L_1(K)$ such that $q \in u\langle K\rangle$. From $u \in L_1(K)$ and $q \in u\langle K\rangle$ we obtain $q^* \in L_{-1}(u^*)$; cf. Lemma 3.2.3. On the other hand, we are assuming that $l \in L_1(q)$, and that implies $q^* \in L_1(l)$; cf. Lemma 2.2.3(i). Thus, by Lemma 2.2.3(ii), $u^* \in L_1(l)$, and this implies $l \in L_1(u)$; cf. Lemma 2.2.3(i).

Now recall that L is assumed to satisfy the exchange condition. Thus, we obtain from $l \in L_1(u)$ and $u \in L_1(K)$ that $lu \subseteq uK$ or $lu \subseteq L_1(K)$. However, as $p \in lq\langle K\rangle$, $q\langle K\rangle = u\langle K\rangle$, and $p \notin q\langle K\rangle$, we cannot have $lu \subseteq uK$. Thus, $lu \subseteq L_1(K)$.

Similarly, we obtain an element t in $L_1(K)$ such that $p \in t\langle K\rangle$, $l \in L_1(t)$, and $lt \subseteq L_1(K)$. Thus, referring to Theorem 3.1.5 we now obtain $lt = lu$. (Note that $lt\langle K\rangle = lu\langle K\rangle$.) Thus, by Lemma 2.4.1, $t = u$. Thus, as $p \in t\langle K\rangle$ and $q \in u\langle K\rangle$, $p \in q\langle K\rangle$. □

Lemma 3.2.5 *Let l be an element in L, and let s be an element in $L_1(K)$. Assume there exists an element r in $s\langle K\rangle$ such that $l \in L_1(r) \setminus \langle K\rangle^{r^*}$. Then $ls \subseteq L_1(K)$.*

PROOF. From $r \in s\langle K\rangle$ we obtain $r\langle K\rangle = s\langle K\rangle$. According to Lemma 1.1.4(ii), this implies $\langle K\rangle r^* = \langle K\rangle s^*$. Thus, by Lemma 1.5.2, $\langle K\rangle^{r^*} = \langle K\rangle^{s^*}$. Thus, as we are assuming that $l \notin \langle K\rangle^{r^*}$, $l \notin \langle K\rangle^{s^*}$. Thus, by definition, $s^*l \not\subseteq \langle K\rangle s^*$. Thus, by Lemma 1.1.4(ii), $ls \not\subseteq s\langle K\rangle$.

From $s \in L_1(K)$ and $r \in s\langle K\rangle$ we obtain $r^* \in L_{-1}(s^*)$; cf. Lemma 3.2.3. Thus, by Lemma 2.2.3(iv), $L_1(r) \subseteq L_1(s)$. Thus, as we are assuming that $l \in L_1(r)$, $l \in L_1(s)$. Thus, as L is assumed to satisfy the exchange condition, we have $ls \subseteq L_1(K)$. □

3.3 Sets of subsets of Coxeter sets

In this section, we deal with sets of subsets of L.

Lemma 3.3.1 *Let $\mathcal{K}$ be a nonempty set of subsets of L. Then we have*

$$\langle \bigcap_{K\in\mathcal{K}} K\rangle = \bigcap_{K\in\mathcal{K}} \langle K\rangle.$$

PROOF. Define H to be the intersection of the elements in $\mathcal{K}$ and T to be the intersection of the sets $\langle K\rangle$ with $K \in \mathcal{K}$. We have to show that $\langle H\rangle = T$.

Assume, by way of contradiction, that $\langle H\rangle \neq T$. Then, as $\langle H\rangle \subseteq T$, $T \not\subseteq \langle H\rangle$. We pick an element t in $T \setminus \langle H\rangle$, and we do this in such a way that $\ell(t)$ is as small as possible.

Since $e \in \langle H\rangle$ and $t \notin \langle H\rangle$, $t \neq e$. Thus, by Lemma 1.6.2, there exist elements s in $\langle L\rangle$ and l in L such that $t \in sl$ and $\ell(t) = \ell(s) + 1$. It follows that $t \in L_{-1}(l)$. Thus, as $t \in T$, $l \in H$; cf. Lemma 3.1.6. Thus, as $t \in sl$ and $t \in T$, $s \in T$. Thus, as $\ell(t) = \ell(s) + 1$, the minimal choice of t forces $s \in \langle H\rangle$. Thus, as $l \in H$ and $t \in sl$, $t \in \langle H\rangle$, contradiction. □

Lemma 3.3.2 *Let $\mathcal{K}$ be a nonempty set of subsets of L. Then*

$$s\langle \bigcap_{K\in\mathcal{K}} K\rangle = \bigcap_{K\in\mathcal{K}} s\langle K\rangle$$

for each element s in $\langle L\rangle$.

Proof. Define R to be the set of all elements in $\langle L\rangle$ which do not satisfy the equation in question. By way of contradiction, we assume that R is not empty. We pick an element r in R, and we do this in such a way that $\ell(r)$ is as small as possible.

By Lemma 3.3.1, $e \notin R$. Thus, as $r \in R$, $r \neq e$. Thus, by Lemma 1.6.2, there exist elements l in L and q in $\langle L\rangle$ such that $r \in lq$ and $\ell(r) = 1 + \ell(q)$.

Define H to be the intersection of the elements in $\mathcal{K}$ and Q to be the intersection of the sets $r\langle K\rangle$ with $K \in \mathcal{K}$. Then, as $r \in R$, $r\langle H\rangle \neq Q$. Thus, as $r\langle H\rangle \subseteq Q$, $Q \not\subseteq r\langle H\rangle$. Thus, we find an element s in Q that is not in $r\langle H\rangle$.

Assume first that $s^* \in L_{-1}(l)$. Then, $\langle L\rangle$ possesses an element p such that $s \in lp$ and $\ell(s) = 1 + \ell(p)$. Thus, $l \in L_1(p)$. Moreover, since $r \in lq$ and $\ell(r) = 1 + \ell(q)$, $l \in L_1(q)$. On the other hand, for each element K in $\mathcal{K}$, we have $s \in r\langle K\rangle$. Thus, for each element K in $\mathcal{K}$, $p \in q\langle K\rangle$; cf. Lemma 3.2.4. Thus, as $\ell(r) = 1 + \ell(q)$, the minimal choice of r forces $p \in q\langle H\rangle$. It follows that $s \in lp \subseteq lq\langle H\rangle = r\langle H\rangle$, contrary to the choice of s.

Assume now that $s^* \notin L_{-1}(l)$. Then, by Lemma 2.5.2, $s^* \in L_1(l)$. Thus, by Lemma 2.2.3(i), $l \in L_1(s)$. Thus, there exists an element t in $\langle L\rangle$ such that $ls = \{t\}$. Since $s \in Q$, we have $s \in r\langle K\rangle$ for each element K in $\mathcal{K}$. Thus, for each element K in $\mathcal{K}$,

$$q \in lr \subseteq ls\langle K\rangle = t\langle K\rangle,$$

and this is equivalent to $t \in q\langle K\rangle$. Thus, as $\ell(r) = 1+\ell(q)$, the minimal choice of r forces $t \in q\langle H\rangle$. Thus, $s \in lt \subseteq lq\langle H\rangle = r\langle H\rangle$, contradiction. □

Corollary 3.3.3 *Let $\mathcal{K}$ be a nonempty set of subsets of L. Then*

$$\langle \bigcap_{K\in\mathcal{K}} K\rangle^s = \bigcap_{K\in\mathcal{K}} \langle K\rangle^s$$

for each element s in $\langle L\rangle$.

PROOF. Define H to be the intersection of the elements in $\mathcal{K}$. Then, for each element K in $\mathcal{K}$, $\langle H\rangle^s \subseteq \langle K\rangle^s$.

Conversely, let r be an element in $\langle L\rangle$ such that, for each element K in $\mathcal{K}$, $r \in \langle K\rangle^s$. Then, for each element K in $\mathcal{K}$, $sr \subseteq \langle K\rangle s$. Thus, by Lemma 3.3.2, $sr \subseteq \langle H\rangle s$, and this means that $r \in \langle H\rangle^s$. □

Recall from Corollary 2.4.4(iii) that L does not contain thin elements if and only if $O_\vartheta(\langle L\rangle) = \{e\}$. We now deal with that case.

Theorem 3.3.4 *Assume that L does not contain thin elements. Then $K \mapsto \langle K\rangle$ is a bijective map from the power set of L to the set of all closed subsets of $\langle L\rangle$.*

PROOF. We are assuming that $O_\vartheta(L)$ is empty. Thus, by Corollary 2.4.3, $\langle L \cap T\rangle = T$ for each closed subset T of $\langle L\rangle$. This shows that the map in question is surjective.

Injectivity follows from Corollary 3.1.3(iii). □

Let

$$T_1, \ldots, T_n$$

be normal closed subsets of S.[1] For each element i in $\{1, \ldots, n\}$, we define $\hat{T}_i$ to be the product of the closed subsets T_j with $i \neq j$. The hypergroup S is called *direct product* of the closed subsets T_1, …, T_n if $T = T_1 \cdots T_n$ and $T_i \cap \hat{T}_i = \{e\}$ for each element i in $\{1, \ldots, n\}$.

If S is the direct product of closed subsets T_1, …, T_n, we indicate this by writing

$$S = T_1 \times \ldots \times T_n.$$

A closed subset T of S different from $\{e\}$ is called *simple* if $\{e\}$ and T are the only normal closed subsets of T.

We now shall see that $\langle L\rangle$ is the direct product of simple closed subsets each of which is generated by the elements of L which it contains.

Theorem 3.3.5 *Assume that L is finite and does not contain thin elements. Then L possesses subsets L_1, …, L_n such that $\{L_1, \ldots, L_n\}$ is a partition of L,*

$$\langle L\rangle = \langle L_1\rangle \times \ldots \times \langle L_n\rangle,$$

and, for each element i in $\{1, \ldots, n\}$, $\langle L_i\rangle$ is simple.

PROOF. Assume that $\langle L\rangle$ is not simple. Then, by definition, $\langle L\rangle$ possesses a normal closed subset T different from $\{e\}$ and $\langle L\rangle$.

[1] Recall that S is an arbitrary hypergroup.

Since we are assuming that L has no thin element, we obtain from Theorem 3.3.4 a subset K of L such that $T = \langle K \rangle$. Since $T \neq \{e\}$, K is not empty, and that means that $L \setminus K \neq L$. Since $T \neq \langle L \rangle$, $K \neq L$.

Since T is normal in $\langle L \rangle$ and $\langle K \rangle = T$, $\langle K \rangle$ is normal in $\langle L \rangle$. Thus, by Theorem 3.1.8, $\langle L \setminus K \rangle$ is normal in $\langle L \rangle$.

From Lemma 3.3.1 we know that $\langle K \rangle \cap \langle L \setminus K \rangle = \{e\}$. Thus, by definition, $\langle L \rangle = \langle K \rangle \times \langle L \setminus K \rangle$.

Now the claim follows by induction. □

Theorem 3.3.5 tells us that, in order to investigate Coxeter sets without thin elements, it is enough to look at Coxeter sets which generate simple closed subsets.

The following lemma is a result about simple closed subsets generated by a Coxeter set.

Lemma 3.3.6 *Assume that $3 \leq |L|$ and that $\langle L \rangle$ is simple. Let l be an element in L. Then there exists an element s in $L_1(l)$ such that, for each subset K of L with $|K| = 2$, $\langle K \rangle s \subseteq L_1(l)$.*

Proof. Since $\{l\} \neq L$ and $\langle L \rangle$ is assumed to be simple, we find an element k in $L \setminus C_S(l)$; cf. Theorem 3.1.8.

Since L is assumed to have at least three elements and $\langle L \rangle$ is assumed to be simple, we find an element h in $L \setminus \{k, l\}$ such that $h \notin C_S(\langle k, l \rangle)$; cf. Theorem 3.1.8.

Let s be the element in hk, let K be a subset of L with $|K| = 2$, and let q be an element in $\langle L \rangle$ such that $\langle K \rangle \subseteq L_1(q)$ and $\langle K \rangle s = \langle K \rangle q$; cf. Lemma 2.2.5.

Since $s \in \langle K \rangle q$, $\langle K \rangle$ possesses an element p such that $s \in pq$. From $p \in \langle K \rangle$ and $\langle K \rangle \subseteq L_1(q)$ we obtain $p \in L_1(q)$. Thus, as $s \in pq$, $\ell(s) = \ell(p) + \ell(q)$. It follows that $s \in L_{-1}(q)$.

From $s \in hk$ and $l \notin \{h, k\}$ we obtain $s \in L_1(l)$; cf. Lemma 3.1.6. It follows that $s \in L_{-1}(q) \cap L_1(l)$. Thus, by Lemma 2.2.3(ii), $q \in L_1(l)$.

Note also that $hkl \not\subseteq \langle K \rangle s$. Thus, by Lemma 3.2.5, $\langle K \rangle \subseteq L_1(ql)$. Thus, by Lemma 2.2.4, $\langle K \rangle q \subseteq L_1(l)$. Thus, as $\langle K \rangle q = \langle K \rangle s$, $\langle K \rangle s \subseteq L_1(l)$. □

3.4 Coxeter sets generating finite sets

In this section, we deal with Coxeter sets generating finite sets.

Lemma 3.4.1 *Let s be an element in $\langle L \rangle$, and set $K := \{l \in L \mid s \in L_{-1}(l)\}$. Then $\langle K \rangle$ is a finite set.*

Proof. We first prove that K is finite, and in order to do so we assume that K is infinite. Then, for each positive integer n, $\langle K\rangle$ possesses an element r with $\ell(r) = n$; cf. Lemma 3.1.6. On the other hand, $s \in L_{-1}(r)$ for each element r in $\langle K\rangle$. Thus, $\ell(r) \leq \ell(s)$ for each element r in $\langle K\rangle$. This contradiction shows that K is finite.

For each element k in K, we have $s \in L_{-1}(k)$. Thus, $s \in L_{-1}(K) = L_{-1}(\langle K\rangle)$; cf. Theorem 3.1.5. Thus, for each element r in $\langle K\rangle$, $s \in L_{-1}(r)$. Thus, for each element r in $\langle K\rangle$ different from s, we have $\ell(r) \leq \ell(s) - 1$.

Let n be a positive integer with $n \leq \ell(s) - 1$, and let q be an element in $\langle K\rangle$ such that $\ell(q) = n$. Then there exist elements p in $\langle K\rangle$ and k in K such that $q \in pk$ and $\ell(q) = \ell(p) + 1$; cf. Lemma 1.6.2. It follows that $p \in L_1(k)$. Thus, as L is assumed to be constrained, $pk = \{q\}$. By induction, we may assume that $\langle K\rangle$ possesses only finitely many elements of length $n - 1$. Thus, as K is finite, $\langle K\rangle$ possesses only finitely many elements of length n. □

Theorem 3.4.2 *The following statements are equivalent.*

(a) *The set $\langle L\rangle$ is finite.*

(b) *The set $\langle L\rangle$ has at least one element of maximal length.*

(c) *The set $\langle L\rangle$ has exactly one element of maximal length.*

(d) *The set $L_{-1}(L)$ is not empty.*

(e) *The set $L_{-1}(L)$ contains exactly one element.*

Proof. (a) $\Rightarrow$ (b) Assume $\langle L\rangle$ to be finite. Then $\langle L\rangle$ possesses an element s such that $\ell(r) \leq \ell(s)$ for each element r in $\langle L\rangle$.

(b) $\Rightarrow$ (d) We are assuming that L is a Coxeter set. In particular, L satisfies the exchange condition. Thus, by Lemma 2.5.2, L is dichotomic. Thus, by Lemma 2.5.1, each element of maximal length in $\langle L\rangle$ is in $L_{-1}(L)$.

(d) $\Rightarrow$ (a) This follows from Lemma 3.4.1.

(d) $\Rightarrow$ (e) This follows from Corollary 3.2.2.

(d) $\Rightarrow$ (c) This follows from Lemma 3.2.1. □

For the remainder of this section, we assume $\langle L\rangle$ to be finite.

Assuming $\langle L\rangle$ to be finite we obtain from Theorem 3.4.2 that $L_{-1}(L)$ contains exactly one element. In the following, we shall denote this element by m_L.

Lemma 3.4.3 *The element m_L is the uniquely determined element in $\langle L\rangle$ of maximal length.*

Proof. This follows from Lemma 3.2.1. □

Recall that $(m_L)^* \in \langle L\rangle$ and $\ell((m_L)^*) = \ell(m_L)$. Thus, by Lemma 3.4.3,

$$(m_L)^* = m_L.$$

Let s be an element in $\langle L\rangle$. Then, as $m_L \in L_{-1}(L)$, $m_L \in L_{-1}(s)$; cf. Theorem 3.1.4. Thus, by definition, $\langle L\rangle$ contains an element r with $m_L \in rs$ and $\ell(m_L) = \ell(r) + \ell(s)$. By Lemma 2.4.1, this element is uniquely determined. For the remainder of this section, we shall denote this element by $s^{(L)}$.

Lemma 3.4.4 *For any two elements p and q in $\langle L\rangle$, we have the following.*

(i) *If $p \neq q$, $p^{(L)} \neq q^{(L)}$.*

(ii) *Let r be an element in pq, and assume that $\ell(r) = \ell(p) + \ell(q)$. Then we have $q^{(L)} \in r^{(L)}p$ and $\ell(q^{(L)}) = \ell(r^{(L)}) + \ell(p)$.*

Proof. (i) Let p and q be elements in $\langle L\rangle$ such that $p^{(L)} = q^{(L)}$. Then, by definition, $m_L \in p^{(L)}p$, $\ell(m_L) = \ell(p^{(L)}) + \ell(p)$, $m_L \in p^{(L)}q$, and $\ell(m_L) = \ell(p^{(L)}) + \ell(q)$. Thus, by Lemma 2.4.1, $p = q$.

(ii) By definition, we have $m_L \in r^{(L)}r$ and $\ell(m_L) = \ell(r^{(L)}) + \ell(r)$. By hypothesis, we have $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$. Thus, by Lemma 1.6.3, $r^{(L)}p$ possesses an element s such that $m_L \in sq$, $\ell(s) = \ell(r^{(L)}) + \ell(p)$, and $\ell(m_L) = \ell(s) + \ell(q)$.

From $m_L \in sq$ and $\ell(m_L) = \ell(s) + \ell(q)$ we obtain $s = q^{(L)}$; cf. Lemma 2.4.1. Thus, the claim follows from $s \in r^{(L)}p$ and $\ell(s) = \ell(r^{(L)}) + \ell(p)$. □

For each element s in $\langle L\rangle$, we now shall write $s^{[L]}$ instead of $s^{(L)(L)}$.

Lemma 3.4.5 *For any two elements p and q in $\langle L\rangle$, we have the following.*

(i) *If $p \neq q$, $p^{[L]} \neq q^{[L]}$.*

(ii) *Let r be an element in pq, and assume that $\ell(r) = \ell(p) + \ell(q)$. Then we have $r^{[L]} \in p^{[L]}q^{[L]}$ and $\ell(r^{[L]}) = \ell(p^{[L]}) + \ell(q^{[L]})$.*

Proof. (i) This follows from Lemma 3.4.4(i). (Apply this lemma twice.)

(ii) This follows from Lemma 3.4.4(ii). (Apply this lemma three times.) □

Lemma 3.4.6 *For each element s in $\langle L\rangle$, the following hold.*

(i) *We have $\ell(s^{[L]}) = \ell(s)$.*

(ii) *We have $s^{[L][L]} = s$.*

Proof. (i) This follows immediately from the definition of $s^{[L]}$.

(ii) Let s be an element in $\langle L\rangle$. Then, by definition, $m_L \in s^{[L](L)}s^{[L]}$. Thus, as $(m_L)^* = m_L$, $m_L \in (s^{[L]})^*(s^{[L](L)})^*$; cf. Lemma 1.1.4(ii). Thus, as $m_L \in s^{(L)}s$, the set

$$(s^{(L)})^*(s^{[L]})^* \cap ss^{[L](L)}$$

is not empty; cf. Lemma 1.1.4(iii). Thus, as $\{m_L\} = (s^{(L)})^*(s^{[L]})^*$, $m_L \in ss^{[L](L)}$. However, by definition, we have

$$m_L \in s^{[L][L]}s^{[L](L)}.$$

Moreover, by (i), $\ell(s^{[L][L]}) = \ell(s)$. Thus, by Lemma 2.4.1, $s^{[L][L]} = s$. □

Recall that, for each subset R of $\langle L\rangle$, $L_1(R)$ is our notation for the intersection of the sets $L_1(r)$ with $r \in R$.

Lemma 3.4.7 *Let K be a nonempty subset of L. Then we have $(m_K)^{(L)} \in L_1(K)$ and $(m_K)^{(L)} \in m_L\langle K\rangle$.*

PROOF. Let k be an element in K. Then $m_K \in L_{-1}(k)$. Thus, by Lemma 2.2.3(iv), $L_1(m_K) \subseteq L_1(k)$. Thus, as $(m_K)^{(L)} \in L_1(m_K)$, $(m_K)^{(L)} \in L_1(k)$.

Since k has been chosen arbitrarily in K, we have shown that $(m_K)^{(L)} \in L_1(K)$.

By definition, $m_L \in (m_K)^{(L)}m_K$. Thus, as $m_K \in \langle K\rangle$, $m_L \in (m_K)^{(L)}\langle K\rangle$, so that the second claim follows from Lemma 1.2.3. □

For each nonempty subset R of $\langle L\rangle$, we define $R^{[L]}$ to be the set of all elements $r^{[L]}$ with $r \in R$.

Lemma 3.4.8 *Let K be a nonempty subset of L. Then the following hold.*

(i) *We have $m_{K^{[L]}} = (m_K)^{[L]}$.*

(ii) *We have $\langle K\rangle \subseteq \langle K^{[L]}\rangle^{((m_K)^{(L)})}$.*

PROOF. (i) From Lemma 3.4.5(ii) one obtains $\langle K\rangle^{[L]} = \langle K^{[L]}\rangle$. Thus, $s \mapsto s^{[L]}$ is a surjective map from $\langle K\rangle$ to $\langle K^{[L]}\rangle$. By Lemma 3.4.5(i), this map is injective, too. Thus, the claim follows from Lemma 3.4.6(i).

(ii) Considering Lemma 1.6.1 we obtain from Lemma 1.5.1(iii) that the set $\langle K^{[L]}\rangle^{((m_K)^{(L)})}$ is closed. Thus, we shall be done if we succeed in showing that $K \subseteq \langle K^{[L]}\rangle^{((m_K)^{(L)})}$.

In order to show this we pick an element k in K. From the first statement of Lemma 3.4.7 we know that

$$(m_K)^{(L)} \in L_1(k).$$

Thus, there exists an element p in $(m_K)^{(L)}k$ such that $\ell(p) = \ell((m_K)^{(L)})+1$. Assume that $K^{[L]} \subseteq L_1(p)$. Then, by Theorem 3.1.4, $\langle K^{[L]}\rangle \subseteq L_1(p)$. Thus, as $m_{K^{[L]}} \in \langle K^{[L]}\rangle$, $m_{K^{[L]}} \in L_1(p)$. Thus, by (i),

$$(m_K)^{[L]} \in L_1(p).$$

Thus, there exists an element q in $(m_K)^{[L]}p$ such that $\ell(q) = \ell((m_K)^{[L]})+\ell(p)$. Thus, as $\ell(p) = \ell((m_K)^{(L)}) + 1$,

$$\ell(q) = \ell((m_K)^{[L]}) + \ell((m_K)^{(L)}) + 1 = \ell(m_L) + 1,$$

contrary to $q \in \langle L \rangle$.

Thus, as L is assumed to satisfy the exchange condition, there exists an element h in $K^{[L]}$ such that

$$h(m_K)^{(L)} = (m_K)^{(L)}k.$$

It follows that

$$(m_K)^{(L)}k \subseteq \langle K^{[L]} \rangle (m_K)^{(L)},$$

and that means that $k \in \langle K^{[L]} \rangle^{((m_K)^{(L)})}$. □

4

Constrained sets of involutions with injections

Throughout this chapter, the letter S stands for a hypergroup. We shall investigate a very specific class of injective maps from closed subsets of S generated by constrained sets of involutions to S.

The injective maps which we shall discuss generalize the map $s \mapsto s^{[L]}$ from $\langle L \rangle$ to itself that had been introduced in Section 3.4. While domain and codomain of the maps in Section 4.3 were equal, this requirement is not anymore imposed on the injective maps of this chapter.

At no point of this chapter, the constrained sets of involutions under investigation are assumed to be Coxeter sets. Two times, in Section 4.2 and in Section 4.4, they will be assumed to be dichotomic.

4.1 Definition and first observations

In this section, the letter L stands for a constrained set of involutions of S, the letter ρ for an injective map from $\langle L \rangle$ to S.

Recall from Lemma 1.2.1(i) that $e \in \langle L \rangle$. Thus, e^{ρ} is defined. For the remainder of this section, we set $m := e^{\rho}$ and assume

$$r^{\rho} r = \{m\}$$

for each element r in $\langle L \rangle$.

The map ρ generalizes the map $s \mapsto s^{[L]}$ from $\langle L \rangle$ to itself that had been introduced in Section 3.4 in the case where L was a Coxeter set and $\langle L \rangle$ finite.

As before, we write ℓ instead of ℓ_L.

Lemma 4.1.1 *Let s be an element in S, and let r be an element in $\langle L \rangle$ such that $sr = \{m\}$. Then $s = r^{\rho}$.*

Proof. The statement is obviously true if $r = e$. Therefore, we assume that $r \neq e$. From $r \neq e$ we obtain elements l in L and u in $\langle L \rangle$ such that $r \in lu$ and $\ell(r) = 1 + \ell(u)$; cf. Lemma 1.6.2.

Since L is assumed to be constrained, we obtain from $r \in lu$ and $\ell(r) = 1+\ell(u)$ that $lu = \{r\}$. Thus, our hypothesis $sr = \{m\}$ leads to $slu = \{m\}$.

Let t be an element in sl. Then, as $slu = \{m\}$, $tu = \{m\}$. Thus, by induction, $t = u^\rho$. Thus, as t has been chosen arbitrarily in sl, we have shown that

$$sl = \{u^\rho\}.$$

From $r^\rho r = \{m\}$ and $lu = \{r\}$ we obtain $r^\rho lu = \{m\}$.

Let t be an element in $r^\rho l$. Then, as $r^\rho lu = \{m\}$, $tu = \{m\}$. Thus, by induction, $t = u^\rho$. Thus, as t has been chosen arbitrarily in $r^\rho l$, we have shown that

$$r^\rho l = \{u^\rho\}.$$

From $sl = \{u^\rho\}$ and $r^\rho l = \{u^\rho\}$ we obtain $sl = r^\rho l$. Thus,

$$r^\rho \in s\langle l \rangle = \{s\} \cup sl = \{s, u^\rho\}.$$

Assume that $r^\rho = u^\rho$. Then the injectivity of ρ forces $r = u$, contrary to $\ell(r) = 1 + \ell(u)$. Thus, $r^\rho = s$. □

Lemma 4.1.2 *Let q be an element in $\langle L \rangle$, let p be an element in $L_1(q)$, and let r be an element in pq. Then $r^{*\rho} q^* = \{p^{*\rho}\}$.*

Proof. Let s be an element in $r^{*\rho} q^*$. We have to show that $s = p^{*\rho}$.

We are assuming L to be constrained. Thus, as $p \in L_1(q)$ and r in pq, $pq = \{r\}$. Thus, by Lemma 1.1.4(ii), $q^* p^* = \{r^*\}$. Thus, as $s \in r^{*\rho} q^*$,

$$sp^* \subseteq r^{*\rho} q^* p^* = r^{*\rho} r^* = \{m\}.$$

Thus, by Lemma 4.1.1, $s = p^{*\rho}$. □

Corollary 4.1.3 *Let l be an element in L, let p be an element in $L_1(l)$, and let q be an element in pl. Then the following hold.*

(i) *If l is thin, $p^{*\rho} l = \{q^{*\rho}\}$.*

(ii) *If l is not thin, $p^{*\rho} l = \{q^{*\rho}, p^{*\rho}\}$.*

Proof. From $p \in L_1(l)$ and $q \in pl$ we obtain $pl = \{q\}$. (Recall that L is assumed to be constrained.) On the other hand, as $p \in L_1(l)$, pl possesses an element of length $\ell(p) + 1$. Thus, $\ell(q) = \ell(p) + 1$. In particular, $q \neq p$. Thus, as ρ is assumed to be injective, $q^{*\rho} \neq p^{*\rho}$.

From $p \in L_1(l)$ and $q \in pl$ we also obtain $q^{*\rho} l = \{p^{*\rho}\}$; cf. Lemma 4.1.2. Thus, the claims follow from Lemma 2.1.2(i), (ii), respectively. □

We define $\langle L\rangle^\rho$ to be the image of $\langle L\rangle$ under ρ. Thus, $\langle L\rangle^\rho$ is the set of all elements r^ρ with $r \in \langle L\rangle$.

Theorem 4.1.4 *We have $\langle L\rangle^\rho = m\langle L\rangle$.*

Proof. For each element r in $\langle L\rangle$, we have $m \in r^\rho r$. Thus, $r^\rho \in mr^*$. Thus, as $r^* \in \langle L\rangle$, $r^\rho \in m\langle L\rangle$. Since r has been chosen arbitrarily in $\langle L\rangle$, this proves $\langle L\rangle^\rho \subseteq m\langle L\rangle$.

Assume, by way of contradiction, that $m\langle L\rangle \not\subseteq \langle L\rangle^\rho$. Then there exists an element r in $\langle L\rangle$ such that $mr \not\subseteq \langle L\rangle^\rho$. Among the elements r in $\langle L\rangle$ with $mr \not\subseteq \langle L\rangle^\rho$ we chose r such that $\ell(r)$ is as small as possible.

Since $mr \not\subseteq \langle L\rangle^\rho$, there exists an element s in mr with $s \notin \langle L\rangle^\rho$.

Define P to be the set of all elements p in $\langle L\rangle$ such that there exists an element q in $\langle L\rangle$ with $r \in pq$, $\ell(r) = \ell(p) + \ell(q)$, and $s \in p^{*\rho}q$.

Since $s \in mr$, $e \in P$. Thus, P is not empty. From $s \notin \langle L\rangle^\rho$ we obtain $r \notin P$. Among the elements in P we fix an element p such that $\ell(p)$ is as large as possible.

Since $p \in P$, there exists an element q in $\langle L\rangle$ with $r \in pq$, $\ell(r) = \ell(p) + \ell(q)$, and $s \in p^{*\rho}q$.

From $p \in P$ and $r \notin P$ we obtain $p \neq r$. Thus, $q \neq e$. Thus, by Lemma 1.6.2, there exist elements l in L and u in $\langle L\rangle$ such that $q \in lu$ and $\ell(q) = 1 + \ell(u)$. Thus, as $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$, there exists an element t in pl such that $r \in tu$, $\ell(t) = \ell(p) + 1$ and $\ell(r) = \ell(t) + \ell(u)$; cf. Lemma 1.6.3.

From $\ell(t) = \ell(p) + 1$ together with the maximal choice of p we obtain $t \notin P$. Thus, as $r \in tu$ and $\ell(r) = \ell(t) + \ell(u)$, we conclude that $s \notin t^{*\rho}u$.

From $s \in p^{*\rho}q$ we obtain $q^* \in s^*p^{*\rho}$, from $q \in lu$, $q^* \in u^*l$. Thus, $q^* \in s^*p^{*\rho} \cap u^*l$. Thus, by Lemma 1.1.4(iii), $su^* \cap p^{*\rho}l$ is not empty.

From Corollary 4.1.3 we know that $p^{*\rho}l \subseteq \{t^{*\rho}, p^{*\rho}\}$. Thus, as $t^{*\rho} \notin su^*$ and $su^* \cap p^{*\rho}l \neq \emptyset$, $p^{*\rho} \in su^*$. It follows that $s \in p^{*\rho}u$.

By definition, $m \in p^{*\rho}p^*$. Thus, $p^{*\rho} \in mp$. Thus, as $s \in p^{*\rho}u$, $s \in mpu$. Thus, there exists an element r' in pu such that $s \in mr'$.

From $r' \in pu$ we obtain

$$\ell(r') \leq \ell(p) + \ell(u) = \ell(p) + \ell(q) - 1 = \ell(r) - 1.$$

Thus, the minimal choice of r forces $mr' \subseteq \langle L\rangle^\rho$. Thus, as $s \in mr'$, $s \in \langle L\rangle^\rho$. This contradiction finishes the proof. $\square$

As a consequence of Theorem 4.1.4 we obtain that the image of ρ is either equal to its domain or disjoint from its domain; cf. Lemma 1.2.2.

4.2 Dichotomy

In this section, the letter L stands for a constrained dichotomic set of involutions of S, the letter ρ for an injective map from $\langle L\rangle$ to S.[1] As before, we set $m := e^\rho$ and assume

$$r^\rho r = \{m\}$$

for each element r in $\langle L\rangle$.

Instead of ℓ_L we simply write ℓ.

Lemma 4.2.1 *Let p and q be elements in $\langle L\rangle$, and let l be an element in L such that $q^{*\rho} \in p^{*\rho}l$. Then $p\langle l\rangle = q\langle l\rangle$.*

Proof. Assume first that $p \in L_1(l)$. Then, by definition, there exists an element r in pl such that $\ell(r) = \ell(p) + 1$.

From $p \in L_1(l)$ and $r \in pl$ we obtain $p^{*\rho}l \subseteq \{r^{*\rho}, p^{*\rho}\}$; cf. Corollary 4.1.3. Thus, as we are assuming that $q^{*\rho} \in p^{*\rho}l$, we have $q^{*\rho} = r^{*\rho}$ or $q^{*\rho} = p^{*\rho}$. It follows that $q = r$ or $q = p$. (Recall that ρ is assumed to be injective.)

If $q = r$, we obtain from $r \in pl$ that $q \in pl$, so that we are done by Lemma 1.2.3. If $q = p$, the statement is trivial.

Assume now that $p \notin L_1(l)$. Then, as $p \in \langle L\rangle$ and L is assumed to be dichotomic, $p \in L_{-1}(l)$. Thus, by definition, $\langle L\rangle$ possesses an element r such that $p \in rl$ and $\ell(p) = \ell(r) + 1$.

From $p \in rl$ and $\ell(p) = \ell(r) + 1$ we obtain $r \in L_1(l)$. Thus, as $p \in rl$, $p^{*\rho}l = \{r^{*\rho}\}$; cf. Lemma 4.1.2. Since we are assuming $q^{*\rho} \in p^{*\rho}l$, this implies $q^{*\rho} = r^{*\rho}$. Now the injectivity of ρ forces $q = r$. Thus, as $p \in rl$, $p \in ql$, so that we are done by Lemma 1.2.3. □

Lemma 4.2.2 *Let p and q be elements in $\langle L\rangle$, and assume that $p^{*\rho} \in mq$. Then $\ell(p) \leq \ell(q)$.*

Proof. If $q = e$, our assumption $p^{*\rho} \in mq$ leads to $p^{*\rho} = m = e^\rho$. Thus, the injectivity of ρ yields $p^* = e$. This implies that $p = e$, and we are done.

Assume that $q \neq e$. Then, by Lemma 1.6.2, there exist elements u in $\langle L\rangle$ and l in L such that $q \in ul$ and $\ell(q) = \ell(u) + 1$.

From $p^{*\rho} \in mq$ we obtain $m \in p^{*\rho}q^*$. Thus, as $q \in ul$, $m \in p^{*\rho}lu^*$. Thus, $p^{*\rho}l$ possesses an element r^* such that $m \in r^*u^*$.

From $m \in r^*u^*$ and $u \in \langle L\rangle$ we obtain

$$r^* \in mu \subseteq m\langle L\rangle = \langle L\rangle^\rho;$$

cf. Theorem 4.1.4. Thus, $\langle L\rangle$ possesses an element t with $r^* = t^{*\rho}$.

[1] Recall that L is called dichotomic if $\langle L\rangle = L_{-1}(l) \cup L_1(l)$ for each element l in L.

From $r^* = t^{*\rho}$ and $m \in r^*u^*$ we obtain $m \in t^{*\rho}u^*$. Thus, $t^{*\rho} \in mu$. Thus, by induction,

$$\ell(t) \leq \ell(u) = \ell(q) - 1.$$

From $r^* = t^{*\rho}$ and $r^* \in p^{*\rho}l$ we obtain $t^{*\rho} \in p^{*\rho}l$. Thus, by Lemma 4.2.1, $p\langle l\rangle = t\langle l\rangle$. In particular, $p \in t\langle l\rangle = \{t\} \cup tl$. It follows that

$$\ell(p) \leq \ell(t) + 1.$$

From this together with $\ell(t) \leq \ell(q) - 1$ we obtain $\ell(p) \leq \ell(q)$. □

Lemma 4.2.3 *Let p and q be elements in $\langle L\rangle$ such that $p^{*\rho} \in mq$ and $\ell(p) = \ell(q)$. Then $p = q$.*

Proof. If $q = e$, our assumption $p^{*\rho} \in mq$ leads to $p^{*\rho} = m = e^{\rho}$. Thus, the injectivity of ρ yields $p^* = e$. This implies that $p = e$, and we are done.

Assume that $q \neq e$. Then, by Lemma 1.6.2, there exist elements u in $\langle L\rangle$ and l in L such that $q \in ul$ and $\ell(q) = \ell(u) + 1$.

From $p^{*\rho} \in mq$ we obtain $m \in p^{*\rho}q^*$. Thus, as $q \in ul$, $m \in p^{*\rho}lu^*$. Thus, $p^{*\rho}l$ possesses an element r^* such that $m \in r^*u^*$.

From $m \in r^*u^*$ and $u \in \langle L\rangle$ we obtain

$$r^* \in mu \subseteq m\langle L\rangle = \langle L\rangle^{\rho};$$

cf. Theorem 4.1.4. Thus, $\langle L\rangle$ possesses an element t with $r^* = t^{*\rho}$.

From $m \in r^*u^*$ we obtain $r^* \in mu$. Thus, as $r^* = t^{*\rho}$, $t^{*\rho} \in mu$. Thus, by Lemma 4.2.2, $\ell(t) \leq \ell(u)$.

From $r^* \in p^{*\rho}l$ and $r^* = t^{*\rho}$ we obtain $t^{*\rho} \in p^{*\rho}l$. Thus, by Lemma 4.2.1, $p\langle l\rangle = t\langle l\rangle$. In particular, $p \in t\langle l\rangle$. Thus, $\ell(p) \leq \ell(t) + 1$. Thus, as we are assuming that $\ell(p) = \ell(q)$,

$$\ell(u) = \ell(q) - 1 = \ell(p) - 1 \leq \ell(t).$$

Earlier, we saw $\ell(t) \leq \ell(u)$, so that we now have $\ell(t) = \ell(u)$. Thus, as $t^{*\rho} \in mu$, induction yields $t = u$.

From $p\langle l\rangle = t\langle l\rangle$ and $t = u$ we obtain $p\langle l\rangle = u\langle l\rangle$. It follows that $p \in u\langle l\rangle = \{q, u\}$. Since $\ell(p) = \ell(q) = \ell(u) + 1$, we cannot have $p = u$. Thus, $p = q$. □

4.3 Twinned constrained sets of involutions

Two constrained sets H and K of involutions of S are said to be *twinned* if there exist injective maps λ from $\langle H\rangle$ to S and ρ from $\langle K\rangle$ to S such that

$$pp^{\lambda} = q^{\rho}q$$

for any two elements p in $\langle H\rangle$ and q in $\langle K\rangle$ and

$$\langle H\rangle e^\lambda = e^\rho \langle K\rangle.$$

The injective maps λ and ρ will be called the *twinning maps* of H and K.

For the remainder of this section, the letters H and K stand for twinned constrained sets of involutions of S with twinning maps λ and ρ.

Lemma 4.3.1 *The following hold.*

(i) *We have $e^\lambda = e^\rho$.*

(ii) *For each element p in $\langle H\rangle$, we have $pp^\lambda = \{e^\lambda\}$.*

(iii) *For each element q in $\langle K\rangle$, we have $q^\rho q = \{e^\rho\}$.*

Proof. (i) Since H and K are twinned with twinning maps λ and ρ, we have

$$pp^\lambda = q^\rho q$$

for any two elements p in $\langle H\rangle$ and q in $\langle K\rangle$. Thus, as $e \in \langle H\rangle$ and $e \in \langle K\rangle$, we have $e^\lambda = ee^\lambda = e^\rho e = e^\rho$.

(ii) From (i) one obtains

$$pp^\lambda = e^\rho e = \{e^\rho\} = \{e^\lambda\}$$

for each element p in $\langle H\rangle$.

(iii) This follows similar to (ii). □

From Lemma 4.3.1(i) we know that $e^\lambda = e^\rho$. We set $m := e^\rho$ and call this element the *conjugating* element of H and K.

From $e^\lambda = e^\rho$ and $m = e^\rho$ we obtain $\langle H\rangle m = m\langle K\rangle$. Thus

$$\langle K\rangle \subseteq \langle H\rangle^m,$$

and that explains our terminology.

Lemma 4.3.2 *We have $\langle H\rangle^\lambda = \langle H\rangle m\langle K\rangle = \langle K\rangle^\rho$.*

Proof. Referring to Lemma 4.3.1(iii) we obtain from Theorem 4.1.4 that $\langle K\rangle^\rho = m\langle K\rangle$. Similarly, referring to Lemma 4.3.1(ii), we obtain $\langle H\rangle m = \langle H\rangle^\lambda$. Finally, as H and K are assumed to be twinned, we have $\langle H\rangle m = m\langle K\rangle$. Thus,

$$\langle H\rangle^\lambda = \langle H\rangle m = \langle H\rangle m\langle K\rangle = m\langle K\rangle = \langle K\rangle^\rho,$$

and that finishes the proof. □

Lemma 4.3.3 *For each element p in $\langle H\rangle$, there exists exactly one element q in $\langle K\rangle$ such that $p^\lambda = q^\rho$.*

Proof. Let p be an element in $\langle H\rangle$. Then $p^\lambda \in \langle H\rangle^\lambda$. Thus, by Lemma 4.3.2, $p^\lambda \in \langle K\rangle^\rho$. Thus, there exists an element q in $\langle K\rangle$ such that $p^\lambda = q^\rho$.

Let q' be an element in $\langle K\rangle$ with $p^\lambda = q'^\rho$. Then, as $p^\lambda = q^\rho$, $q'^\rho = q^\rho$. Thus, as ρ is assumed to be injective, $q' = q$. □

Similarly to Lemma 4.3.3 one obtains that, for each element q in $\langle K\rangle$, there exists exactly one element p in $\langle H\rangle$ with $p^\lambda = q^\rho$. We shall refer to this result also as Lemma 4.3.3.

Lemma 4.3.4 *Let p be an element in $\langle H\rangle$, and let q denote the uniquely determined element in $\langle K\rangle$ such that $p^\lambda = q^\rho$. Then the following hold.*

(i) *We have $pm = mq$.*

(ii) *We have $p^{*\lambda} \in mq$ and $q^{*\rho} \in pm$.*

(iii) *If p is thin, $p^{*\lambda} = q^{*\rho}$.*

(iv) *If p is thin, so is q.*

Proof. (i) Since $pp^\lambda = \{m\} = q^\rho q$, we obtain from $p^\lambda = q^\rho$ that

$$pm = pq^\rho q = pp^\lambda q = mq.$$

(ii) From $m \in p^*p^{*\lambda}$ we obtain $p^{*\lambda} \in pm$. Thus, the first claim follows from (i). The second statement follows similarly.

(iii) Assume p to be thin. Then, by definition, $p^*p = \{e\}$. Thus, by (i),

$$p^*mq = p^*pm = \{m\}.$$

Let s be an element in mq. Then, as $p^*mq = \{m\}$, $p^*s = \{m\}$. Thus, by Lemma 4.1.1, $s = p^{*\lambda}$. Thus, as s has been chosen arbitrarily in mq, we have $mq = \{p^{*\lambda}\}$. Thus, as $q^{*\rho} \in mq$, $p^{*\lambda} = q^{*\rho}$.

(iv) Assume p to be thin. Then, by (i) and (iii),

$$\{m\} = p^*pm = p^*mq = mq^*q.$$

Let r be an element in q^*q. Then, as $mq^*q = \{m\}$, $mr = \{m\}$. Thus, by Lemma 2.1.1, $m = r^\rho$. Thus, as $m = e^\rho$, $r^\rho = e^\rho$. Thus, as ρ is assumed to be injective, $r = e$. Thus, as r has been chosen arbitrarily in q^*q, we conclude that $q^*q = \{e\}$, and this means that q is thin. □

4.4 Twinning and dichotomy

In this section, the letters H and K stand for twinned constrained dichotomic sets of involutions of S with twinning maps λ and ρ.

Lemma 4.4.1 *Let p be an element in $\langle H\rangle$, and let q denote the uniquely determined element in $\langle K\rangle$ such that $p^\lambda = q^\rho$. Then $p^{*\lambda} = q^{*\rho}$.*

Proof. From Theorem 4.1.4 we know that $\langle K\rangle^\rho = m\langle K\rangle$. Thus, by Lemma 4.2.2 and Lemma 4.2.3, $\langle K\rangle$ possesses a subset R such that

$$mq = \{q^{*\rho}\} \cup \{r^{*\rho} \mid r \in R\}$$

and $\ell_K(r) \leq \ell_K(q) - 1$ for each element r in R. On the other hand, we know from Lemma 4.3.4(ii) that $p^{*\lambda} \in mq$. Thus $\{q\} \cup R$ possesses an element u with $p^{*\lambda} = u^{*\rho}$.

Assume that $u \neq q$. Then $u \in R$, and that means that $\ell_K(u) \leq \ell_K(q) - 1$. It follows that $\ell_K(u^*) \leq \ell_K(q) - 1$. Thus, as $p^{*\lambda} = u^{*\rho}$, induction yields $p^\lambda = u^\rho$. Now recall that we are assuming $p^\lambda = q^\rho$. Thus, $u^\rho = q^\rho$. Thus, as ρ is assumed to be injective, $u = q$, contrary to $\ell_K(u) \leq \ell_K(q) - 1$.

Thus, we must have $u = q$. Thus, as $p^{*\lambda} = u^{*\rho}$, $p^{*\lambda} = q^{*\rho}$. □

For the remainder of this section, we assume that K does not contain thin elements. (By Lemma 4.3.4(iv), this implies that also H does not contain thin elements.)

Lemma 4.4.2 *Let p be an element in $\langle H\rangle$, and let q denote the uniquely determined element in $\langle K\rangle$ such that $p^\lambda = q^\rho$. Then $\ell_H(p) = \ell_K(q)$.*

Proof. Assume first that $q = e$. Then, as $p^\lambda = q^\rho$, $p^\lambda = e^\rho = e^\lambda$; cf. Lemma 4.3.1(i). Thus, as λ is assumed to be injective, $p = e$, and we are done.

Assume now that $q \neq e$. Then there exist elements u in $\langle K\rangle$ and k in K such that $q \in uk$ and $\ell_K(q) = \ell_K(u) + 1$; cf. Lemma 1.6.2.

We are assuming that K does not contain thin elements. Thus, as $k \in K$, k is not thin. Thus, by Corollary 4.1.3(ii), $u^{*\rho}k = \{q^{*\rho}, u^{*\rho}\}$. In particular, $u^{*\rho} \in u^{*\rho}k$. Since $m \in u^{*\rho}u^*$, we also have $u^{*\rho} \in mu$. Thus,

$$u^{*\rho} \in muk = mq.$$

Assume that $u^{*\rho} = q^{*\rho}$. Then, as ρ is injective, $u^* = q^*$. It follows that $u = q$, contrary to $\ell_K(q) = \ell_K(u) + 1$. Thus, $u^{*\rho} \neq q^{*\rho}$. Thus, as $u^{*\rho} \in mq$,

$$u^{*\rho} \in mq \setminus \{q^{*\rho}\}.$$

From Lemma 4.3.4(i) we know that $pm = mq$, from Lemma 4.4.1 that $p^{*\lambda} = q^{*\rho}$. Thus,

$$u^{*\rho} \in pm \setminus \{p^{*\lambda}\}.$$

Thus, $\langle H\rangle$ possesses an element t such that $t^{*\lambda} = u^{*\rho}$ and $\ell_H(t) \leq \ell_H(p) - 1$; cf. Lemma 4.2.2 and Lemma 4.2.3. From $t^{*\lambda} = u^{*\rho}$ we obtain $t^\lambda = u^\rho$; cf.

Lemma 4.4.1. Thus, as $\ell_K(u) = \ell_K(q) - 1$, induction allows us to assume that $\ell_H(t) = \ell_K(u)$.

From $\ell_H(t) = \ell_K(u)$, $\ell_K(u) = \ell_K(q) - 1$, and $\ell_H(t) \leq \ell_H(p) - 1$ we obtain $\ell_K(q) \leq \ell_H(p)$.

Similarly one proves that $\ell_H(p) \leq \ell_K(q)$. Thus, $\ell_H(p) = \ell_K(q)$. □

Let h be an element in H. From Lemma 4.4.2 we obtain that the uniquely determined element q in $\langle K \rangle$ satisfying $h^\lambda = q^\rho$ belongs to K.

Theorem 4.4.3 *Let p, r, and t be elements in $\langle H \rangle$ such that $t \in pr$ and $\ell_H(t) = \ell_H(p) + \ell_H(r)$. Let q, s, and u denote the uniquely determined elements in $\langle K \rangle$ such that $p^\lambda = q^\rho$, $r^\lambda = s^\rho$, and $t^\lambda = u^\rho$. Then $u \in qs$ and $\ell_K(u) = \ell_K(q) + \ell_K(s)$.*

Proof. From $p^\lambda = q^\rho$ we obtain $\ell_H(p) = \ell_K(q)$; cf. Lemma 4.4.2. Similarly, we obtain $\ell_H(r) = \ell_K(s)$ from $r^\lambda = s^\rho$ and $\ell_H(t) = \ell_K(u)$ from $t^\lambda = u^\rho$. Thus, as $\ell_H(t) = \ell_H(p) + \ell_H(r)$,

$$\ell_K(u) = \ell_K(q) + \ell_K(s).$$

From Lemma 4.3.4(i) we obtain

$$mu = tm = prm = pms = mqs.$$

From $m \in u^{*\rho}u^*$ we obtain $u^{*\rho} \in mu$. Thus, as $mu = mqs$, $u^{*\rho} \in mqs$. Thus, qs possesses an element u' such that $u^{*\rho} \in mu'$.

From $u' \in qs$ we obtain $\ell_K(u') \leq \ell_K(q) + \ell_K(s) = \ell_K(u)$. On the other hand, as $u^{*\rho} \in mu'$, $\ell_K(u) \leq \ell_K(u')$; cf. Lemma 4.2.2. Thus, $\ell_K(u) = \ell_K(u')$. Thus, by Lemma 4.2.3, $u = u'$. Thus, as $u' \in qs$, $u \in qs$. □

5

Association schemes

It is the purpose of this chapter to apply the results on hypergroups which we obtained in the first four chapters to association schemes. In order to do so we first recall the definition of a scheme. We follow [9].

Let X be a set. We write 1_X to denote the set of all pairs (x, x) with $x \in X$. For each subset r of the cartesian product $X \times X$, we define r^* to be the set of all pairs (y, z) with $(z, y) \in r$. For x an element of X and r a subset of $X \times X$, we write xr to denote the set of all elements y in X with $(x, y) \in r$.

We fix a partition S of $X \times X$ with $1_X \in S$ and assume that $s^* \in S$ for each element s in S. The set S is called an *association scheme* or simply a *scheme on* X if, for any three elements p, q, and r in S, there exists a cardinal number a_{pqr} such that $|yp \cap zq^*| = a_{pqr}$ for any two elements y in X and z in yr. This latter condition will be referred to as the *regularity condition*. The cardinal numbers showing up in the regularity conditions are called the *structure constants* of S.

Let p and q be elements of a scheme S. We define $\mu(p, q)$ to be the set of all elements r in S which satisfy $a_{pqr} \neq 0$. From [9; Lemma 1.3.1], [9; Lemma 1.3.3(ii)], and [9; Lemma 1.3.3(i)] we obtain that S is a hypergroup with respect to μ and with neutral element 1_X.

For the remainder of this chapter, the letter X stands for a set, the letter S for a scheme on X. Instead of 1_X we just write 1, and, for any two elements p and q in S, we write pq instead of $\mu(p, q)$.

There is one equation which says something about the structure constants of S. It is the following. For any four elements p, q, t, and u in S, we have

$$\sum_{s \in S} a_{pqs} a_{stu} = \sum_{s \in S} a_{psu} a_{qts}.$$

This equation is obtained by counting in two different ways the elements in $(yp \times zt^*) \cap u$, where y is an element in X and z an element in yu.

Let W be a nonempty subset of X, and let R be a nonempty subset of S. We define WR to be the union of the sets wr with $w \in W$ and $r \in R$. If W consists of a single element, say w, we write wR instead of WR. Similarly, if R consists of a single element r, we write Wr instead of WR.

Note that $W(PQ) = (WP)Q$ for each nonempty subset W of X and for any two nonempty subsets P and Q of S.

5.1 Involutions

We collect a few elementary facts about involutions of schemes.

Lemma 5.1.1 *Let p and q be elements in S such that $p \neq q$, and let l be an involution of S such that $pl = \{q\}$. Then $a_{plq} = 1$.*

Proof. Let y be an element in X, let z be an element in yq, and assume that $2 \leq a_{plq}$. Then there exist elements v and w in $yp \cap zl$ such that $v \neq w$.

From $v \in yp$ we obtain $y \in vp^*$. Thus, as $w \in yp$, $w \in vp^*p$. From $v \in zl$ we obtain $z \in vl$. Thus, as $w \in zl$, $w \in vll$. Thus, as $v \neq w$, $w \in vl$.

From $w \in vp^*p$ and $w \in vl$ we obtain $l \in p^*p$. Thus, $p \in pl = \{q\}$, contrary to $p \neq q$. □

For the remainder of this section, the letters p and q stand for two elements in S with $p \neq q$, the letters h and k for involutions of S satisfying

$$hp = \{q\} = pk.$$

We have the following.

Lemma 5.1.2 *Let y be an element in X, let z be an element in yq, and let x be an element in $y\langle h\rangle$. Then $|xp \cap z\langle k\rangle| = 1$.*

Proof. Since $x \in y\langle h\rangle$, $y \in x\langle h\rangle$. Thus, as $z \in yq$, $z \in x\langle h\rangle q$. On the other hand, as $hp = \{q\}$,

$$\langle h\rangle q = \langle h\rangle hp = \langle h\rangle p = \{1, h\}p = \{p\} \cup hp = \{p, q\}.$$

Thus, $z \in xp$ or $z \in xq$.

From $pk = \{q\}$ and $p \neq q$ we obtain $a_{pkp} = 0$. Thus, if $z \in xp$, we must have $xp \cap z\langle k\rangle = \{z\}$.

If $z \in xq$, the claim follows from $pk = \{q\}$; cf. Lemma 5.1.1. □

Lemma 5.1.3 *Let y be an element in X, and let z be an element in yq. For each element x in $y\langle h\rangle$, we define $\phi(x)$ to be the unique element in $xp \cap z\langle k\rangle$. Then ϕ is a bijective map from $y\langle h\rangle$ to $z\langle k\rangle$.*

Proof. We first show that ϕ is injective. We fix elements v and w in $y\langle h\rangle$ and assume that $v \neq w$.

From $v \in y\langle h\rangle$ and $w \in y\langle h\rangle$ we obtain $w \in v\langle h\rangle$. Thus, as $v \neq w$, $w \in vh$.

By definition, we have $\phi(w) \in wp$. Thus, as $w \in vh$ and $hp = \{q\}$, $\phi(w) \in vq$. Thus, as $\phi(v) \in vp$ and $p \neq q$, $\phi(v) \neq \phi(w)$.

This proves that ϕ is injective.

In order to show that ϕ is surjective we fix an element in $z\langle k\rangle$ and call it w.

Since $z \in yq$, we obtain from $w \in z\langle k\rangle$ that $w \in yq\langle k\rangle$. Thus, as $\{q\} = pk$, $w \in y\{p, q\}$; cf. Lemma 2.1.2(i), (ii). It follows that $w \in yp$ or $w \in yq$.

If $w \in yp$, $w \in yp \cap z\langle k\rangle$. Thus, $\phi(y) = w$.

If $w \in yq$, we obtain from our hypothesis $hp = \{q\}$ that $w \in yhp$. Thus, yh possesses an element v such that $w \in vp$. Thus, as $w \in z\langle k\rangle$, $w \in vp \cap z\langle k\rangle$. Thus, as $v \in y\langle h\rangle$, $\phi(v) = w$. $\square$

Corollary 5.1.4 *We have $n_h = n_k$.*

Proof. This is an immediate consequence of Lemma 5.1.3. $\square$

5.2 Constrained sets of involutions and injections

In this section, the letter L stands for a constrained set of involutions of S, the letter ρ for an injective map from $\langle L\rangle$ to S satisfying

$$r^\rho r = \{1^\rho\}$$

for each element r in $\langle L\rangle$.[1]

We set $m := 1^\rho$ and $\ell := \ell_L$.

Lemma 5.2.1 *Let q be an element in $\langle L\rangle$, let p be an element in $L_1(q)$, and let r be an element in pq. Then $a_{r^{*\rho}q^*p^{*\rho}} = 1$.*

Proof. The statement is trivial if $q = 1$. Thus, we assume that $q \neq 1$.

Assuming $q \neq 1$ we obtain elements t in $\langle L\rangle$ and l in L such that $q \in tl$ and $\ell(q) = \ell(t) + 1$; cf. Lemma 1.6.2. Thus, as $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$, pt possesses an element u such that $r \in ul$, $\ell(u) = \ell(p)+\ell(t)$, and $\ell(r) = \ell(u)+1$; cf. Lemma 1.6.3.

From $r \in ul$ and $\ell(r) = \ell(u) + 1$ we obtain $u \in L_1(l)$. Thus, as $r \in ul$, $r^{*\rho}l = \{u^{*\rho}\}$; cf. Lemma 4.1.2. Thus, by Lemma 5.1.1, $a_{r^{*\rho}lu^{*\rho}} = 1$.

[1] We emphasize that in this section (as well as for the following section) the constrained set L is never assumed to satisfy the exchange condition (to be a Coxeter set). The strongest assumption which we will impose on L will be dichotomy.

From $u \in pt$ and $\ell(u) = \ell(p) + \ell(t)$ we obtain $p \in L_1(t)$. Thus, as $u \in pt$, induction yields $a_{u^{*\rho}t^*p^{*\rho}} = 1$. Thus, as $r^{*\rho}l = \{u^{*\rho}\}$,

$$\sum_{s \in S} a_{r^{*\rho}ls} a_{st^*p^{*\rho}} = a_{r^{*\rho}lu^{*\rho}} a_{u^{*\rho}t^*p^{*\rho}} = 1.$$

On the other hand, recall that $q \in tl$ and $\ell(q) = \ell(t)+1$. Thus, as L is assumed to be constrained, $tl = \{q\}$. Thus, $lt^* = \{q^*\}$ and, by Lemma 5.1.1, $a_{tlq} = 1$. The latter equation yields $a_{lt^*q^*} = 1$. It follows that

$$\sum_{s \in S} a_{r^{*\rho}sp^{*\rho}} a_{lt^*s} = a_{r^{*\rho}q^*p^{*\rho}} a_{lt^*q^*} = a_{r^{*\rho}q^*p^{*\rho}}.$$

Thus, the claim follows from the equation on the structure constants that we mentioned in the introduction of this chapter. □

Corollary 5.2.2 *For each element r in $\langle L \rangle$, we have $a_{r^{*\rho}r^*m} = 1$.*

Proof. This is the case $p = 1$ in Lemma 5.2.1. □

5.3 Apartments

In this section, the letter L stands for a constrained set of involutions of S, the letter ρ for an injective map from $\langle L \rangle$ to S. We assume that

$$r^\rho r = \{1^\rho\}$$

for each element r in $\langle L \rangle$, we set $m := 1^\rho$, we fix elements y in X and z in ym, and we define C_{yz} to be the union of the sets

$$yr^{*\rho} \cap zr$$

with $r \in \langle L \rangle$.

Recall from Corollary 5.2.2 that, for each element r in $\langle L \rangle$, $yr^{*\rho} \cap zr$ contains exactly one element.

Let T be a closed subset of S. A subset W of X is called *apartment* of T if $|W \cap wt| = 1$ for any two elements $w \in W$ and $t \in T$.

It is the goal of this section to show that C_{yz} is an apartment of $\langle L \rangle$.

Since $z \in ym$ and $m = 1^\rho$, $z \in y1^\rho$. Thus, $z \in C_{yz}$.

Lemma 5.3.1 *For each element r in $\langle L \rangle$, $|C_{yz} \cap zr| = 1$.*

Proof. Let r be an element in $\langle L \rangle$. The definition of C_{yz} yields

$$C_{yz} \cap zr = yr^{*\rho} \cap zr,$$

and from Lemma 5.2.2 we know that $|yr^{*\rho} \cap zr| = 1$. Thus, $|C_{yz} \cap zr| = 1$. □

Lemma 5.3.2 *Let p and q be elements in $\langle L \rangle$ such that p in $L_1(q)$. Let v be an element in zp, and let w be an element in vq. Assume that $w \in C_{yz}$. Then $v \in C_{yz}$.*

Proof. From $w \in vq$ and $v \in zp$ we obtain $w \in zpq$. Thus, pq possesses an element r such that $w \in zr$. Thus, as $w \in C_{yz}$, $w \in yr^{*\rho}$. Thus, as $v \in wq^*$, $v \in yr^{*\rho}q^*$.

From $p \in L_1(q)$ and r in pq we obtain $r^{*\rho}q^* = \{p^{*\rho}\}$; cf. Lemma 4.1.2. Thus, as $v \in yr^{*\rho}q^*$, $v \in yp^{*\rho}$. Thus, as $v \in zp$, $v \in C_{yz}$. □

As earlier, we shall now write ℓ instead of ℓ_L.

Lemma 5.3.3 *Let p and q be elements in $\langle L \rangle$ such that p in $L_1(q)$. Let x be an element in $C_{yz} \cap zp$. Then $|C_{yz} \cap xq| = 1$.*

Proof. We are assuming that $p \in L_1(q)$. Thus, pq possesses an element r with $\ell(r) = \ell(p) + \ell(q)$. Let w denote the uniquely determined element in $C_{yz} \cap zr$; cf. Lemma 5.3.1.

From $w \in zr$ and $r \in pq$ we obtain $w \in zpq$. Thus, zp possesses an element v such that $w \in vq$. Thus, as $p \in L_1(q)$ and $w \in C_{yz}$, $v \in C_{yz}$; cf. Lemma 5.3.2. It follows that $v \in C_{yz} \cap zp$. Thus, as $x \in C_{yz} \cap zp$, $v = x$; cf. Lemma 5.3.1.

From $w \in vq$ and $v = x$ we obtain $w \in xq$. Thus, as $w \in C_{yz}$,

$$w \in C_{yz} \cap xq.$$

In order to show $|C_{yz} \cap xq| = 1$ we fix an element w' in $C_{yz} \cap xq$. We shall see that $w' = w$.

From $w' \in xq$ and $x \in zp$ we obtain $w' \in zpq$. From $p \in L_1(q)$ and $r \in pq$ we obtain $pq = \{r\}$. (Recall that L is assumed to be constrained.) Thus, $w' \in zr$. Thus, as $w' \in C_{yz}$, $w' \in C_{yz} \cap zr$. Thus, as $C_{yz} \cap zr = \{w\}$, $w' = w$. □

For the remainder of this section, the (constrained) set L is assumed to be dichotomic.

Lemma 5.3.4 *For any two elements x in C_{yz} and l in L, we have $|C_{yz} \cap xl| = 1$.*

Proof. Let x be an element in C_{yz}. Then, by definition, $\langle L \rangle$ possesses an element r such that

$$x \in yr^{*\rho} \cap zr.$$

Let l be an element in L. If $r \in L_1(l)$, we are done by Lemma 5.3.3. Therefore, we assume that $r \notin L_1(l)$.

Assuming that $r \notin L_1(l)$, we obtain $r \in L_{-1}(l)$. (Recall that L is assumed to be dichotomic.) This means that $\langle L \rangle$ possesses an element q such that $r \in ql$ and $\ell(r) = \ell(q) + 1$.

From $x \in zr$ and $r \in ql$ we obtain $x \in zql$. Thus, zq possesses an element w such that $x \in wl$. From $x \in wl$ we obtain $w \in xl$.

From $r \in ql$ and $\ell(r) = \ell(q) + 1$ we obtain $q \in L_1(l)$. Thus, as $w \in zq$, $x \in wl$, and $x \in C_{yz}$, we must have $w \in C_{yz}$; cf. Lemma 5.3.2. Thus, as $w \in xl$,

$$w \in C_{yz} \cap xl.$$

In order to show $|C_{yz} \cap xl| = 1$ we fix an element w' in $C_{yz} \cap xl$. We shall see that $w' = w$.

From $w \in xl$ and $w' \in xl$ we obtain $w' = w$ or $w' \in wl$; cf. Lemma 2.1.1(ii).

Assume, by way of contradiction, that $w' \in wl$. Then, as $w \in zq$, $w' \in zql$. On the other hand, as L is assumed to be constrained, we obtain from $q \in L_1(l)$ and $r \in ql$ that $ql = \{r\}$. Thus, $w' \in zr$. Thus, as $w' \in C_{yz}$, $w' \in yr^{*\rho}$.

From $w' \in xl$ and $x \in yr^{*\rho}$ we obtain $w' \in yr^{*\rho}l$. From $q \in L_1(l)$ and $r \in ql$ we obtain $r^{*\rho}l = \{q^{*\rho}\}$; cf. Lemma 4.1.2. Thus, $w' \in yq^{*\rho}$.

From $w' \in yr^{*\rho}$ and $w' \in yq^{*\rho}$ we obtain $q^{*\rho} = r^{*\rho}$. Thus, as ρ is assumed to be injective, we conclude that $q = r$, contrary to $\ell(r) = \ell(q) + 1$. □

Lemma 5.3.5 *For any two elements x in C_{yz} and r in $\langle L \rangle$, we have $1 \leq |C_{yz} \cap xr|$.*

Proof. Let x be an element in C_{yz}, and let r be an element in $\langle L \rangle$. There is nothing to show if $r = 1$. Therefore, we assume that $r \neq 1$. Thus, by Lemma 1.6.2, there exist elements q in $\langle L \rangle$ and l in L such that $r \in ql$ and $\ell(r) = \ell(q) + 1$.

From $r \in ql$ and $\ell(r) = \ell(q) + 1$ we obtain that $q \in L_1(l)$. Thus, as L is assumed to be constrained, $ql = \{r\}$.

By induction, we may assume that $C_{yz} \cap xq$ is not empty. Let w be an element in $C_{yz} \cap xq$. From Lemma 5.3.4 we obtain an element v in $C_{yz} \cap wl$. From $v \in wl$ and $w \in xq$ we obtain $v \in xql$. Thus, as $ql = \{r\}$, $v \in xr$. Thus, $v \in C_{yz} \cap xr$. □

Lemma 5.3.6 *Let p and q be elements in $\langle L \rangle$ with $p \neq q$, and let l be an element in L such that $q \in pl$. Let v denote the uniquely determined element in $C_{yz} \cap zp$, let w denote the uniquely determined element in $C_{yz} \cap zq$. Then $w \in vl$.*

Proof. Assume first that $p \in L_1(l)$. Then, as $q \in pl$, $pl = \{q\}$. (Recall that L is assumed to be constrained.)

Since $p \in L_1(l)$, $|C_{yz} \cap vl| = 1$; cf. Lemma 5.3.3. Let w' denote the element in $C_{yz} \cap vl$. From $w' \in vl$ and $v \in zp$ we obtain $w' \in zpl = zq$. Thus, $w' \in C_{yz} \cap zq = \{w\}$. Thus, as $w' \in vl$, $w \in vl$.

Assume now that $p \notin L_1(l)$. Then, as $p \in \langle L \rangle$ and L is assumed to be dichotomic, $p \in L_{-1}(l)$. Thus, by definition, $\langle L \rangle$ possesses an element r such that $p \in rl$ and $\ell(p) = \ell(r) + 1$. It follows that $r \in L_1(l)$.

From $v \in zp$ and $p \in rl$ we obtain $v \in zrl$. Thus, zr possesses an element w' such that $v \in w'l$. From $w' \in zr$, $r \in L_1(l)$, $v \in w'l$, and $v \in C_{yz}$ we obtain $w' \in C_{yz}$; cf. Lemma 5.3.2.

Since $p \in rl$, $r \in pl = \{p, q\}$. If $r = p$, $w' \in C_{yz} \cap zp = \{v\}$, contrary to $v \in w'l$. Thus, $r = q$. It follows that $w' \in C_{yz} \cap zq = \{w\}$. Thus, $w' = w$. Thus, as $v \in w'l$, $v \in wl$, and this implies $w \in vl$. □

The following lemma generalizes Lemma 5.3.2. In fact, Lemma 5.3.2 is the case $x = z$ of Lemma 5.3.7.

Lemma 5.3.7 *Let p and q be elements in $\langle L \rangle$ such that p in $L_1(q)$. Let x be an element in C_{yz}, let v be an element in xp, and let w be an element in vq. Assume that $w \in C_{yz}$. Then $v \in C_{yz}$.*

Proof. We define r to be the uniquely determined element in $\langle L \rangle$ which satisfies $w \in xr$ and set $n := \ell(r)$.

There is nothing to show if $r = 1$. Thus, we assume that $r \neq 1$. Then $1 \leq n$. Thus, there exist elements

$$b_0, \ldots, b_n$$

in $x\langle L \rangle$ and $l_1, \ldots, l_n$ in L such that $x = b_0$, $v = b_j$ for some element j in $\{1, \ldots, n\}$, $w = b_n$, and

$$b_i \in b_{i-1} l_i$$

for each element i in $\{1, \ldots, n\}$.

For each element i in $\{0, \ldots, n\}$, we define r_i to be the uniquely determined element in $\langle L \rangle$ which satisfies

$$b_i \in zr_i.$$

(Then $x \in zr_0$, $v \in zr_j$, and $w \in zr_n$.)

Let i be an element in $\{1, \ldots, n\}$. Then $b_{i-1} \in zr_{i-1}$ and $b_i \in b_{i-1}l_i$. It follows that $b_i \in zr_{i-1}l_i$. Thus, as $b_i \in zr_i$,

$$r_i \in r_{i-1} l_i.$$

For each element i in $\{0, \ldots, n\}$, we define a_i to be the uniquely determined element in $C_{yz} \cap zr_i$. (Then $x = a_0$ and $w = a_n$.) We claim that

$$a_i \in a_{i-1} \langle l_i \rangle$$

for each element i in $\{1, \ldots, n\}$.

In order to show this, we fix an element in $\{1, \ldots, n\}$ and call it i.

If $r_i = r_{i-1}$, the definition of a_{i-1} and a_i yields $a_i = a_{i-1}$. In this case, the claim is obvious. If $r_i \neq r_{i-1}$, we obtain from $r_i \in r_{i-1}l_i$ that $a_i \in a_{i-1}l_i$; cf. Lemma 5.3.6.

Since $\ell(r) = n$, $a_0 = x = b_0$, $a_n = w = b_n$, and $a_i \in a_{i-1}\langle l_i \rangle$ for each element i in $\{1, \ldots, n\}$, we must have $a_i = b_i$ for each element i in $\{0, \ldots, n\}$. (Here we use the fact that L is assumed to be constrained.) In particular, $v = b_j = a_j \in C_{yz}$. □

Theorem 5.3.8 *The set C_{yz} is an apartment of $\langle L \rangle$.*

Proof. Let x be an element in C_{yz}, and let r be an element in $\langle L \rangle$. We have to show $|C_{yz} \cap xr| = 1$.

Due to Lemma 5.3.5, it is enough to show that $|C_{yz} \cap xr| \leq 1$. In order to show this, we fix elements w and w' in $C_{yz} \cap xr$. We shall see that $w = w'$.

There is nothing to show if $r = 1$. Thus, we assume that $r \neq 1$.

From $r \neq 1$ we obtain elements q in $\langle L \rangle$ and l in L such that $r \in ql$ and $\ell(r) = \ell(q) + 1$; cf. Lemma 1.6.2. It follows that $q \in L_1(l)$.

From $w \in xr$ and $r \in ql$ we obtain $w \in xql$. Thus, xq possesses an element v such that $w \in vl$. Thus, as $q \in L_1(l)$ and $w \in C_{yz}$, $v \in C_{yz}$; cf. Lemma 5.3.7.

Similarly, we obtain an element v' in xq such that $w' \in v'l$ and $v' \in C_{yz}$.

Since $\ell(q) = \ell(r) - 1$, we may assume, by induction, that $|C_{yz} \cap xq| \leq 1$. Thus, as $v \in C_{yz} \cap xq$ and $v' \in C_{yz} \cap xq$, $v = v'$. Thus, as $w \in C_{yz} \cap vl$ and $w' \in C_{yz} \cap v'l$, $w = w'$; cf. Lemma 5.3.4. □

5.4 Twinned constrained sets of involutions

In this section, the letters H and K stand for constrained sets of involutions of S.[2] We assume H and K to be twinned. The twinning maps from $\langle K \rangle$ and $\langle H \rangle$ to S will be denoted by ρ and λ, respectively, and we define m to be the conjugating element of H and K.

Lemma 5.4.1 *Let h be an element in H, and let k denote the uniquely determined element in K such that $h^\lambda = k^\rho$. Then $n_h = n_k$.*

Proof. By definition, we have $hh^\lambda = \{m\} = k^\rho k$. Thus, as $h^\lambda = k^\rho$, we obtain the desired equation $n_h = n_k$ from Corollary 5.1.4. □

[2] Also here, we emphasize that in this section the constrained sets H and K are never assumed to satisfy the exchange condition (to be Coxeter sets). The strongest assumption which we will impose on L will be dichotomy.

We now come to apartments. Therefore, we assume the (constrained) sets H and K to be dichotomic. We also assume that K (and hence H) does not contain thin elements.

We fix elements y and z in X satisfying $z \in ym$. We define A_{yz} to be the union of the sets $yp \cap zp^{\lambda *}$ with $p \in \langle H \rangle$. By B_{yz} we mean the union of the sets $yq^{*\rho} \cap zq$ with $q \in \langle K \rangle$. From Theorem 5.3.8 we know that A_{yz} is an apartment of $\langle H \rangle$ and B_{yz} an apartment of $\langle K \rangle$.

Proposition 5.4.2 *Let p be an element in $\langle H \rangle$, let q be an element in $\langle K \rangle$, let v denote the uniquely determined element in $yp \cap A_{yz}$, let w denote the uniquely determined element in $B_{yz} \cap zq$. Then $p^\lambda = q^\rho$ if and only if $w \in vm$.*

Proof. Assume first that $p^\lambda = q^\rho$. Then, as $v \in yp \cap A_{yz}$, $v \in zq^{\rho *}$. From this we obtain $z \in vq^\rho$. Thus, as $w \in zq$, $w \in vq^\rho q$. Thus, as $q^\rho q = \{m\}$, $w \in vm$.

Assume now that $w \in vm$. Then, as $v \in yp$, $w \in ypm$. On the other hand, we are assuming that $w \in B_{yz} \cap zq$, so that $w \in yq^{*\rho}$. Thus, $q^{*\rho} \in pm$.

Let t denote the uniquely determined element in $\langle H \rangle$ satisfying $t^\lambda = q^\rho$; cf. Lemma 4.3.3. We shall be done if we succeed in showing that $t = p$.

From $t^\lambda = q^\rho$ we obtain $t^{*\lambda} = q^{*\rho}$; cf. Lemma 4.4.1. (Recall that H and K are assumed to be dichotomic.) Thus, as $q^{*\rho} \in pm$, $t^{*\lambda} \in pm$. Thus, by Lemma 4.2.2, $\ell_H(t) \leq \ell_H(p)$.

From $t^\lambda = q^\rho$ we also obtain $\ell_H(t) = \ell_K(q)$; cf. Lemma 4.4.2. Thus, $\ell_K(q) \leq \ell_H(p)$.

Similarly, one proves $\ell_H(p) \leq \ell_K(q)$, so that $\ell_H(p) = \ell_K(q)$. Thus, as $\ell_H(t) = \ell_K(q)$, $\ell_H(p) = \ell_H(t)$. Thus, as $t^{*\lambda} \in pm$, Lemma 4.2.3 yields $t = p$. □

Theorem 5.4.3 *For each element v in A_{yz}, we have $|vm \cap B_{yz}| = 1$.*

Proof. Let v be an element in A_{yz}, let p denote the uniquely determined element of $\langle H \rangle$ satisfying $v \in yp$, let q denote the uniquely determined element of $\langle K \rangle$ satisfying $p^\lambda = q^\rho$, and let w denote the uniquely determined element of $B_{yz} \cap zq$. Then, by Proposition 5.4.2, $w \in vm$.

In order to show uniqueness we fix an element w' in $vm \cap B_{yz}$. We shall be done if we succeed in showing that $w = w'$.

Let q' denote the uniquely determined element of $\langle K \rangle$ satisfying $w' \in zq'$. Then, as $w' \in vm$, $p^\lambda = q'^\rho$; cf. Proposition 5.4.2. Thus, as $p^\lambda = q^\rho$, $q'^\rho = q^\rho$. Thus, as ρ is assumed to be injective, $q' = q$. Thus, as $w' \in zq'$, $w' \in zq$. Thus, as $B_{yz} \cap zq = \{w\}$, $w = w'$. □

Similar to Theorem 5.4.3 one can show that, for each element w in B_{yz}, there exists exactly one element v in A_{yz} such that $w \in vm$. Thus, the restriction of m to $A_{yz} \times B_{yz}$ establishes a bijective map between A_{yz} and B_{yz}.

6

Buildings

In this chapter, we connect Coxeter sets of hypergroups, Coxeter schemes, buildings, and twin buildings.

Section 6.2 is devoted to the proof of the fact that Coxeter sets of hypergroups and Coxeter groups are equivalent notions. We shall prove this fact in Theorem 6.2.6. In Section 6.3, we associate a building to each Coxeter scheme. (This will be done in Theorem 6.3.1.) In Section 6.4 (more specifically, in Theorem 6.4.9) we, conversely, will associate a Coxeter scheme to each building.

That the two constructions given by Theorem 6.3.1 and Theorem 6.4.9 are inverse to each other will be the contents of Section 6.5. In fact, Theorem 6.5.2 will show that Coxeter schemes and buildings are equivalent notions.[1] Finally, Theorem 6.6.2 shows that twinned Coxeter sets give rise to twin buildings. Whether each twin building arises from our construction seems to be unknown.

We shall now fix terminology and notation which will be needed in this chapter. In order to do so we fix a set and call it I.

We define $\mathbf{F}(I)$ to be the free monoid over I and denote by $*$ the multiplication of $\mathbf{F}(I)$.

Let i and j be elements in I such that $i \neq j$, let n be a positive integer, and let i_1, …, i_n be elements in I such that, for each element m in $\{1, \ldots, n\}$, $i_m = i$ if m is odd and $i_m = j$ if m is even. We define

$$\mathbf{f}_n(i,j) := i_1 * \cdots * i_n$$

and will use this notation at several instances throughout this chapter.

[1] The reader might argue that this was shown already in [8; Theorem E]. This is true. However, the definition of Coxeter schemes that was suggested in [8; Section 5.1] (and used in the proof of [8; Theorem E]) was not entirely independent from scheme theory, it was based on the free monoid over the generating set of involutions. Of course, the definition from [8] and the one used in the present notes are equivalent.

We denote by λ_I the uniquely determined monoid homomorphism from $\mathbf{F}(I)$ to the additive monoid of the nonnegative integers which sends each element i of I to 1.[2]

Let c be a Coxeter matrix over I (in the sense of [4; 2.11]).

The set of all elements in $\mathbf{F}(I)$ which are reduced with respect to c (in the sense of [5; Section 3.1]) will be denoted by $\mathbf{F}_c(I)$. We shall always speak about c-reduced elements if we mean elements in $\mathbf{F}(I)$ that are reduced with respect to c.

If two elements $\mathbf{e}$ and $\mathbf{f}$ of $\mathbf{F}(I)$ are c-homotopic (in the sense of [5; Section 3.4.1]), we indicate this by writing $\mathbf{e} \sim_c \mathbf{f}$.

Note that $\sim_c$ is an equivalence relation on $\mathbf{F}(I)$. From the definition of $\sim_c$ one obtains that each equivalence class of $\sim_c$ is either subset of $\mathbf{F}_c(I)$ or disjoint from $\mathbf{F}_c(I)$. We define $\mathcal{H}_c(I)$ to be the set of the equivalence classes of $\sim_c$ which are subsets of $\mathbf{F}_c(I)$.

For each element $\mathbf{f}$ in $\mathbf{F}_c(I)$, the equivalence class of $\sim_c$ which contains $\mathbf{f}$ will be denoted by $[\mathbf{f}]_c$.

6.1 Hypergroups and free monoids

In this short section, the letter S stands for a hypergroup.

Let I be a set, and let ρ be a monoid homomorphism from $\mathbf{F}(I)$ to the monoid of all nonempty subsets of $\langle L\rangle$. Assume that $|\rho(i)| = 1$ for each element i in I and that $\rho|_I$ is injective. Define R to be the set of all elements s in S for which there exists an element i in I with $\rho(i) = \{s\}$.

Lemma 6.1.1 *Set $\ell := \ell_R$ and $\lambda = \lambda_I$. Let s be an element in $\langle R\rangle$. Then the following hold.*

(i) *Let $\mathbf{f}$ be an element in $\mathbf{F}(I)$ with $s \in \rho(\mathbf{f})$. Then $\ell(s) \leq \lambda(\mathbf{f})$.*

(ii) *There exists an element $\mathbf{f}$ in $\mathbf{F}(I)$ such that $s \in \rho(\mathbf{f})$ and $\ell(s) = \lambda(\mathbf{f})$.*

Proof. (i) Set $n := \lambda(\mathbf{f})$. There is nothing to show if $n = 0$. Thus, we assume that $1 \leq n$.

From $1 \leq n$ we obtain elements $i_1, \ldots, i_n$ in I such that $\mathbf{f} = i_1 * \cdots * i_n$. Thus, as $s \in \rho(\mathbf{f})$,

$$s \in \rho(i_1 * \cdots * i_n) = \rho(i_1) \cdots \rho(i_n) \subseteq R^n.$$

Thus, $\ell(s) \leq \lambda(\mathbf{f})$.

(ii) Set $n := \ell(s)$. Then there exist elements $r_1, \ldots, r_n$ in R such that

$$s \in r_1 \cdots r_n.$$

[2] The map λ can be thought of as a "length function" on $\mathbf{F}(I)$.

Then there exist elements i_1, …, i_n in I such that $\rho(i_m) = \{r_m\}$ for each element m in $\{1, \ldots, n\}$.

Set $\mathbf{f} := i_1 * \cdots * i_n$. Then

$$s \in \rho(i_1) \cdots \rho(i_n) = \rho(i_1 * \cdots * i_n) = \rho(\mathbf{f})$$

and $\ell(s) = \lambda(\mathbf{f})$. □

6.2 Coxeter sets of hypergroups and free monoids

Throughout this section, the letter S stands for a hypergroup, the letter L for a Coxeter set of S. Instead of ℓ_L we shall write ℓ.

Let c_L denote the restriction of c to $L \times L$.[3] Then c_L is a Coxeter matrix over L. For the remainder of this section, we shall write $\sim_L$ instead of $\sim_{c_L}$.

We denote by ρ_L the uniquely determined monoid homomorphism from $\mathbf{F}(L)$ to the monoid of all nonempty subsets of $\langle L \rangle$ which sends each element l in L to $\{l\}$. Within this section, however, we set $\rho := \rho_L$ so that we have

$$\rho(l) = \{l\}$$

for each element l in L.

Lemma 6.2.1 *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be elements in* $\mathbf{F}(L)$ *such that* $\mathbf{d} \sim_L \mathbf{e}$*. Then* $\rho(\mathbf{d}) = \rho(\mathbf{e})$*.*

Proof. Let h and k be elements in L such that $h \neq k$, assume that $c(h,k)$ is an integer, and set $n := c(h,k)$. Then, by Lemma 2.3.4, $R_n(h,k) \cap R_n(k,h)$ is not empty.[4]

On the other hand, by Lemma 2.4.5, $|R_n(h,k)| = 1$ and $|R_n(k,h)| = 1$. Thus,

$$\rho(\mathbf{f}_n(h,k)) = R_n(h,k) = R_n(k,h) = \rho(\mathbf{f}_n(k,h)).$$

Now the statement follows by induction. □

Recall that λ_L stands for the uniquely determined monoid homomorphism from $\mathbf{F}(L)$ to the additive monoid of the nonnegative integers which sends each element l of L to 1. Within this section, we shall write λ instead of λ_L.

Theorem 6.2.2 *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be elements in* $\mathbf{F}(L)$*. Assume that* $\lambda(\mathbf{d}) = \lambda(\mathbf{e})$ *and that* $\rho(\mathbf{d}) \cap \rho(\mathbf{e})$ *possesses an element* s *with* $\ell(s) = \lambda(\mathbf{d})$*. Then* $\mathbf{d} \sim_L \mathbf{e}$*.*

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[3] Recall from Section 2.3 that, for any two involutions h and k of S, $c(h,k)$ stands for the Coxeter number of h and k.

[4] Recall from Section 2.3 that $R_n(h,k)$ is our notation for the set $l_1 \cdots l_n$ where, for each element i in $\{1, \ldots, n\}$, $l_i = h$ if i is odd and $l_i = k$ if i is even.

Proof. The statement is obviously true if $\lambda(\mathbf{d}) = 0$. Therefore, we assume that $1 \leq \lambda(\mathbf{d})$.

From $1 \leq \lambda(\mathbf{d})$ we obtain elements $\mathbf{d}'$ in $\mathbf{F}(L)$ and h in L such that $\mathbf{d} = \mathbf{d}' * h$. Thus, as $s \in \rho(\mathbf{d})$, $s \in \rho(\mathbf{d}')h$. Thus, $\rho(\mathbf{d}')$ possesses an element p such that $s \in ph$.

Since $p \in \rho(\mathbf{d}')$, $\ell(p) \leq \lambda(\mathbf{d}')$; cf. Lemma 6.1.1(i). Thus, as $\ell(s) = \lambda(\mathbf{d}) = \lambda(\mathbf{d}') + 1$, $\ell(p) \leq \ell(s) - 1$. Since $s \in ph$, $\ell(s) - 1 \leq \ell(p)$. Thus, $\ell(p) = \ell(s) - 1$. Thus, as $s \in ph$, $s \in L_{-1}(h)$.

Let $\mathbf{e}'$ be an element in $\mathbf{F}(L)$, and let k be an element in L such that $\mathbf{e} = \mathbf{e}' * k$. Then, as $s \in \rho(\mathbf{e})$, $s \in \rho(\mathbf{e}')k$. Thus, $\rho(\mathbf{e}')$ possesses an element q such that $s \in qk$. Similar to the reasoning in the previous paragraph, we obtain $s \in L_{-1}(k)$.

Assume that $h = k$. Then, as L is assumed to be constrained, we obtain $p = q$; cf. Lemma 2.4.1. Thus, by induction, $\mathbf{d}' \sim_L \mathbf{e}'$. Thus, as $h = k$, $\mathbf{d} \sim_L \mathbf{e}$, and we are done.

Assume now that $h \neq k$, and set $M := \{h, k\}$. Then, as $s \in L_{-1}(h)$ and $s \in L_{-1}(k)$, $\langle M \rangle$ is finite; cf. Lemma 3.4.1. Thus, by Theorem 3.4.2, $M_{-1}(M)$ contains exactly one element. Staying consistent with the notation of Section 3.4 we denote this element by m_M.

Set $m := m_M$. Then, by Proposition 2.3.6, $\ell_M(m) = c(h, k)$. Now recall from Lemma 3.1.1 that $\ell_M(m) = \ell(m)$. Thus,

$$\ell(m) = c(h, k).$$

Since $s \in L_{-1}(h)$ and $s \in L_{-1}(k)$, $s \in L_{-1}(M)$. Thus, by Theorem 3.1.5, $s \in L_{-1}(\langle M \rangle)$. Thus, as $m \in \langle M \rangle$, $s \in L_{-1}(m)$. Thus, $\langle L \rangle$ possesses an element r such that $s \in rm$ and $\ell(s) = \ell(r) + \ell(m)$.

Since $m \in M_{-1}(h)$, $\langle M \rangle$ possesses an element a such that $m \in ah$ and $\ell(m) = \ell(a) + 1$.

From $s \in rm$, $\ell(s) = \ell(r) + \ell(m)$, $m \in ah$, and $\ell(m) = \ell(a) + 1$ we obtain an element p' in ra such that $s \in p'h$, $\ell(p') = \ell(r) + \ell(a)$, and $\ell(s) = \ell(p') + 1$; cf. Lemma 1.6.3.

Now recall that $s \in ph$ and $\ell(s) = \ell(p)+1$. Thus, as $s \in p'h$ and $\ell(s) = \ell(p')+1$, we must have $p' = p$; cf. Lemma 2.4.1. It follows that

$$p \in ra \quad \text{and} \quad \ell(p) = \ell(r) + \ell(a).$$

Set $\mu := \ell(m)$. Then $\ell(a) = \mu - 1$. Thus, by Lemma 2.2.2, $a \in R_{\mu-1}(h, k)$ or $a \in R_{\mu-1}(k, h)$. It follows that $a \in \rho(\mathbf{f}_{\mu-1}(h, k))$ or $a \in \rho(\mathbf{f}_{\mu-1}(k, h))$. Set

$$\mathbf{b} := \begin{cases} \mathbf{f}_{\mu-1}(h, k) & \text{if } \mu \text{ is odd,} \\ \mathbf{f}_{\mu-1}(k, h) & \text{if } \mu \text{ is even.} \end{cases}$$

Then, as $m \in ah$ and $\ell(m) = \ell(a) + 1$,

$$a \in \rho(\mathbf{b}) \quad \text{and} \quad \ell(a) = \lambda(\mathbf{b}).$$

Now recall that $r \in \langle L \rangle$. Thus, $\mathbf{F}(L)$ possesses an element $\mathbf{a}$ such that

$$r \in \rho(\mathbf{a}) \quad \text{and} \quad \ell(r) = \lambda(\mathbf{a});$$

cf. Lemma 6.1.1(ii).
From $p \in ra$, $r \in \rho(\mathbf{a})$, and $a \in \rho(\mathbf{b})$ we obtain

$$p \in \rho(\mathbf{a})\rho(\mathbf{b}) = \rho(\mathbf{a} * \mathbf{b}).$$

From $\ell(p) = \ell(r) + \ell(a)$, $\ell(r) = \lambda(\mathbf{a})$, and $\ell(a) = \lambda(\mathbf{b})$ we obtain

$$\lambda(\mathbf{d}') = \ell(p) = \lambda(\mathbf{a}) + \lambda(\mathbf{b}) = \lambda(\mathbf{a} * \mathbf{b}).$$

Thus, as $p \in \rho(\mathbf{d}')$, induction yields

$$\mathbf{d}' \sim_L \mathbf{a} * \mathbf{b}.$$

It follows that

$$\mathbf{d} \sim_L \mathbf{a} * \mathbf{b} * h.$$

Similarly,

$$\mathbf{e} \sim_L \mathbf{a} * \mathbf{c} * k,$$

where

$$\mathbf{c} := \begin{cases} \mathbf{f}_{\mu-1}(k,h) \text{ if } \mu \text{ is odd,} \\ \mathbf{f}_{\mu-1}(h,k) \text{ if } \mu \text{ is even.} \end{cases}$$

Now recall that $\mu = c(h,k)$. Thus, $\mathbf{b} * h \sim_L \mathbf{c} * k$. It follows that $\mathbf{d} \sim_L \mathbf{e}$. $\square$

We set

$$\mathbf{F}_c(L) := \mathbf{F}_{c_L}(L).$$

(Recall that c_L is a Coxeter matrix over L.)

Proposition 6.2.3 *Let $\mathbf{f}$ be an element in $\mathbf{F}(L)$. Then the following statements are equivalent.*

(a) *The set $\rho(\mathbf{f})$ possesses an element s such that $\ell(s) = \lambda(\mathbf{f})$.*

(b) *We have $\mathbf{f} \in \mathbf{F}_c(L)$.*

Proof. (a) $\Rightarrow$ (b) Assume that $\mathbf{f} \notin \mathbf{F}_c(L)$. Then there exist elements $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{F}_c(L)$ and l in L such that

$$\mathbf{f} \sim_L \mathbf{a} * l * l * \mathbf{b};$$

cf. [5; Proposition 2(i)]. From $\mathbf{f} \sim_L \mathbf{a} * l * l * \mathbf{b}$ we obtain $\rho(\mathbf{f}) = \rho(\mathbf{a})ll\rho(\mathbf{b})$; cf. Lemma 6.2.1. Thus, for each element s in $\rho(\mathbf{f})$,

$$s \in \rho(\mathbf{a})ll\rho(\mathbf{b}) \subseteq \rho(\mathbf{a})\rho(\mathbf{b}) \cup \rho(\mathbf{a})l\rho(\mathbf{b});$$

cf. 2.1.1(ii). It follows that $\ell(s) \leq \lambda(\mathbf{f}) - 1$ for each element s in $\rho(\mathbf{f})$; cf. Lemma 6.1.1(i).

(b) $\Rightarrow$ (a) Assume the statement to be false. Among the elements in $\mathbf{F}_c(L)$ violating the implication we fix $\mathbf{f}$ such that $\lambda(\mathbf{f})$ is as small as possible.

The choice of $\mathbf{f}$ forces $1 \leq \lambda(\mathbf{f})$. Thus, there exist elements $\mathbf{e}$ in $\mathbf{F}_c(L)$ and l in L such that $\mathbf{f} = \mathbf{e} * l$. From $\mathbf{f} = \mathbf{e} * l$ we obtain $\lambda(\mathbf{e}) = \lambda(\mathbf{f}) - 1$. Thus, the minimal choice of $\mathbf{f}$ yields an element q in $\rho(\mathbf{e})$ such that $\ell(q) = \lambda(\mathbf{e})$.

Assume that $q \in L_1(l)$. Then, by definition, ql possesses an element s such that $\ell(s) = \ell(q) + 1$. Thus, as $\ell(q) = \lambda(\mathbf{f}) - 1$, $\ell(s) = \lambda(\mathbf{f})$. On the other hand, $s \in ql \subseteq \rho(\mathbf{e})l = \rho(\mathbf{e} * l) = \rho(\mathbf{f})$, contrary to the choice of $\mathbf{f}$.

Assume that $q \notin L_1(l)$. Then, by Lemma 2.5.2, $q \in L_{-1}(l)$. Thus, $\langle L \rangle$ possesses an element p such that $q \in pl$ and $\ell(q) = \ell(p)+1$. Since $p \in \langle L \rangle$, there exists an element $\mathbf{d}$ in $\mathbf{F}_c(L)$ such that $p \in \rho(\mathbf{d})$ and $\ell(p) = \lambda(\mathbf{d})$; cf. Lemma 6.1.1(ii). Then $\lambda(\mathbf{d} * l) = \lambda(\mathbf{f})$ and

$$q \in \rho(\mathbf{d} * l) \cap \rho(\mathbf{e}).$$

Thus, by Theorem 6.2.2, $\mathbf{d} * l \sim_L \mathbf{e}$. Thus, as $\mathbf{f} = \mathbf{e} * l$, $\mathbf{f} \notin \mathbf{F}_c(L)$, contrary to the choice of $\mathbf{f}$. □

Corollary 6.2.4 *Let* $\mathbf{f}$ *be an element in* $\mathbf{F}_c(L)$*. Then* $|\rho(\mathbf{f})| = 1$.

PROOF. Since we have $\rho(l) = \{l\}$ for each element l in L, we may assume that $\mathbf{f} \notin L$.

Since $\mathbf{f} \in \mathbf{F}_c(L)$, $\rho(\mathbf{f})$ possesses an element q such that $\ell(q) = \lambda(\mathbf{f})$; cf. Proposition 6.2.3. Since $\mathbf{f} \notin L$, there exist elements $\mathbf{e}$ in $\mathbf{F}_c(L)$ and l in L such that $\mathbf{f} = \mathbf{e} * l$. From $q \in \rho(\mathbf{f})$ and $\mathbf{f} = \mathbf{e} * l$ we obtain $q \in \rho(\mathbf{e})l$.

By induction (on $\lambda(\mathbf{f})$), $\rho(\mathbf{e})$ possesses an element p such that $\rho(\mathbf{e}) = \{p\}$. Thus, as $q \in \rho(\mathbf{e})l$, $q \in pl$.

From $p \in \rho(\mathbf{e})$ we also obtain $\ell(p) \leq \lambda(\mathbf{e})$; cf. Lemma 6.1.1(i). Thus, as $\ell(q) = \lambda(\mathbf{f})$ and $q \in pl$, $p \in L_1(l)$. Thus, as L is assumed to be constrained, $q \in pl$ yields $pl = \{q\}$. Thus, as $\rho(\mathbf{e}) = \{p\}$, $\rho(\mathbf{f}) = \{q\}$. □

The following result is a partial converse of Lemma 6.2.1.

Corollary 6.2.5 *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be elements in* $\mathbf{F}_c(L)$ *such that* $\rho(\mathbf{d}) = \rho(\mathbf{e})$*. Then* $\mathbf{d} \sim_L \mathbf{e}$*.*

PROOF. Since $\mathbf{d} \in \mathbf{F}_c(L)$, $\rho(\mathbf{d})$ possesses an element p such that $\ell(p) = \lambda(\mathbf{d})$; cf. Proposition 6.2.3. Similarly, $\lambda(\mathbf{e})$ possesses an element q such that $\ell(q) = \lambda(\mathbf{e})$.

On the other hand, by Corollary 6.2.4, $|\rho(\mathbf{d})| = 1$ and $|\rho(\mathbf{e})| = 1$. Thus, $p = q$. It follows that $\lambda(\mathbf{d}) = \ell(p) = \ell(q) = \lambda(\mathbf{e})$ and, by Theorem 6.2.2, $\mathbf{d} \sim_L \mathbf{e}$. □

For the remainder of this section, we set

$$\mathcal{H}(L) := \mathcal{H}_{c_L}(L).$$

Thus, $\mathcal{H}(L)$ is the set of all equivalence classes of $\sim_L$ which are subsets of $\mathbf{F}_c(L)$.

We also set

$$[\mathbf{f}] := [\mathbf{f}]_{c_L}$$

for each element $\mathbf{f}$ in $\mathbf{F}_c(L)$. Thus, for each element $\mathbf{f}$ in $\mathbf{F}_c(L)$, $[\mathbf{f}]$ is the equivalence class of $\sim_L$ which contains $\mathbf{f}$.

We are now in the position to prove the main result of this section. Recall that the letter L stands for a Coxeter set of the hypergroup S.

Theorem 6.2.6 *There exists a bijective map ω from $\langle L\rangle$ to $\mathcal{H}(L)$ such that, for any two elements s in $\langle L\rangle$ and $\mathbf{f}$ in $\mathbf{F}_c(L)$, $\omega(s) = [\mathbf{f}]$ if and only if $s \in \rho(\mathbf{f})$.*

Proof. Let s be an element in $\langle L\rangle$. Then $\mathbf{F}_c(L)$ possesses an element $\mathbf{f}$ such that $s \in \rho(\mathbf{f})$; cf. Lemma 6.1.1(ii) and Proposition 6.2.3. We define

$$\omega(s) := [\mathbf{f}].$$

Let $\mathbf{d}$ and $\mathbf{e}$ be elements in $\mathbf{F}_c(L)$ such that $s \in \rho(\mathbf{d})$ and $s \in \rho(\mathbf{e})$. Then, by Corollary 6.2.4, $\rho(\mathbf{d}) = \rho(\mathbf{e})$. Thus, by Corollary 6.2.5, $\mathbf{d} \sim_L \mathbf{e}$. This shows that ω is well defined.

That ω is surjective is obvious. That ω is injective follows from Corollary 6.2.4.

That $\omega(s) = [\mathbf{f}]$ if and only if $s \in \rho(\mathbf{f})$ for any two elements s in $\langle L\rangle$ and $\mathbf{f}$ in $\mathbf{F}_c(L)$ follows right from the definition of ω. □

Since there is a natural bijection between $\mathcal{H}(L)$ and the Coxeter group of type c_L, Theorem 6.2.6 establishes a bijection between Coxeter sets of hypergroups and Coxeter groups.

The bijective map ω which was established in Theorem 6.2.6 will be called the *signature* of L.

6.3 From Coxeter schemes to buildings

It is the purpose of this section to associate to each Coxeter scheme a building.

First we give the definition of a Coxeter scheme.

Let S be a scheme, and let L be a Coxeter set of S. The scheme S is called a *Coxeter scheme with respect to L* if $S = \langle L\rangle$.

And here is the definition of a building. We take it from [7].

Let c be a Coxeter matrix with vertex set I, and let (W, r) be a Coxeter system of type c. A *building of type* c is a pair (X, δ), where X is a chamber system whose index set is I and δ is a function from $X \times X$ to W, such that

$$\delta(y, z) = r_{\mathbf{f}} \quad \Leftrightarrow \quad \text{there is a gallery in } X \text{ of type } \mathbf{f} \text{ from } y \text{ to } z$$

for each c-reduced word $\mathbf{f}$ in $\mathbf{F}(I)$ and for each ordered pair (y, z) of chambers.[5]

A chamber system X is called *regular* if any two equivalence classes of a given equivalence relation of X have the same cardinality. For the remainder of these notes, all buildings are assumed to be regular.

Let X be a set, let S be a scheme on X, let L be a set of involutions of S, and assume that S is a Coxeter scheme with respect to L. We shall now associate a building to S.

Let l be an element in L. Then, by Lemma 2.1.1(i), l is a symmetric binary relation and, by Lemma 2.1.1(ii), l is transitive. Thus, $1_X \cup l$ is an equivalence relation. It follows that

$$(X, \{1_X \cup l \mid l \in L\})$$

is a regular chamber system (whose index set is L).

Let (W, r) be a Coxeter system of type c_L.[6]

We denote by ψ the signature of $\{r_l \mid l \in L\}$ and by ω the signature of L. For any two elements y and z in X, we define $\sigma(y, z)$ to be the uniquely determined element s in S satisfying $(y, z) \in s$. Then

$$\psi^{-1} \circ \omega \circ \sigma$$

is a map from $X \times X$ to W. It will be called the *distance function associated to* the Coxeter set L.

Theorem 6.3.1 *Let X be a set, let S be a scheme on X, let L be a set of involutions of S such that S is a Coxeter scheme with respect to L, and let δ denote the distance function associated to L. Then (X, δ) is a building of type c_L.*

Proof. Let (W, r) be a Coxeter system of type c_L, let ψ denote the signature of the set $\{r_l \mid l \in L\}$, and let ω denote the signature of L. For any two elements

[5] Here r stands for the uniquely determined monoid homomorphism from $\mathbf{F}(I)$ to W which sends each element i of I to r_i. - Note that, in contrast to [7], where chamber systems are viewed as graphs, we consider chamber systems to be sets endowed with a collection of equivalence relations. Since we adopt the notion of a chamber system from [7], each equivalence class of each equivalence relation of a chamber system has at least two elements. - Note also that this definition does not distinguish between a chamber system and the underlying set of a chamber system. We shall have to take that into account in our further consideration.

[6] Recall that c_L is a Coxeter matrix over L.

y and z in X, define $\sigma(y,z)$ to be the uniquely determined element s in S satisfying $(y,z) \in s$. Then, as δ is the distance function associated to L, we have

$$\delta = \psi^{-1} \circ \omega \circ \sigma.$$

Let y and z be elements in X, and let $\mathbf{f}$ be a c_L-reduced element in $\mathbf{F}(L)$. We have to show that

$$\delta(y,z) = r_{\mathbf{f}} \quad \Leftrightarrow \quad \text{there is a gallery in } X \text{ of type } \mathbf{f} \text{ from } y \text{ to } z.$$

Set $s := \sigma(y,z)$. Then

$$\delta(y,z) = (\psi^{-1} \circ \omega \circ \sigma)(y,z) = \psi^{-1}(\omega(\sigma(y,z))) = \psi^{-1}(\omega(s)).$$

Thus, setting $[\mathbf{f}] := [\mathbf{f}]_{c_L}$, we have

$$\delta(y,z) = r_{\mathbf{f}} \quad \Leftrightarrow \quad \psi^{-1}(\omega(s)) = r_{\mathbf{f}} \quad \Leftrightarrow \quad \omega(s) = [\mathbf{f}].$$

Following our notation of Section 6.2 we now denote by ρ_L the uniquely determined monoid homomorphism from $\mathbf{F}(L)$ to the monoid of all nonempty subsets of $\langle L \rangle$ which sends each element l in L to $\{l\}$. Set $\rho := \rho_L$. Then, by Theorem 6.2.6,

$$\omega(s) = [\mathbf{f}] \quad \Leftrightarrow \quad s \in \rho(\mathbf{f}).$$

Since $(y,z) \in s$, this last statement means that there is a gallery in X of type $\mathbf{f}$ from y to z.

That (X,δ) has type c_L follows from the definition of δ. □

The building (X,δ) which was constructed in Theorem 6.3.1 is called the *building associated to* the Coxeter scheme S.

6.4 From buildings to Coxeter schemes

In this section, we associate to each building a Coxeter scheme.

Let (X,δ) be a (regular) building. We denote by c the type of (X,δ) and by I the vertex set of c. Then, by definition, there exists a Coxeter system (W,r) of type c such that δ is a map from $X \times X$ to W satisfying

$$\delta(y,z) = r_{\mathbf{f}} \quad \Leftrightarrow \quad \text{there is a gallery in } X \text{ of type } \mathbf{f} \text{ from } y \text{ to } z$$

for any three elements y and z in X and $\mathbf{f}$ in $\mathbf{F}_c(I)$.

For each element w in W, we set

$$s_w := \{(y,z) \in X \times X \mid \delta(y,z) = w\}.$$

We define

$$S := \{s_w \mid w \in W\}.$$

Note that, for each element w in W, s_w is the set of all preimages of w under δ. Thus, $w \mapsto s_w$ is a bijective map from W to S.

Proposition 6.4.1 *The set S is a scheme on X.*

Proof. For any two elements y and z in X, we have $(y,z) \in s_{\delta(y,z)}$. For any four elements u and v in W and y and z in X with $(y,z) \in s_u \cap s_v$, we have $u = \delta(y,z) = v$ and then $s_u = s_v$. Thus, S is a partition of $X \times X$.

Note also that $s_{1_W} = 1_X$ (where 1_W is the identity of the group W and 1_X the identity on X) and that $s_{w^{-1}} = (s_w)^*$ for each element w in W. Thus, in order to show that S is a scheme, we just have to verify the regularity condition for S.

Let u, v, and w be elements in W, let y be an element in X, let z be an element in ys_w, and set

$$A := ys_u \cap zs_{v^*}.$$

We have to show that the cardinality a of A does not depend on y or z.

Let $\mathbf{d}$ be an element in $\mathbf{F}_c(I)$ such that $u = r_{\mathbf{d}}$, let $\mathbf{e}$ be an element in $\mathbf{F}_c(I)$ such that $v = r_{\mathbf{e}}$, and let $\mathbf{f}$ be an element in $\mathbf{F}_c(I)$ such that $w = r_{\mathbf{f}}$. By induction, we may assume that $\mathbf{e} \in I$. We set $i := \mathbf{e}$.

Let x be an element in A. From $x \in ys_u$ and $u = r_{\mathbf{d}}$ we obtain a gallery of type $\mathbf{d}$ from y to x. Moreover, as $\mathbf{d} \in \mathbf{F}_c(I)$, there exists at most one such gallery; cf. [5; Theorem 2]. From $z \in xs_v$ and $v = r_{\mathbf{i}}$ we obtain a gallery of type i from x to z. This shows that a is equal to the number of galleries of type $\mathbf{d} * i$ from y to z.

Assume, firstly, that $\mathbf{d} * i \in \mathbf{F}_c(I)$. Then, by [5; Theorem 2], there is at most one gallery of type $\mathbf{d} * i$ from y to z. This means that $a = 0$ if $\mathbf{d} * i$ and $\mathbf{f}$ are not c-homotopic and $a = 1$ if they are. Thus, we are done in this case.

Assume, secondly, that $\mathbf{f} * i \in \mathbf{F}_c(I)$. Assume that $a \neq 0$. From $x \in ys_u$ and $u = r_{\mathbf{d}}$ we obtain a gallery of type $\mathbf{d}$ from y to x. From $z \in ys_w$, $w = r_{\mathbf{f}}$, $z \in xs_v$, and $v = r_i$ we obtain a gallery of type $\mathbf{f} * i$ from y to x. Thus, as $\mathbf{d}$ and $\mathbf{f} * i$ both are c-reduced, $\mathbf{d} \sim_c \mathbf{f} * i$.

Thus, for each element x in zs_v, there exists a gallery of type $\mathbf{d}$ from y to x, and that means that $x \in ys_u$. Thus, a is equal to the size of the i-equivalence classes minus 1.

Assume, finally, that

$$\mathbf{d} * i \notin \mathbf{F}_c(I) \quad \text{and} \quad \mathbf{f} * i \notin \mathbf{F}_c(I).$$

From $\mathbf{d} \in \mathbf{F}_c(I)$ and $\mathbf{d} * i \notin \mathbf{F}_c(I)$ we obtain an element $\mathbf{d}'$ in $\mathbf{F}_c(I)$ such that $\mathbf{d} \sim_c \mathbf{d}' * i$; cf. [4]. Similarly, as $\mathbf{f} \in \mathbf{F}_c(I)$ and $\mathbf{f} * i \notin \mathbf{F}_c(I)$, there exists an element $\mathbf{f}'$ in $\mathbf{F}_c(I)$ such that $\mathbf{f} \sim_c \mathbf{f}' * i$.

Assume $a \neq 0$. From $x \in ys_u$ and $u = r_{\mathbf{d}}$ we obtain a gallery of type $\mathbf{d}' * i$ from y to x. Thus, there exists an element x' in X with a gallery of type $\mathbf{d}'$ from y to x' and a gallery of type i from x' to x. Similarly, there exists an element z' in X with a gallery of type $\mathbf{f}'$ from y to z' and a gallery of type i from z' to z.

The two galleries of type i yield a gallery of type $i * i$ from x' to z (via x). Thus, $x' = z$ or there is a gallery of type i from x' to z.

Assume that $x' = z$. Then there exist galleries of type $\mathbf{d}'$ and of type $\mathbf{f}$ from y to z. Thus, $\mathbf{d}' \sim_c \mathbf{f}$, and from this we obtain

$$\mathbf{d} \sim_c \mathbf{d}' * i \sim_c \mathbf{f} * i \sim_c \mathbf{f}' * i * i,$$

contrary to $\mathbf{d} \in \mathbf{F}_c(I)$.

Thus, there is a gallery of type i from x' to z. Thus, there exist galleries of type $\mathbf{d}$ and of type $\mathbf{f}$ from y to z. It follows that $\mathbf{d} \sim_c \mathbf{f}$, and that implies that $\mathbf{d}' \sim_c \mathbf{f}'$. Since $\mathbf{d}$ and $\mathbf{f}$ both are reduced, this implies $x' = z'$. Thus, A consists of exactly the elements in the i-equivalence class of z that are different from x' and z. Thus, a is equal to the size of the i-equivalence classes minus 2. □

We set

$$L := \{s_{r_i} \mid i \in I\}.$$

Then we have the following.

Lemma 6.4.2 *The set L consists of involutions of S.*

Proof. The set X is assumed to be a chamber system with index set I. Thus, for each element i in I, $1_X \cup s_{r_i}$ is an equivalence relation.[7] Thus, for each element i in I, s_{r_i} is an involution. □

We define ρ_I to be the uniquely determined monoid homomorphism from $\mathbf{F}(I)$ to the monoid of all nonempty subsets of S which sends each element i of I to $\{s_{r_i}\}$. For the remainder, however, of this section, we set $\rho := \rho_I$. Thus, we have

$$\rho(i) = \{s_{r_i}\}$$

for each element i in I.

Lemma 6.4.3 *Let y and z be elements in X, and let $\mathbf{f}$ be an element in $\mathbf{F}(I)$. Then $z \in y\rho(\mathbf{f})$ if and only if there exists a gallery in X of type $\mathbf{f}$ from y to z.*

Proof. We may assume that $\mathbf{f}$ is not the neutral element of $\mathbf{F}(I)$. Thus, there exist elements $i_1, \ldots, i_n$ in I such that $\mathbf{f} = i_1 * \ldots * i_n$. Thus,

[7] By 1_X we mean the identity on X.

$$\rho(\mathbf{f}) = \rho(i_1 * \ldots * i_n) = \rho(i_1)\ldots\rho(i_n) = s_{r_{i_1}} \ldots s_{r_{i_n}}.$$

Thus, we have $z \in y\rho(\mathbf{f})$ if and only if $z \in y(s_{r_{i_1}} \ldots s_{r_{i_n}})$, and this means that there exist elements x_0, x_1, …, x_n in X such that $y = x_0$, $z = x_n$, and $x_m \in x_{m-1}s_{r_{i_m}}$ for each element m in $\{1, \ldots, n\}$. This latter condition says that there exists a gallery in X of type $\mathbf{f}$ from y to z. □

Lemma 6.4.4 *We have $S = \langle L \rangle$.*

Proof. Let s be an element in S. Then there exists an element w in W such that $s = s_w$. Let y be an element in X, and let z be an element in ys. Then $z \in ys_w$. Thus, by definition, $\delta(y, z) = w$.

Let $\mathbf{f}$ be an element in $\mathbf{F}_c(I)$ such that $w = r_{\mathbf{f}}$. Then, as $\delta(y, z) = w$, $\delta(y, z) = r_{\mathbf{f}}$. Thus, by definition, there is a gallery in X of type $\mathbf{f}$ from y to z. Thus, by Lemma 6.4.3, $z \in y\rho(\mathbf{f})$.

Let i_1, …, i_n be elements in I such that $\mathbf{f} = i_1 * \cdots * i_n$. Then, as $z \in y\rho(\mathbf{f})$,

$$z \in y\rho(i_1 * \cdots * i_n) = y(\rho(i_1)\cdots\rho(i_n)) = y(s_{r_{i_1}} \cdots s_{r_{i_n}}).$$

Thus, as $z \in ys$, $s \in s_{r_{i_1}} \cdots s_{r_{i_n}}$.

Since s has been chosen arbitrarily from S, the claim now follows from Lemma 1.6.1. □

Lemma 6.4.5 *Let $\mathbf{f}$ be an element in $\mathbf{F}_c(I)$. Then $\rho(\mathbf{f}) = \{s_{r_{\mathbf{f}}}\}$.*

Proof. Let y and z be elements in X. Then $z \in y\rho(\mathbf{f})$ if and only if there is a gallery in X of type $\mathbf{f}$ from y to z; cf. Lemma 6.4.3. Thus,

$$z \in ys_{r_{\mathbf{f}}} \quad \Leftrightarrow \quad \delta(y, z) = r_{\mathbf{f}} \quad \Leftrightarrow \quad z \in y\rho(\mathbf{f}).$$

It follows that $\rho(\mathbf{f}) = \{s_{r_{\mathbf{f}}}\}$. □

Recall that λ_I is our notation for the uniquely determined monoid homomorphism from $\mathbf{F}(I)$ to the additive monoid of the nonnegative integers which sends each element i of I to 1. For the remainder, however, of this section, we set $\lambda := \lambda_I$. Thus, we have

$$\lambda(i) = 1$$

for each element i in I.

For the remainder of this section, we set $\ell := \ell_L$.

Lemma 6.4.6 *Let $\mathbf{f}$ be an element in $\mathbf{F}(I)$. Then the following hold.*

(i) *Assume that $\rho(\mathbf{f})$ possesses an element s such that $\ell(s) = \lambda(\mathbf{f})$. Then $\mathbf{f} \in \mathbf{F}_c(I)$.*

(ii) *Let s be an element in $\rho(\mathbf{f})$, and assume that $\mathbf{F}_c(I)$. Then $\ell(s) = \lambda(\mathbf{f})$.*

Proof. (i) Assume, by way of contradiction, that $\mathbf{f} \notin \mathbf{F}_c(I)$. Then there exist elements $\mathbf{d}$ and $\mathbf{e}$ in $\mathbf{F}(I)$ and i in I such that

$$\mathbf{f} \sim_c \mathbf{d} * i * i * \mathbf{e}.$$

Thus, by Lemma 6.4.3,

$$\rho(\mathbf{f}) = \rho(\mathbf{d} * i * i * \mathbf{e}).$$

Thus, as $s \in \rho(\mathbf{f})$,

$$s \in \rho(\mathbf{d} * i * i * \mathbf{e}) = \rho(\mathbf{d})\rho(i)\rho(i)\rho(\mathbf{e}).$$

Thus, there exists elements p in $\rho(\mathbf{d})$ and q in $\rho(\mathbf{e})$ such that

$$s \in ps_{r_i}s_{r_i}q \subseteq pq \cup ps_{r_i}q.$$

From $p \in \rho(\mathbf{d})$ we obtain $\ell(p) \leq \lambda(\mathbf{d})$; cf. Lemma 6.1.1(i). Similarly, $q \in \rho(\mathbf{e})$ yields $\ell(q) \leq \lambda(\mathbf{e})$. Thus, $\ell(s) \leq \lambda(\mathbf{f}) - 1$, contradiction.

(ii) From Lemma 6.1.1(ii) we know that there exists an element $\mathbf{e}$ in $\mathbf{F}(I)$ with $s \in \rho(\mathbf{e})$ and $\ell(s) = \lambda(\mathbf{e})$.

From $s \in \rho(\mathbf{e})$ and $\ell(s) = \lambda(\mathbf{e})$ we now obtain $\mathbf{e} \in \mathbf{F}_c(I)$; cf. (i). Thus, by Lemma 6.4.5, $\rho(\mathbf{e}) = \{s\}$. From Lemma 6.4.5 we also know that $\rho(\mathbf{f}) = \{s\}$. Thus, we have $\rho(\mathbf{e}) = \rho(\mathbf{f})$, so that, by Lemma 6.4.3, $\mathbf{e} \sim_c \mathbf{f}$. In particular, $\lambda(\mathbf{e}) = \lambda(\mathbf{f})$. Thus, as $\ell(s) = \lambda(\mathbf{e})$, $\ell(s) = \lambda(\mathbf{f})$. □

Proposition 6.4.7 *The set L is constrained.*

Proof. Let q be an element in S, and let p be an element in $L_1(q)$. We shall be done if we succeed in showing that $|pq| = 1$.

From $p \in L_1(q)$ we obtain an element r in pq such that $\ell(r) = \ell(p) + \ell(q)$.

Let $\mathbf{c}$ be an element in $\mathbf{F}_c(I)$ such that $p \in \rho(\mathbf{c})$ and $\ell(p) = \lambda(\mathbf{c})$, let $\mathbf{d}$ be an element in $\mathbf{F}_c(I)$ such that $q \in \rho(\mathbf{d})$ and $\ell(q) = \lambda(\mathbf{d})$; cf. Lemma 6.1.1(ii). Then

$$r \in pq \subseteq \rho(\mathbf{c})\rho(\mathbf{d}) = \rho(\mathbf{c} * \mathbf{d})$$

and

$$\ell(r) = \ell(p) + \ell(q) = \lambda(\mathbf{c}) + \lambda(\mathbf{d}) = \lambda(\mathbf{c} * \mathbf{d}).$$

Thus, by Lemma 6.4.6(i), $\mathbf{c} * \mathbf{d} \in \mathbf{F}_c(I)$. It follows that $\rho(\mathbf{c} * \mathbf{d}) = \{s_{r_{\mathbf{c}*\mathbf{d}}}\}$; cf. Lemma 6.4.5. Thus, as $pq \subseteq \rho(\mathbf{c} * \mathbf{d})$, $|pq| = 1$. □

Proposition 6.4.8 *The set L satisfies the exchange condition.*

Proof. Let k be an element in L, let s be an element in $L_1(k)$, and let h be an element in L with $h \in L_1(s)$. We have to show that

$$hs = sk \quad \text{or} \quad hs \subseteq L_1(k).$$

Let $\mathbf{f}$ be an element in $\mathbf{F}_c(L)$ such that $s \in \rho(\mathbf{f})$ and $\ell(s) = \lambda(\mathbf{f})$; cf. Lemma 6.1.1(ii). From $\mathbf{f} \in \mathbf{F}_c(L)$ and $s \in \rho(\mathbf{f})$ we obtain $\rho(\mathbf{f}) = \{s\}$; cf. Lemma 6.4.5. Let j be an element in I such that $k = s_{r_j}$. Then

$$sk = \rho(\mathbf{f})\rho(j) = \rho(\mathbf{f} * j).$$

Thus, as $s \in L_1(k)$, $\mathbf{f} * j \in \mathbf{F}_c(I)$; cf. Lemma 6.4.6(i).

Similarly, I possesses an element i such that $h = s_{r_i}$,

$$hs = \rho(i * \mathbf{f}),$$

and $i * \mathbf{f} \in \mathbf{F}_c(I)$.

From $\mathbf{f} * j \in \mathbf{F}_c(I)$ and $i * \mathbf{f} \in \mathbf{F}_c(I)$ we obtain

$$i * \mathbf{f} \sim_c \mathbf{f} * j \quad \text{or} \quad i * \mathbf{f} * j \in \mathbf{F}_c(I);$$

cf. [1].

Assume that $i * \mathbf{f} \sim_c \mathbf{f} * j$. Then, by Lemma 6.4.3, $\rho(i * \mathbf{f}) = \rho(\mathbf{f} * j)$. Thus, as $hs = \rho(i * \mathbf{f})$ and $sk = \rho(\mathbf{f} * j)$, $hs = sk$, and we are done.

Assume that $i * \mathbf{f} * j \in \mathbf{F}_c(I)$, and let t be an element in hsk. Then, by Lemma 6.4.6(ii),

$$\ell(t) = \lambda(i * \mathbf{f} * j) = \lambda(i) + \lambda(\mathbf{f}) + \lambda(j) = 1 + \lambda(\mathbf{f}) + 1 = \ell(s) + 2,$$

and we are done. □

Theorem 6.4.9 *The set S is a Coxeter scheme with respect to L.*

Proof. From Lemma 6.4.2 we know that the elements in L are involutions, from Lemma 6.4.4 that $S = \langle L \rangle$. From Proposition 6.4.7 we know that L is constrained, from Proposition 6.4.8 that L satisfies the exchange condition. □

The scheme S is called the *Coxeter scheme associated to* the building (X, δ), the set L of involutions of S is called the *Coxeter set associated to* (X, δ).

Let i and j be elements in I such that $i \neq j$, let n be a positive integer, and let $i_1, \ldots, i_n$ be elements in I such that, for each element m in $\{1, \ldots, n\}$, $i_m = i$ if m is odd and $i_m = j$ if m is even. Recall from Section 2.3 that

$$R_n(s_{r_i}, s_{r_j}) := s_{r_{i_1}} \cdots s_{r_{i_n}}.$$

Recall also that

$$\mathbf{f}_n(i, j) := i_1 * \cdots * i_n.$$

Note that

$$\rho(\mathbf{f}_n(i, j)) = R_n(s_{r_i}, s_{r_j})$$

for any two elements i and j in I with $i \neq j$.

Recall now that c stands for the type of the building (X, δ).

Lemma 6.4.11 *We have $c_L = c$.*

Proof. Let i and j be elements in I such that $i \neq j$, and let n be a positive integer. Then, by Lemma 6.4.3,

$$\mathbf{f}_n(i,j) \sim_c \mathbf{f}_n(j,i)$$

if and only if

$$\rho(\mathbf{f}_n(i,j)) = \rho(\mathbf{f}_n(j,i)).$$

Since $\rho(\mathbf{f}_n(i,j)) = R_n(s_{r_i}, s_{r_j})$ and $\rho(\mathbf{f}_n(j,i)) = R_n(s_{r_j}, s_{r_i})$, this latter equation is equivalent to

$$R_n(s_{r_i}, s_{r_j}) = R_n(s_{r_j}, s_{r_i}).$$

Setting $1 := 1_X$, we obtain that $R_n(s_{r_i}, s_{r_j}) = R_n(s_{r_j}, s_{r_i})$ is equivalent to

$$1 \in R_n(s_{r_i}, s_{r_j}) R_n(s_{r_j}, s_{r_i})^* = (s_{r_i} s_{r_j})^n.$$

Thus, $C(s_{r_i}, s_{r_j})$ is not empty if and only $c(i,j)$ is an integer, and, if that is the case, Lemma 2.3.2 yields $c_L(s_{r_i}, s_{r_j}) = c(i,j)$. □

6.5 Coxeter schemes and buildings

In this section, we shall see that Coxeter schemes and buildings are the same thing.

Theorem 6.5.1 *Let X be a set, let S be a scheme on X, let L be a set of involutions of S such that S is a Coxeter scheme with respect to L. Let δ be the distance function associated to L, let S_δ be the Coxeter scheme associated to the building (X, δ), and let L_δ be the Coxeter set associated to (X, δ). Then $S = S_\delta$ and $L = L_\delta$.*

Proof. We first prove that that $L = L_\delta$.

Let (W, r) be a Coxeter system of type c_L, let ψ be the signature of $\{r_l \mid l \in L\}$, and let ω be the signature of L.[8] For any two elements y and z in X, we define $\sigma(y,z)$ to be the uniquely determined element s in S satisfying $(y,z) \in s$. Then, as δ is the distance function associated to L,

$$\delta = \psi^{-1} \circ \omega \circ \sigma.$$

For each element l in L, we have $\rho_L(l) = \{l\}$. Thus, $\omega(l) = [l]_{c_L}$ and $\psi(l) = [l]_{c_L}$. It follows that

$$\psi^{-1}(\omega(l)) = \psi^{-1}([l]_{c_L}) = r_l.$$

[8] Recall that c_L is a Coxeter matrix over L.

Note also that

$$\delta(y,z) = (\psi^{-1} \circ \omega \circ \sigma)(y,z) = \psi^{-1}(\omega(\sigma(y,z)))$$

for any two elements y and z in X. Thus, as ψ and ω are bijective, we have

$$\sigma(y,z) = l \quad \Leftrightarrow \quad \delta(y,z) = r_l$$

for any three elements y, z in X and l in L. However, we also have

$$(y,z) \in l \quad \Leftrightarrow \quad \sigma(y,z) = l$$

and

$$(y,z) \in s_{r_l} \quad \Leftrightarrow \quad \delta(y,z) = r_l$$

for any three elements y, z in X and l in L. Thus, $l = s_{r_l}$ for each element l in L. Thus, as $L_\delta := \{s_{r_l} \mid l \in L\}$, this yields $L = L_\delta$.

Now we prove that $S = S_\delta$.

We set $1 := 1_X$ and fix an element s in S. We shall see that $s \in S_\delta$.

If $s = 1$, $s \in S_\delta$. Assume that $s \neq 1$. Then, by Lemma 1.6.2, there exists elements r in S and l in L such that $s \in rl$ and $\ell_L(s) = \ell_L(r) + 1$. Since L is constrained, this implies $rl = \{s\}$. By induction, we may assume that $r \in S_\delta$. Thus, as $l \in L_\delta$, there exists an element s' in S_δ such that $s' \in rl$.

Since $L = L_\delta$, $\ell_L = \ell_{L_\delta}$. Thus, as $\ell_L(s) = \ell_L(r) + 1$, $\ell_{L_\delta}(s') = \ell_{L_\delta}(r) + 1$. Now recall from Theorem 6.4.9 that S_δ is a Coxeter scheme. Thus, as $s' \in rl$, $rl = \{s'\}$. Thus, as $rl = \{s\}$, $s = s' \in S_\delta$. □

Theorem 6.5.2 *Let (X, δ) be a building. Let L be the Coxeter set associated to (X, δ), and let δ_L denote the distance function associated to L. Then $\delta = \delta_L$.*

Proof. Let c denote the type of the building (X, δ), and let I denote the vertex set of c. Then, by definition, there exists a Coxeter system (W, r) of type c such that δ is a map from $X \times X$ to W satisfying

$$\delta(y,z) = r_{\mathbf{f}} \quad \Leftrightarrow \quad \text{there is a gallery in } X \text{ of type } \mathbf{f} \text{ from } y \text{ to } z$$

for any three elements y and z in X and $\mathbf{f}$ in $\mathbf{F}_c(I)$.

For each element w in W, we set

$$s_w := \{(y,z) \in X \times X \mid \delta(y,z) = w\}.$$

Then, as L is the Coxeter set associated to (X, δ),

$$L = \{s_{r_i} \mid i \in I\}.$$

Let (W, r) be a Coxeter system of type c_L, let ψ be the signature of $\{r_l \mid l \in L\}$, and let ω be the signature of L.[9] For any two elements y and z in X, we define

[9] Recall that c_L is a Coxeter matrix over L.

$\sigma(y,z)$ to be the uniquely determined element s in S satisfying $(y,z) \in s$. Then, as δ_L is the distance function associated to L, we have

$$\delta_L = \psi^{-1} \circ \omega \circ \sigma.$$

Let y and z be elements in X, and set $w := \delta(y,z)$. We shall be done if we succeed in showing that $\delta_L(y,z) = w$.

From $\delta(y,z) = w$ we obtain $(y,z) \in s_w$. Thus, $\sigma(y,z) = s_w$. Thus

$$\delta_L(y,z) = (\psi^{-1} \circ \omega \circ \sigma)(y,z) = \psi^{-1}(\omega(\sigma(y,z))) = \psi^{-1}(\omega(s_w)).$$

Let $\mathbf{f}$ be an element in $\mathbf{F}_c(L)$ such that $\omega(s_w) = [\mathbf{f}]_{c_L}$. Then, as ω is the signature of L, $\{s_w\} = \rho_L(\mathbf{f})$.

On the other hand, as $\delta_L(y,z) = \psi^{-1}(\omega(s_w))$ and $\omega(s_w) = [\mathbf{f}]_{c_L}$, $\delta_L(y,z) = \psi^{-1}([\mathbf{f}]_{c_L}) = r_{\mathbf{f}}$.

From $z \in ys_w$ and $\{s_w\} = \rho_L(\mathbf{f})$ we obtain $z \in y\rho_L(\mathbf{f})$. Thus, as $\mathbf{f} \in \mathbf{F}(L)$, there exists a gallery in X of type $\mathbf{f}$ from y to z. Thus, $\delta(y,z) = r_{\mathbf{f}}$. Thus, as $\delta(y,z) = w$, $w = r_{\mathbf{f}}$. Thus, as $\delta_L(y,z) = r_{\mathbf{f}}$, $\delta_L(y,z) = w$. $\square$

Theorem 6.5.1 and Theorem 6.5.2 establish a bijective map from the class of all buildings to the class of all Coxeter schemes.

6.6 Coxeter sets and twin buildings

In this section, we shall prove that twinned Coxeter sets of a scheme give rise to twinned buildings. We first recall the definition of twinned buildings. Our definition is taken from [6].

Let (W,r) be a Coxeter system. A pair $((C_+,\delta_+),(C_-,\delta_-))$ of buildings of type (W,r) is called *twinned* if there exists a map

$$\delta^* : (C_+ \times C_-) \cup (C_- \times C_+) \rightarrow W$$

satisfying the following conditions, where $\epsilon \in \{+,-\}$, $y \in C_\epsilon$, $z \in C_{-\epsilon}$, and $w := \delta^*(y,z)$:

- T1 $\delta^*(z,y) = w^{-1}$.
- T2 If $z' \in C_{-\epsilon}$ satisfies $\delta_{-\epsilon}(z,z') = s$ with s in S and $\ell(ws) = \ell(w) - 1$, then $\delta^*(y,z') = ws$.
- T3 For any element s in S, there exists a chamber z' in $C_{-\epsilon}$ with $\delta_{-\epsilon}(z,z') = s$ and $\delta^*(y,z') = ws$.

Now we are ready to prove that twinned Coxeter sets give rise to twinned buildings. In fact, we do not need that the twinned sets are Coxeter sets, it seems that it is enough to assume that they are constrained and dichotomic and that they do not contain thin elements. Of course, we do not necessarily get twinned buildings then, but (possibly) more general structures.

Theorem 6.6.1 *Let S be a scheme, let H and K be twinned Coxeter sets of S with conjugating element m, and let y and z be elements in X such that $(y, z) \in m$. Assume that H and K do not have thin elements. Then $y\langle H\rangle$ and $z\langle K\rangle$ are twinned buildings.*

Proof. Let v be an element in $y\langle H\rangle$, and let w be an element in $z\langle K\rangle$. From $w \in z\langle K\rangle$ and $z \in ym$ we obtain $w \in ym\langle K\rangle$. From $v \in y\langle H\rangle$ we obtain $y \in v\langle H\rangle$. Thus, $w \in v\langle H\rangle m\langle K\rangle$. Thus, by Lemma 4.3.2, $w \in v\langle K\rangle^{\rho}$. Thus, $\langle K\rangle$ possesses an element q such that $w \in vq^{\rho}$. We define

$$\delta^*(v, w) := q$$

and $\delta^*(w, v) := q^*$. Then

$$\delta^* : \ (y\langle H\rangle \times z\langle H\rangle) \cup (z\langle K\rangle \times y\langle K\rangle) \ \rightarrow \ \langle K\rangle$$

is a map.

Condition T1 in the definition of twinned buildings is trivial. Condition T2 is equivalent to Lemma 4.1.2, and Condition T3 follows from Lemma 4.1.2 and Corollary 4.1.3. That the buildings $y\langle H\rangle$ and $z\langle K\rangle$ have the same type follows from Lemma 4.4.2 together with Theorem 4.4.3. □

Let S be a scheme, let L be a Coxeter set which is twinned to itself with conjugating element m, and let y and z be elements in X such that $(y, z) \in m$. Then $\langle L\rangle m = m\langle L\rangle$, and from Theorem 6.6.1 one obtains that $y\langle L\rangle$ and $z\langle L\rangle$ are twinned buildings.

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