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Pi. Cassou-Noguès
C. Eyral
M. Oka

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Pi. Cassou-Noguès
C. Eyral
M. Oka

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institut de Mathématiques de Bordeaux
Université Bordeaux I
351 Cours de la Libération
33405 Talence Cedex
France

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka
Shinjuku-ku
Tokyo 162-8601
Japan

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TOPOLOGY OF SEPTICS WITH THE SET OF SINGULARITIES $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$ AND π_1-EQUIVALENT WEAK ZARISKI PAIRS

PI. CASSOU-NOGUÈS, C. EYRAL AND M. OKA

Abstract. We study the topology of the moduli space of septics with the set of singularities $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$. In particular, we construct a new π_1-equivalent weak Zariski pair.

1. Introduction

The (topological) fundamental group $\pi_1(V \setminus H)$ of a complex quasi-projective variety $V \setminus H$ plays an important role in the study of the (finite) algebraic coverings of V that are ramified along H. For example, when V is non-singular and irreducible, a general version of the Riemann existence theorem shows that the subgroups of $\pi_1(V \setminus H)$ with a finite index correspond to connected coverings (cf. [12, 15, 4] and references therein). In the case where V is the complex projective space $\mathbb{P}^n$ and H a hypersurface in it, the calculation of $\pi_1(\mathbb{P}^n \setminus H)$ reduces to the calculation of the fundamental group of the complement of a plane curve. Indeed, by the Zariski hyperplane section theorem [16] (see also Hamm–Lê [7] and Chéniot [3]), we can see easily that $\pi_1(\mathbb{P}^n \setminus H)$ is isomorphic to the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$, where $\mathbb{P}^2$ is a generic 2-dimensional projective subspace in $\mathbb{P}^n$ and C the plane curve given by $C = H \cap \mathbb{P}^2$. This shows how important the knowledge of the group $\pi_1(\mathbb{P}^2 \setminus C)$ is. Note that when this group is abelian we completely know its structure. Indeed, in this case, the Hurewicz theorem shows that $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to first integral homology group $H_1(\mathbb{P}^2 \setminus C)$. Now, by Lefschetz duality, it is not difficult to see that $H_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the product $\mathbb{Z}^{r-1} \times (\mathbb{Z}/d_0\mathbb{Z})$, where r is the number of irreducible components of C and $d_0 = \gcd(d_1, \ldots, d_r)$. (Here, d_i is the degree of the i-th irreducible component, $1 \leq i \leq r$.) See Serre [12].

The Zariski–van Kampen theorem [15, 8] (see also Chéniot [3]) gives a presentation by generators and relations of $\pi_1(\mathbb{P}^2 \setminus C)$.[1] The generators are loops in a generic line around its intersection points with the curve. The relations are obtained by considering a generic pencil

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[1] A generalization of this theorem to (possibly singular) quasi-projective varieties is given in [5].

containing this line: each loop must be identified with its transforms by monodromy around the singular lines of the pencil. In practice, the monodromy relations are not so easy to calculate, especially when the curve and the singular lines of the pencil are not defined over the real numbers (i.e., by complex-valued polynomials with real coefficients). In this case, we try to deform the curve within the corresponding equisingular moduli space onto one which is defined by a polynomial having only real coefficients, and at the same time we strive to choose a pencil with as many real singular lines as possible. However this is not always possible. For example, if the coefficients of the polynomial are uniquely determined up to the standard action of $PGL(3,\mathbb{C})$, then in general it is not possible to deform the initial curve onto one defined over $\mathbb{R}$. On the other hand, when we have a certain freedom in the choice of the coefficients, the possibility to find a curve topologically equivalent to the initial one (and more suitable for the calculation of the fundamental group) increases substantially. Notice that, in general, different choices of coefficients lead to curves with different embedded topologies. This is usually, but not always, the case when the equisingular moduli space is not path-connected. When the moduli space is path-connected, any choice of coefficients leads to topologically equivalent curves.

The question of distinguishing the path-connected components of a moduli space is therefore essential, although it is usually extremely difficult. It has been studied extensively over the last decade, especially through the works on *weak Zariski pairs*. We recall that a weak Zariski pair is a pair of curves (C, C') such that C and C' have the same degree, the same set of singularities but not the same embedded topology (i.e., the pairs $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic). A moduli space possessing such a pair is not path-connected. (Note that the converse is not true.) In general, to show that C and C' form a weak Zariski pair, one tries to show that the Alexander polynomials of C and C' are different. But it may happen that these polynomials are the same while the topologies of the pairs $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are different (so-called *Alexander-equivalent* weak Zariski pairs). In this case, we look at the fundamental groups $\pi_1(\mathbb{P}^2 \setminus C)$ and $\pi_1(\mathbb{P}^2 \setminus C')$ which are much stronger invariants. If these groups are also isomorphic and the topologies of $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are still different, then one says that (C, C') is a π_1*-equivalent* weak Zariski pair. The first example of such a pair was given in [6]. It involves reducible curves of degree 6 with the set of singularities $\mathbf{D}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$.

There is an abundant literature dealing with the topology of curves of degree 6. (For curves of degree less than or equal to 5, almost everything is known, in particular there is no weak Zariski pairs in these degrees.) In this paper, we investigate a special class of curves of degree 7, namely those with the set of singularities $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$. The equisingular moduli space of septics associated with this configuration of singularities has interesting features. Certainly, the most striking one is that it contains a π_1-equivalent weak Zariski pair. In fact, we shall see that this moduli space has at least three path-connected components: one corresponding to irreducible curves and two associated with reducible curves of the form $C = C_3 \cup C_4$, where C_3 is an irreducible cubic and C_4 an irreducible quartic. In each case the fundamental group is abelian but the (embedded) topology differs from one component to another.

Throughout we work over the complex number field $\mathbb{C}$.

2. Statements of the results

In the projective space of septics (i.e., curves of degree 7), we consider the equisingular stratum $\mathscr{S}$ consisting of the curves with the set of singularities $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$. There is a right action of $PGL(3, \mathbb{C})$ on $\mathscr{S}$ given by

$$\begin{aligned} PGL(3,\mathbb{C}) \times \mathscr{S} &\to \mathscr{S} \\ (\psi, C) &\mapsto C \cdot \psi, \end{aligned}$$

where $C \cdot \psi$ is the curve given by $F \circ \psi(X, Y, Z) = 0$ with F a defining polynomial for C. (Here, X, Y and Z are homogeneous coordinates on $\mathbb{P}^2$.) The quotient space

$$\mathscr{M} := \mathscr{S} \big/ PGL(3, \mathbb{C})$$

is called the *moduli space* of septics with the set of singularities $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$. This space is not path-connected. In fact, the configuration $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$ is realizable by both irreducible and reducible septics. A standard way to construct two such curves is described below using the approaches introduced in [1, 2, 10, 11].

Let us denote by $x = X/Z$ and $y = Y/Z$ the affine coordinates on the chart $\mathbb{P}^2 \setminus \{Z = 0\}$. We consider the family of polynomials

$$p_b(t) := t^4(t-1)^2(t-b), \quad b \in \mathbb{C} \setminus \{0, 1\},$$

and we look at the *join* type projective curves $C_{a,a',b,b'}$ defined by the affine equations

$$a\, p_b(x) - a'\, p_{b'}(y) = 0,$$

where a, $a' \neq 0$. Generically, the polynomials $p_b(t)$ have three distinct critical values corresponding to the following four values of t:

$$0,\ 1,\ \frac{5}{14} + \frac{3}{7}\, b \pm \frac{1}{14}\sqrt{25 - 52\, b + 36\, b^2}$$

($t = 0$, 1 gives the critical value 0). The corresponding (generic) curves $C_{a,a',b,b'}$ are irreducible with four singularities: a singularity of type $\mathbf{B}_{4,4}$ (normal form $x^4 - y^4$) at $(0, 0)$, two singularities of type $\mathbf{A}_3$ (normal form $x^2 - y^4$) at $(1, 0)$ and $(0, 1)$ respectively, and a singularity of type

$\mathbf{A}_1$ (normal form $x^2 - y^2$) at $(1,1)$. As these singular points are contained in the intersection of lines:

$$x^4(x-1)^2(x-b) = y^4(y-1)^2(y-b) = 0,$$

it follows from [11, Theorem (1.3) and Example (6.3)] that, generically,

$$\pi_1(\mathbb{P}^2 \setminus C_{a,a',b,b'}) \approx \mathbb{Z}/7\mathbb{Z}.$$

Let us now consider 'degenerations' of these curves. First, by [13], we know that there exist two real numbers $b_1 > 1$ and $b_2 < 0$ such that $p_{b_1}(t)$ and $p_{b_2}(t)$ are polynomials of Chebycheff type. (We say that a polynomial is of *Chebycheff* type if it has exactly two critical values.) Actually, one can take the two real roots

$$b_1 = \frac{3+2\sqrt{21}}{9} \quad \text{and} \quad b_2 = \frac{3-2\sqrt{21}}{9}$$

of the polynomial

$$27b^2 - 18b - 25 = 0.$$

Next, we choose real numbers a_1 and a_2 so that the critical values of $a_1\, p_{b_1}(t)$ and $a_2\, p_{b_2}(t)$ are the same. Explicitly, we take

$$a_k = \frac{250047}{400000} - \frac{38257191}{800000}\, b_k, \quad k = 1,2.$$

The corresponding (degenerate) curve C_{a_1,a_2,b_1,b_2} obtains four extra singularities of type $\mathbf{A}_1$ located at (a_+, b_+), (a_+, b_-), (a_-, b_+) and (a_-, b_-), where

$$a_\pm = \frac{1}{2} + \frac{2}{21}\sqrt{21} \pm \frac{1}{42}\sqrt{441 - 56\sqrt{21}},$$
$$b_\pm = \frac{1}{2} - \frac{2}{21}\sqrt{21} \pm \frac{1}{42}\sqrt{441 + 56\sqrt{21}}.$$

It turns out that C_{a_1,a_2,b_1,b_2} is also irreducible. (This technique is similar to that of [10] where Chebycheff polynomials are used for the construction of nodal curves.)

Actually, we find another degeneration C_{a_3,a_4,b_3,b_4} which is not over $\mathbb{R}$. It is given by the two complex roots

$$b_3 = \frac{-1+i\sqrt{7}}{4} \quad \text{and} \quad b_4 = \frac{-1-i\sqrt{7}}{4}$$

of the polynomial

$$2b^2 + b + 1 = 0,$$

and by the complex numbers

$$a_k = \frac{10633}{256} + \frac{4459}{128}\, b_k, \quad k = 3,4.$$

Here too, the degenerate curve C_{a_3,a_4,b_3,b_4} obtains four extra singularities of type $\mathbf{A}_1$ at (c_+, d_+), (c_+, d_-), (c_-, d_+) and (c_-, d_-), where

$$c_\pm = \frac{1}{4} + \frac{3}{28}\, i\,\sqrt{7} \pm \frac{1}{28}\sqrt{98 - 70\, i\,\sqrt{7}},$$

$$d_\pm = \frac{1}{4} - \frac{3}{28}\, i\, \sqrt{7} \pm \frac{1}{28}\sqrt{98 + 70\, i\, \sqrt{7}}.$$

However, in this case, the curve splits into an irreducible cubic and an irreducible quartic. The cubic has a singularity of type $\mathbf{A}_1$ at $(0,0)$, while the quartic has two singularities of this type: one at $(0,0)$ and one at $(1,1)$. They intersect at $(0,0)$ with intersection multiplicity 4, they are tangent at $(1,0)$ and $(0,1)$, and they meet transversally at (c_+, d_+), (c_+, d_-), (c_-, d_+) and (c_-, d_-).

Hereafter, for simplicity, we shall note

$$f(x,y) := a_1\, p_{b_1}(x) - a_2\, p_{b_2}(y),$$
$$g(x,y) := a_3\, p_{b_3}(x) - a_4\, p_{b_4}(y),$$

and we shall write $C(f)$ and $C(g)$ instead of C_{a_1,a_2,b_1,b_2} and C_{a_3,a_4,b_3,b_4} respectively. Explicitly,

$$f(x,y) = \left(-\frac{12252303}{800000} - \frac{4250799}{400000}\sqrt{21}\right) x^4\,(x-1)^2\left(x - \frac{1}{3} - \frac{2}{9}\sqrt{21}\right) -$$
$$\left(-\frac{12252303}{800000} + \frac{4250799}{400000}\sqrt{21}\right) y^4\,(y-1)^2\left(y - \frac{1}{3} + \frac{2}{9}\sqrt{21}\right),$$
$$g(x,y) = \left(\frac{16807}{512} + i\,\frac{4459}{512}\sqrt{7}\right) x^4\,(x-1)^2\left(x + \frac{1}{4} - i\,\frac{1}{4}\sqrt{7}\right) -$$
$$\left(\frac{16807}{512} - i\,\frac{4459}{512}\sqrt{7}\right) y^4\,(y-1)^2\left(y + \frac{1}{4} + i\,\frac{1}{4}\sqrt{7}\right).$$

Note that $g(x,y) = -\frac{343}{262144}\,(49 + 13\, i\sqrt{7})\, g_3(x,y)\, g_4(x,y)$, where

$$g_3(x,y) = -16\,x^3 + 16\,x^2 - 20\,yx^2 + 4\,i\,x^2\sqrt{7}y + 20\,yx - 20\,y^2x -$$
$$4\,i\,x\sqrt{7}y + 4\,i\,x\sqrt{7}y^2 - 9\,y^3 + 9\,y^2 - 5\,i\sqrt{7}y^2 + 5\,i\sqrt{7}y^3,$$
$$g_4(x,y) = 32\,x^4 - 31\,y^4 - 8\,x^2 + 13\,y^2 + 18\,y^3 - 24\,x^3 - 4\,y^2x^2 +$$
$$44\,yx^2 - 40\,yx^3 - 4\,yx + 44\,y^3x - 40\,y^2x + 8\,i\,x^3\sqrt{7}y -$$
$$12\,i\,x\sqrt{7}y + 7\,i\sqrt{7}y^2 + 8\,i\,x\sqrt{7}y^2 - 8\,i\,x^3\sqrt{7} + 8\,i\,x^2\sqrt{7} +$$
$$4\,i\,x^2\sqrt{7}y - 10\,i\sqrt{7}y^3 - 12\,i\,x^2\sqrt{7}y^2 + 4\,i\,x\sqrt{7}y^3 + 3\,i\,y^4\sqrt{7}.$$

We sum up this study in the following theorem.

Theorem 2.1. *The curves $C(f)$ and $C(g)$ are both septics with the set of singularities $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$; $C(f)$ is irreducible while $C(g)$ is a union of an irreducible cubic and an irreducible quartic. In particular, the topologies of the pairs $(\mathbb{P}^2, C(f))$ and $(\mathbb{P}^2, C(g))$ are different, so that $(C(f), C(g))$ is a weak Zariski pair and the moduli space $\mathscr{M}$ is not path-connected.*

Concerning the fundamental group, we will show the next result.

Theorem 2.2. *The groups $\pi_1(\mathbb{P}^2 \setminus C(f))$ and $\pi_1(\mathbb{P}^2 \setminus C(g))$ are abelian, isomorphic to $\mathbb{Z}/7\mathbb{Z}$ and $\mathbb{Z}$ respectively.*

Remark 2.3. Note that as soon as we know that the group $\pi_1(\mathbb{P}^2\setminus C(f))$ is abelian, we know that $\pi_1(\mathbb{P}^2\setminus C(g))$ is not isomorphic to it. Hereafter, we will construct another component of the moduli space $\mathscr{M}$ and show that for any curve $C(l)$ in it the fundamental group $\pi_1(\mathbb{P}^2 \setminus C(l))$ is isomorphic to $\pi_1(\mathbb{P}^2 \setminus C(g))$.

Theorem 2.2 can be proved easily for the group $\pi_1(\mathbb{P}^2 \setminus C(f))$ using the Zariski–van Kampen theorem (cf. section 3.1). On the other hand, as the polynomial g has non-real coefficients, it is extremely difficult to calculate the group $\pi_1(\mathbb{P}^2 \setminus C(g))$ directly from the theorem of Zariski and van Kampen. In such a situation, as we explained it in the introduction, we try to deform the curve $C(g)$ (within its equisingular stratum) onto one which is defined over $\mathbb{R}$. It turns out that in our case this is possible and there are even several ways to do it. For example, we shall see in section 5 that $C(g)$ can be deformed onto the curve $C(h)$ defined by the equation

$$h(x,y) := h_3(x,y)\cdot h_4(x,y) = 0, \tag{2.1}$$

where

$$\begin{aligned} h_3(x,y) &= y^3 + y^2(x-1) + y(x^2-x) - x^3 + x^2,\\ h_4(x,y) &= y^4 + y^3(x-3) + y^2(x^2-2\,x+2)+\\ &\quad y\,(-2\,x^3+x^2+x) + x^4 - x^2. \end{aligned}$$

More precisely, we shall prove the following result.

Theorem 2.4. *Let $\mathscr{N}$ be the moduli subspace of $\mathscr{M}$ consisting of the reducible septics C obtained as union of an irreducible cubic C_3 and an irreducible quartic C_4 satisfying the following conditions:*

1. *C_3 has a singularity of type $\mathbf{A}_1$ (say, at a point P_1);*
2. *C_4 has two singularities of type $\mathbf{A}_1$: one at P_1 and one at another point P_2;*
3. *C_3 and C_4 intersect at P_1 with intersection multiplicity 4, they are tangent at two other points (say, P_3 and P_4), and they intersect transversally at four other points (say, P_5, P_6, P_7 and P_8);*

where $P_i \neq P_j$ for $i \neq j$.

The moduli subspace $\mathscr{N}$ so-defined is path-connected and it contains the curves $C(g)$ and $C(h)$.

Theorem 2.4 implies that $C(g)$ and $C(h)$ have the same embedded topology. Indeed, it is well known that the topological type of a pair $(\mathbb{P}^2, C)$ remains constant when C moves inside a path-connected component. (This follows easily from the Lê–Ramanujam theorem [9] modulo a partition of unity argument.) In particular, $C(g)$ and $C(h)$ have the same fundamental group[2], and we will see in section 3.2 that

[2]By the fundamental group of a curve we always mean the fundamental group of the complement of the curve.

$\pi_1(\mathbb{P}^2 \setminus C(h))$ can be calculated easily from the theorem of Zariski and van Kampen.

The curve $C(l)$ defined by the equation

$$l(x,y) := l_3(x,y) \cdot l_4(x,y) = 0, \tag{2.2}$$

where

$$l_3(x,y) = -2\,y^3 + y^2 + y\,(x+1) - 2\,x^3 + x^2 + x,$$
$$l_4(x,y) = -y^4 + y^3(x+1) - \frac{1}{3}\,y^2x + y\left(-\frac{1}{3}\,x^3 + x^2\right) - x^4 + x^3,$$

is also a reducible septic (union of an irreducible cubic C_3 and an irreducible quartic C_4) with the set of singularities $\mathbf{B}_{4,4} \oplus 2\mathbf{A}_3 \oplus 5\mathbf{A}_1$. The singularity of type $\mathbf{B}_{4,4}$ is located at $(0,0)$, the $\mathbf{A}_3$-singularities at $(0,1)$ and $(1,0)$, and the singularities of type $\mathbf{A}_1$ at five other points. In this case, however, C_4 has only one singularity of type $\mathbf{B}_{3,3}$ (normal form $x^3 - y^3$) located at $(0,0)$, while C_3 is smooth. These two curves intersect at $(0,0)$ with intersection multiplicity 3, they are tangent at $(0,1)$ and $(1,0)$, and they meet transversally at five other points.

In section 4, we shall prove the following theorem.

Theorem 2.5. *The curves $C(h)$ and $C(l)$ are not homeomorphic, while the fundamental groups $\pi_1(\mathbb{P}^2 \setminus C(h))$ and $\pi_1(\mathbb{P}^2 \setminus C(l))$ are isomorphic. In particular, $(C(h), C(l))$ is a π_1-equivalent weak Zariski pair.*

Note that Theorems 2.1, 2.4 and 2.5 imply the next result.

Corollary 2.6. *The curves $C(f)$, $C(h)$ and $C(l)$ form a weak Zariski triple. In particular, the moduli space $\mathscr{M}$ has at least three path-connected components (two of them corresponding to reducible curves of the form $C = C_3 \cup C_4$, where C_3 is an irreducible cubic and C_4 an irreducible quartic).*

We recall that a *weak Zariski k-ple* is a k-ple $(C_1, \dots, C_k)$ of curves with the same degree, the same set of singularities but such that for any $1 \leq i < j \leq k$ the pairs of spaces $(\mathbb{P}^2, C_i)$ and $(\mathbb{P}^2, C_j)$ are not homeomorphic. (Note that $C_1, \dots, C_k$ may have different component type.)

3. Proof of Theorem 2.2

3.1. **The fundamental group $\pi_1(\mathbb{P}^2 \setminus C(f))$.** The real plane section of $C(f)$, namely $\{(x,y) \in \mathbb{R}^2\,;\, f(x,y) = 0\}$, is shown in Figure 1. As we explained it in the introduction, it suffices to show that $\pi_1(\mathbb{P}^2 \setminus C(f))$ is abelian. To do that, we apply the Zariski–van Kampen theorem with the pencil given by the horizontal lines $L_\eta\colon y = \eta$, $\eta \in \mathbb{C}$. We take the point $(1\colon 0\colon 0)$ as base point for the fundamental group. This point is nothing but the axis of the pencil, which is also the point at infinity of the lines L_η. Notice that it does not belong to the curve. This pencil has 8 singular lines $L_{\eta_1}, \dots, L_{\eta_8}$ with respect to $C(f)$. They correspond to the roots

$$\eta_1 \approx -0.6850, \quad \eta_2 = b_- \approx -0.5653$$
$$\eta_3 \approx -0.0225 - i\,0.4199, \quad \eta_4 = \bar{\eta}_3 \approx -0.0225 + i\,0.4199,$$
$$\eta_5 \approx 0, \quad \eta_6 = b_+ \approx 0.6924, \quad \eta_7 = 1, \quad \eta_8 \approx 1.1058$$

of the discriminant of f as a polynomial in x. The lines L_{η_2}, L_{η_5}, L_{η_6} and L_{η_7} pass through a singular point of $C(f)$, while L_{η_1}, L_{η_3}, L_{η_4} and L_{η_8} are tangent to the curve. Note that, at $P = (0, \eta_1)$, L_{η_1} is tangent to $C(f)$ with intersection multiplicity 4, and therefore, by the implicit function theorem, the germ $(C(f), P)$ is given by

$$y - \eta_1 = a_4\, x^4 + \text{higher terms}, \tag{3.1}$$

where $a_4 \neq 0$. Similarly, at the point $Q = (1, \eta_1)$, L_{η_1} is tangent to $C(f)$ with intersection multiplicity 2, and the germ $(C(f), Q)$ is given by

$$y - \eta_1 = a_2\,(x-1)^2 + \text{higher terms}, \tag{3.2}$$

where $a_2 \neq 0$. We consider the generic line $L_{\eta_1+\varepsilon}$ and we choose generators $\xi_1, \dots, \xi_7$ of the fundamental group $\pi_1(L_{\eta_1+\varepsilon} \setminus C(f))$ as indicated in Figure 2, where $\varepsilon > 0$ is small enough. The position of the generators is easily determined using equations (3.1) and (3.2). They are *lassos* oriented counter-clockwise around the intersection points of $L_{\eta_1+\varepsilon}$ with $C(f)$. (In the figures, a lasso is represented by a path ending with a bullet.) Note that

$$\xi_7 \cdot \ldots \cdot \xi_1 = e, \tag{3.3}$$

where e is the unit element (vanishing relation at infinity).

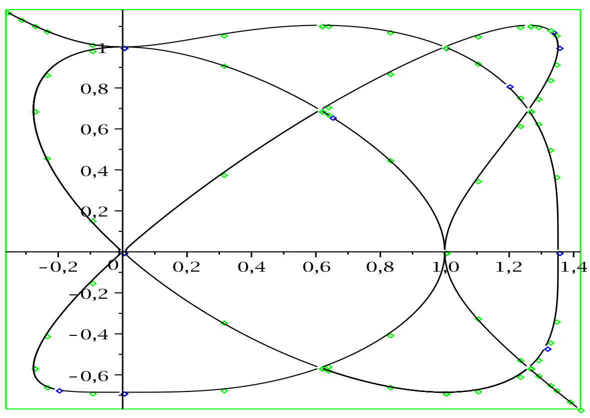

FIGURE 1. Real plane section of $C(f)$

The Zariski–van Kampen theorem says that

$$\pi_1(\mathbb{P}^2 \setminus C(f)) \simeq \pi_1(L_{\eta_1+\varepsilon} \setminus C(f)) \Big/ M(f),$$

where $M(f)$ is the normal subgroup of $\pi_1(L_{\eta_1+\varepsilon} \setminus C(f))$ generated by the monodromy relations associated with the singular lines of the pencil. To find these relations we fix a 'standard' system of generators $\sigma_1, \ldots, \sigma_8$ for the fundamental group $\pi_1(\mathbb{C} \setminus \{\eta_1, \ldots, \eta_8\})$ with base point $\eta_1 + \varepsilon$. Here, each σ_j is a lasso oriented counter-clockwise around η_j with base point $\eta_1 + \varepsilon$. For $j \neq 3, 4$, the tail of σ_j is a union of real segments and half-circles around the exceptional parameters η_l $(l \neq j)$ located in the real axis between the base point $\eta_1 + \varepsilon$ and η_j. Its head is the circle $\mathbb{S}_\varepsilon(\eta_j)$ with centre η_j and radius ε. (For $j = 1$, the tail of σ_1 is reduced to the single point $\eta_1 + \varepsilon$, so σ_1 coincide with $\mathbb{S}_\varepsilon(\eta_1)$.) The lasso σ_3 corresponding to the non-real root η_3 is given by $\zeta\theta\zeta^{-1}$, where θ is the loop obtained by moving y once on the circle $\mathbb{S}_\varepsilon(\eta_3)$ starting at $\Re(\eta_3) + i\,(\Im(\eta_3) + \varepsilon)$, while ζ is the path obtained when y moves on the real axis from $\eta_1 + \varepsilon$ to $\eta_2 - \varepsilon$, makes a half-turn on the circle $\mathbb{S}_\varepsilon(\eta_2)$ from $\eta_2 - \varepsilon$ to $\eta_2 + \varepsilon$, then moves on the real axis from $\eta_2 + \varepsilon$ to $\Re(\eta_3)$, and finally moves in a straight line from $\Re(\eta_3)$ to $\Re(\eta_3) + i\,(\Im(\eta_3) + \varepsilon)$. (Here, $\Re(\eta_3)$ and $\Im(\eta_3)$ denote the real and imaginary parts of η_3.) The lasso σ_4 is defined similarly from a loop θ' and a path ζ' meeting at $\Re(\eta_4) + i\,(\Im(\eta_4) - \varepsilon)$. The monodromy relations around the singular line L_{η_j} are obtained by moving the generic fibre $F \simeq L_{\eta_1+\varepsilon} \setminus C(f)$ isotopically above σ_j and by identifying each ξ_k with its image by the terminal homeomorphism of this isotopy.

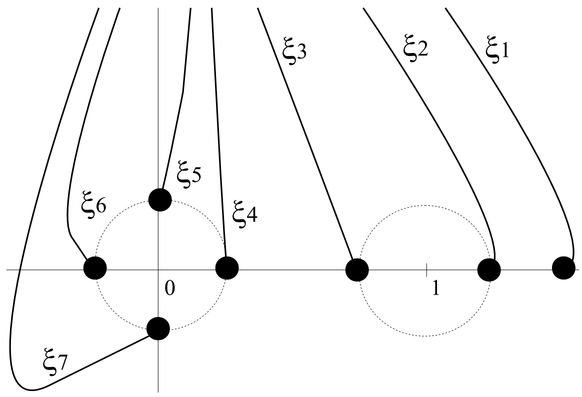

FIGURE 2. Generators at $y = \eta_1 + \varepsilon$

By (3.1) and (3.2), when y runs once counter-clockwise on the circle $\mathbb{S}_\varepsilon(\eta_1)$ starting at $\eta_1 + \varepsilon$, the four generators ξ_4, ξ_5, ξ_6 and ξ_7 make a (1/4)-turn counter-clockwise around 0, while the generators ξ_2 and ξ_3 make a half-turn counter-clockwise around 1 (cf. Figure 2). (In the figures we do not respect the numerical scale, we even zoom on the 'monodromy' parts represented here by dotted circles.) The monodromy relations around the singular line L_{η_1} are then given by

$$\xi_4 = \xi_5, \tag{3.4}$$

(3.5) $$\xi_5 = \xi_6,$$

(3.6) $$\xi_6 = \xi_7$$

(contribution of the singular point P), and

(3.7) $$\xi_2 = \xi_3$$

(contribution of the singular point Q).

In Figure 3, we show how the ξ_k's are deformed when y moves on the real axis from $\eta_1 + \varepsilon$ to $\eta_2 - \varepsilon$. At $(a_\pm, b_-) = (a_\pm, \eta_2)$, the curve has a singularity of type $\mathbf{A}_1$ and the Newton principal part has the form

$$\alpha_\pm (y - \eta_2)^2 + \alpha'_\pm (x - a_\pm)^2$$

where $\alpha_\pm, \alpha'_\pm \neq 0$. The monodromy relations at $y = \eta_2$ are then given by

(3.8) $$\xi_4 \xi_2 = \xi_2 \xi_4 \quad \text{(contribution of } (a_-, b_-)),$$

(3.9) $$\xi_2 \xi_1 = \xi_1 \xi_2 \quad \text{(contribution of } (a_+, b_-)).$$

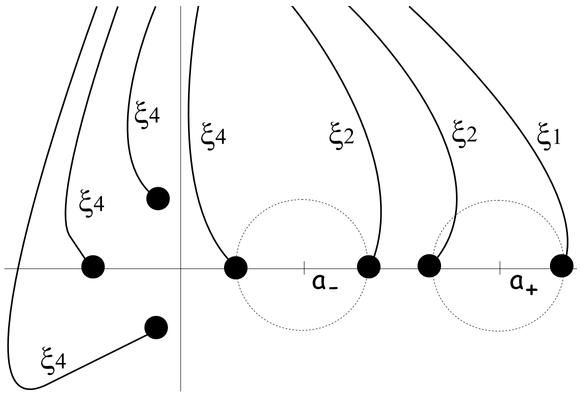

FIGURE 3. Generators at $y = \eta_2 - \varepsilon$

The relations obtained above are enough to conclude. (We do not need to look at the monodromy relations around the singular lines $L_{\eta_3}, \ldots, L_{\eta_8}$.) Indeed, by (3.3)–(3.7), we have

$$\xi_1 = (\xi_4^4 \xi_2^2)^{-1}.$$

Therefore the fundamental group $\pi_1(\mathbb{P}^2 \setminus C(f))$ is generated by only two elements, ξ_2 and ξ_4, and (3.8) says that these elements commute.

3.2. **The fundamental group $\pi_1(\mathbb{P}^2 \setminus C(g))$.** As we explained it in section 2 (see the comment following Theorem 2.4), it suffices to prove that $\pi_1(\mathbb{P}^2 \setminus C(h))$ is abelian, where $C(h)$ is the curve defined in (2.1). We recall that $C(h)$ is reducible, union of an irreducible cubic C_3 and an irreducible quartic C_4 defined by $h_3 = 0$ and $h_4 = 0$ respectively. Like $C(g)$, the curve $C(h)$ has a singularity of type $\mathbf{B}_{4,4}$ at $(0, 0)$, two singularities of type $\mathbf{A}_3$ at $(0,1)$ and $(1,0)$, and five singularities of type $\mathbf{A}_1$ at $(1,1)$ and four other points. We show its real plane section in Figure 4. (The gray color corresponds to the curve C_3 and the black one to the curve C_4.) Note that since the Newton principal part of h at $(1,1)$, given by $(x-1)^2+(y-1)^2$, has no real factorization, the point $(1,1)$ appears as an isolated point in $\{(x,y)\in\mathbb{R}^2\ ;\ h(x,y)=0\}$.

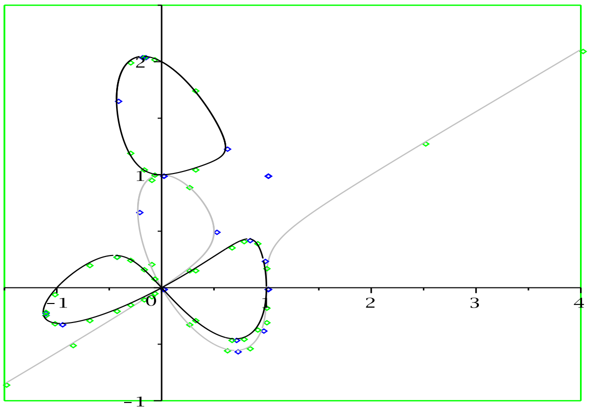

FIGURE 4. Real plane section of $C(h)$

To find $\pi_1(\mathbb{P}^2\setminus C(h))$ we use again the pencil given by the horizontal lines $L_\eta\colon y=\eta$, $\eta\in\mathbb{C}$. Note that its axis does not belong to the curve. This pencil has 8 *real* singular lines $L_{\eta_1},\dots,L_{\eta_8}$ with respect to $C(h)$, where

$$\eta_1\approx -0.5552,\quad \eta_2\approx -0.4524,$$
$$\eta_3\approx -0.3163,\quad \eta_4=0,\quad \eta_5\approx 0.2864,$$
$$\eta_6\approx 0.4328,\quad \eta_7=1,\quad \eta_8\approx 2.0478.$$

(There are also non-real singular lines but we will not need them.) The lines L_{η_4} and L_{η_7} pass through a singular point of $C(h)$. All the other real singular lines are tangent to the curve. We consider the generic line $L_{\eta_4-\varepsilon}$ and we choose generators $\xi_1,\dots,\xi_7$ of $\pi_1(L_{\eta_4-\varepsilon}\setminus C(h))$ as in Figure 5, where $\varepsilon>0$ is small enough. (The gray color corresponds to generators around the intersection points of $L_{\eta_4-\varepsilon}$ with the curve C_3 and the black one to generators around the intersection points of $L_{\eta_4-\varepsilon}$ with C_4.) To find the exact position of the roots of $h(x,\eta_4-\varepsilon)$, note that at $(0,0)$ the curve $C(h)$ has four branches (two corresponding to C_3 and two for C_4) given by

$$C_3(0,0)^-:\quad x=(1/2)(1-\sqrt{5})\,y+\text{higher terms},\tag{3.10}$$

$$C_3(0,0)^+:\quad x=(1/2)(1+\sqrt{5})\,y+\text{higher terms},\tag{3.11}$$

$$C_4(0,0)^-:\quad x=-y+\text{higher terms},\tag{3.12}$$

$$C_4(0,0)^+:\quad x=2y+\text{higher terms},\tag{3.13}$$

while at $(1,0)$ it has two branches (one for C_3 and one for C_4) given by

(3.14) $$C_3(1,0): \quad x = 1 + y^3 + \text{higher terms},$$

(3.15) $$C_4(1,0): \quad x = 1 - (1/2)\, y^2 + \text{higher terms}.$$

The Zariski–van Kampen theorem says that

$$\pi_1(\mathbb{P}^2 \setminus C(h)) \simeq \pi_1(L_{\eta_4-\varepsilon} \setminus C(h)) \Big/ M(h),$$

where $M(h)$ is the normal subgroup of $\pi_1(L_{\eta_4-\varepsilon} \setminus C(h))$ generated by the monodromy relations associated with the singular lines of the pencil.

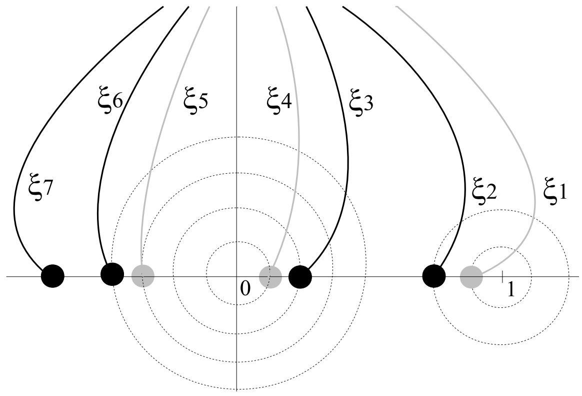

FIGURE 5. Generators at $y = \eta_4 - \varepsilon$

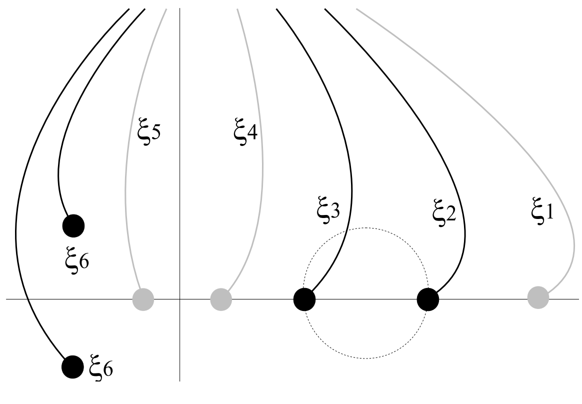

FIGURE 6. Generators at $y = \eta_2 + \varepsilon$

The monodromy relations around the singular lines L_{η_3}, L_{η_2} and L_{η_1} are multiplicity-2 tangent relations given by

(3.16) $$\xi_7 = \xi_6,$$

(3.17) $$\xi_3 = \xi_2,$$

(3.18) $$\xi_4 = \xi_2 \xi_1 \xi_2^{-1},$$

respectively. The relation (3.16) is clear from Figures 4 and 5. As for (3.17) and (3.18), use Figures 6 and 7 respectively. Note that, with these relations, the vanishing relation at infinity is written as

$$\xi_6\xi_6\xi_5\xi_2\xi_1\xi_2\xi_1 = e, \tag{3.19}$$

where e is the unit element.

To read the monodromy relations around L_{η_4} (contribution of $(1,0)$), we use the equations (3.14) and (3.15). We find

$$\xi_2 = (\xi_2\xi_1)^2 \cdot \xi_2 \cdot (\xi_2\xi_1)^{-2},$$

that is,

$$\xi_1\xi_2\xi_1\xi_2 = \xi_2\xi_1\xi_2\xi_1.$$

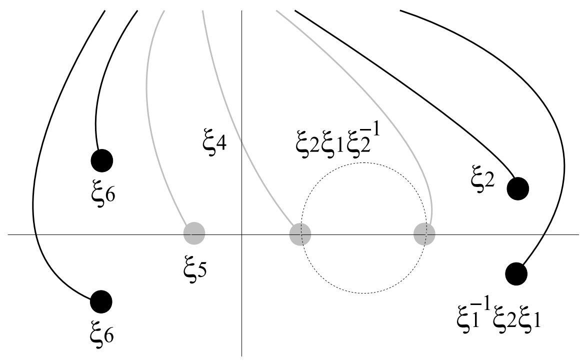

Figure 7. Generators at $y = \eta_1 + \varepsilon$

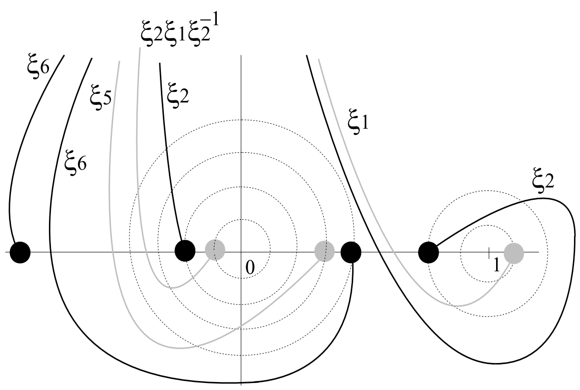

Figure 8. Generators at $y = \eta_4 + \varepsilon$

The monodromy relations around L_{η_4} (contribution of $(0,0)$) can be found by using the equations (3.10)–(3.13). They are given by

$$\xi_i = \omega \cdot \xi_i \cdot \omega^{-1} \quad (3 \le i \le 6) \tag{3.20}$$

where $\omega := \xi_6\xi_5\xi_4\xi_3$. Now, by (3.17) and (3.18), we have $\omega = \xi_6\xi_5\xi_2\xi_1$ and the relations (3.20) turn into

$$\xi_i = \omega \cdot \xi_i \cdot \omega^{-1} \quad (i = 2,\, 5,\, 6), \tag{3.21}$$

$$\xi_2\xi_1\xi_2^{-1} = \omega \cdot \xi_2\xi_1\xi_2^{-1} \cdot \omega^{-1}. \tag{3.22}$$

To determine the monodromy relations around L_{η_5}, we need to get to know how the generators are deformed when y makes a half-turn counter-clockwise on the circle $\mathbb{S}_\varepsilon(\eta_4)$ from $\eta_4 - \varepsilon$ to $\eta_4 + \varepsilon$, and then moves on the real axis from $\eta_4 + \varepsilon$ to $\eta_5 - \varepsilon$. This is shown in Figures 8 and 9 respectively. Then, clearly, the monodromy relation at $y = \eta_5$ (multiplicity-2 tangent relation) is given by

$$\xi_6 = \xi_2. \tag{3.23}$$

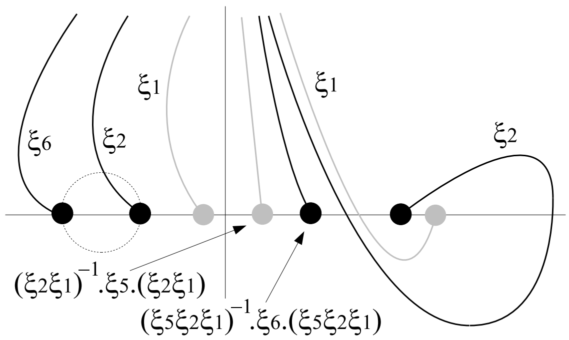

FIGURE 9. Generators at $y = \eta_5 - \varepsilon$

The relations given above are enough to conclude. Indeed, by (3.23), the relation (3.19) can be written under the form

$$\xi_5 = (\xi_2\xi_1\xi_2\xi_1\xi_2\xi_2)^{-1}, \tag{3.24}$$

or equivalently,

$$\xi_6\xi_5\xi_2\xi_1 = (\xi_2\xi_1\xi_2)^{-1}.$$

Combined with (3.21) when $i = 2$, we get

$$\xi_1\xi_2 = \xi_2\xi_1. \tag{3.25}$$

It follows from (3.16)–(3.18), (3.23) and (3.24) that the fundamental group $\pi_1(\mathbb{P}^2 \setminus C(h))$ is generated by only two elements, ξ_1 and ξ_2, and the relation (3.25) says that these two elements commute.

4. Proof of Theorem 2.5

4.1. **The topologies of $C(h)$ and $C(l)$.** Here we show that the curves $C(h)$ and $C(l)$ are not homeomorphic. By the genus formula, the curves $h_3 = 0$ and $l_4 = 0$ have genus 0, while $h_4 = 0$ and $l_3 = 0$ have genus 1. Therefore $h_3 = 0$ is homeomorphic to the sphere $\mathbb{S}^2$ with two points identified, $l_4 = 0$ is homeomorphic to $\mathbb{S}^2$ with three points identified, $l_3 = 0$ is homeomorphic to the torus $\mathbb{T} := \mathbb{S}^1 \times \mathbb{S}^1$, and $h_4 = 0$ is homeomorphic to $\mathbb{T}$ with two pairs of points identified. The sets of regular points of $C(h)$ and $C(l)$ are homeomorphic to

$$(\mathbb{S}^2 \setminus \{8 \text{ points}\}) \sqcup (\mathbb{T} \setminus \{10 \text{ points}\})$$
$$\text{and} \quad (\mathbb{T} \setminus \{8 \text{ points}\}) \sqcup (\mathbb{S}^2 \setminus \{10 \text{ points}\})$$

respectively. Clearly, the component $\mathbb{S}^2 \setminus \{8 \text{ points}\}$ is not homeomorphic neither to $\mathbb{T} \setminus \{8 \text{ points}\}$ nor to $\mathbb{S}^2 \setminus \{10 \text{ points}\}$, so $C(h)$ and $C(l)$ are not homeomorphic.

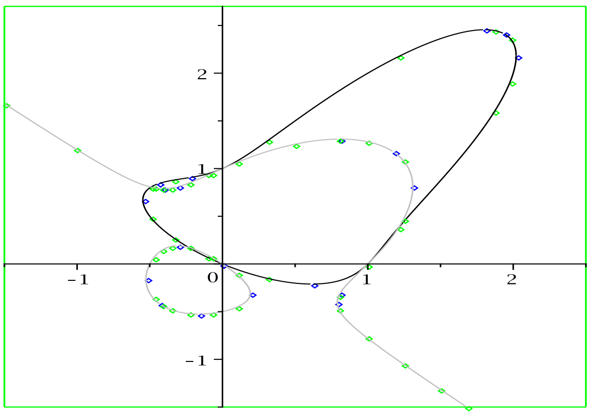

Figure 10. Real plane section of $C(l)$

4.2. **The fundamental group $\pi_1(\mathbb{P}^2 \setminus C(l))$.** We show the real plane section of $C(l)$ in Figure 10. (The gray color corresponds to the curve $C_3 := \{l_3 = 0\}$, the black one to the curve $C_4 := \{l_4 = 0\}$.) We are going to show that $\pi_1(\mathbb{P}^2 \setminus C(l))$ is isomorphic to $\mathbb{Z}$. Here, we use the pencil given by the vertical lines $L_\eta \colon x = \eta$, $\eta \in \mathbb{C}$. Note that its axis $(0\colon 1\colon 0)$ does not belong to the curve. We take this point as base point for the fundamental group. This pencil has 11 *real* singular lines $L_{\eta_1}, \dots, L_{\eta_{11}}$ with respect to $C(l)$, where

$$\eta_1 \approx -0.5475, \quad \eta_2 \approx -0.5262, \quad \eta_3 \approx -0.4918,$$
$$\eta_4 \approx -0.2344, \quad \eta_5 = 0, \quad \eta_6 \approx 0.1910, \quad \eta_7 = 0.7900, \quad \eta_8 = 1,$$
$$\eta_9 \approx 1.2116, \quad \eta_{10} \approx 1.3090, \quad \eta_{11} \approx 2.0201.$$

(We will not use the non-real lines.) The lines L_{η_3}, L_{η_4}, L_{η_5}, L_{η_8} and L_{η_9} pass through one or two singular points of $C(l)$, while L_{η_1}, L_{η_2}, L_{η_6}, L_{η_7}, $L_{\eta_{10}}$ and $L_{\eta_{11}}$ are tangent to the curve. We consider the generic line $L_{\eta_2+\varepsilon}$ and we choose generators $\xi_1, \ldots, \xi_7$ of $\pi_1(L_{\eta_2+\varepsilon} \setminus C(l))$ as in Figure 11, where $\varepsilon > 0$ is small enough. (The gray color corresponds to generators around the intersection points of $L_{\eta_2+\varepsilon}$ with the curve C_3 while the black one corresponds to generators around the intersection points of $L_{\eta_2+\varepsilon}$ with C_4.) By the Zariski–van Kampen theorem,

$$\pi_1(\mathbb{P}^2 \setminus C(l)) \simeq \pi_1(L_{\eta_2+\varepsilon} \setminus C(l)) \big/ M(l),$$

where $M(l)$ is the normal subgroup of $\pi_1(L_{\eta_2+\varepsilon} \setminus C(l))$ generated by the monodromy relations associated with the singular lines of the pencil.

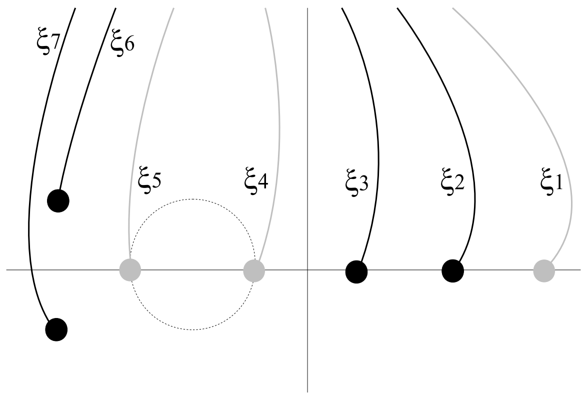

FIGURE 11. Generators at $x = \eta_2 + \varepsilon$

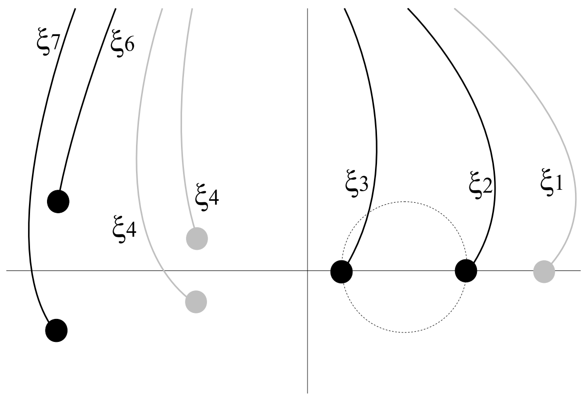

FIGURE 12. Generators at $x = \eta_1 + \varepsilon$

The monodromy relations around L_{η_2} and L_{η_1} are multiplicity-2 tangent relations given by

$$\xi_5 = \xi_4 \quad \text{and} \quad \xi_3 = \xi_2 \tag{4.1}$$

respectively. (At $x = \eta_1 + \varepsilon$, the generators are as in Figure 12.)

At $x = \eta_3 - \varepsilon$ the ξ_k's are deformed as in Figure 13, while at $x = \eta_4 - \varepsilon$ they are as in Figure 14. The monodromy relations around L_{η_3} and L_{η_4} are node relations given by

$$\xi_1\xi_2 = \xi_2\xi_1 \quad \text{and} \quad \xi_4\xi_2 = \xi_2\xi_4 \tag{4.2}$$

respectively.

Notice that the vanishing relation at infinity shows that

$$\xi_7 = (\xi_6\xi_4^2\xi_2^2\xi_1)^{-1}. \tag{4.3}$$

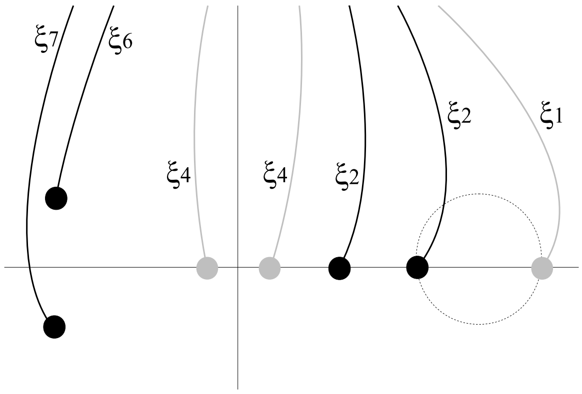

FIGURE 13. Generators at $x = \eta_3 - \varepsilon$

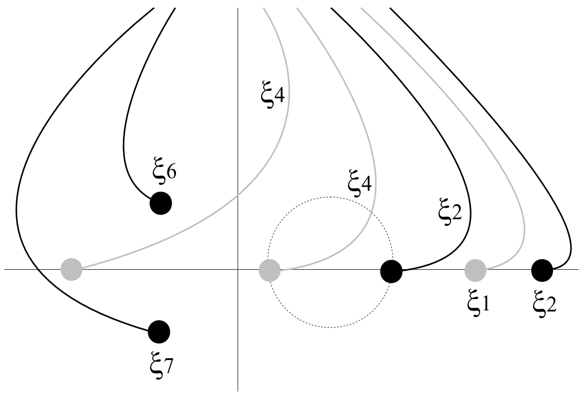

FIGURE 14. Generators at $x = \eta_4 - \varepsilon$

At $(0,0)$ the curve $C(l)$ has four branches (one corresponding to C_3, and three from C_4):

$$C_3(0,0): \quad y = -x + \text{higher terms},$$
$$C_4^{\text{real}}(0,0): \quad y \approx -0.6252\,x + \text{higher terms},$$
$$C_4^{\pm i}(0,0): \quad y \approx (0.4793 \pm i\,1.1703)\,x + \text{higher terms},$$

while at $(0,1)$ it has two branches (one for C_3, one for C_4):

$$C_3(0,1): \quad y = 1 + \frac{2}{3}\,x - \frac{5}{27}\,x^2 + \text{higher terms},$$
$$C_4(0,1): \quad y = 1 + \frac{2}{3}\,x + \frac{11}{9}\,x^2 + \text{higher terms}.$$

It follows that, at $x = \eta_5 - \varepsilon$, the ξ_k's are as in Figure 15, where

$$\mu := (\xi_6\xi_4\xi_2\xi_2\xi_1\xi_6\xi_4\xi_6^{-1})^{-1},$$

and the monodromy relations around L_{η_5} (contribution of the singular point $(0,0)$) are given by

$$\xi_j = \omega \cdot \xi_j \cdot \omega^{-1} \quad \text{for} \quad j = 2,\, 4,\, 6, \tag{4.4}$$

where $\omega := \mu \cdot \xi_6\xi_2\xi_4$, that is, $\omega = (\xi_2\xi_1\xi_6\xi_4\xi_6^{-1})^{-1}$ by (4.2). (Notice that, as $\xi_1\xi_2 = \xi_2\xi_1$, the contribution of the singular point $(0,1)$ does not give any information.)

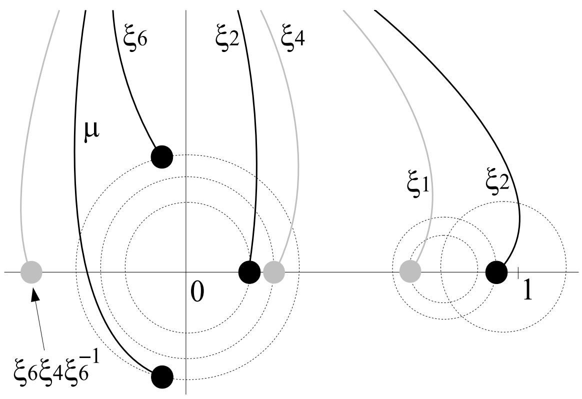

Figure 15. Generators at $x = \eta_5 - \varepsilon$

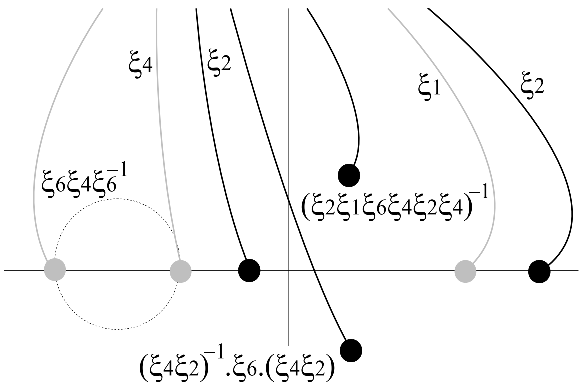

Figure 16. Generators at $x = \eta_6 - \varepsilon$

At $x = \eta_6 - \varepsilon$, the ξ_k's are deformed as in Figure 16, and the monodromy relation around L_{η_6} is a multiplicity-2 tangent relation:

$$\xi_6\xi_4 = \xi_4\xi_6. \tag{4.5}$$

It follows that

$$\omega = (\xi_2\xi_1\xi_4)^{-1},$$

and therefore the relations (4.4) can be written as

$$\xi_1\xi_2 = \xi_2\xi_1 \quad (j = 2), \tag{4.6}$$

$$\xi_1\xi_4 = \xi_4\xi_1 \quad (j = 4), \tag{4.7}$$

$$\xi_2\xi_1\xi_6 = \xi_6\xi_2\xi_1 \quad (j = 6). \tag{4.8}$$

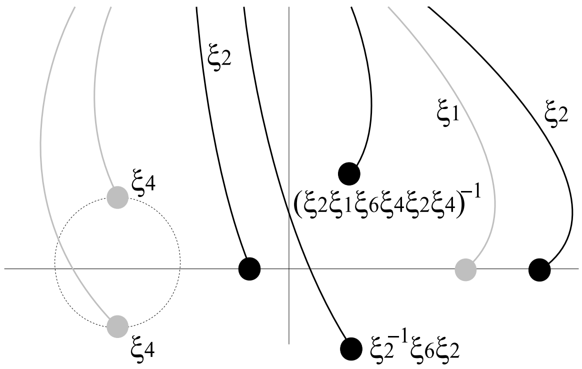

FIGURE 17. Generators at $x = \eta_7 - \varepsilon$

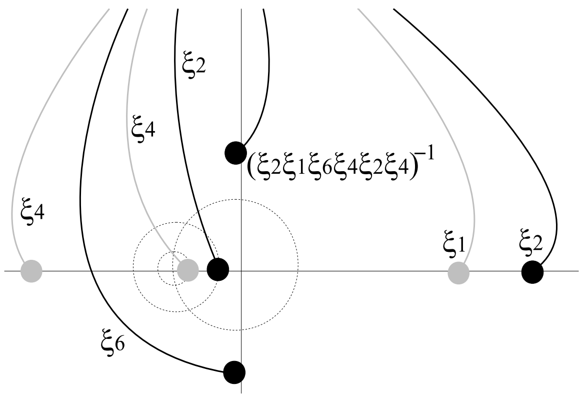

FIGURE 18. Generators at $x = \eta_8 - \varepsilon$

At $x = \eta_7 - \varepsilon$ (respectively $x = \eta_8 - \varepsilon$ and $x = \eta_9 - \varepsilon$), the ξ_k's are deformed as in Figure 17 (respectively Figures 18 and 19), and the monodromy relations around L_{η_7}, L_{η_8} and L_{η_9} do not give any new information. (To find the exact position of the generators at $x = \eta_8 - \varepsilon$, one may notice that at $(1, 0)$ the curve has two branches: one corresponding to C_3 given by

$$y = \frac{3}{2}(x-1) + \frac{5}{8}(x-1)^2 + \text{higher terms},$$

and one corresponding to C_4 given by

$$y = \frac{3}{2}(x-1) + \frac{27}{8}(x-1)^2 + \text{higher terms}.$$

To find the generators in Figure 19, use the relations (4.6)–(4.8).)

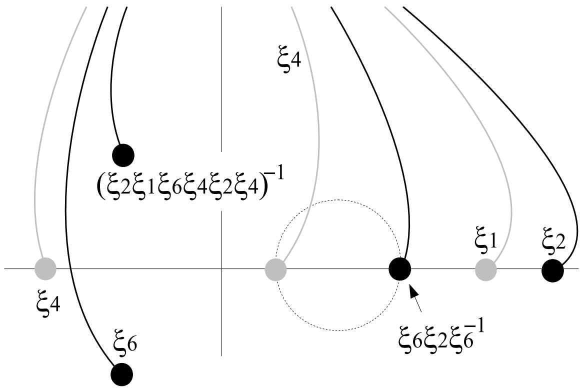

FIGURE 19. Generators at $x = \eta_9 - \varepsilon$

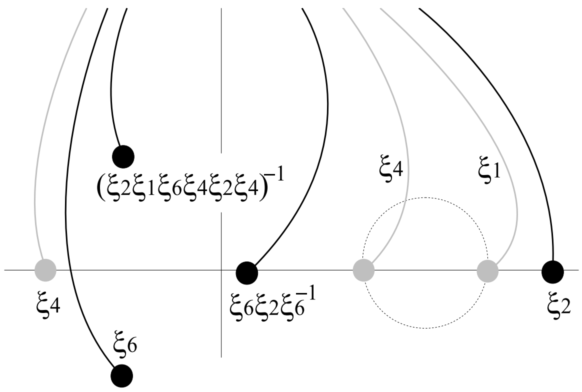

FIGURE 20. Generators at $x = \eta_{10} - \varepsilon$

Finally, at $x = \eta_{10} - \varepsilon$, the ξ_k's are as in Figure 20, and the monodromy relation around $L_{\eta_{10}}$ is a multiplicity-2 tangent relation given by

$$\xi_4 = \xi_1. \tag{4.9}$$

It follows from this relation and (4.5) that

$$\xi_6\xi_1 = \xi_1\xi_6. \tag{4.10}$$

Combined with (4.8), this relation shows that

$$\xi_6\xi_2 = \xi_2\xi_6. \tag{4.11}$$

Altogether, we have proved that the fundamental group $\pi_1(\mathbb{P}^2 \setminus C(l))$ is generated by four elements, ξ_1, ξ_2, ξ_4 and ξ_6, and (4.2), (4.5) (4.7), (4.10) and (4.11) say that these elements commute.

5. Proof of Theorem 2.4

By [14, Chapter 2, Lemma 2I], it suffices to prove that $\mathscr{N}$ is a Zariski-open subset in a linear space. Let $\mathscr{N}'$ be the moduli space of reducible septics of the form $C = C_3 \cup C_4$, where C_3 and C_4 are irreducible curves of degree 3 and 4, respectively, satisfying the conditions (1), (2) and (3) of the theorem. (Note that $\mathscr{N}$ is a Zariski-open subset in $\mathscr{N}'$.) By the action of $PGL(3,\mathbb{C})$, we can assume that $P_1 = (0,0)$, $P_3 = (0,1)$ and $P_4 = (1,0)$. We can also assume that $P_2 = (t,t)$ with $t \neq 0$. Indeed, if the points $P_1,\dots,P_4$ are generic (i.e., no three of them are collinear), then we can take any $t \neq 0, \frac{1}{2}$. (Given two sets $\{A_1,\dots,A_4\}$ and $\{A'_1,\dots,A'_4\}$ consisting of four generic points in $\mathbb{P}^2$, there always exists $\psi \in PGL(3,\mathbb{C})$ such that $\psi(A_i) = A'_i$ for $1 \le i \le 4$.) If $P_1,\dots,P_4$ are not generic, then, by Bezout's theorem, P_1, P_3, P_4 (respectively, P_1, P_2, P_3 and P_1, P_2, P_4) cannot be collinear. Thus the only possibility is that P_2, P_3, P_4 are collinear, and in this case we take $t = \frac{1}{2}$.

Let $\mathscr{N}_4'$ be the moduli space of quartics with a singularity of type $\mathbf{A}_1$ at P_1 and at P_2, and passing through P_3 and P_4.

Claim 5.1. $\mathscr{N}_4'$ is a Zariski-open subset in $\mathbb{C}^7$.

Proof. Let C_4 be a curve in $\mathscr{N}_4'$, and q_4 a defining polynomial for it:

$$\begin{aligned} q_4(x,y) = {} & b_{04}\,y^4 + (b_{13}\,x + b_{03})\,y^3 + \big(b_{22}\,x^2 + b_{12}\,x + b_{02}\big)\,y^2 + \\ & \big(b_{31}\,x^3 + b_{21}\,x^2 + b_{11}\,x + b_{01}\big)\,y + b_{40}\,x^4 + b_{30}\,x^3 + \\ & b_{20}\,x^2 + b_{10}\,x + b_{00}. \end{aligned}$$

The b_i's must satisfy the following 8 linear equations:

- $b_{00} = b_{01} = b_{10} = 0$

  (because P_1 is a singular point),
- $b_{04} = -b_{02} - b_{03}, \quad b_{40} = -b_{20} - b_{30}$

  (because the curve passes through P_3 and P_4),
- $b_{31} = (b_{03}\,t^2 - b_{21}\,t + b_{02}\,t^2 - b_{13}\,t^2 - b_{03}\,t - b_{30}\,t - b_{11} + b_{30}\,t^2 + b_{20}\,t^2 - b_{02} - b_{20} - b_{22}\,t^2 - b_{12}\,t)/\,t^2,$

  $b_{12} = (-2\,b_{03}\,t - b_{22}\,t^2 - b_{30}\,t^2 - b_{02} + b_{30}\,t + b_{20} + 3\,b_{02}\,t^2 -$

$$b_{20}\,t^2 - 2\,b_{13}\,t^2 + 3\,b_{03}\,t^2)/\,t,$$
$$b_{22} = (3\,b_{02}\,t^2 - b_{20}\,t^2 + 3\,b_{03}\,t^2 - b_{03}\,t + 3\,b_{20} + b_{21}\,t - 2\,b_{13}\,t^2 + 2\,b_{11} + b_{02} - b_{30}\,t^2 + 2\,b_{30}\,t)/\,t^2$$
(because P_4 is a singular point).

The polynomial q_4 has then 7 free coefficients:

$$b_{02},\ b_{03},\ b_{11},\ b_{13},\ b_{20},\ b_{21},\ b_{30}.$$

To get a singularity of type $\mathbf{A}_1$ at P_i $(i = 1,\,2)$, we need to remove the algebraic subsets in $\mathbb{C}^7$ defined by $\operatorname{Hess}(q_4, P_i) = 0$, so the moduli space $\mathscr{N}_4'$ is a Zariski-open subset in $\mathbb{C}^7$. □

Consider the projection $\pi\colon \mathscr{N}' \to \mathscr{N}_4'$ defined by $\pi(C_3 \cup C_4) = C_4$. For any fixed C_4 in $\mathscr{N}_4'$, the fibre $\pi^{-1}(C_4)$ is isomorphic to the moduli space $\mathscr{N}_3'$ of cubics with a singularity of type $\mathbf{A}_1$ at P_1, and tangent to C_4 at P_3 and P_4.

Claim 5.2. $\mathscr{N}_3'$ is a Zariski-open subset in $\mathbb{C}^3$.

Proof. Let C_3 be a curve in $\mathscr{N}_3'$, and q_3 a defining polynomial for it:

$$q_3(x,y) = a_{03}\,y^3 + (a_{12}\,x + a_{02})\,y^2 + \big(a_{21}\,x^2 + a_{11}\,x + a_{01}\big)\,y + a_{30}\,x^3 + a_{20}\,x^2 + a_{10}\,x + a_{00}.$$

The a_i's must satisfy the following 5 linear equations:

- $a_{00} = a_{01} = a_{10} = 0$
  (because P_1 is a singular point),
- $a_{03} = -a_{02}, \quad a_{30} = -a_{20}$
  (because the curve passes through P_3 and P_4).

Moreover, as C_3 and C_4 are tangent at P_3 and P_4, the a_i's should also satisfy the following two equations:

$$\frac{\partial q_4}{\partial x}\,(P_i)\,\frac{\partial q_3}{\partial y}\,(P_i) - \frac{\partial q_4}{\partial y}\,(P_i)\,\frac{\partial q_3}{\partial x}\,(P_i) = 0 \quad (i = 3,\,4)$$

that is,

$$-a_{12}\,t\,(b_{03} + 2\,b_{02}) = -a_{02}\,b_{13}\,t + 2\,a_{02}\,b_{20} + a_{02}b_{30}\,t + a_{02}\,b_{03}\,t + 2\,a_{02}\,b_{02} + 2\,a_{02}\,b_{11} + a_{02}\,b_{21}\,t - a_{02}\,b_{11}\,t + t\,b_{03}\,a_{11} + 2\,t\,b_{02}\,a_{11}$$

and

$$-a_{21}\,t^2(b_{30} + 2\,b_{20}) = 2\,a_{20}\,b_{20} + t^2 b_{30}\,a_{11} - a_{20}\,b_{11}\,t^2 + 2\,t^2 b_{20}\,a_{11} + 2\,a_{20}\,b_{30}\,t + 2\,a_{20}\,b_{03}\,t^2 - a_{20}\,b_{13}\,t^2 + a_{20}\,b_{21}\,t - a_{20}\,b_{03}\,t - 2\,a_{20}\,b_{20}\,t^2 + a_{20}\,b_{11} - 2\,a_{20}\,b_{30}\,t^2 + 2\,a_{20}\,b_{02}\,t^2 - a_{20}\,b_{21}\,t^2.$$

Note that these equations are linear in the a_i's. The polynomial q_3 has then 3 free coefficients:

$$a_{02},\ a_{11},\ a_{20}.$$

To get a singularity of type $\mathbf{A}_1$ at P_1, we must remove the algebraic subset in $\mathbb{C}^3$ defined by Hess $(q_3, P_1) = 0$, so $\mathscr{N}_3'$ is a Zariski-open subset in $\mathbb{C}^3$. □

It follows from Claims 5.1 and 5.2 that the moduli space $\mathscr{N}'$ is a Zariski-open subset in $\mathbb{C}^{10}$. Now, the resultant of q_3 and q_4 as polynomials in y has the form

$$x^6\,(x-1)^2\,R(x),$$

where R is a degree-4 polynomial. This shows that, generically, $q_3 \cdot q_4$ has also four singularities of type $\mathbf{A}_1$ at four other points P_5, P_6, P_7 and P_8. In other words, $\mathscr{N}$ is also a Zariski-open subset in $\mathbb{C}^{10}$.

Remark 5.3. The curve $C(h)$ introduced in (2.2) corresponds to $t = 1$ and the following choice of coefficients:

$$a_{02} = -1,\ a_{11} = -1,\ a_{20} = 1,\ b_{02} = 2,\ b_{03} = -3,$$
$$b_{11} = 1,\ b_{13} = 1,\ b_{20} = -1,\ b_{21} = 1,\ b_{30} = 0.$$

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Pi. Cassou-Noguès, Institut de Mathématiques de Bordeaux, Université Bordeaux I, 351 cours de la Libération, 33405 Talence cedex, France

E-mail address: Pierrette.Cassou-nogues@math.u-bordeaux1.fr

C. Eyral, Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

E-mail address: eyralchr@yahoo.com

M. Oka, Department of Mathematics, Tokyo University of Science, 1–3 Kagurazaka, Shinjuku–ku, Tokyo 162–8601, Japan

E-mail address: oka@rs.kagu.tus.ac.jp