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by

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THE THIRD HOMOTOPY GROUP AS A \( \pi_1 \)-MODULE

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Abstract. It is well-known how to compute the structure of the second homotopy group of a space, \( X \), as a module over the fundamental group, \( \pi_1 X \), using the homology of the universal cover and the Hurewicz isomorphism. We describe a new method to compute the third homotopy group, \( \pi_3 X \), as a module over \( \pi_1 X \). Moreover, we determine \( \pi_3 X \) as an extension of \( \pi_1 X \)-modules derived from Whitehead’s Certain Exact Sequence. Our method is based on the theory of quadratic modules. Explicit computations are carried out for pseudo-projective 3-spaces \( X = S^1 \cup e^2 \cup e^3 \) consisting of exactly one cell in each dimension \( \leq 3 \).

1. Introduction

Given a connected 3-dimensional CW-complex, \( X \), with universal cover, \( \tilde{X} \), Whitehead’s Certain Exact Sequence \([W2]\) yields the short exact sequence

\[
\begin{array}{cccccc}
\Gamma \pi_2 X & \longrightarrow & \pi_3 X & \longrightarrow & H_3 \tilde{X} & \text{as a sequence of abelian groups.}
\end{array}
\]

of \( \pi_1 \)-modules, where \( \pi_1 = \pi_1(X) \). As a group, the homology \( H_3 \tilde{X} \) is a subgroup of the free abelian group of cellular 3-chains of \( \tilde{X} \), and thus itself free abelian. Hence the sequence splits as a sequence of abelian groups. This raises the question whether \((1.1)\) splits as a sequence of \( \pi_1 \)-modules – there are no examples known in the literature.

It is well-known how to compute \( \pi_2(X) \cong H_2 \tilde{X} \) as a \( \pi_1 \)-module, using the Hurewicz isomorphism, and how to compute \( H_3 \tilde{X} \) using the cellular chains of the universal cover. In this paper we compute \( \pi_3(X) \) as a \( \pi_1 \)-module and \((1.1)\) as an extension over \( \pi_1 \). We answer the question above by providing an infinite family of examples where \((1.1)\) does not split over \( \pi_1 \), as well as an infinite family of examples where it does split over \( \pi_1 \). As a first surprising example we obtain

**Theorem 1.1.** **There is a connected 3-dimensional CW-complex** \( X \) **with fundamental group** \( \pi_1 = \pi_1 X = \mathbb{Z}/2\mathbb{Z} \), **such that** \( \pi_1 \) **acts trivially on both** \( \Gamma \pi_2 X \) **and** \( H_3 \tilde{X} \), **but non-trivially on** \( \pi_3 X \). **Hence**

\[
\begin{array}{cccccc}
\Gamma \pi_2 X & \longrightarrow & \pi_3 X & \longrightarrow & H_3 \tilde{X} & \text{does not split as a sequence of} \ \pi_1 \text{-modules.}
\end{array}
\]

Below we describe examples for all finite cyclic fundamental groups, \( \pi_1 \), of even order, where \((1.1)\) does not split over \( \pi_1 \). The examples we consider are CW-complexes,

\[
X = S^1 \cup e^2 \cup e^3,
\]

with precisely one cell, \( e^i \), in every dimension \( i = 0, 1, 2, 3 \). In general, we obtain such a CW-complex, \( X \), by first attaching the 2-cell \( e_2 \) to \( S^1 \) via \( f \in \pi_1 S^1 = \mathbb{Z} \). We assume \( f > 0 \). This yields the 2-skeleton of \( X \), \( X^2 = P_f \), which is a pseudo-projective plane, see [O]. Then \( \pi_1 = \pi_1 X = \pi_1 P_f = \mathbb{Z}/f\mathbb{Z} \) is a cyclic group of order \( f \). We write \( R = \mathbb{Z}[\pi_1] \) for the integral group ring of \( \pi_1 \) and \( K \) for the kernel of the augmentation \( \varepsilon : R \rightarrow \mathbb{Z} \). Then the pseudo-projective 3-space, \( X = P_f \cup \varepsilon \), is determined by the pair, \((f, x)\), of attaching maps, where \( x \in \pi_2 P_f = K \) is the attaching map of the 3-cell \( e_3 \). In this case

\[
\pi_2(X) = H_2(\tilde{X}) = K/xR,
\]

and

\[
H_3 \tilde{X} = \ker(d_x : R \rightarrow R, x \mapsto xy),
\]

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where \( xy \) is the product of \( x, y \in R \).

A splitting function \( u \) for the exact sequence (1.1) is a function between sets, \( u : H_3\hat{X} \to \pi_3X \), such that \( u(0) = 0 \) and the composite of \( u \) and the projection \( \pi_3X \to H_3\hat{X} \) is the identity. Such a splitting function determines maps \( A = A_u : H_3\hat{X} \times H_3\hat{X} \to \Gamma(\pi_2X) \) and \( B = B_u : H_3\hat{X} \to \Gamma(\pi_2X) \), by the cross–effect formulæ

\[
A(y, z) = u(y + z) - (u(y) + u(z)) \quad \text{and} \quad B(y) = (u(y))^1 - u(y^1).
\]

Here \( B \) is determined by the action of the generator 1 in the cyclic group \( \pi_1 \), denoted by \( y \mapsto y^1 \).

**Remark 1.2.** The functions \( A \) and \( B \) determine \( \pi_3X \) as a \( \pi_1 \)-module. In fact, the bijection \( H_3\hat{X} \times \Gamma(\pi_2X) = \pi_3(P_{f,x}) \), which assigns to \( (y, v) \) the element \( u(y) + v \) is an isomorphism of \( \pi_1 \)-modules, where the left hand side is an abelian group by

\[
(y, v) + (z, w) = (y + z, v + w + A(y, z))
\]

and a \( \pi_1 \)-module by

\[
(y, v)^1 = (y^1, v^1 + B(y)).
\]

The cross–effect of \( B \) satisfies

\[
B(y + z) - (B(y) + B(z)) = (A(y, z))^1 - A(y^1, z^1),
\]

such that \( B \) is a homomorphism of abelian groups if \( A = 0 \).

In this paper we describe a method to determine a splitting function \( u = u_x \), which, a priori, is not a homomorphism of abelian groups. We investigate the corresponding functions \( A \) and \( B \) and compute them for a family of examples.

**Theorem 1.3.** Let \( X = P_{f,x} \) be a pseudo–projective 3–space with \( x = \hat{x}(1-0) \in K, \hat{x} \in \mathbb{Z}, \hat{x} \neq 0 \) and \( f > 1 \). Let \( N = \sum_{i=0}^{f-1} [i] \) be the norm element in \( R \). Then

\[
H_3(\hat{P}_{f,x}) = \{ \hat{y}N \mid \hat{y} \in \mathbb{Z} \} \cong \mathbb{Z}
\]

is a \( \pi_1 \)-module with trivial action of \( \pi_1 \), and

\[
\pi_2(P_{f,x}) = (\mathbb{Z}/\hat{x}\mathbb{Z}) \otimes_\mathbb{Z} K,
\]

with the action of \( \pi_1 \) induced by the \( \pi_1 \)-module \( K \). There is a splitting function \( u = u_x \) such that, for \( y = \hat{y}N \) and \( z \in H_3(\hat{P}_{f,x}) \), the functions \( A \) and \( B \) are given by

\[
A(y, z) = 0 \quad \text{and} \quad B(y) = -\hat{x}\hat{y}q(1-[0]),
\]

where \( \gamma : \pi_2(P_{f,x}) \to \Gamma(\pi_2(P_{f,x})) \) is the universal quadratic map for the Whitehead functor \( \Gamma \) and \( q : K \to \pi_2(P_{f,x}), k \mapsto 1 \otimes k. \) As in 1.2, the pair \( A, B \) computes \( \pi_3X \) as a \( \pi_1 \)-module.

As \( H_3(\hat{X}) \) is free abelian, the exact sequence (1.1) always allows a splitting function which is a homomorphism of abelian groups. This leads, for \( X = P_{f,x} \), to the injective function

\[
\tau : \text{Ext}_{\pi_1}(H_3(\hat{X}), \Gamma(\pi_2X)) \hookrightarrow \text{coker}(\beta),
\]

with

\[
\beta : \text{Hom}_Z(H_3(\hat{X}), \Gamma(\pi_2X)) \to \text{Hom}_Z(H_3(\hat{X}), \Gamma(\pi_2X)), t \mapsto \beta_t,
\]

given by

\[
\beta_t(\ell) = -t(\ell^1) + (t(\ell))^1.
\]

The function \( \tau \) maps the equivalence class of an extension to the element in \( \text{coker}(\beta) \) represented by \( B = B_u \), where \( u \) is a \( \mathbb{Z} \)-homomorphic splitting function for the extension. Hence the equivalence class, \( \{ \pi_3X \} \), of the extension \( \pi_3X \) in (1.1) is determined by the image \( \tau\{ \pi_3X \} \in \text{coker}(\beta) \). For the family of examples in 1.3 we show
Theorem 1.4. Let \( X = P_{f,x} \) be a pseudo–projective 3–space with \( x = \tilde{x}([1] - [0]), \tilde{x} \in \mathbb{Z}, \tilde{x} \neq 0 \) and \( f > 1 \). Then \( \beta : \Gamma((\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_2 K) \to \Gamma((\mathbb{Z}/\tilde{x}\mathbb{Z}) \otimes_2 K) \) maps \( \ell \to -\ell + \ell^1 \) and \( \tau\{\pi_3X\} \in \text{coker}(\beta) \) is represented by \( \tilde{x}\gamma([1] - [0]) \in \Gamma(\pi_2) \). Hence \( \tau\{\pi_3X\} = 0 \) if \( \tilde{x} \) is odd, so that, in this case, \( \pi_3X \) in (1.1) is a split extension over \( \pi_1 \). If both \( \tilde{x} \) and \( f \) are even, then \( \tau\{\pi_3X\} \) is a non–trivial element of order 2, and the extension \( \pi_3X \) in (1.1) does not split over \( \pi_1 \). Moreover, \( \tau\{\pi_3X\} \) is represented by \( B \) in 1.3. If \( \tilde{x} \) is even and \( f \) is odd, then \( \tau\{\pi_3X\} \) is trivial and the extension \( \pi_3X \) in (1.1) does split over \( \pi_1 \).

This result is a corollary of 1.3, the computations are contained at the end of Section 8.

We recall the notions of pre-crossed module, Peiffer commutator, crossed module and nil(2)–module, which are ingredients of algebraic models of 2– and 3–dimensional CW–complexes used in the proofs of our results, see [B] and [BHS]. In particular, Theorem 2.2 provides an exact sequence in the algebraic context of a nil(2)–module equivalent to Whitehead’s Certain Exact Sequence (1.1).

A pre–crossed module is a homomorphism of groups, \( \partial : M \to N \), together with an action of \( N \) on \( M \), such that, for \( x \in M \) and \( \alpha \in N \),

\[
\partial(x^\alpha) = -\alpha + \partial x + \alpha.
\]

Here the action is given by \( (\alpha, x) \mapsto x^\alpha \) and we use additive notation for group operations even where the group fails to be abelian. The Peiffer commutator of \( x, y \in M \) in such a pre–crossed module is given by

\[
\langle x, y \rangle = -x - y + x + y \partial x.
\]

The subgroup of \( M \) generated by all iterated Peiffer commutators \( \langle x_1, \ldots, x_n \rangle \) of length \( n \) is denoted by \( P_n(\partial) \) and a nil(n)–module is a pre–crossed module \( \partial : M \to N \) with \( P_{n+1}(\partial) = 0 \). A crossed module is a nil(1)–module, that is, a pre–crossed module in which all Peiffer commutators vanish. We also consider nil(2)–modules, that is, pre–crossed modules for which \( P_3(\partial) = 0 \).

A morphism or map \( (m, n) : \partial \to \partial' \) in the category of pre–crossed modules is given by a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{m} & M' \\
\downarrow \partial & & \downarrow \partial' \\
N & \xrightarrow{n} & N'
\end{array}
\]

in the category of groups, where \( m \) is \( n \)–equivariant, that is, \( m(x^\alpha) = m(x)^{n(\alpha)} \), for \( x \in M \) and \( \alpha \in N \). The categories of crossed modules and nil(2)–modules are full subcategories of the category of pre–crossed modules.

Note that \( P_{n+1}(\partial) \subseteq \ker \partial \) for any pre–crossed module, \( \partial : M \to N \). Thus we obtain the associated nil(n)–module \( r_n(\partial) : M/P_{n+1}(\partial) \to N \), where the action on the quotient is determined by demanding that the quotient map \( q : M \to M/P_{n+1}(\partial) \) be equivariant. For \( n = 1 \) we write \( \partial' = r_1(\partial) : M' = M/P_2(\partial) \to N \) for the crossed module associated to \( \partial \).

Given a set, \( Z \), let \( \langle Z \rangle \) denote the free group generated by \( Z \). Now take a group, \( N \), and a group homomorphism, \( f : F = \langle Z \rangle \to N \). Then the free \( N \)–group generated by \( Z \) is the free...
group, \((Z \times N)\), generated by elements denoted by \(x^\alpha = ((x, \alpha))\) with \(x \in Z\) and \(\alpha \in N\). These are elements in the product \(Z \times N\) of sets. The action is determined by
\[
((x, \alpha))^\beta = ((x, \alpha + \beta)).
\]

Define the group homomorphism \(\partial_f : (Z \times N) \to N\) by \((- \alpha + f(x) + \alpha\) for generators \(((x, \alpha)) \in Z \times N\), to obtain the pre-crossed module \(\partial_f\) with associated \(n\)-module \(r_n(\partial_f) : (Z \times N)/P_{n+1}(\partial_f) \to N\). Note that \(r_n(\partial_f)w = f\), where \(w = p\ell_{F}\) is the composition of the inclusion \(\ell_{F} : F = \langle Z \rangle \to \langle Z \times N\rangle\) and the projection \(p : \langle Z \times N \rangle \to M = \langle Z \times N\rangle/P_{n+1}(\partial_f)\) onto the quotient.

**Remark 2.1.** The \(n\)-module, \(r_n(\partial_f) : M = (Z \times N)/P_{n+1}(\partial_f) \to N\), satisfies the following universal property: For every \(n\)-module, \(\partial' : M' \to N'\), and every pair of group homomorphisms, \(m_F : F = \langle Z \rangle \to M'\), and \(n : N \to N'\) with \(\partial'm_F = nf\), there is a unique group homomorphism, \(m : M \to M'\), such that \(m_{\ell} = m_{F}\), and \((m,n) : r_n(\partial_f) \to \partial'\) is a map of \(n\)-modules.

Thus \(r_n(\partial_f)\) is called the **free \(n\)-module with basis \(f\)**. A free \(n\)-module is **totally free** if \(N\) is a free group.

Given a path connected space \(Y\) and a space \(X\) obtained from \(Y\) by attaching 2-cells, let \(Z_2\) be the set of 2-cells in \(X - Y\), and let \(f : Z_2 \to \pi_1(Y)\) be the attaching map. J.H.C. Whitehead [W1] showed that
\[
\partial : \pi_2(X,Y) \to \pi_1(Y)
\]
is a free crossed module with basis \(f\). Then \(\ker \partial = \pi_2(X), \text{coker} \partial = \pi_1(X)\) and \(\partial\) is totally free if \(Y\) is a one-point union of 1-spheres. Whitehead also proved that the abelianisation of the group \(\pi_2(X,Y)\) is the free \(R\)-module \(\langle Z_2 \rangle_R\) generated by the set \(Z_2\), where \(R = Z[\pi_1(X)]\) is the group ring \([W1]\).

Now take a totally free \(2\)-module \(\partial : M \to N\) with associated crossed module \(\partial^{cr} : M^{cr} \to N\). Let
\[
M \xrightarrow{q} M^{cr} \xrightarrow{h_2} C = (M^{cr})^{ab}
\]
be the composition of projections. Put \(K = h_2(\ker(\partial^{cr}))\). Further, let \(\Gamma\) be **Whitehead’s quadratic functor** and \(\tau : \Gamma(K) \to K \otimes K \subset C \otimes C\) the composition of the injective homomorphism induced by the quadratic map \(K \to K \otimes K, k \mapsto k \otimes k\) and the inclusion. The **Pfeiffer commutator map**, \(w : C \otimes C \to M\), is given by \(w\{x \otimes y\} = \langle x, y \rangle\), for \(x, y \in M\) with \(\{x\} = h_2(q(x)), \{y\} = h_2(q(y))\).

Lemma (IV.1.6) and Theorem (IV.1.8) in [B] imply

**Theorem 2.2.** Let \(\partial : M \to N\) be a totally free \(2\)-module. Then the sequence
\[
\Gamma(K) \xrightarrow{\tau} C \otimes C \xrightarrow{w} M \xrightarrow{q} M^{cr}
\]
is exact and the image of \(w\) is central in \(M\).

3. **Pseudo–Projective Spaces in Dimensions 2 and 3**

Real projective \(n\)-space \(\mathbb{R}P^n\) has a cell structure with precisely one cell in each dimension \(\leq n\). More generally, a CW–complex,
\[
X = S^1 \cup e^2 \cup \ldots \cup e^n,
\]
with precisely one cell in each dimension $\leq n$, is called a pseudo–projective $n$–space. For $n = 2$ we obtain pseudo–projective planes, see [O]. In this section we fix notation and consider pseudo–projective spaces in dimensions 2 and 3. In particular, we determine the totally free crossed module associated with a pseudo–projective plane and begin to investigate the totally free nil(2)–module associated with a pseudo–projective 3–space.

The fundamental group of a pseudo–projective plane $P_f = S^1 \cup e^2$, with attaching map $f \in \pi_1(S^1) = \mathbb{Z}$, is the cyclic group $\pi_1 = \pi_1(P_f) = \mathbb{Z}/f\mathbb{Z}$. We obtain $\pi_1 = \mathbb{Z}$ for $f = 0$, $\pi_1 = \{0\}$ for $f = 1$, and the bijection of sets

$$\{0, 1, 2, \ldots, f - 1\} \to \pi_1 = \mathbb{Z}/f\mathbb{Z}, \quad k \mapsto \overline{k} = k + f\mathbb{Z},$$

for $1 < f$. Addition in $\pi_1$ is given by

$$\overline{k} + \overline{\ell} = \begin{cases} k + \ell & \text{for } k + \ell < f; \\ k + \ell - f & \text{for } k + \ell \geq f. \end{cases}$$

Denoting the integral group ring of the cyclic group $\pi_1$ by $R = \mathbb{Z}[\pi_1]$, an element $x \in R$ is a linear combination

$$x = \sum_{\alpha \in \pi_1} x_\alpha [\alpha] = \sum_{k=0}^{f-1} x_\overline{k} \overline{k},$$

with $x_\alpha, x_\overline{k} \in \mathbb{Z}$. Note that $1_R = [\overline{0}]$ is the neutral element with respect to multiplication in $R$ and, for $x = \sum_{\alpha \in \pi_1} x_\alpha [\alpha], y = \sum_{\beta \in \pi_1} y_\beta [\beta]$,

$$xy = \sum_{\alpha, \beta \in \pi_1} x_\alpha y_\beta [\alpha + \beta] = \sum_{\ell=0}^{f-1} \left( \sum_{k=0}^{\ell} x_\overline{k} y_{\ell - k} + \sum_{k=\ell+1}^{f-1} x_\overline{k} y_{f - \ell + k - k} \right) \overline{\ell}.$$

The augmentation $\varepsilon = \varepsilon_R : R \to \mathbb{Z}$ maps $\sum_{\alpha \in \pi_1} x_\alpha [\alpha]$ to $\sum_{\alpha \in \pi_1} x_\alpha$. The augmentation ideal, $K$, is the kernel of $\varepsilon$. For a right $R$–module, $C$, we write the action of $\alpha \in \pi_1$ on $x \in C$ exponentially as $x^{\alpha} = x[\alpha]$.

Given a pseudo–projective plane $P_f = S^1 \cup e^2$ with attaching map $f \in \pi_1(S^1) = \mathbb{Z}$, Whitehead’s results on the free crossed module (2.2) imply that

$$\partial : \pi_2(P_f, S^1) \to \pi_1(S^1)$$

is a totally free crossed module with one generator, $e_i$, in dimensions $i = 1, 2$, and basis $\tilde{f} : Z_2 = \{ e_2 \} \to \pi_1(S^1)$ given by $\tilde{f}(e_2) = f e_1$. Note that $\partial$ has cokernel $\pi_1(P_f) = \mathbb{Z}/f\mathbb{Z} = \pi_1$ and kernel $\pi_2(P_f)$.

**Lemma 3.1.** The diagram

$$\begin{array}{ccc}
\pi_2(P_f, S^1) & \xrightarrow{\partial} & \pi_1(S^1) \\
\cong \downarrow & & \downarrow \cong \\
R & \xrightarrow{f \varepsilon_R} & \mathbb{Z}
\end{array}$$

is an isomorphism of crossed modules, where $\varepsilon_R : R \to \mathbb{Z}$ is the augmentation.

**Proof.** By Whitehead’s results [W1] on the free crossed module (2.2), it is enough to show that $\pi_2(P_f, S^1)$ is abelian. As $\partial$ is a totally free crossed module with basis $\tilde{f}$, $\pi_2(P_f, S^1)$ is generated by elements $e^n = ((e_2, n), n = 0$ and $\partial(e^n) = -n + \partial e + n = \partial e$ as $\pi_1(S^1) = \mathbb{Z}$ is abelian. We obtain

$$\langle e^n, e^m \rangle - \langle e^m, e^n \rangle = -e^n - e^m + e^n + (e^m)\partial(e^n) - (-e^m - e^n + e^m + (e^m)\partial(e^m)) = -e^n - e^m + e^n + (e^m)f - (e^m)f + e^m = (e^n, e^m),$$

where $(a, b) = -a - b + a + b$ denotes the commutator of $a$ and $b$. Thus commutators of generators are sums of Peiffer commutators which are trivial in a crossed module. □
With the notation of Theorem 2.2 and \( M = \pi_2(P_f, S^1) \), Lemma 3.1 shows that \( M = M^{cr} = (M^{cr})^{ab} = R \) and that \( \pi_2(P_f) = \ker \partial = \ker \partial^{cr} = \ker(f \cdot \varepsilon) = K \) is the augmentation ideal of \( R \), for \( f \neq 0 \). Thus the homotopy type of a pseudo-projective 3-space,
\[
P_{f,x} = S^1 \cup e^2 \cup e^3,
\]
is determined by the pair \((f, x)\) of attaching maps, \( f \in \pi_1(S^1) = \mathbb{Z} \) of the 2-cell \( e^2 \), and \( x \in \pi_2(P_f) = K \subseteq R \) of the 3-cell \( e^3 \). We obtain the totally free nil(2)–module
\[
M = \pi_2(P_{f,x}, S^1) \xrightarrow{\partial} N = \pi_1(S^1).
\]
In the next section we use Theorem 2.2 to describe the group structure of \( \pi_2(P_{f,x}, S^1) \), as well as the action of \( N \) on \( \pi_2(P_{f,x}, S^1) \). The formulæ we derive are required to compute the homotopy group \( \pi_3(P_{f,x}) \) as a \( \pi_1 \)–module.

4. Computations in nil(2)–Modules

In this Section we consider totally free nil(2)–modules, \( \partial : M \to N \), generated by one element, \( e_i \), in dimensions \( i = 1, 2 \), with basis \( \hat{f} : \{e_2\} \to N \cong \mathbb{Z} \). Then \( \pi_1 = \coker \partial = \mathbb{Z}/f\mathbb{Z} \) and, with \( R = \mathbb{Z}[\pi_1] \), we obtain \( (M^{cr})^{ab} = C = R \). Thus Theorem 2.2 yields the short exact sequence
\[
(R \otimes R)/\Gamma(K) \xrightarrow{w} M \xrightarrow{q} R
\]
with the image of \((R \otimes R)/\Gamma(K)\) central in \( M \). This allows us to compute the group structure of \( M \), as well as the action of \( N = \mathbb{Z} \) on \( M \), by computing the cross–effects of a set–theoretic splitting \( s \) of (4.1) with respect to addition and the action of \( N \), even though here \( M \) need not be commutative.

The element \( x \otimes y \in R \otimes R \) represents an equivalence class in \( R \otimes R/\Gamma(K) \), also denoted by \( x \otimes y \), so that \( w(x \otimes y) = (\hat{x}, \hat{y}) \) is the Peiffer commutator for \( x, y \in R \), with \( x = q(\hat{x}) \) and \( y = q(\hat{y}) \). As a group, \( M \) is generated by elements \( e^n = ((e_2, n)) \), in particular, \( e = e^0 = ((e_2, 0)) \), see (2.1).

We write
\[
ke^n = \begin{cases} e^n + \ldots + e^n \ (k \text{ summands}) & \text{for } k > 0, \\ 0 & \text{for } k = 0 \text{ and } \\ -e^n - \ldots - e^n \ (-k \text{ summands}) & \text{for } k < 0,
\end{cases}
\]
and define the set-theoretic splitting \( s \) of (4.1) by
\[
s : R \to M, \quad \sum_{k=0}^{f-1} x_{\Pi[k]} \mapsto x_{\Pi[0]}e^0 + x_{\Pi[1]}e^1 + \ldots + x_{\Pi[f-1]}e^{f-1}.
\]
Then every \( m \in M \) can be expressed uniquely as a sum \( m = s(x) + w(m^\oplus) \) with \( x \in R \) and \( m^\oplus \in (R \otimes R)/\Gamma(K) \). The following formulæ for the cross–effects of \( s \) with respect to addition and the action provide a complete description of the nil(2)–module \( M \) in terms of \( R \) and \( R \otimes R/\Gamma(K) \).

Given a function, \( f : G \to H \), between groups, \( G \) and \( H \), we write
\[
f(x|y) = f(x + y) - (f(x) + f(y)), \quad \text{for } x, y \in G.
\]

Lemma 4.1. Take \( x = \sum_{m=0}^{f-1} x_{\Pi[m]}y_{\Pi[m]} \in R \). Then
\[
s(x|y) = w(\nabla(x, y)),
\]
where
\[
\nabla(x, y) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{\Pi[m]}y_{\Pi[n]}w([\Pi] \otimes [\Pi] - [\Pi] \otimes [\Pi]).
\]
Thus \( \nabla(x, y) \) is linear in \( x \) and \( y \), yielding a homomorphism \( \nabla : R \otimes R \to R \otimes R \).
Proof. First note that, by definition, \( \nabla(k[|m|], \ell[|n|]) = 0 \) unless \( m > n \). To deal with the latter case, recall that commutators are central in \( M \) and use induction, first on \( k \), then on \( \ell \), to show that
\[
(ke^n, \ell e^n) = ke^{\ell e^n},
\]
for \( k, \ell > 0 \). To show equality for negative \( k \) or \( \ell \), replace \( e^m \) or \( e^n \) by \(-e^m\) and \(-e^n\), respectively. Furthermore, note that the equality
\[
(e^n, e^m) = -e^n - e^m + e^n + e^m = \langle e^n, e^m \rangle - \langle e^m, e^n \rangle
\]
for commutators of generators of totally free cyclic crossed modules derived in the proof of Lemma 3.1 holds in any totally free nil\((n)\)-module generated by one element in each dimension. Taking \( x = \sum_{m=0}^{f-1} x_{m} [m] \) and \( y = \sum_{n=0}^{f-1} y_{n} [n] \), we obtain
\[
s(x + y)
= (x_{m} + y_{m}) e + \ldots + (x_{m} + y_{m}) e^m + \ldots + (x_{f-1} + y_{f-1}) e^{f-1}
= (x e + \ldots + x_{f-1} e^{f-1}) + (y e + \ldots + y_{f-1} e^{f-1}) + \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{m} y_{n} (e^n, e^m)
= s(x) + s(y) + \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{m} y_{n} (\langle e^n, e^m \rangle - \langle e^m, e^n \rangle)
= s(x) + s(y) + \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} x_{m} y_{n} w([m] \otimes [m] - [m] \otimes [m]).
\]

\[\square\]

Corollary 4.2. Take \( x \in R \) and \( r \in \mathbb{Z} \). Then
\[
s(rx) = rs(x) + \binom{r}{2} w(\nabla(x, x)), \quad \text{where} \quad \binom{r}{2} = \frac{r(r-1)}{2}.
\]

As \( N = \mathbb{Z} \) is cyclic, the action of \( N \) on \( M \) is determined by the action of the generator, \( 1 \in \mathbb{Z} \). The formula for general \( k \in \mathbb{Z} \) provided in the next lemma is required for the definition of the set-theoretic splitting \( u_k \) of (1.1) and the explicit computation of \( A \) and \( B \) in Theorem 1.3.

Lemma 4.3. Take \( x = \sum_{n=0}^{f-1} x_{n} [n] \in R \) and \( k \in \pi_1 \). Write \( R = \mathbb{Z}[^0, \ldots, f-1] = R_k \times \hat{R}_k \), where \( R_k = \mathbb{Z}[^0, \ldots, f-k-1] \) and \( \hat{R}_k = \mathbb{Z}[^f-k, \ldots, f-1] \). Then
\[
(s(x))^k = s(x^k) + w(\nabla_k(a, b)),
\]
where \( x = (a, b) \) and
\[
\nabla_k : R_k \times \hat{R}_k \to R \otimes R, \quad (a, b) \mapsto Q_k(a, b) + L_k(b)
\]
with
\[
Q_k(a, b) = \sum_{p=0}^{f-k-1} \sum_{q=0}^{f-k-1} x_{p+q} [p+q] \otimes [q] - [q] \otimes [p] \]
\[
L_k(b) = \sum_{q=0}^{f-k-1} x_{p+q} [q] \otimes [q].
\]
Thus \( Q_k \) is linear in \( a \) and \( b \) and \( L_k \) is linear in \( b \).

Proof. For \( f \in \pi_1 \) and \( p \in \mathbb{Z} \),
\[
e^{j+f} = (e^{j})^{\tilde{\partial}(e)}
= e^j + (e^j, e) + \langle e, e^j \rangle
= e^j - \langle e, e^j \rangle - \langle e^j, e \rangle + \langle e, e^j \rangle
= e^j + \langle e^j, e \rangle.
\]
Thus, for $\pi, \bar{k} \in \pi_1$, with $\pi + \bar{k} = \bar{j}$,
\[
(s(\pi))_k^k = \begin{cases} 
  e_j, & \text{for } 0 \leq n < f - k, \\
  e_j + (e_j, e_j), & \text{for } f - k \leq n < f
\end{cases} 
= \begin{cases} 
  s(\pi\bar{k}), & \text{for } 0 \leq n < f - k, \\
  s(\pi\bar{k}) + w([\bar{j}] \otimes [\bar{j}]), & \text{for } f - k \leq n < f.
\end{cases}
\]

Hence, for $x = \sum_{p=0}^{f-1} x_\pi[p]$,
\[
(s(x))_k^k = x_\pi s([0])^k + x_\pi s([1])^k + \ldots + x_{f-n} s([f-n])^k \\
= x_\pi s([\bar{k}])^k + x_\pi s([\bar{1}])^k + \ldots + x_{f-n} s([f-n])^k + \sum_{n=f-k}^{f-1} x_\pi w([n+k-f] \otimes [n+k-f]) \\
= x_{f-k} s([\bar{k}])^k + \ldots + x_{f-n} s([f-n])^k + x_\pi s([\bar{0}])^k + \ldots + x_{f-k} s([f-k])^k \\
+ \sum_{p=0}^{f-k-1} \sum_{n=f-k}^{f-1} x_\pi s(p+k) s([\pi+n])^k + \sum_{q=0}^{k-1} x_{q+f-k} w([q] \otimes [q]) \\
= s(x^k) + \sum_{p=0}^{f-k-1} \sum_{q=0}^{k-1} x_\pi x_{q+f-k} w([p+k] \otimes [q] - [q] \otimes [q]) + \sum_{q=0}^{k-1} x_{q+f-k} w([q] \otimes [q]).
\]

\[\square\]

**Remark 4.4.** We use the final results of this section to define and establish the properties of the set-theoretic splitting $u_x$ of (1.1). The next result shows how the cross-effects interact with multiplication in $R$.

**Lemma 4.5.** Take $x, y \in R$. Then
\[
\sum_{i=0}^{f-1} y_i(s(x))^i = s(xy) + w(\mu(x, y)),
\]
where $\mu : R \times R \to R \otimes R$ is given by
\[
\mu(x, y) = -\sum_{i<j} y_i y_j \nabla(x^i, x^j) + \sum_{i=0}^{f-1} \left( \nabla_i(y^i x^i) - \frac{y_i}{2} \nabla(x^i, x^j) \right).
\]

**Proof.** By Lemmata 4.1 and 4.3 and Corollary 4.2, we obtain, for $x, y \in R$,
\[
\sum_{i=0}^{f-1} y_i(s(x))^i = \sum_{i=0}^{f-1} (y_i s(x))^i \\
= \sum_{i=0}^{f-1} s(y_i x^i) - \left( \frac{y_i}{2} \right) w(\nabla(x, x))^i \\
= \sum_{i=0}^{f-1} s(y_i x^i) + w(\nabla_i(y_i x^i)) - \left( \frac{y_i}{2} \right) w(\nabla(x, x))^i \\
= s(\sum_{i=0}^{f-1} y_i x^i) - \sum_{i<j} w(\nabla(y_i x^i, y_j x^j)) + \sum_{i=0}^{f-1} w(\nabla_i(y_i x^i)) - \left( \frac{y_i}{2} \right) w(\nabla(x, x)^i).
\]

\[\square\]

Finally, the definitions and a simple calculation yield
Lemma 4.6. For \( x, y, z \in R \) and with the notation in (4.2),

\[
\mu(x, y | z) = - \sum_{i < j} (y_i z_j + z_i y_j) \nabla(x^i, x^j) + 2 \sum_{i=1}^{f-1} y_i z_i Q_i(x) - \sum_{i=0}^{f-1} y_i z_i \nabla(x, x^i).
\]

Hence, for fixed \( x \in R, \mu(x, ) : R \times R \to R \otimes R, (y, z) \mapsto \mu(x, y | z) \) is bilinear.

5. Quadratic Modules

In dimension 3, quadratic modules assume the role played by crossed modules in dimension 2. We recall the notion of quadratic modules and totally free quadratic modules, see [B], which we require for the description of the third homotopy group \( \pi_3(P_{f,x}) \) of a 3–dimensional pseudo–projective space \( P_{f,x} \), as in (3.2).

A quadratic module \((\omega, \delta, \partial)\) consists of a commutative diagram of group homomorphisms

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & C \\
\downarrow{\delta} & & \downarrow{\partial} \\
L & \rightarrow & M & \rightarrow & N,
\end{array}
\]

such that

- \( \partial : M \to N \) is a nil(2)–module with quotient map \( M \to C = (M^e)^{ab}, x \mapsto \{ x \} \), and Peiffer commutator map \( w \) given by \( w(\{ x \} \otimes \{ y \}) = \langle x, y \rangle \);
- the boundary homomorphisms \( \partial \) and \( \delta \) satisfy \( \partial \delta = 0 \), and the quadratic map \( \omega \) is a lift of \( w \), that is, for \( x, y \in M \),
  \[ \delta \omega(\{ x \} \otimes \{ y \}) = \langle x, y \rangle; \]
- \( N \) acts on \( L \), all homomorphisms are equivariant with respect to the action of \( N \) and, for \( a \in L \) and \( x \in M \),
  \[ a^{\partial(x)} = a + \omega(\{ \delta a \} \otimes \{ x \} + \{ x \} \otimes \{ \delta a \}); \]
- finally, for \( a, b \in L \),
  \[ (a, b) = -a - b + a + b = \omega(\{ \delta a \} \otimes \{ \delta b \}). \]

A map \( \varphi : (\omega, \delta, \partial) \to (\omega', \delta', \partial') \) of quadratic modules is given by a commutative diagram

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & L \\
\downarrow{\varphi \otimes \varphi} & & \downarrow{l} \\
C' \otimes C' & \xrightarrow{\omega'} & L' \\
\end{array}
\]

where \( l \) is \( n \)–equivariant, and \( (m, n) \) is a map between pre–crossed modules inducing \( \varphi_* : C \to C' \).

Given a nil(2)–module \( \partial : M \to N \), a free group \( F \) and a homomorphism \( \bar{f} : F \to M \) with \( \partial \bar{f} = 0 \), a quadratic module \( (\omega, \delta, \partial) \) is free with basis \( \bar{f} \), if there is a homomorphism \( i : F \to L \) with \( \delta i = \bar{f} \), such that the following universal property is satisfied: For every quadratic module \( (\omega', \delta', \partial') \) and map \( (m, n) : \partial \to \delta' \) of nil(2)–modules and every homomorphism \( l_F : F \to L' \) with \( mf = \delta' l_F \), there is a unique map \( (l, m, n) \) of quadratic modules with \( li = l_F \).
For $F = \langle Z \rangle$, the homomorphism $\hat{f}$ is determined by its restriction $\hat{f}|_{Z}$ which is then called a basis for $(\omega, \delta, \partial)$. A quadratic module $(\omega, \delta, \partial)$ is totally free if it is free, if $\partial$ is a free nil(2)-module and if $N$ is a free group.

6. The Homotopy Group $\pi_3$ of a Pseudo–Projective 3–Space and the Associated Splitting Function $u_x$

In this section we return to pseudo–projective 3–spaces

$$P_{f,x} = S^1 \cup e^2 \cup e^3,$$

determined by the pair $(f, x)$ of attaching maps, $f \in \pi_1(S^1) = Z$ and $x \in \pi_2(P_f) = K \subseteq R$, as in (3.2). Using results on totally free quadratic modules in [B], we investigate the structure of the third homotopy group $\pi_3(P_{f,x})$ as a $\pi_1$–module by defining a set–theoretic splitting $u_x$ of J.H.C. Whitehead's Certain Exact Sequence of the universal cover, $\hat{P}_{f,x},$

$$\Gamma(\pi_2(P_{f,x})) \longrightarrow \pi_3(P_{f,x}) \longrightarrow \text{H}_3(\hat{P}_{f,x}).$$

Recall that $\pi_1 = \pi_1(P_f) = Z/fZ$ with augmentation ideal $K = \ker f \varepsilon$, and let $B$ be the image of $d_x : R \rightarrow R, y \mapsto xy$. Then

$$\pi_2(P_{f,x}) = \text{H}_2(\hat{P}_{f,x}) = K/B = (\ker f \varepsilon)/xR.$$

The functor $\sigma$ in (IV 6.8) in [B] assigns a totally free quadratic module $(\omega, \delta, \partial)$ to the pseudo–projective 3–space $P_{f,x}$ and we obtain the commutative diagram

$$\Gamma(\pi_2(P_{f,x})) \longrightarrow R \otimes R/\Delta_B \longrightarrow R \otimes R/\Gamma(K)$$

$$\pi_3(P_{f,x}) \longrightarrow M \longrightarrow N$$

$$\text{H}_3(\hat{P}_{f,x}) \longrightarrow R \otimes R \longrightarrow \text{Z}$$

of straight arrows. Here the generators $e_3 \in L, e_2 \in M$ and $e_1 = 1 \in N = Z$ correspond to the cells of $P_{f,x}$ and $\partial$ is the totally free nil(2)–module of Lemma 3.1. The right hand column is the short exact sequence (4.1) with the set theoretic splitting $s$ defined in Section 4. The short exact sequence in the middle column is described in (IV 2.13) in [B], where the product $[\alpha, \beta]$ of $\alpha \in K$ and $\beta \in B$ is given by $[\alpha, \beta] = \alpha \otimes \beta + \beta \otimes \alpha \in R \otimes R$ and

$$\Delta_B = \Gamma(B) + [K, B].$$

By Corollary (IV 2.14) in [B], taking kernels yields Whitehead's short exact sequence (6.1) in the left hand column of the diagram, that is, $\ker q = \Gamma(\pi_2(\hat{P}_{f,x}))$, $\ker \delta = \pi_3(P_{f,x})$ and $\ker d_x = \text{H}_3(\hat{P}_{f,x})$. As $(\omega, \delta, \partial)$ is a quadratic module associated to $P_{f,x}$, we may assume that $\delta(e_3) = s(x)$.

In Section 4 we determined the structure of $M$ as an $N$–module by computing the cross–effects of the set–theoretic splitting $s$ with respect to addition and the action. Analogously to the definition of $s$, we now define a set-theoretic splitting of the short exact sequence in the second column of this diagram by

$$t_x : R \longrightarrow L, \quad \sum_{k=0}^{f-1} y_{f-1-k} e_3^k \longrightarrow y_{f-1} e_3^{f-1}.$$

The cross–effects of $t_x$ with respect to addition and the action determine the $N$–module structure of $L$, but we want to determine the module structure of $\pi_3(P_{f,x})$. To obtain a set theoretic splitting of the first column which will allow us to do so, we must adjust $t_x$, such that the image of $\text{H}_3(\hat{P}_{f,x})$ under the new splitting is contained in $\ker \delta = \pi_3(P_{f,x})$. Recall that $\delta$ is a homomorphism which
is equivariant with respect to the action of $N$ and $\delta(e_3) = s(x)$. Thus Lemma 4.5 yields, for $y \in H_3(\hat{P},x) = \ker d_x$, that is, for $d_x(y) = xy = 0$,
\[
\delta(t_x(y)) = \delta\left(\sum_{i=0}^{f-1} y_i e_3^i\right) = \sum_{i=0}^{f-1} y_i \delta(e_3)^i = \sum_{i=0}^{f-1} y_i (s(x))^i = s(x)^y + w(\mu(x,y)) = \delta \omega \mu(x,y).
\]
Hence $t_x(y) - \omega \mu(x,y) \in \ker \delta = \pi_3(P,f_x)$, giving rise to the set theoretic splitting
\[
\pi_3 = \ker \delta \oplus \pi_3(P,f_x), \quad y \mapsto t_x(y) - \omega \mu(x,y)
\]
of the Hurewicz map $\pi_3 \to H_3$. The cross–effects of $u_x$ with respect to addition and the action determine (6.1) as a short exact sequence of $\pi_1$–modules. In Section 7 we determine the cross–effects of $t_x$ and investigate the properties of the functions $A$ and $B$ describing the cross–effects of $u_x$.

7. Computations in Free Quadratic Modules

The first two results of this Section describe the cross–effects of $t_x$ with respect to addition and the action, respectively. We then turn to the properties of the cross–effects of $u_x$.

**Lemma 7.1.** Take $z, y \in R$. Then, with the notation in (4.2),
\[
t_x(z|y) = \omega(\Psi(z, y)),
\]
where
\[
\Psi(z, y) = \sum_{mn=1}^{f-1} \sum_{n=0}^{m-1} z^m y^n x[\bar{m}] \otimes x[\bar{n}].
\]
Thus $\Psi(z, y)$ is linear in $z$ and $y$, yielding a homomorphism $\Psi : R \otimes R \to R \otimes R$.

**Proof.** As in the proof of Lemma 4.1, we obtain
\[
t_x(z|y) = \sum_{mn=1}^{f-1} \sum_{n=0}^{m-1} z^m y^n (e_3^m, e_3^n).
\]
Noting that $\{\delta(e_3^m)\} = \{\delta(t_x([\bar{m}]))\} = d_x([\bar{m}]) = x[\bar{m}]$. Thus (5.2) yields
\[
t_x(z|y) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} z^m y^n \omega(\{\delta(e_3^m)\}) \otimes \{\delta(e_3^n)\}) = \sum_{m=1}^{f-1} \sum_{n=0}^{m-1} z^m y^n \omega(x[\bar{m}] \otimes x[\bar{n}]).
\]

As $N = \mathbb{Z}$ is cyclic, the action of $N$ on $L$ is determined by the generator $1 \in \mathbb{Z}$.

**Lemma 7.2.** Take $x \in R$. Then
\[
(t_x(y))^1 = t_x(y^\top) + \omega(\overline{\Psi}_4(a, b)),
\]
where
\[
\overline{\Psi}_4 = \sum_{p=0}^{f-2} y_p y_{f-1-p} x[\bar{m}+1] \otimes x[\bar{0}] + y_{f-1-p} (x \otimes [\bar{0}] + [\bar{0}] \otimes x).
\]

**Proof.** With $\{\delta(e_3^m)\} = x[\bar{m}]$ from above and (5.1), we obtain
\[
e_3^1 + f = (e_3^m)^f = (e_3^m) \delta(e) = e^1 + \omega(\{\delta(e_3^m)\} \otimes \{e\} + \{e\} \otimes \{\delta(e_3^m)\})
= t_x([\bar{m}]^\top) + \omega(x[\bar{1}] \otimes [\bar{0}] + [\bar{0}] \otimes x[\bar{1}]).
\]
Thus, for $n \in \pi$,
\[
(t_x([\bar{m}]))^n = \begin{cases} 
\omega(t_x([\bar{m}])^\top) & \text{for } 0 \leq n < f - 1, \\
\omega(t_x([\bar{m}])^\top + x[\bar{1}] \otimes [\bar{0}] + [\bar{0}] \otimes x[\bar{1}]) & \text{for } n = f - \ell.
\end{cases}
\]
With (5.2), we obtain, for $y = \sum_{n=0}^{f-1} y_n[n]$
\[
(t_x(y))^1 = y_f e_4^1 + y_{f-1} e_2^1 \ldots + y_1 e_3^1 + y_0 e_3^0
\]
\[
= y_f t_x([0]^1) + \ldots + y_{f-2} t_x([f-1]^1) + y_{f-1} t_x([f-1]^0) + y_{f-2} \omega(x \otimes [0] + [0] \otimes x)
\]
\[
= t_x(y^1) + \sum_{p=0}^{f-2} y_p y_{f-p} \sum_{e_3} (e_3^{p+1}, e_3) + y_{f-1} \omega(x \otimes [0] + [0] \otimes x)
\]
\[
= t_x(y^1) + \sum_{p=0}^{f-2} y_p y_{f-p} x[p + 1] \otimes x[0] + y_{f-1} \omega(x \otimes [0] + [0] \otimes x)
\]

The next two results concern the properties of the maps $A$ and $B$ which describe the cross–effects of $u_x$ with respect to addition and the action, respectively.

**Lemma 7.3.** For $x \in K$ the map
\[ A : H_3 \hat{P}_{f,x} \times H_3 \hat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), (y, z) \mapsto u_x(y|z) \]

is bilinear.

**Proof.** Take $x \in K$ and $y, z \in H_3 \hat{P}_{f,x}$. By definition
\[ A(y, z) = u_x(y|z) = t_x(y|z) - \omega \mu(x, y|z) = \omega(\Psi(y, z) - \mu(x, y|z)). \]

Thus Lemmata 4.6 and 7.1 imply that $A$ is bilinear. \qed

**Lemma 7.4.** For $x \in K$ define
\[ B : H_3 \hat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), y \mapsto (u_x(y))^1 - u_x(y^1) \]

Then
\[ H_3 \hat{P}_{f,x} \times H_3 \hat{P}_{f,x} \to \Gamma(\pi_2 P_{f,x}), (y, z) \mapsto B(y|z) \]

is bilinear.

**Proof.** Take $x \in K$ and $y, z \in H_3 \hat{P}_{f,x}$. Then
\[ (A(y, z))^1 = (u_x(y + z) - (u_x(y) + u_x(z))^1 \]
\[ = (u_x(y + z))^1 - (u_x(y))^1 - (u_x(z))^1 \]
\[ = B(y + z) + u_x((y + z)^1) - (B(y) + u_x(y^1) + B(z) + u_x(z^1)). \]
\[ = B(y|z) + A(y^1, z^1) \]

Thus
\[ (7.1) \quad B(y|z) = (A(y, z))^1 - A(y^1, z^1) \]

and bilinearity follows from that of $A$ and the properties of an action. \qed

8. **Examples of Pseudo–Projective 3–Spaces**

In this Section we provide explicit computations for examples of pseudo–projective 3–spaces, including proofs for Theorem 1.1, Theorem 1.3 and Theorem 1.4.

Note that, as abelian group, the augmentation ideal $K$ of a pseudo–projective 3–space $P_{f,x}$, as in (3.2), is freely generated by \{ $[1] - [0], \ldots, [f-1] - [0]$ \}. We consider pseudo–projective 3–spaces, $P_{f,x}$, with $x = \hat{x}([1] - [0])$ and $\hat{x} \in \mathbb{Z}$. We compute $\pi_2(P_{f,x}), \pi_3(P_{f,x})$, as well as the cross–effects of $u_x$ for this special case. For $f = 2$, the general case coincides with the special case and provides an example where $\pi_1$ acts trivially on $\Gamma\pi_2(P_{2,\hat{x}})$ and on $H_3(P_{2,\hat{x}})$, but non–trivially on $\pi_3(P_{2,\hat{x}})$.
Lemma 8.1. For \( x = \hat{x}([1] - [0]) \) with \( \hat{x} \in \mathbb{Z} \),
\[
H_3(\hat{P}_{f,x}) = \{ \hat{y}N \mid \hat{y} \in \mathbb{Z} \} \cong \mathbb{Z},
\]
is generated by the norm element \( N = \sum_{k=0}^{f-1} [k] \). Hence \( \pi_1 \) acts trivially on \( H_3(\hat{P}_{f,x}) \). Furthermore,
\[
\pi_2(P_{f,x}) = (\mathbb{Z}/\hat{x}\mathbb{Z}) \otimes K.
\]
Hence \( \hat{x}^2\ell = 0 \) for every \( \ell \in \Gamma(\pi_2(P_{f,x})) \).

Proof. Take \( x = \hat{x}([1] - [0]) \) with \( \hat{x} \in \mathbb{Z} \) and \( y = \sum_{k=0}^{f-1} y_{[k]} \in \ker d_x \). Then
\[
d_x(y) = xy = 0 \iff \hat{x} \sum_{k=0}^{f-1} y_{[k]}([1] - [k]) = 0
\]
\[
\iff y_{[f-1]} - y_0 = y_1 = y_2 = \ldots = y_{f-2} = \hat{y},
\]
for some \( \hat{y} \in \mathbb{Z} \). Hence \( y = \hat{y}N \).

By (6.2), \( \pi_2(P_{f,x}) = K/xR \). As abelian group, \( K = \ker \varepsilon \) is freely generated by \( \{ [k] - [0] \} \) for \( 1 \leq k \leq f-1 \) and hence also by \( \{ [k] - [k-1] \} \) for \( 1 \leq k \leq f-1 \). For \( y = \sum_{i=0}^{f-1} y_i[1] \in R \) we obtain
\[
xy = \hat{x} \sum_{i=1}^{f-1} y_i([i] - [i-1]) + \hat{x}y_{f-1}([0] - [f-1])
\]
\[
= \hat{x} \sum_{i=1}^{f-1} y_i([i] - [i-1]) - \hat{x}y_{f-1} \sum_{i=1}^{f-1} ([i] - [i-1])
\]
\[
= \hat{x} \sum_{i=1}^{f-1} (y_i - y_{f-1})([i] - [i-1]).
\]
As \( \hat{x}K \subseteq xR \), we obtain \( xR = \hat{x}K \) and hence
\[
\pi_2(P_{f,x}) = K/xR = K/\hat{x}K = (\mathbb{Z}/\hat{x}\mathbb{Z}) \otimes K.
\]
If \( \hat{x} \) is odd, then every element \( \ell \in \Gamma(\pi_2(P_{f,x})) \) has order \( \hat{x} \). If \( \hat{x} \) is even, an element \( \ell \in \Gamma(\pi_2(P_{f,x})) \) has order \( 2\hat{x} \) or \( \hat{x} \). In either case, \( \hat{x}^2\ell = 0 \) for every \( \ell \in \Gamma(\pi_2(P_{f,x})) \). \( \square \)

Lemma 8.2. Take \( x = \hat{x}([1] - [0]) \) and \( y, z \in H_3(\hat{P}_{f,x}) \). Then
\[
A(y, z) = 0.
\]

Proof. By definition,
\[
A(y, z) = u_x(y|z) = t_x(y|z) - \omega\mu(x, y|z) = \omega(\Psi(y, z) - \mu(x, y|z)).
\]
The definition of \( \Psi \) and Lemma 4.6 yield
\[
\Psi(y, z) - \mu(x, y|z) = \hat{y}z \left( \sum_{p=1}^{f-1} \sum_{q=0}^{p-1} x[\eta] \otimes x[\rho] + 2 \sum_{q=1}^{f-1} \sum_{p=0}^{q-1} \nabla(x^p, x^q) - 2 \sum_{p=0}^{f-1} Q_p(x) + \sum_{p=0}^{f-1} (\nabla(x, x))^p \right).
\]
Recall that \( \hat{x}^2\ell = 0 \) for every \( \ell \in \Gamma(\pi_2(P_{f,x})) \) and note that, by the properties of \( Q \) and \( \nabla \), each summand in the above sum has a factor of \( \hat{x}^2 \). \( \square \)

Lemma 8.3. Let \( \gamma : \pi_2(P_{f,x}) \to \Gamma(\pi_2(P_{f,x})) \) be the universal quadratic map for the Whitehead functor \( \Gamma \). Take \( q : K \to \pi_2(P_{f,x}), k \mapsto 1 \otimes k, x = \hat{x}([1] - [0]) \) and \( y = \hat{y}N \). Then
\[
B(y) = -\hat{x}\hat{y}\gamma([1] - [0]).
\]
By Lemma 8.3 and (8.1), the action of \( \pi \) that is, 3 trivial. The group \( H \)

\[
\begin{align*}
\Psi_1(y) &= \sum_{p=0}^{f-2} \hat{y}^2 ((\hat{x}([T] - [0]) \otimes (\hat{x}[T] - [0])) + \hat{y}(\hat{x}([T] - [0]) \otimes [0] + [0] \otimes \hat{x}([T] - [0]))) \\
&= \hat{x}\hat{y}(([[T] - [0]) \otimes [0] + [0] \otimes ([T] - [0]))
\end{align*}
\]

Lemma 4.5 yields

\[
\mu(x, y) = -\sum_{q=0}^{f-1} \sum_{p=0}^{f-2} \hat{y}^2 \nabla((|p+1| - |p|), (|q+1| - |q|)) + \sum_{p=0}^{f-1} \nabla_p (\hat{y}\hat{x}([T] - [0]))
\]

\[
= \nabla_{f-1} (\hat{x}\hat{y}([T] - [0]))
\]

\[
= -\hat{x}^2\hat{y}^2 ([f-1] \otimes [0] - [0] \otimes [0]) + \hat{x}\hat{y} [0] \otimes [0]
\]

Thus

\[
B(y) = (u_x(y))^1 - u_x(y^T) = \omega(\Psi_1(y) - (\mu(x, y))^1 + \mu(x, y)) = -\hat{x}\hat{y}\gamma q([T] - [0]).
\]

Together Lemmata 8.1, 8.2 and 8.3 provide a proof of Theorem 1.3.

For \( f = 2 \) the special case coincides with the general case and we obtain

**Theorem 8.4.** Let \( X = P_{2,x} \) be a pseudo–projective 3–space with \( x = \hat{x}([T] - [0]), \) for \( \hat{x} \in \mathbb{Z} \) and \( \hat{x} \neq 0. \) Then \( u_x \) is a homomorphism and the fundamental group \( \pi_1 = \mathbb{Z} / 2\mathbb{Z} \) acts trivially on \( \Gamma(\pi_2 P_{2,x}) \) and on \( H_3 P_{2,x}. \) The action of \( \pi_1 \) on \( \pi_3 P_{2,x} \) is non–trivial if and only if \( \hat{x} \) is even.

**Proof.** For \( f = 2 \) the augmentation ideal \( K \) is generated by \( k = [T] - [0]. \) Since \( k[T] = -k, \) the action of \( \pi_1 = \mathbb{Z} / 2\mathbb{Z} \) on \( K \) and hence on \( \pi_2 P_{2,x} = K / xR = \mathbb{Z} / \hat{x}\mathbb{Z} \) is multiplication by \(-1\). As the \( \Gamma \)–functor maps multiplication by \(-1\) to the identity morphism, the action on \( \pi_1 \) on \( \Gamma(\pi_2 P_{2,x}) \) is trivial. The group \( H_3 P_{2,x} \) is generated by the norm element \( N = [0] + [T]. \) As \( N[T] = N, \pi_1 \) acts trivially on \( H_3 P_{2,x}. \) As \( \pi_2 = \mathbb{Z} / \hat{x}\mathbb{Z} \) is cyclic, \( \Gamma\pi_2 = \pi_2 \) if \( \hat{x} \) is odd and \( \Gamma\pi_2 = \mathbb{Z} / 2\hat{x}\mathbb{Z} \) if \( \hat{x} \) is even, that is,

\[
(8.1) \quad \Gamma\pi_2 = \mathbb{Z} / \gcd(\hat{x}, 2)\hat{x}\mathbb{Z}.
\]

By Lemma 8.3 and (8.1), the action of \( \pi_1 \) on \( \pi_3 X \) is non–trivial if and only if \( \hat{x} \) is even. \( \square \)

Theorem 1.1 is a corollary to Theorem 8.4.

**Proof of 1.4.** Note that \( \mathbb{Z} / \hat{x}\mathbb{Z} \otimes_k \mathbb{Z} \) is generated by \( \{\alpha_k = q([k] - [k-1])\}_{0 < k < f}, \) where \( q : K \rightarrow \mathbb{Z} / \hat{x}\mathbb{Z} \otimes_k \mathbb{Z}, k \mapsto 1 \otimes k. \) Thus \( \Gamma\pi_2(P_{f,x}) = \Gamma(\mathbb{Z} / \hat{x}\mathbb{Z} \otimes_k \mathbb{Z}) \subseteq (\mathbb{Z} / \hat{x}\mathbb{Z} \otimes_k \mathbb{Z}) \otimes (\mathbb{Z} / \hat{x}\mathbb{Z} \otimes_k \mathbb{Z}) \) is generated by \( \{q(\alpha_k), q(\alpha_{=0}), q(\alpha_{<0})\}_{0 < k < f}. \) With \( \alpha_k = \alpha_{k+1} \) for \( 1 < k < f - 1 \) and \( \alpha_{f-1} = [0] - [f-1] = -\sum_{i=1}^{f-1} \alpha_i, \) we obtain, for \( \ell = \sum_{k=1}^{f-1} \ell_k \gamma(\alpha_k) + \sum_{k=1}^{f-1} \sum_{j=1}^{f-k-1} \ell_{j,k} [\alpha_j, \alpha_k] \in \mathbb{Z} / \hat{x}\mathbb{Z} \),

\[
\begin{align*}
\ell \equiv -\sum_{i=1}^{f-1} \alpha_i & \pmod{\hat{x}\mathbb{Z}}.
\end{align*}
\]
\(\Gamma(\pi_2(P_{f,\hat{x}}))\),

\[
\ell^i - \ell = \sum_{k=1}^{f-1} \sum_{j=1}^{k-1} \ell_{j,k} q(\alpha_k) + \sum_{k=2}^{f-2} \sum_{j=1}^{k-1} \ell_{j,k} q(\alpha_k) - \sum_{k=2}^{f-1} \sum_{j=1}^{k-1} \ell_{j,k} q(\alpha_k)
\]

Thus the sequence (1.1) splits if and only if there is at least one solution of the system of equations

\[
\ell_{f-1} - \ell_1, \quad \ell_{k-1} - \ell_{k-1, f-1}, \quad \ell_{f-1, k-1} - \ell_{f-1, f-1, k-1}, \quad \ell_{f-1, k-1, f-1}
\]

For odd \(f\), a solution of the system is given by \(\ell_{j,k} = 0\) for \(1 \leq j \leq k - 1, 1 < k < f - 1, \ell_k = 0\) for \(k\) odd, and \(\ell_k = \hat{x}\) for \(k\) even. Hence (1.1) splits if \(f\) is odd. It remains to show that there are no solutions for even \(f > 2\).

For \(2 \leq j < \frac{1}{2}(f - 2)\), subtract the equation \((D_{j, f-j+i})\) from the equation \((D_{j, f-j+i-1})\) for \(2 \leq i < j\). Add \((D_{j, f-j})\) and \((C_{f-j}),\) then subtract \((C_{f-j+i})\). Adding the resulting equations yields

\[
\ell_{f-1} - \ell_1 - \ell_{f-1, f-1} - \ell_{f-1, f-1, f-1} = 0 \quad \text{mod} \ 2\hat{x}
\]

For \(2 \leq j < \frac{1}{2}(f - 2)\), multiply the equations \((C_{f-j}), 2 \leq j \leq \frac{1}{2}(f - 2)\) by 2 and adding them we obtain

\[
0 = (f - 2)\ell_{f-1} - 2\sum_{j=1}^{f-2} \ell_{j,f-1} \quad \text{mod} \ 2\hat{x}.
\]

On the other hand, adding the equations \((A)\) and \((B_k), 1 < k < f - 1\), the resulting equation is

\[
\hat{x} = (f - 2)\ell_{f-1} - 2\sum_{j=1}^{f-2} \ell_{j,f-1} \quad \text{mod} \ 2\hat{x}.
\]

Hence there are no solutions for \(f\) even.
Theorem 9.1. implies that every element of \( \pi_3(P_{f,x}) \) may be expressed uniquely as a sum \( u_x(v) + \beta \) with \( v \in H_3(\bar{P}_{f,x}) \), that is, \( xv = 0 \), and \( \beta \in \Gamma(\pi_2(P_{f,x})) = \Gamma((\ker f) / xR) \), see (6.2). Using \( u_x(y) + \alpha \in \pi_3(P_{f,x}) \) to attach a 4–cell to \( P_{f,x} \) we obtain the 4–dimensional pseudo–projective space,

\[
P = P_{f,x,y,\alpha} = S_1 \cup e^2 \cup e^3 \cup e^4.
\]

Note that the homotopy type of \( P = P_{f,x,y,\alpha} \) is determined by \( (f, x, y, \alpha) \) and that every 4–dimensional pseudo–projective space is of this form. The cellular chain complex, \( C_*(\bar{P}) \), of the universal cover, \( \bar{P} = \bar{P}_{f,x,y,\alpha} \), is the complex of free \( R \)–modules,

\[
\langle e_4 \rangle_R \xrightarrow{d_4} \langle e_3 \rangle_R \xrightarrow{d_3} \langle e_2 \rangle_R \xrightarrow{d_2} \langle e_1 \rangle_R \xrightarrow{d_1} \langle e_0 \rangle_R,
\]

given by \( d_1(e_1) = e_0([\Gamma] - [\emptyset]) \), \( d_2(e_2) = e_1N \), that is, multiplication by the norm element, \( N = \sum_{t=0}^{f-1} \emptyset \), \( d_3(e_3) = e_2x \), and \( d_4(e_4) = e_3y \). Let \( b : R \rightarrow \pi_3P_{f,x} \) be the homomorphism of \( R \)–modules which maps the generator \( [\emptyset] \in R \) to \( b([\emptyset]) = u_x(y) + \alpha \), so that composition with the projection onto \( H_3\bar{P}_{f,x} \) yields the homomorphism of \( R \)–modules induced by the boundary operator \( d_4 \). Thus we obtain the commutative diagram

\[
\begin{array}{cccc}
\text{H}_4\bar{P} & \xrightarrow{b} & \Gamma\pi_2P & \xrightarrow{j} & \pi_3P \\
\downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \\
\text{H}_3\bar{P}_{f,x} & \xrightarrow{h} & \pi_3P_{f,x} & \xrightarrow{h} & \text{H}_3\bar{P}
\end{array}
\]

in the category of \( R \)–modules, where the middle column is the short exact sequence (6.1) and

\[
(9.1) \quad H_4\bar{P} \xrightarrow{b} \Gamma\pi_2P \xrightarrow{j} \pi_3P \xrightarrow{h} H_3\bar{P}
\]

is Whitehead’s Certain Exact Sequence of the universal cover, \( \bar{P} = \bar{P}_{f,x,y,\alpha} \).

Now we restrict attention to the case \( f = 2 \). Then \( \pi_1 = \pi_1P = \mathbb{Z}/2\mathbb{Z} \) and the augmentation ideal, \( K \) is generated by \( [\Gamma] - [\emptyset] \). Thus

\[
x = \hat{x}([\Gamma] - [\emptyset]) \quad \text{and} \quad y = \hat{y}([\Gamma] + [\emptyset]), \quad \text{for some} \ \hat{x}, \hat{y} \in \mathbb{Z}.
\]

We assume that \( x \) and \( y \) are non–trivial, that is, \( \hat{x}, \hat{y} \neq 0 \).

Theorem 9.1. For \( P = P_{2,x,y,\alpha} \), with \( x \) and \( y \) as above, \( \pi_1P = \mathbb{Z}/2\mathbb{Z} \) acts on \( \pi_2P = \mathbb{Z}/\hat{x}\mathbb{Z} \) via multiplication by \(-1\), trivially on \( H_3\bar{P} = \mathbb{Z}/\hat{y}\mathbb{Z} \) and via multiplication by \(-1\) on \( H_3\bar{P} = \mathbb{Z} = (\langle \Gamma \rangle - [\emptyset]) \). The exact sequence (9.1) is given by

\[
(9.2) \quad H_4\bar{P} = \mathbb{Z} \xrightarrow{b} \Gamma\pi_2P = \Gamma(\mathbb{Z}/\hat{x}\mathbb{Z}) \xrightarrow{j} \pi_3P \xrightarrow{h} H_3\bar{P} = \mathbb{Z}/\hat{y}\mathbb{Z}.
\]

Denoting the generator of \( \Gamma\pi_2P \) by \( \xi \), the boundary \( b \) is determined by

\[
b([\Gamma] - [\emptyset]) = \hat{x}\hat{y}\xi,
\]

and the action of \( \pi_1P \) on \( \pi_3P \) is trivial. As abelian group, \( \pi_3P \) is the extension of \( H_3\bar{P} \) by \( \text{coker} b \) given by the image of \(-\alpha \in \Gamma\pi_2 \) under the homomorphism

\[
\tau : \Gamma\pi_2 \rightarrow \text{coker} b \rightarrow \text{coker} b / \hat{y}\text{coker} b = \text{Ext}(\mathbb{Z}/\hat{y}\mathbb{Z}, \text{coker} b).
\]

Hence the extension \( \pi_3P \) over \( \mathbb{Z} \) determines \( \alpha \) modulo \( \text{ker} \tau \).

Theorem 1.5 is a corollary to Theorem 9.1.
Proof. As the augmentation ideal $K \cong \mathbb{Z}$ is generated by $k = [\mathbb{T}] - [\emptyset]$, the action of $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ on $K = \pi_2 P_2$ and hence on $\pi_2 P = K/xR = \mathbb{Z}/\hat{x}\mathbb{Z}$ is multiplication by $-1$, since $k[\mathbb{T}] = -k$. But the $\Gamma$–functor maps multiplication by $-1$ to the identity morphism, so that $\pi_1$ acts trivially on $\Gamma(\pi_2 P)$.

As $d_3(e_3) = e_2 x$, we obtain $H_3 \hat{P}_{2,x} \cong \mathbb{Z}$, generated by the norm element $N = [\mathbb{T}] + [\emptyset]$. Since $N[\mathbb{T}] = N$, the action of $\pi_1$ on $H_3 \hat{P}_{2,x}$ is trivial.

As $d_4(e_4) = e_3 y$, we obtain $H_4 \hat{P} \cong \mathbb{Z}/\hat{y}\mathbb{Z}$ and $H_4 \hat{P} \cong \mathbb{Z}$, generated by $k = [\mathbb{T}] - [\emptyset]$. Hence the action of $\pi_1$ on $H_4 \hat{P}$ is multiplication by $-1$.

Now let $\xi = ([\mathbb{T}] - [\emptyset]) \otimes ([\mathbb{T}] - [\emptyset])$ be the generator of $\Gamma(K)$. Note that $v[\mathbb{T}] = v$ and $\beta[\mathbb{T}] = \beta$, for $v \in H_3 \hat{P}_{2,x}$ and $\beta \in \Gamma(\pi_2 P)$, since $\pi_1$ acts trivially on both $H_3 \hat{P}_{2,x}$ and $\Gamma(\pi_2 P)$. Lemma 8.3 implies
\[
(u(v) + \beta)[\mathbb{T}] = -\hat{x}\hat{y}\omega(\xi) + u(v[\mathbb{T}]) + \omega(\beta)[\mathbb{T}] = -\hat{x}\hat{y}\omega(\xi) + u(v) + \omega(\beta).
\]
We obtain
\[
\tilde{b}(e_4([\mathbb{T}] - [\emptyset])) = (u(y) + \omega(\alpha))(\mathbb{T} - [\emptyset]) = -\hat{x}\hat{y}\omega(\xi) + u(y) + \omega(\alpha) - (u(y) + \omega(\alpha)) = -\hat{x}\hat{y}\omega(\xi).
\]
By definition of $\tilde{b}$,
\[
\pi_3 P = \pi_3 P_{2,x}/\text{im} \tilde{b}.
\]
Hence $\pi_1$ acts trivially on $\pi_3(P)$.

Sequence (9.1) yields the short exact sequence
\[
(9.3) \quad G = \text{coker} b \longrightarrow \pi_3 P \xrightarrow{h} H_3 \hat{P} \cong \mathbb{Z}/\hat{y}\mathbb{Z},
\]
which represents $\pi_3 P$ as an extension of $\mathbb{Z}/\hat{y}\mathbb{Z}$ by $G = \text{coker} b$. Thus the extension $\pi_3 P$ over $\mathbb{Z}$ determines $\gamma$ modulo the kernel of the map
\[
\tau : \Gamma \pi_2 \longrightarrow \text{coker} b \longrightarrow \text{coker} b/\gamma \text{coker} b = \text{Ext}(\mathbb{Z}/\hat{y}\mathbb{Z}, \text{coker} b).
\]
\[\square\]

References


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