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Space Adaptive Finite Element Methods for Dynamic Signorini Problems

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Abstract Space adaptive techniques for dynamic Signorini problems are discussed. For discretisation, the Newmark method in time and low order finite elements in space are used. For the global discretisation error in space, an a posteriori error estimate is derived on the basis of the semi-discrete problem in mixed form. This approach relies on an auxiliary problem, which takes the form of a variational equation. An adaptive method based on the estimate is applied to improve the finite element approximation. Numerical results illustrate the performance of the presented method.

Keywords Dynamic Signorini problem · A posteriori error estimation · Mesh refinement · Finite element method

1 Introduction

Dynamic Signorini problems arise in many engineering processes, e.g., in milling and grinding processes, vehicle design and ballistics. In these processes, the main effects result from the contact at the surface of the bodies under consideration. Typical examples for engineering processes, where contact problems play a dominant role, are grinding processes. The workpiece interacts with the grinding wheel only in a small contact zone. However, the behaviour of the grinding machine is strongly affected by the resulting contact forces. A detailed study of this engineering problem is found in [1]. For the reliable simulation of such a process, a precise prediction is required of the contact forces, the contact zone and their effects onto the whole body. Furthermore, the contact zone and the contact forces are strongly depending on time. Hence, the precise consideration of these dependences is essential in the numerical simulation.

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An adequate technique, which gives rise to a flexible and efficient finite element discretisation, is based on a posteriori error control and resulting adaptive mesh refinement. In general, a posteriori error estimates for second order hyperbolic problems are possible for two different discretisation approaches. One of them uses space time Galerkin methods for discretisation and applies similar techniques for error control as in the static case ([2–5]). The other one is based on finite differences in time and finite elements in space. Here, separate error estimators are used for the space and time direction ([6–8]) or error estimates for the whole problem ([9, 10]) are derived.

In this article, finite differences in time and finite elements in space are used to discretise the dynamic Signorini problems. Because only the data of the current time step comes into play, the error estimator can be evaluated efficiently. However, the separation of the space and time direction complicates the consideration of space time effects. The aim of this article is to derive an error estimator for the finite element discretisation in space direction. Therefore, an error control technique for static contact problems is applied to the semi discrete spatial problem. This technique goes back to Braess [11] and Schröder [12]. Other approaches to a posteriori error control for static contact problems are discussed in [13–20]. In particular, an adaptive scheme for two-body contact is contained in [21]. Convergence results for adaptive algorithms in the context of obstacle problems are proven in [22].

Adaptivity in time direction is not taken into account in this article for notational simplicity, although it is easy to incorporate. One can do this on the basis of error estimators, which are known from the literature of second order hyperbolic problems [6].
The temporal discretisation of dynamic contact problems is a difficult task. Several approaches based on different problem formulations have been presented in literature. In [23] the penalty-method is used to solve the discrete problems. Special contact elements in combination with Lagrange multipliers are presented in [24]. Other techniques for smoothing and stabilizing the computation with special finite elements, e.g. Mortar finite elements, are presented in [25–27]. In [28], the Newmark scheme is used with an additional L^2-projection for stabilization. Algorithms for dynamic contact/impact problems based on the energy- and momentum conservation are derived in [29,30]. An additive splitting of the acceleration into two parts, representing the interior forces and the contact forces, is the basis of the methods introduced in [31,32]. In [33–35] algorithms based on variational inequalities and optimisation algorithms are presented. Detailed surveys of this topic can be found in the monographs [36,37].

The article is organised as follows: In the next two sections, the strong and weak formulations of the dynamic simplified Signorini problem are introduced and the discretisation of the problem is discussed. In Section 4, the spatial error estimator is derived serving as basis of an adaptive algorithm. In Section 6, two examples illustrate the application of the mentioned techniques. The article concludes with a discussion of the results.

2 Continuous Formulation

In this section, the strong and the weak formulation of the dynamic simplified Signorini problem are presented. Let $\Omega \subset \mathbb{R}^2$ be the basic domain and $I := [0,T] \subset \mathbb{R}$ a time interval. The boundary $\partial\Omega$ of Ω is divided into three mutually disjoint parts Γ_D, Γ_C and Γ_N with positive measure. Homogeneous Dirichlet and Neumann boundary conditions are prescribed on the closed set Γ_D and on Γ_N, respectively. Contact may take place on the sufficiently smooth set Γ_C, $\overline{\Gamma_C} \subset \complement\Gamma_D$. See, e.g., [38], Section 5.3 for more details. The time dependent rigid foundation is parameterised by a sufficiently smooth function $g : \Gamma_C \times I \to \mathbb{R} \cup \{-\infty\}$. Here, the restriction $u \geq g$ on Γ_C is considered, $u \leq g$ can be treated analogously.

The initial displacement u_0 is in

$$H^1(\Omega,\Gamma_D) := \{v \in H^1(\Omega) | \gamma_{|\Gamma_D}(v) = 0\}$$

and the initial velocity v_0 is in $L^2(\Omega)$. Here, γ denotes the trace operator of functions in $H^1(\Omega,\Gamma_D)$ onto the boundary $\partial\Omega$. See, e.g., [39] for more details. The gradient of the displacement u in space direction is denoted by ∇u and Δu is the usual Laplace operator. The first and second time derivatives are denoted by $\dot{u}$ and $\ddot{u}$, respectively. In the following, all relations have to be understood almost everywhere.

We choose the unconstrained trial space

$$V := W^{2,\infty}\left(I;L^2(\Omega)\right) \cap L^\infty\left(I;H^1(\Omega,\Gamma_D)\right)$$

for notational convenience, although the existense of a solution in V can not be proven, even in the contact free case [39]. The set of admissible displacements is

$$K := \left\{\varphi \in V \,\middle|\, \gamma_{|\Gamma_C}(\varphi) \geq g \text{ on } \Gamma_C \times I\right\}.$$

The L^2-scalar product is defined by $(u,v) = \int_\Omega uv\,dx$ for $u,v \in L^2(\Omega)$. The density is set equal to 1 for notational simplicity. Eventually, the weak formulation of the simplified dynamic Signorini problem, see e.g. [40], reads

Problem 2.1 Find a function $u \in K$ with $u(t=0) = u_0$ and $\dot{u}(t=0) = v_0$ for which

$$\begin{aligned} &(\ddot{u}(t), \varphi(t) - u(t)) + (\nabla u(t), \nabla(\varphi(t) - u(t))) \\ \geq\; &(f(t), \varphi(t) - u(t)) \end{aligned}$$

holds for all $\varphi \in K$ and all $t \in I$.

Throughout this article, we assume $f \in L^\infty\left(I;L^2(\Omega)\right)$.
If the solution is sufficiently smooth, we obtain the equivalent strong formulation (see [40])

$$\begin{aligned} \ddot{u} - \Delta u &= f && \text{in } \Omega \times I \\ u &= 0 && \text{on } \Gamma_D \times I \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma_N \times I \\ u - g &\geq 0 && \text{on } \Gamma_C \times I \\ \frac{\partial u}{\partial \nu} &\leq 0 && \text{on } \Gamma_C \times I \\ \frac{\partial u}{\partial \nu}(u-g) &= 0 && \text{on } \Gamma_C \times I. \end{aligned}$$

3 Discretisation

We use Rothe's method to discretise the dynamic simplified Signorini problem. First, the problem is discretised in temporal direction by the Newmark method (see [41]). The resulting spatial problems are approximately solved by low order finite elements.

3.1 Temporal Discretisation

The time interval I is split into N equidistant subintervals $I_n := (t_{n-1}, t_n]$ of length $k = t_n - t_{n-1}$ with $0 =: t_0 < t_1 < \ldots < t_{N-1} < t_N := T$. The value of a function w at a time instance t_n is approximated by w^n. We use the notation $v = \dot{u}$ and $a = \ddot{u}$ for the velocity and the acceleration, respectively.

In the Newmark method, v and a are approximated by

$$a^n = \frac{1}{\beta k^2}\left(u^n - u^{n-1}\right) - \frac{1}{\beta k} v^{n-1} - \left(\frac{1}{2\beta} - 1\right) a^{n-1}, \quad (3.1)$$

$$v^n = v^{n-1} + k\left[(1-\alpha)a^{n-1} + \alpha a^n\right]. \quad (3.2)$$

Here, α and β are free parameters in the interval $[0,2]$. For second order convergence, $\alpha = \frac{1}{2}$ is required. Furthermore, the inequality $2\beta \geq \alpha \geq \frac{1}{2}$ has to be valid for unconditional stability (see [42]). For dynamic contact problems, the choice $\alpha = \beta = \frac{1}{2}$ is recommended to guarantee conservation of energy and momentum (see [24,35]). For starting the Newmark method the initial acceleration a_0 is needed. It can be calculated on the basis of the initial displacement and velocity (see [42]). The semi-discrete problem then reads as follows:

Problem 3.1 Find u with $u^0 = u_0$, $v^0 = v_0$ and $a^0 = a_0$, such that in every time step $n \in \{1,2,\ldots,N\}$, the function $u^n \in K^n$ is the solution of the variational inequality

$$\left(a^n, \varphi - u^n\right) + \left(\nabla u^n, \nabla\left(\varphi - u^n\right)\right) \geq \left(f\left(t_n\right), \varphi - u^n\right), \quad (3.3)$$

for all $\varphi \in K^n$. Moreover u^n, v^n and a^n have to fulfill the equations (3.1) and (3.2).

The set $K^n := \left\{\varphi \in H^1\left(\Omega, \Gamma_D\right) \middle| \gamma_{|\Gamma_C}(\varphi) \geq g^n \text{ on } \Omega\right\}$ is the time discretized set of the admissible displacements. Substituting the equation (3.1) with $\gamma = \beta = \frac{1}{2}$ in the inequality (3.3) leads to

$$\begin{aligned} &\left(u^n, \varphi - u^n\right) + \tfrac{1}{2}k^2\left(\nabla u^n, \nabla\left(\varphi - u^n\right)\right) \\ \geq &\left(\tfrac{1}{2}k^2 f\left(t_n\right) + u^{n-1} + k v^{n-1}, \varphi - u^n\right). \end{aligned}$$

This can be written as

$$c\left(u^n, \varphi - u^n\right) \geq \left(F^n, \varphi - u^n\right), \quad (3.4)$$

where c is defined by

$$c(\omega, \varphi) := (\omega, \varphi) + \frac{1}{2}k^2\left(\nabla\omega, \nabla\varphi\right)$$

and F^n as

$$F^n := \frac{1}{2}k^2 f\left(t_n\right) + u^{n-1} + k v^{n-1}.$$

The bilinear form c is uniformly elliptic, continuous and symmetric. Thus, an elliptic variational inequality has to be solved in each time step. An efficient way for solving variational inequalities is given by their mixed formulation. The Lagrange parameters may be interpreted as contact forces. The variational inequality (3.4) is equivalent to the following mixed problem:

Problem 3.2 Find (u, λ) with $u^0 = u_0$, $v^0 = v_0$ and $a^0 = a_0$, such that $(u^n, \lambda^n) \in V^n \times \Lambda^n$ is the solution of the system

$$c\left(u^n, \varphi\right) + \left\langle \lambda^n, \gamma_{|\Gamma_C}(\varphi)\right\rangle = \left(F^n, \varphi\right) \quad (3.5)$$

$$\left\langle \mu - \lambda^n, \gamma_{|\Gamma_C}\left(u^n\right) - g^n\right\rangle \leq 0, \quad (3.6)$$

for all $\varphi \in V^n$, all $\mu \in \Lambda^n$ and all $n \in \{1,2,\ldots,N\}$. Based on the equations (3.1) and (3.2), the functions v^n and a^n are calculated in a postprocessing step.

Here, Λ^n is the dual cone of the set

$$G := \left\{\mu \in H^{1/2}\left(\Gamma_C\right) \middle| \mu \leq 0\right\}.$$

The dual pairing is expressed by $\langle\cdot,\cdot\rangle$. The set $V^n := H^1\left(\Omega, \Gamma_D\right)$ is the time discretised unconstrained trial space.
The equivalence of the two formulations is a well-known conclusion from the general theory of minimisation problems in Hilbert spaces presented, e.g., in [43,44].

3.2 Spatial Discretisation

A finite element approach is applied to discretise the mixed problem 3.2. We use adaptive algorithms with dynamic meshes. Therefore, the trial spaces V_h^n and Λ_H^n may vary from time step to time step. Bilinear basis functions on the mesh $\mathbb{T}^n$ are used for the finite element space V_h^n. The discrete Lagrange multipliers are piecewise constant and are contained in the set Λ_H^n. The index H indicates that coarser meshes may be chosen for the Lagrange multipliers. In our calculations, we use $H = 2h$ for stability reasons. A detailed study of the stability properties of this discretisation can be found in [12].

Because of the varying meshes, FE-functions have to be transfered to the mesh of the current time step. This process is denoted by I_h and is realized by an L^2-projection. One might also consider standard interpolation as a transfer operator, which needs less effort, but can lead to instabilities. The space and time discrete problem is

Problem 3.3 Find $\left(u_h^n, \lambda_H^n\right) \in V_h^n \times \Lambda_H^n$ with $u_h^0 = I_h u_0$, $v_h^0 = I_h v_0$ and $a_h^0 = I_h a_0$, such that the system

$$c\left(u_h^n, \varphi_h\right) + \left\langle \lambda_H^n, \gamma_{|\Gamma_C}\left(\varphi_h\right)\right\rangle = \left(F_h^n, \varphi_h\right) \quad (3.7)$$

$$\left\langle \mu_H - \lambda_H^n, \gamma_{|\Gamma_C}\left(u_h^n\right) - g^n\right\rangle \leq 0 \quad (3.8)$$

is valid for all $\varphi_h \in V_h^n$ and $\mu_H \in \Lambda_H^n$, $n \in \{1,2,\ldots,N\}$. Additionally, the equations (3.1) and (3.2) determine v_h^n and a_h^n.

Here, F_h^n is given by

$$F_h^n := \frac{1}{2}k^2 f\left(t_n\right) + I_h u_h^{n-1} + k I_h v_h^{n-1}.$$

The system (3.7-3.8) leads to the following saddle point problem in $\mathbb{R}^m$, where m depends on n:

$$\begin{aligned} A^n\bar{u}^n + B^n\bar{\lambda}^n &= \bar{F}^n \\ \left(\bar{\mu} - \bar{\lambda}^n\right)^T \left((B^n)^T \bar{u}^n - \bar{g}^n\right) &\le 0, \end{aligned}$$

which must hold for all $\bar{\mu} \in \mathbb{R}^{\tilde{m}}_{\le 0}$. Here, $A^n := M^n + \frac{1}{2}k^2 K^n$ is the generalised stiffness matrix, $M^n \in \mathbb{R}^{m\times m}$ is the mass matrix and $K^n \in \mathbb{R}^{m\times m}$ is the stiffness matrix. The matrix $B^n \in \mathbb{R}^{m\times \tilde{m}}$ represents the dual pairing in (3.8). Notice, that all matrices may change from time step to time step.
The saddle point problem can be rewritten as a quadratic optimisation problem, which can be solved by substituting $\bar{u}^n := (A^n)^{-1}\left(\bar{F}^n - B^n\bar{\lambda}^n\right)$ and applying SQP methods.
More details can be found in [12].

4 Spatial Error Estimation

In this section, an error estimation is derived for the spatial error in every time step. The estimation is easy to implement and can be evaluated fast. The temporal error is not considered. The idea of the error estimation goes back to Braess [11], who presented it for static obstacle problems. This idea was extended by Schröder [12] to static Signorini problems even with friction by introducing a general framework for error control of variational inequalities in Hilbert spaces. In order to apply this framework here, we consider the following saddlepoint problem:

Problem 4.1 Find $\left(\tilde{u}^n, \tilde{\lambda}^n\right) \in V^n \times \Lambda^n$, such that

$$\begin{aligned} c(\tilde{u}^n, \varphi) + \langle \tilde{\lambda}^n, \gamma_{|\Gamma_C}(\varphi)\rangle &= (F_h^n, \varphi) \\ \langle \mu - \tilde{\lambda}^n, \gamma_{|\Gamma_C}(\tilde{u}^n) - g^n\rangle &\le 0 \end{aligned}$$

for all $\varphi \in V^n$ and all $\mu \in \Lambda^n$.

An essential part of the general framework in [12] is the formulation of the auxiliary problem:

Problem 4.2 Find $u_\star^n \in V^n$, such that the variational equation

$$c(u_\star^n, \varphi) = (F_h^n, \varphi) - \left\langle \lambda_H^n, \gamma_{|\Gamma_C}(\varphi)\right\rangle$$

holds for all $\varphi \in V^n$.

Problem 4.2 corresponds to the first line of Problem 4.1, but with the discrete Lagrangian multiplier λ_H^n instead of $\tilde{\lambda}^n$. Applying Lemma IV.2 in [12] yields

Lemma 4.1 *There are constants $C', C'' \in \mathbb{R}_{>0}$, such that*

$$\begin{aligned} &\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \\ \le\ &C'\, \|u_\star^n - u_h^n\|^2 + C''\langle \tilde{\lambda}^n - \lambda_H^n, \gamma_{|\Gamma_C}(u_h^n) - g^n\rangle. \end{aligned}$$

Here, $\|\cdot\|$ denotes the norm correponding to the related function spaces. We use the $H^1(\Omega)$-norm for V^n. Taking into account, that the discrete solution u_h^n is also a discrete solution of Problem 4.2, we are able to get rid of the term $\|u_\star^n - u_h^n\|$ by using an appropriate error estimator for the auxiliary problem: Let $\eta_\star^n > 0$ be an error estimator of Problem 4.2, i.e., there exists a constant $C_\star > 0$ independent of V_h^n and Λ_H^n, such that

$$\|u_\star^n - u_h^n\|^2 \le C_\star (\eta_\star^n)^2.$$

Then, Lemma 4.1 leads to

$$\begin{aligned} &\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \\ \le\ &C' C_\star (\eta_\star^n)^2 + C''\langle \lambda^n - \lambda_H^n, \gamma_{|\Gamma_C}(u_h^n) - g^n\rangle. \end{aligned}$$

The remaining term $\langle \tilde{\lambda}^n - \lambda_H^n, \gamma_{|\Gamma_C}\left(u_h^n\right) - g^n\rangle$ is estimated by

Lemma 4.2 *Let $\nu_0 > 0$ be the constant of continuity of c. Furthermore, let*

$$d \in \tilde{K}^n := \left\{ v \in V^n | g^n - \gamma_{|\Gamma_C}(u_h^n) - \gamma_{|\Gamma_C}(v) \le 0 \right\}$$

and $\varepsilon > 0$. Then, there holds

$$\begin{aligned} &\langle \tilde{\lambda}^n - \lambda_H^n, \gamma_{|\Gamma_C}\left(u_h^n\right) - g^n\rangle \\ \le\ &\tfrac{\varepsilon}{2}\left\|\tilde{u}^n - u_h^n\right\|^2 + \tfrac{(1+\varepsilon)\nu_0^2}{2\varepsilon}\|d\|^2 \\ &+ \tfrac{1}{2}\left\|u_\star^n - u_h^n\right\|^2 + |(\lambda_H^n, \gamma_{|\Gamma_C}(d))|. \end{aligned}$$

Proof. Inserting 0 and $2\lambda_H^n$ in (3.8) yields

$$\langle \lambda_H^n, \gamma_{|\Gamma_C}(u_h^n) - g^n\rangle = 0.$$

Furthermore, we get

$$\begin{aligned} &\langle \tilde{\lambda}^n, \gamma_{|\Gamma_C}(u_h^n) - g^n\rangle \\ =\ &-\langle \tilde{\lambda}^n, g^n - \gamma_{|\Gamma_C}(u_h^n) - \gamma_{|\Gamma_C}(d)\rangle - \langle \tilde{\lambda}^n, \gamma_{|\Gamma_C}(d)\rangle \\ \le\ &c(\tilde{u}^n, d) - (F_h^n, d) \\ =\ &c(\tilde{u}^n - u_h^n, d) + c(u_h^n, d) - (F_h^n, d) \\ \le\ &\nu_0\, \|\tilde{u}^n - u_h^n\|\, \|d\| + c(u_h^n, d) - (F_h^n, d) \\ \le\ &\frac{\varepsilon}{2}\|\tilde{u}^n - u_h^n\|^2 + \frac{\nu_0^2}{2\varepsilon}\|d\|^2 + c(u_h^n, d) - (F_h^n, d). \end{aligned}$$

Here, we have used Young's inequality. The term $c(u_h^n, d) - (F_h^n, d)$ is estimated as follows:

$$\begin{aligned} &c(u_h^n, d) - (F_h^n, d) \\ =\ &c(u_h^n - u_\star^n, d) - (\lambda_H^n, \gamma_{|\Gamma_C}(d)) \\ \le\ &\nu_0\, \|u_\star^n - u_h^n\|\, \|d\| - (\lambda_H^n, \gamma_{|\Gamma_C}(d)) \\ \le\ &\frac{1}{2}\|u_\star^n - u_h^n\|^2 + \frac{\nu_0^2}{2}\|d\|^2 + |(\lambda_H^n, \gamma_{|\Gamma_C}(d))|. \end{aligned}$$

□

Eventually, we obtain an a posteriori error estimate by the following proposition:

Proposition 4.1 *There exists a constant $C > 0$ independent of V_h^n and Λ_H^n, such that*

$$\begin{aligned}
&\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \\
\leq\, & C\left((\eta_\star^n)^2 + \left\|\left(g^n - \gamma_{|\Gamma_C}\left(u_h^n\right)\right)_+\right\|^2\right) \\
&+ C\left|\left(\lambda_H^n, \left(g^n - \gamma_{|\Gamma_C}\left(u_h^n\right)\right)_+\right)\right|
\end{aligned}$$

holds. Here, f_+ denotes the positive part of a function f, which means

$$f_+(x) = \begin{cases} f(x), & f(x) \geq 0, \\ 0, & f(x) < 0. \end{cases}$$

Proof. Combining Lemma 4.1 and Lemma 4.2 yields

$$\begin{aligned}
&\|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \\
\leq\, & C' C_\star (\eta_\star^n)^2 + C'' \langle \tilde{\lambda}^n - \lambda_H^n, \gamma_{|\Gamma_C}(u_h^n) - g^n \rangle \\
\leq\, & \left(C' + \frac{1}{2}C''\right) C_\star (\eta_\star^n)^2 + C'' \frac{\varepsilon}{2} \|\tilde{u}^n - u_h^n\|^2 \\
&+ C'' \left(\frac{(1+\varepsilon)\nu_0^2}{2\varepsilon} \|d\|^2 + |(\lambda_H^n, \gamma_{|\Gamma_C}(d))|\right).
\end{aligned}$$

Choosing $0 < \varepsilon < 2/C''$, we get

$$\begin{aligned}
&\left(1 - \frac{C''\varepsilon}{2}\right) \|\tilde{u}^n - u_h^n\|^2 + \left\|\tilde{\lambda}^n - \lambda_H^n\right\|^2 \\
\leq\, & \tilde{C}\left((\eta_\star)^2 + \|d\|^2 + |(\lambda_H^n, \gamma_{|\Gamma_C}(d))|\right)
\end{aligned}$$

with

$$\tilde{C} := \max\left\{\left(C' + \frac{C''}{2}\right) C_\star, \frac{C''(1+\varepsilon)\nu_0^2}{2\varepsilon}, C''\right\}.$$

The function d in Lemma 4.2 can be chosen as the harmonic continuation $\hat{d}^n$ of $\left(g^n - \gamma_{|\Gamma_C}\left(u_h^n\right)\right)_+$, which is characterised by the minimisation problem

$$\left\|\hat{d}^n\right\|_1^2 = \inf_{v \in W^n} \|v\|_1^2$$

with

$$W^n := \left\{ v \in V^n \,\middle|\, \gamma_{|\Gamma_C}(v) = \left(g^n - \gamma_{|\Gamma_C}\left(u_h^n\right)\right)_+ \right\}.$$

For more details and alternatives in the choice of d see [12]. Since there holds

$$g^n - \gamma_{|\Gamma_C}(u_h^n) - \gamma_{|\Gamma_C}\left(\hat{d}^n\right) \leq 0,$$

$\hat{d}^n$ is an element of $\tilde{K}^n$. From the definition of the norm $\|\cdot\|_{1/2,\Gamma_C}$, it follows

$$\left\|\hat{d}^n\right\|_1 = \left\|\left(g^n - \gamma_{|\Gamma_C}\left(u_h^n\right)\right)_+\right\|_{1/2,\Gamma_C}.$$

□

Remark 4.1 All terms in the error estimate of Proposition 4.1 can be interpreted as typical sources of errors in contact problems. The term $\|(g^n - \gamma_{|\Gamma_C}\left(u_h^n\right))_+\|$ measures the error of the geometrical contact condition and the term $|(\lambda_H^n, (g^n - \gamma_{|\Gamma_C}\left(u_h^n\right))_+)|$ measures the violation of the complementarity condition.

Remark 4.2 In our numerical tests, the term

$$\|(g^n - \gamma_{|\Gamma_C}(u_h^n))_+\|$$

always turned out to be of higher order in h, see [12]. Since it is difficult to split this term into its elementwise contributions, it is neglected in the numerical realisation.

In order to apply the error estimation of Proposition 4.1, we have to specify an appropriate error estimator $\eta_\star^n$ for Problem 4.2. In principle, each error estimator known from literature of variational equations is possible to be used. See [45] or [46] for an overview. For the sake of completeness, a residual based error estimator for Problem 4.2 is specified:

$$\begin{aligned}
(\eta_\star^n)^2 &:= \sum_{K \in \mathbb{T}^n} \eta_K^2 \\
\eta_K^2 &:= h_K^2 \|r\|_{L^2(K)}^2 + h_K \|R\|_{L^2(\partial K)}^2
\end{aligned}$$

with

$$r := F_h^n + \frac{1}{2}k^2 \triangle u_h^n - u_h^n$$

$$R := \begin{cases} -\frac{1}{2}k^2 \left(q - \frac{\partial u_h^n}{\partial \nu}\right) & \text{on } \partial\Omega \\ -\frac{1}{4}k^2 \left[\frac{\partial u_h^n}{\partial \nu}\right] & \text{else.} \end{cases}$$

The quantity R represents the jump discontinuity in the approximation to the normal flux on the interface. We set $q = 0$ on Γ_N and $q = -\lambda_H^n$ on Γ_C. See [46], Section 2.2, for more details.

Remark 4.3 We have used the discrete value F_h^n instead of F^n in Problem 4.1, i.e., Proposition 4.1 specifies a temporal local error estimator for the spatial discretisation error. This technique is commonly used in the derivation of error estimators for numerical methods for ordinary differential equations, see, e.g., [47]. The presented error estimator expresses the spatial error distribution in the single time steps. But it only provides information about the global error under the assumption $F_h^n \approx F^n$, which should hold for small k and h. An a priori error analysis of the Newmark method in the context of dynamic contact problems is needed to make a precise statement. To the best of the authors' knowledge, this analysis does not exist and cannot be derived by standard techniques due to the low regularity of the continuous solution.

Remark 4.4 The presented error estimate is not restricted to the Newmark method. It can easily be used for other similar time stepping schemes. It was tested by the authors for the Generalized-α method, the application of which to dynamic contact problems is presented in [34,35].

5 Adaptive Algorithm

In general, adaptive algorithms for dynamic problems are based on refinement strategies, which are known from static problems, see, e.g., [45,48] for a survey of adaptive algorithms for static problems. Commonly used adaptive algorithms for time dependent problems, see, e.g., [3,49], perform an adaptive refinement process using a prescribed tolerance in every time step. This refinement process is independent of previous and subsequent time steps. Here, the crucial point is, that the time interval is passed only once. The tolerance limit cannot be reached, if the solution in the previous time step has not been calculated exactly enough. Moreover, the difference of the meshes of two succsesive time steps may significantly increase the error. Usually, rapid changes of the problem parameters are the reason for this behaviour.

In dynamic Signorini problems, the problem parameters change rapidly and, thus, the above mentioned algorithms are not appropriate. An alternative is given by algorithms based on the ideas in [50,51]. The refinement procedure is split into several cycles. The whole time interval is passed in every cycle. A cycle constists of two steps: First, the approximated solution of the whole problem is determined, the error is estimated and the mesh is refined via a usual refinement strategy, e.g. a fixed fraction strategy, see [48,45]. Multiple hanging nodes in space and time may be generated by this refinement. In a second step, such nodes are removed. The removal of hanging nodes in time closely connects the meshes of different time steps. A detailed presentation of this adaptive algorithm and its extensions will be given in a seperate article.

6 Numerical Results

The error estimator and the adaptive algorithm are tested for two examples. The first one is a simplified Signorini problem and the second one is a full 2D Signorini problem.

6.1 A simplified Signorini example

We set $\Omega := [0,1]^2$ and $I := [0,1]$. The initial values are $u_0(x_1,x_2) := 0$ and $v_0(x_1,x_2) := -\sin\left(\frac{1}{2}\pi x_1\right)$. Furthermore, we set $\Gamma_D := \{x \in \Omega \,|x_1 = 0\}$, $\Gamma_C := \{x \in \Omega \,|x_1 = 1\}$ and $\Gamma_N := \partial\Omega \backslash (\Gamma_C \cap \Gamma_D)$. The rigid foundation is

$$g := \sin(\pi x_2) - 1.05.$$

The length of the time steps k is chosen as 0.0025. The initial mesh size h_0 is 0.0625. Five refinement cycles are performed, whereas a fixed fraction strategy with constant refinement fraction of 50% and no coarsening is used.

Fig. 6.1 Geometry of the simplified Signorini example

The geometry of the presented problem is illustrated in Figure 6.1. Meshes for different time steps are presented in Figure 6.2, the corresponding movie is shown in Animation 1. The mesh in Figure 6.2 (a) corresponds to the time step, immediately before the first contact between the membrane and the rigid foundation takes places. In Figure 6.2 (b)-(i), the membrane gets into contact with the obstacle and the contact zone is adaptively refined. We observe a moving front, which is resolved by the adaptive meshes. In Figure 6.3 the estimated convergence for adaptive and uniform refinement is compared. The estimated error is measured by

$$\eta = \max_{1\leq n\leq N} \eta^n,$$

where η^n is given in Proposition 4.1. The number of degrees of freedom is the sum of the number of degrees of freedom of all single time steps. It is obvious, that the adaptive refinement is more efficient than the uniform refinement. One achieves the same accuracy with nearly a factor of 10 less unknowns.

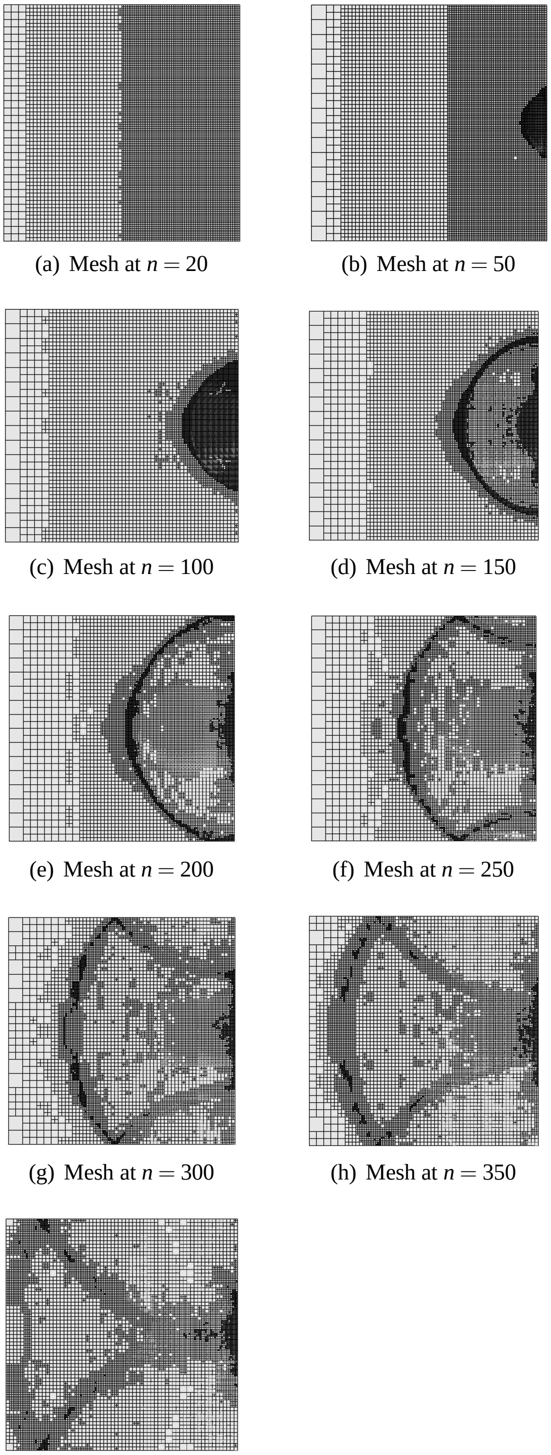
6.2 A Signorini example

Here, a bar of length 0.2 m and height 0.05 m is considered. The domain is $\Omega := [0,0.2] \times [0,0.05]$ and the time interval is $I := \left[0, 2.5\cdot 10^{-3}\right]$. The bar is modelled using a linear elastic material law in a plain strain situation with $E := 73\cdot 10^9\,\mathrm{MPa}$ and $\nu := 0.33$. The density is $\rho := 2770\,\mathrm{kg/m^2}$. The bar is fixed at the left boundary $\Gamma_D = \{x \in \Omega \,|x_1 = 0\}$. The possible contact surface is given by the set

$$\Gamma_C = \{x \in \Omega \,|x_1 \geq 0.15 \wedge x_2 = 0\}.$$

There are nonhomogeneous Neumann boundary conditions on $\Gamma_N = \{x \in \Omega \,|x_1 \geq 0.1 \wedge x_2 = 0.05\}$ with

$$q := 3.75\cdot 10^7\,\mathrm{N/m^2}.$$



(a) Mesh at $n = 20$ (b) Mesh at $n = 50$

(c) Mesh at $n = 100$ (d) Mesh at $n = 150$

(e) Mesh at $n = 200$ (f) Mesh at $n = 250$

(g) Mesh at $n = 300$ (h) Mesh at $n = 350$

(i) Mesh at $n = 400$

Fig. 6.2 Meshes for different time steps of the simplified Signorini example

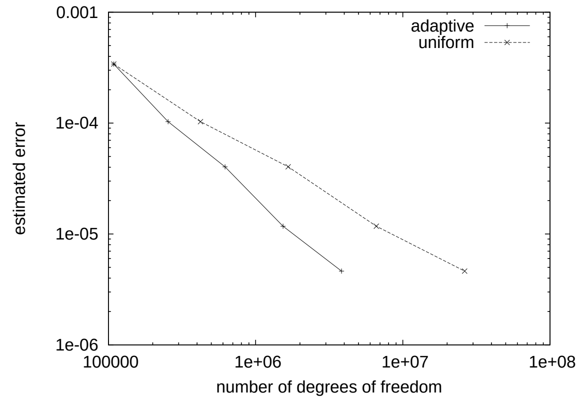


Fig. 6.3 Estimated error for adaptive and uniform refinement in the simplified Signorini example

The rigid foundation is given by the set

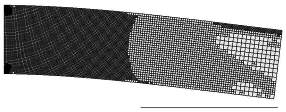
$$\left\{x \in \mathbb{R}^2 \,|\, 0 \leq x_1 \leq 0.2 \wedge x_2 \leq -0.005\right\}.$$

The length of the time steps is 10^{-5} and the initial mesh size h_0 is $6.25 \cdot 10^{-3}$. Again, five refinement cycles are performed, whereas a fixed fraction strategy with constant refinement fraction of 50% without coarsening is used.

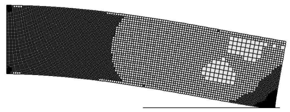
In Figure 6.4 meshes for different time steps are presented the corresponding movie is contained in Animation 2. The displacement is scaled by a factor of 5. During the calculation, contact between the bar and the rigid foundation occurs several times. In the Figures 6.4 (a)-(c) and (d)-(f) a sequence is depicted, which starts before contact takes places and ends after contact. The influence of contact to the mesh is obvious. The last sequence (g)-(i) shows the change of the mesh during contact. The refined zone and the contact zone grow and shrink simultanously. The performance of the adaptive refinement is compared with the uniform refinement in Figure 6.5, where the same variables are used as in Figure 6.3. As in the example above, the application of the adaptive method is more efficient.

7 Conclusions

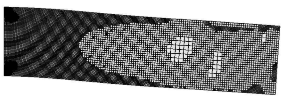
The presented space adaptive scheme for dynamic Signorini problems shows a significant improvement. More sophisticated refinement strategies can further enhance the efficiency. However, not every strategy known from non contact problems seems to be suited for adaptive schemes for contact problems. E.g. the refinement strategy presented in [51] compares the refinement indicators over all time steps. The method has been tested by the authors, but the results are not satisfactory. The contact zone is not resolved, before the first contact takes place. Thus, the algorithm is not able to detect the moment of the first contact exactly, so that the error increases significantly.



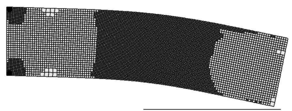
(a) Mesh at $n = 20$



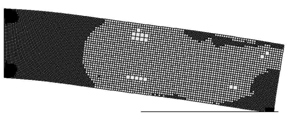
(b) Mesh at $n = 25$



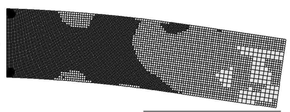
(c) Mesh at $n = 50$



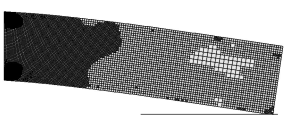
(d) Mesh at $n = 80$



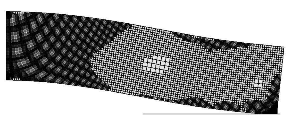
(e) Mesh at $n = 90$



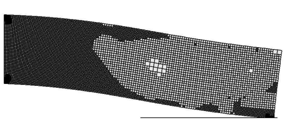
(f) Mesh at $n = 100$



(g) Mesh at $n = 211$



(h) Mesh at $n = 215$



(i) Mesh at $n = 217$

Fig. 6.4 Meshes for different time steps of the 2D Signorini example

Another method to improve the discretisation is given by time adaptivity; error estimators for the Newmark method as presented in [6] can be used. This technique will be considered in future works.

The difficulties discussed in Remark 4.3 and the separation of the spatial and temporal discretisation complicate the derivation of rigorous a posteriori error estimators. A

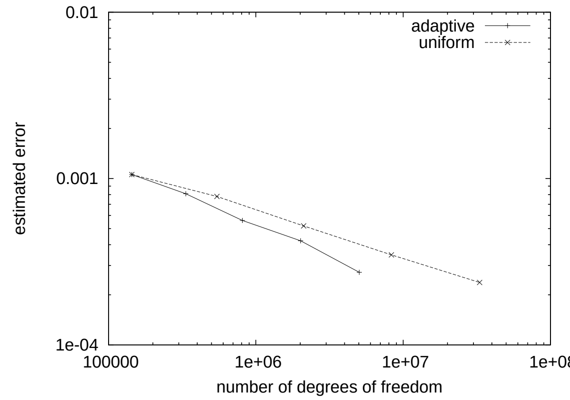


Fig. 6.5 Estimated error for adaptive and uniform refinement in the Signorini example

way out could be the application of a space-time Galerkin method [52] and of DWR techniques [48].

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