hp-Adaptive Finite Element Methods for Variational Inequalities

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In this work, we combine an hp-adaptive strategy with a posteriori error estimates for variational inequalities, which are given by contact problems. The a posteriori error estimates are obtained using a general approach based on the saddle point formulation of contact problems and making use of a posteriori error estimates for variational equations. Error estimates are presented for obstacle problems and Signorini problems with friction.
Numerical experiments confirm the reliability of the error estimates for finite elements of higher order. The use of the hp-adaptive strategy leads to meshes with the same characteristics as geometric meshes and to exponential convergence.

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1 Introduction

Variational inequalities play an important role in many fields of mathematical modelling. In this note, the main focus is on contact problems. A natural starting point of modelling unilateral contact problems in structural mechanics is to consider an energy minimization problem over a convex and closed subset K of a reflexive Banach space V. The set K contains the admissable displacements of a body being in contact with an obstacle. In linear elasticity, the energy functional is given by $E(v) := \frac{1}{2}a(v,v) - \langle \ell, v\rangle + j(v)$ with a symmetric, continuous, and V-elliptic bilinearform a and a convex and continuous functional j. The bilinearform a represents the constitutive law describing the material behavior. The functional ℓ is an element of the topological dual space V^* and describes volume or surface forces. The functional j models friction conditions. According to the principle of minimum potential energy, the body is in a stable, static equilibrium whenever the total potential energy is a minimum. That is, a static equilibrium is characterized by the solution $u \in K$ fulfilling $E(u) = \min_{v\in K} E(v)$. Under these assumptions, it can be shown that the solution u is equivalently characterized by the variational inequality

Problem 1.1 Find $u \in K$ such that: $\forall v \in K: \; a(u, v-u) + j(v) - j(u) \geq \langle \ell, v-u\rangle$.

Problem 1.1 and its various discretisations are widely studied in literature. We refer to the monographs of Kikuchi, Oden [1], Haslinger, et al. [2], and Glowinski, et al. [3] for an overview. In this note, we present a general framework for deriving a posteriori error estimates for contact problems which are included in Problem 1.1. The approach is based on the discretisation via a saddle point formulation. Finite element methods are used to discretise the space V. The key point is to reduce the error estimation for Problem 1.1 to an error estimation for variational equations. The advantage of this approach is that in principle each error estimator known from literature of variational equations can be used in order to obtain an error estimation for Problem 1.1. This technique goes back to Braess [4] who presented it for static obstacle problems. The extension to the general problem 1.1 is presented in [5] and represents the basis of this note.
The note is organised as follows: In Section 2, we briefly introduce a saddle point formulation of Problem 1.1 and its discretisation. This approach allows a proper extension to finite elements of higher order. The general framework of error estimation is presented in Section 3. The techniques are applied to simplified Signorini problems, model friction problems, Signorini problems with given friction and obstacle problems. The results are stated in Section 4. Whenever a posteriori error estimates are used in adaptive schemes, one has to deal with an appropriate refinement strategy and, in the case of hp finite element methods, one has to decide, whether the local polynomial degree should be increased. In Section 5, an hp-adaptive strategy is proposed which is based on the estimation of the local regularity determined by two estimation steps. This strategy goes back to Süli et al. in [6]. Eventually, numerical experiments shows the reliability of the error estimates and their applicability to h- and hp-adaptive schemes.

2 Saddle Point Formulation and its Discretisation

Geometric contact conditions modelled by the set K are typically resolved by either introducing a Lagrangian multiplier or by using an additional penalty term in the definition of E. The penalty method has the disadvantage that the solution is strongly affected by the penalty parameter. In this sense, the Lagrangian method based on a saddle point formulation is often preferable. However, in many cases it is difficult to ensure the stability of discretisation schemes based on it. Here, we use the saddle point formulation, since it is the basis of the error estimation approach of Section 3.
Let U_0 be a reflexive Banach space. We define linearised geometric contact conditions by setting $K := \{v \in V \mid g - \beta_0(v) \in G\}$, where $G \subset U_0$ is a closed and convex cone, g is in U_0 and $\beta_0 \in L(V, U_0)$ is a surjective mapping. We assume that the

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functional j is given by $j(v) = \sup_{\mu_1 \in \Lambda_1} \langle \mu_1, \beta_1(v) \rangle$ for a surjective mapping $\beta_1 \in L(V, U_1)$. Here, U_1 is a Banach Space and Λ_1 is a closed and bounded subset of U_1^*. The definition of j generalises the friction condition based on reduced friction problems or problems with given friction (see, e.g., [1]). An extension to Coulomb friction can be made by using a fix point method. By introducing $\Lambda_0 := G^*$ as the dual cone of G, we obtain the following saddle point problem:

Problem 2.1 Find $(u, \lambda_0, \lambda_1) \in V \times \Lambda_0 \times \Lambda_1$ such that

$$\forall v \in V : \ a(u, v) = \langle \ell, v \rangle - \langle \lambda_0, \beta_0(v) \rangle - \langle \lambda_1, \beta_1(v) \rangle,$$
$$\forall (\mu_0, \mu_1) \in \Lambda_0 \times \Lambda_1 : \ \langle \mu_0 - \lambda_0, \beta_0(u) - g \rangle + \langle \mu_1 - \lambda_1, \beta_1(u) \rangle \leq 0.$$

It can be proven by standard arguments of convex analysis, that a unique solution $(u, \lambda_0, \lambda_1)$ of Problem 2.1 exists. Furthermore, the first component u is the solution of Problem 1.1 (cf., e.g., [1], [7], [8]).
As usual, a discretisation is obtained by introducing finite dimensional linear subspaces $V_h \subset V$ and $U_{i,H}^* \subset U_i^*$, $i = 0, 1$. Here, the subscripts h and H denote the discretisation level. In finite element discretisations, h and H are the mesh sizes. By setting $\Lambda_{i,H} := \Lambda_i \cap U_{i,H}^*$, the discretisation is given by

Problem 2.2 Find $(u_h, \lambda_{0,H}, \lambda_{1,H}) \in V_h \times \Lambda_{0,H} \times \Lambda_{1,H}$ such that

$$\forall v_h \in V_h : \ a(u_h, v_h) = \langle \ell, v_h \rangle - \langle \lambda_{0,H}, \beta_0(v_h) \rangle - \langle \lambda_{1,H}, \beta_1(v_h) \rangle,$$
$$\forall (\mu_{0,H}, \mu_{1,H}) \in \Lambda_{0,H} \times \Lambda_{1,H} : \ \langle \mu_{0,H} - \lambda_{0,H}, \beta_0(u_h) - g \rangle + \langle \mu_{1,H} - \lambda_{1,H}, \beta_1(u_h) \rangle \leq 0.$$

It is well-known that a unique solution $(u_h, \lambda_{0,H}, \lambda_{1,H})$ of Problem 2.2 exists if the discrete Babuška-Brezzi-Condition is fulfilled: $\exists \alpha > 0$ (indep. of the discret.): $\alpha(\|\mu_{0,H}\| + \|\mu_{1,H}\|) \leq \sup_{v_h \in V_h \setminus \{0\}} (\langle \mu_{0,H}, \beta_0(v_h) \rangle + \langle \mu_{1,H}, \beta_1(v_h) \rangle) \|v_h\|^{-1}$. For simplicity, the norm symbols always denote the norms of the corresponding Banach spaces.

3 A Posteriori Error Estimation

The general problem of a posteriori error estimation is to find a number $\eta \geq 0$ which only depends on the computable parameters u_h, $\lambda_{0,H}$, $\lambda_{1,H}$ such that $\|u - u_h\|^2 + \|\lambda_0 - \lambda_{0,H}\|^2 + \|\lambda_1 - \lambda_{1,H}\|^2 \leq C\eta^2$, where the constant $C > 0$ is independent of the discretisation level. The number η is called error estimator.
The key for error estimation based on a saddle point formulation is the observation that the discrete solution u_h is also a discrete solution of the following auxiliary problem:

Problem 3.1 Find $u_\star \in V$ such that: $\forall v \in V : \ a(u_\star, v) = \langle \ell, v \rangle - \langle \lambda_{0,H}, \beta_0(v) \rangle - \langle \lambda_{1,H}, \beta_1(v) \rangle$.

The discretisation error of Problem 3.1 can be estimated by a standard error estimator $\eta_\star > 0$ for variational equations, that is $\|u_\star - u_h\|^2 \leq C_\star \eta_\star^2$ with a constant $C_\star > 0$. The essential approach is to prove the following estimation, which is a consequence of Lemma IV.2 in [5]:

$$\|u - u_h\|^2 + \|\lambda_0 - \lambda_{0,H}\|^2 + \|\lambda_1 - \lambda_{1,H}\|^2 \leq C' \eta_\star^2 + C''(\langle \lambda_0, \beta_0(u_h) - g \rangle + \langle \lambda_1 - \lambda_{1,H}, \beta_1(u_h) \rangle). \tag{1}$$

On the basis of (1) we state the main result (Theorem IV.3 in [5]):

Theorem 3.2 *Let* $d \in \tilde{K} := \{v \in V \mid g - \beta_0(u_h + v) \in G\}$ *and* $\eta^2 := \eta_\star^2 + \|d\|^2 + |\langle \lambda_{0,H}, \beta_0(d) \rangle| + |j(d) - \langle \lambda_{1,H}, \beta_1(d) \rangle| + |j(u_h) - \langle \lambda_{1,H}, \beta_1(u_h) \rangle|)$*, then, there holds*

$$\|u - u_h\|^2 + \|\lambda_0 - \lambda_{0,H}\|^2 + \|\lambda_1 - \lambda_{1,H}\|^2 \leq C\eta^2 \tag{2}$$

with a constant $C > 0$ *independent of* V_h*,* $U_{0,H}^*$ *and* $U_{1,H}^*$*.*

We point out, that the a posteriori error estimation in Theorem 3.2 is a general estimation for arbitrary reflexive Banach spaces and discretisations. By specifying $\eta_\star$ in the context of finite element discretisations, we obtain a concrete error estimation for the Problems 1.1 and 2.1. All terms in the error estimation (2) can be interpreted as typical sources of errors in contact problems. The term $\|d\|^2$ measures the error of the geometric contact condition. The remaining terms measure the violation of the geometric complementary condition and of the friction condition.

4 Application to Various Contact Problems

The general result of Theorem 3.2 can be applied to various contact problems. In this note, we consider simplified Signorini problems, idealised friction problems and obstacle problems, which can be seen as model problems. Furthermore, we apply our results to linear elastic Signorini problems with given friction. Extensions to dynamic contact problems are studied in [9] and [10].
Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a domain and let Γ_0 be a closed subset of the boundary $\Gamma := \partial\Omega$ with $|\Gamma_0| > 0$. Moreover, let Γ_1 be a (sufficiently smooth) subset of Γ with $\overline{\Gamma}_1 \subset \Gamma_2 := \Gamma \backslash \Gamma_0$. In the following, the Banach space V is defined as

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$V := (H_0^1(\Omega, \Gamma_0))^k$, $k \in \{1, 2, 3\}$, where $H_0^1(\Omega, \Gamma_0)$ is the closure of $\{v \in C^\infty(\Omega) \mid v(x) = 0,\ x \in \Gamma_0\}$ with respect to the usual Sobolev-norm $\|\cdot\|_1$. The mapping $\gamma : V \to (L^2(\Gamma))^k$ denotes the trace operator. The Banach spaces U_0 and U_1 are defined as $U_0 := H^{1/2}(\Gamma_1)$ and $U_1 := (H^{1/2}(\Gamma_1))^l$, $l \in \{1, 2\}$. The finite element discretisation is based on finite element meshes $\mathcal{T}_h$ and $\mathcal{T}_{1,H}$ of Ω and Γ_1, respectively. Let $\Psi_T : [-1, 1]^n \to T \in \mathcal{T}_h$, $\Psi_{1,T} : [-1, 1]^{n-1} \to T \in \mathcal{T}_{1,H}$ be bijective and sufficiently smooth transformations, and let $p_T, p_{1,T} \in \mathbb{N}$ be degree distributrions. By using the polynomial space S_n^q of order q on a reference element $[-1, 1]^n$, we set $V_h := (S^p(\mathcal{T}_h))^k$, $U_{0,H}^* := M^{p_1}(\mathcal{T}_{1,H})$ and $U_{1,H}^* := M^{p_1}(\mathcal{T}_{1,H})^l$ where $S^p(\mathcal{T}_h) := \{v \in H^1(\Omega, \Gamma_0) \mid \forall T \in \mathcal{T}_h :\ v_{|T} \circ \Psi_T \in S_n^{p_T}\}$ and $M^{p_1}(\mathcal{T}_{1,H}) := \{v \in L^2(\Gamma_1) \mid \forall T \in \mathcal{T}_{1,H} :\ v_{|T} \circ \Psi_{1,T} \in S_{n-1}^{p_{1,T}}\}$. This discretisation is based on the finite element schemes proposed by Haslinger et al. in [11] and [12]. For uniform meshes and $p_T \equiv 1$ and $p_{1,T} \equiv 0$, it can be proven that the discrete Babuška-Brezzi-Condition is fulfilled, if the quotient h/H is sufficiently small.

The **simplified Signorini** problem is given by the variational inequality $\forall v \in K :\ (\nabla u, \nabla(v-u))_0 \geq (f, v-u)_0 + (q, \gamma(v-u))_{0,\Gamma_2}$, where $K := \{v \in H^1(\Omega, \Gamma_0) \mid \gamma(v) \geq g\}$ and $(\cdot,\cdot)_{0,\Gamma_1}$ denotes the L^2-scalar product on Γ_1. The problem decribes the deformation of a membrane restricted by an obstacle at the boundary Γ_1. In Figure 1a, the solution of the simplified Signorini problem is depicted for $\Omega := (-1, 1)^2$, $\Gamma_2 := (-1, 1) \times \{-1\} \cup \{-1\} \times (-1, 1)$, $f := -1$, $q(x_0, x_1) := -\frac{1}{4}x_1^3$ and $g(x_0, x_1) := -x_0^2$. The two curves are the obstacle g and the Lagrangian multiplier $\lambda = q - \partial_n u$. The saddle point formulation is $\forall v \in H^1(\Omega, \Gamma_0) :\ (\nabla u, \nabla v)_0 = (f, v)_0 + (q, \gamma(v))_{0,\Gamma_2} - \langle \lambda_0, \gamma(v)\rangle$ and $\forall \mu_0 \in \Lambda_0 :\ \langle \mu_0 - \lambda_0, \gamma(u) - g\rangle \leq 0$. Applying Theorem 3.2, we obtain the a posteriori error estimation

$$\|u - u_h\|_1^2 + \|\lambda_0 - \lambda_{0,H}\|_{-1/2,\Gamma_1}^2 \leq C(\eta_\star^2 + \|(g - \gamma(u_h))_+\|_{1/2,\Gamma_1}^2 + |(\lambda_{0,H}, (g - \gamma(u_h))_+)_{0,\Gamma_1}|)$$

where $(\cdot)_+$ denotes the positive part of a function. We choose the harmonic extension $d \in W := \{v \in H^1(\Omega, \Gamma_0) \mid \gamma(v) = (g - \gamma(u_h))_+$ on $\Gamma_1\}$ in Theorem 3.2.

The **idealised friction problem** is described by the inequality $\forall v \in H^1(\Omega, \Gamma_0) :\ (\nabla u, \nabla(v-u))_0 + (s, |\gamma(v)|)_{0,\Gamma_1} - (s, |\gamma(u)|)_{0,\Gamma_1} \geq (f, v-u)_0 + (q, \gamma(v-u))_{0,\Gamma_2}$. The problem models a membrane with the idealised frictional contact conditions: $|q - \partial_n u| < s \Rightarrow \gamma(u) = 0$, $(q - \partial_n u) = s \Rightarrow \gamma(u) \geq 0$, $-(q - \partial_n u) = s \Rightarrow \gamma(u) \leq 0$ (see, e.g., [3]). In Figure 1e, the solution of this problem is depicted where the two curves are the function s and the Lagrangian multiplier $\lambda = q - \partial_n u$. Here, we set $s(x_0, x_1) := 1 - x_0^2$. The saddle point formulation reads $\forall v \in H^1(\Omega, \Gamma_0) :\ (\nabla u, \nabla v)_0 = (f, v)_0 + (q, \gamma(v))_{0,\Gamma_2} - (\lambda_1, \gamma(v))_{0,\Gamma_1}$ and $\forall \mu_1 \in \Lambda_1 :\ (\mu_1 - \lambda_1, \gamma(u))_{0,\Gamma_1} \leq 0$ with $\Lambda_1 := \{\mu_1 \in L^2(\Gamma_1) \mid |\mu_1| \leq s$ a.e. on $\operatorname{supp}(s),\ \mu_1 = 0$ on $\Gamma_1 \setminus \operatorname{supp}(s)\}$. Theorem 3.2 leads to the a posteriori error estimation

$$\|u - u_h\|_1^2 + \|\lambda_1 - \lambda_{1,H}\|_{-1/2,\Gamma_1}^2 \leq C(\eta_1^2 + |j(u_h) - (\lambda_{1,H}, \gamma(u_h))_{0,\Gamma_1}|).$$

The variational inequality of the **Signorini problem with given friction** is given by $\forall v \in K :\ (\sigma(u), \varepsilon(v-u))_0 + (s, |\delta_t(v)|)_{0,\Gamma_1} - (s, |\delta_t(u)|)_{0,\Gamma_1} \geq (f, v-u)_0 + (q, \delta(v-u))_{0,\Gamma_2}$ with $K := \{v \in H^1(\Omega, \Gamma_0)^3 \mid g - \delta_n(v) \leq 0\}$ and $\delta := (\delta_n, \delta_t)$ as the normal and tangential components of γ. The usual stress and strain tensors of linear elasticity are denoted by σ and ϵ, respectivily. The saddle point formulation reads $\forall v \in H^1(\Omega, \Gamma_0)^3 :\ (\sigma(u), \varepsilon(v))_0 = (f, v)_0 + (q, \delta(v))_{0,\Gamma_2} - \langle \lambda_0, \delta_n(v)\rangle - (\lambda_1, \delta_t(v))_{0,\Gamma_1}$ and $\forall (\mu_0, \mu_1) \in \Lambda_0 \times \Lambda_1 :\ \langle \mu_0 - \lambda_0, \delta_n(v) - g\rangle + (\mu_1 - \lambda_1, \delta_t(u))_{0,\Gamma_1} \leq 0$ with $\Lambda_1 := \{\mu_1 \in L^2(\Gamma_1)^2 \mid |\mu_1| \leq s$ a.e. on $\operatorname{supp}(s),\ \mu_1 = 0$ on $\Gamma_1 \setminus \operatorname{supp}(s)\}$. From Theorem 3.2, we obtain the a posteriori error estimation

$$\begin{aligned}\|u - u_h\|_1^2 + \|\lambda_0 - \lambda_{0,H}\|_{-1/2,\Gamma_1}^2 + \|\lambda_1 - \lambda_{1,H}\|_{-1/2,\Gamma_1}^2 \leq{}& C(\eta_\star^2 + \|(g - \delta_n(u_h))_+\|_{1/2,\Gamma_1} \\ &+ |(\lambda_{0,H}, (g - \delta_n(u_h))_+)_{0,\Gamma_1}| + |j(u_h) - (\lambda_{1,H}, \delta_t(u_h))_{0,\Gamma_1}|).\end{aligned}$$

The **obstacle problem** is given by the same variational inequality as the simplified Signorini problem is, but with $K := \{v \in H^1(\Omega, \Gamma_0) \mid v \geq g\}$. It models a membrane with an obstacle in the interior of the domain. Also, the saddle point formulation is similar to the formulation of the simplified Signorini problem. Here, the set Λ_0 is the dual cone of $G := \{v \in H_0^1(\Omega, \Gamma_0) \mid v \leq 0\}$. Eventually, Theorem 3.2 gives the following a posteriori estimation

$$\|u - u_h\|_1^2 + \|\lambda_0 - \lambda_{0,H}\|_{-1}^2 \leq C(\eta_0^2 + \|(g - u_h)_+\|_1 + |(\lambda_{0,H}, (g - u_h)_+)_0|).$$

We choose $U_0 := V$ and $U_{0,H}^* := \{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_{1,H} :\ v_{|T} \circ \Psi_{1,T} \in S_n^{p_{1,T}}\}$ for the finite element discretisation.

5 An hp-Adaptivity Strategy and Numerical Results

For an hp-adaptive strategy, we consider the contributions of two error estimators $(\eta^0)^2 := \sum_{T \in \mathcal{T}_h} (\eta_T^0)^2$ and $(\eta^1)^2 := \sum_{T \in \mathcal{T}_h} (\eta_T^1)^2$ on the same mesh, but for different degree distributions p^0 and p^1. By assuming $\eta_T^0 \approx C_T(p_T^0)^{-\rho_T+1}$ and $\eta_T^1 \approx C_T(p_T^1)^{-\rho_T+1}$ for $T \in \mathcal{T}_h$ with $p_T^0 < p_T^1$, we can approximately calculate the local regularity by $\varrho_T = \log(\eta_T^1/\eta_T^0)/\log(p_T^0/p_T^1) + 1$. If $\varrho_T < p_T^1$, the mesh element T should be divided, otherwise the local polynomial degree should be increased. The complete strategy is given as follows: (1) Determine $(\eta^0)^2$ and sort $\eta_{T_1}^0 \leq \eta_{T_2}^0 \leq \ldots$. (2) With

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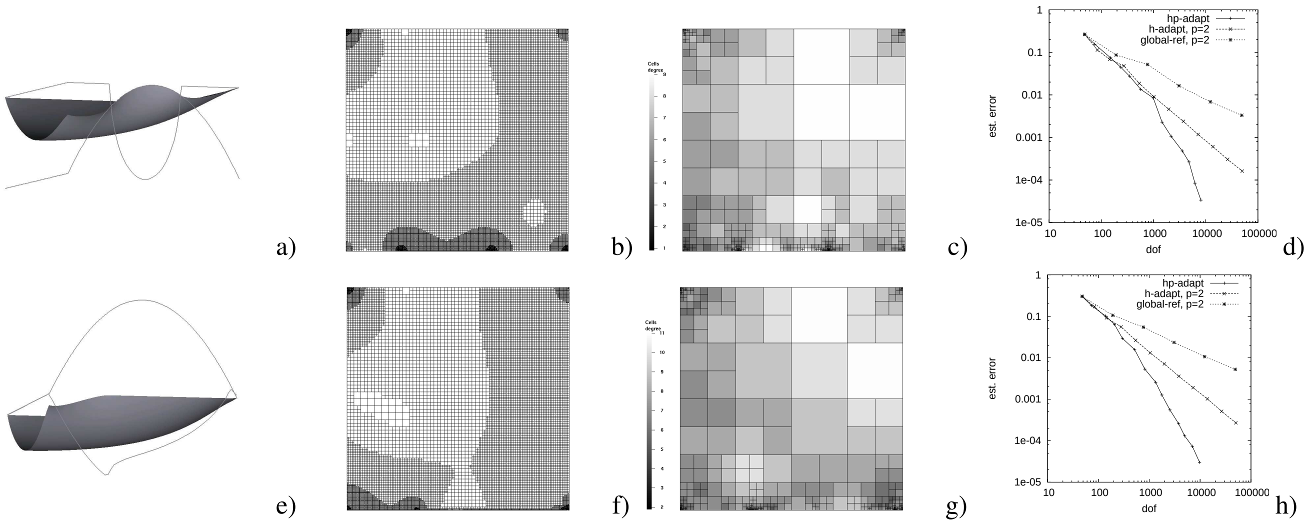

Fig. 1 **a**-**d**: simplified Signorini problem: deformed membrane, h- and hp-adaptive meshes, estimated convergence. **e**-**h**: idealised friction problem.

$k^\star := \lfloor(1-\kappa)|\mathcal{T}_h|\rfloor+1$, $0<\kappa<1$, define $\mathcal{K} := \{T \in \mathcal{T}_h \mid \eta_T^0 \geq \eta^0_{T_{k^\star}}\}$ and $\mathcal{W} := \{T \in \mathcal{T}_h\backslash\mathcal{K} \mid \exists T_0 \in \mathcal{K} :\ \emptyset \neq \overline{T}\cap\overline{T_0} \not\in \mathcal{V}\}$ ($\mathcal{V}$ is the set of all nodes in $\mathcal{T}_h$). (3) Set $p_T^1 := p_T^0+1$ for all $T \in \mathcal{K}\cup\mathcal{W}$ and $p_T^1 := p_T^0$ for all $T \in \mathcal{T}_h\backslash(\mathcal{K}\cup\mathcal{W})$ and determine $(\eta^1)^2$. (4) Divide $T \in \mathcal{K}$, if $\varrho_T < p_T^1$ and set $p^0_{\tilde{T}} := p_T^0$ for the new elements $\tilde{T}$, otherwise set $p_T^0 := p_T^1$. By increasing the local polynomial degree for the neighbouring elements $T \in \mathcal{W}$, we ensure that the complete local polynomial space is used for all $T \in \mathcal{K}$ to determine η^1.

In Figures 1c and 1g, hp-adaptive meshes are shown for the simplified Signorini problem and the idealised friction problem specified in Section 4. The grey values represent different local polynomial degrees (white=high polynomial degree). In Figures 1c and 1g, the estimated convergence is plotted for h- and hp-adaptive refinements. As mentioned in Section 3, an a posteriori error estimator for variational equations is needed to establish the estimation for variational inequalities. Here, we use the a posteriori error estimator proposed by Melenk [13]. The corresponding constant $C_\star$ is independent of the mesh size as well as the local polynomial degree, making it suitable for hp-adaptive schemes.

As we can see in the pictures, we obtain an optimal algebraic convergence order, using an h-adaptive strategy (here, we use a simple fixed fraction strategy) and exponentiell convergence using the hp-adaptive strategy, described above. As expected, the contact zone is refined. Moreover, the use of the hp-adaptive strategy leads to meshes with the characteristics of geometric meshes.

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