Projective SOR-Procedures for Signorini Problems in Linear Elasticity

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Abstract. In this work, we present a new SOR-procedure with projection for solving quadratic optimization problems resulting from Signorini Problems in linear elasticity. The procedure is based on a variable transformation and exploits the sparsity structure of problems which are obtained by finite element discretizations. An accelerated variant of the procedure is presented which is based on a CG-like method. The convergence of the procedures is proven and numerical results are discussed in the context of contact problems.

1. Introduction

Quadratic optimization problems have the structure: Find $x \in K$, such that

$$E(x) = \min_{y \in K} E(y) \tag{1}$$

with

$$E(y) := \frac{1}{2} y^\top A y - y^\top L,$$
$$K := \{v \in \mathbb{R}^n \mid Bv \leq g\},$$
$$A \in \mathbb{R}^{n \times n},\ L \in \mathbb{R}^n,\ B \in \mathbb{R}^{m \times n},\ g \in \mathbb{R}^m.$$

In the following, we assume the matrix A to be symmetric and positive definite. The relation $x \leq y$ with $x, y \in \mathbb{R}^m$ is defined componentwise, i.e.

$$x \leq y \Leftrightarrow \forall i \in \mathbb{Z}_m :\ x_i \leq y_i.$$

Here and in what follows, we use $\mathbb{Z}_m := \{0, \ldots, m-1\}$ for notational convenience. It is well-known that a unique solution $x \in K$ of (1) exists. Many algorithms for solving such general quadratic optimization problems are described in literature. We refer to [2], [4], [8] and [10] for an overview.

In many cases, the matrix B has a special structure which can be exploit for efficient algorithms. In this work, we consider quadratic optimization problems in which the matrix B has the following structure: There exist numbers $\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{m-1} \in \mathbb{N}$, such that

$$\forall j \in \mathbb{Z}_m : \ \alpha_j < \beta_j \leq n, \tag{2}$$

$$\forall j, j' \in \mathbb{Z}_m, j \neq j' : \ \beta_j \leq \alpha_{j'} \vee \beta_{j'} \leq \alpha_j, \tag{3}$$

$$\forall j \in \mathbb{Z}_m : \ \forall i \in \mathbb{Z}_n \backslash \{\alpha_j, \ldots, \beta_j - 1\} : \ B_{ji} = 0. \tag{4}$$

It is clear that each matrix with at the most one non-zero component in each column can be transformed into a matrix with such a structure by permuting the columns.

An important class of quadratic optimization problems with linear constraints described by the conditions (2)-(4) are Signorini problems in linear elasticity (cf., e.g.,[7], [9]) : Each row of the matrix B contains a vector which is the outer normal vector of a certain boundary point. Each column has at the most one non-zero component (see Section 5).

A simple procedure which is frequently proposed for quadratic optimization problems with box constraints is the SOR-procedure with projection which we call PSOR-procedures in the following (Projective Successive OverRelaxation). Especially, model contact problems as obstacle problems or simplified Signorini problems can efficiently be solved by applying such methods. We refer to [5] (p.67), [6] (p.40), and [7] (p.20) for obstacle problems and to [7] (p.20) and [9] (p.128) for simplified Signorini problems. In [1], an accelerated variant has been developed which is included in a cascadic multigrid scheme.

In this work, we present a new PSOR-procedure which includes the matrix conditions (2)-(4). By applying the finite element method for discretization, the matrix A has a sparsity structure. The presented approach provides for minimal cost due to matrix-vector multiplications of sparse matrices. The main results concerning this procedure and its convergence are described in Section 4. Furthermore, we present an accelerated variant which is based on a CG-like method. The presented approach is based on a variable transformation which has originally been proposed by Hlavacek et al. in [7] and which is extended to the conditions (2)-(4) in Section 3. The main idea is to transform the quadratic optimization problem with conditions (2)-(4) to a quadratic optimization problem with box constraints so that the standard PSOR-procedure can be applied.

In Section 2, we briefly sum up the main results dealing with the PSOR-procedure for box constraints. In the last section, numerical results are presented in the context of contact problems.

2. The PSOR-Procedure for Box Constraints

The PSOR-procedure is based on a successive-overrelaxation procedure with an additional projection onto the set K in each iteration step.

The set K is given by box constraints, if

$$K = K_0 \times K_1 \times \cdots \times K_{n-1} \tag{5}$$

with $K_i := [v_i, w_i]$ for $v, w \in (\mathbb{R} \cup \{\infty\})^n$, $v \leq w$. In the cases $v_i = -\infty$ or/and $w_i = \infty$ the interval K_i is unbounded. It is clear that box constraints can be described by setting

$$B := \begin{pmatrix} \tilde{I}_0 \\ -\tilde{I}_1 \end{pmatrix}, \quad g := \begin{pmatrix} \tilde{w} \\ -\tilde{v} \end{pmatrix}$$

where the submatrices $\tilde{I}_0$ and $\tilde{I}_1$ result from cancelling the i-th row of the identity $I \in \mathbb{R}^{n \times n}$ with $v_i = -\infty$ or $w_i = \infty$, respectively. In the same manner, the vectors $\tilde{v}$ and $\tilde{w}$ are defined. Obviously, the conditions (2)-(4) only generalize unilateral box constraints.

The PSOR-procedure is usually introduced by defining a sequence $\{x^\ell\}_{\ell \in \mathbb{N}}$ with $x^0 \in \mathbb{R}^n$ and

$$x_i^{\ell+1} := P_i\Big((1-\omega)x_i^\ell + \omega(\frac{1}{A_{ii}}(L_i - (\sum_{j=0}^{i-1} A_{ij}x_j^{\ell+1} + \sum_{j=i+1}^{n-1} A_{ij}x_j^\ell))\Big),$$
$$i = 0, \ldots, n-1$$

with the relaxation parameter $0 < \omega < 2$ and the projection $P_i : \mathbb{R} \to \mathbb{R}$ given as

$$P_i(a) := \begin{cases} v_i, & a < v_i \\ w_i, & a > w_i \\ a, & \text{else.} \end{cases}$$

In many cases, the matrix A is implicitly given, so that only the matrix vector product is available. Here it is useful to rewrite the PSOR scheme in the following way: For $i \in \{0, \dots, n-1\}$, we define a mapping $S_i : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(S_i(x))_j := \begin{cases} P_i(x_i + \delta_i(L_i - A_{i,\cdot}x)), & i = j \\ x_j, & \text{else} \end{cases}$$

Here, $A_{i,\cdot}$ denotes the i-th row vector of A and $\delta \in \mathbb{R}^n$ is defined by $\delta_i := \omega A_{ii}^{-1}$. By introducing the mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ through

$$S := S_{n-1} \circ S_{n-2} \circ \ldots \circ S_1 \circ S_0,$$

the PSOR-sequence $\{x^\ell\}_{\ell \in \mathbb{N}}$ fulfills $x^{\ell+1} = S(x^\ell)$. Therefore, we obtain:

Theorem 1. *Let the assumption* (5) *be fulfilled and* $\{x^\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R}^n$ *be given by* $x^{\ell+1} := S(x^\ell)$. *For the solution* $x \in K$ *of* (1), *there holds*

$$\lim_{\ell \to \infty} x^\ell = x.$$

Proof. *Ch.2, Th.1.3 in [5].* □

An efficient computer routine can be implemented as follows: For an integer `i` and a real valued array `x`, we define the routine

$$\begin{aligned} &\mathtt{PSOR}[\mathtt{i},\mathtt{x}]: \\ &\quad \mathtt{x_i} \leftarrow \mathtt{x_i} + \delta_{\mathtt{i}}(\mathtt{L_i} - \mathtt{A_{i,\cdot}}\mathtt{x}); \\ &\quad \text{if } \mathtt{x_i} > \mathtt{w_i} : \ \mathtt{x_i} \leftarrow \mathtt{w_i} \\ &\quad \text{else if } \mathtt{x_i} < \mathtt{v_i} : \ \mathtt{x_i} \leftarrow \mathtt{v_i}. \end{aligned}$$

The notation $\mathtt{a} \leftarrow \mathtt{b}$ means that the variable `a` is assigned to the value of the variable `b`, and $\mathtt{ROUTINE}[\mathtt{x},\mathtt{y},\ldots]$ can be understood as a function header with variables $\mathtt{x},\mathtt{y},\ldots$ (in the sense of 'call by reference'). The use of such a notation is adapted to a direct implementation and, especially, to the reuse of auxiliary variables by overwriting their data contents.
The routine $\mathtt{PSOR}[\mathtt{i},\mathtt{x}]$ corresponds to the mapping S_i in such a way that `x` contains the vector $S_i(x)$ after calling this routine. Letting `i` pass from `0` to $\mathtt{n}-\mathtt{1}$ calling $\mathtt{PSOR}[\mathtt{i},\mathtt{x}]$, we get a routine $\mathtt{PSOR}[\mathtt{x}]$, which corresponds to the mapping S. Of course, the complete PSOR-procedure is the iteration of $\mathtt{PSOR}[\mathtt{x}]$ until a prescribed stopping criterion is reached.
Similarly, we obtain a further procedure by the routine $\mathtt{PSSOR}[\mathtt{x}]$

(Projective Symmetric SOR), where we pass $\mathtt{i}$ from $\mathtt{0}$ to $\mathtt{n}-\mathtt{1}$ and, thereafter, from $\mathtt{n}-\mathtt{2}$ to $\mathtt{0}$ calling $\mathtt{PSOR}[\mathtt{i},\mathtt{x}]$.

The possible rate of convergence of the PSOR- and PSSOR-procedure is limited by the order of convergence of the unconstrained SOR-procedures. However, we obtain an significantly accelerated variant of the PSSOR-procedure by searching a minimizer in the affine subspace $x^{\ell+1}+\mathrm{span}\{r,s\}$ spanned by $r := x^{\ell}-x^{\ell-1}$ and $s := x^{\ell+1}-x^{\ell}$. The minimizer can simply be determined by solving the two dimensional unconstrained quadratic problem

$$\min_{(\alpha,\beta)\in\mathbb{R}^2}\frac{1}{2}(x^{\ell+1}+\alpha r+\beta s)^\top A(x^{\ell+1}+\alpha r+\beta s)-(x^{\ell+1}+\alpha r+\beta s)^\top L,$$

which reduces to a linear system of equations in two variables. For these purposes, we introduce the following routine:

$$\begin{aligned}
&\mathtt{MIN}[\mathtt{A},\mathtt{x},\mathtt{r},\mathtt{s}]:\\
&\qquad \mathtt{d}_0 \leftarrow \mathtt{r}^\top(\mathtt{L}-\mathtt{Ax});\\
&\qquad \mathtt{d}_1 \leftarrow \mathtt{r}^\top\mathtt{As};\\
&\qquad \mathtt{d}_2 \leftarrow \mathtt{r}^\top\mathtt{Ar};\\
&\qquad \beta \leftarrow ((\mathtt{s}^\top(\mathtt{L}-\mathtt{Ax}))\mathtt{d}_2-\mathtt{d}_0\mathtt{d}_1)/((\mathtt{s}^\top\mathtt{As})\mathtt{d}_2-\mathtt{d}_1^2);\\
&\qquad \alpha \leftarrow (\mathtt{d}_0-\beta\mathtt{d}_1)/\mathtt{d}_2;\\
&\qquad \mathtt{x} \leftarrow \mathtt{x}+\alpha\mathtt{r}+\beta\mathtt{s}.
\end{aligned}$$

Let $\mathfrak{M}:\mathbb{R}^{n\times n}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be defined in such a way, that $\mathfrak{M}(A,x,r,s)$ is equal to $\mathtt{x}$ after calling the routine `MIN[A,x,r,s]`. Then, obviously

$$E(\mathfrak{M}(A,x,r,s)) = \min_{y\in x+\mathrm{span}\{r,s\}} E(y). \tag{6}$$

The accelerated procedure APSSOR (Accelerated PSSOR) is given by

$$\begin{aligned}
&\mathtt{APSSOR}[\mathtt{x},\mathtt{y}]:\\
&\qquad \mathtt{r} \leftarrow \mathtt{x}-\mathtt{y};\\
&\qquad \mathtt{y} \leftarrow \mathtt{x};\\
&\qquad \mathtt{PSSOR}[\mathtt{x}];\\
&\qquad \mathtt{s} \leftarrow \mathtt{x}-\mathtt{y};\\
&\qquad \mathtt{MIN}[\mathtt{A},\mathtt{x},\mathtt{r},\mathtt{s}];\\
&\qquad \mathtt{i}=\mathtt{0},\ldots,\mathtt{n}-\mathtt{1}:\\
&\qquad\qquad \text{if } \mathtt{x_i}>\mathtt{w_i}:\ \mathtt{x_i}\leftarrow\mathtt{w_i}\\
&\qquad\qquad \text{else if } \mathtt{x_i}<\mathtt{v_i}:\ \mathtt{x_i}\leftarrow\mathtt{v_i}.
\end{aligned}$$

The complete APSSOR-procedure is obtained by iterating the routine `APSSOR[x,y]` until a prescribed stopping criterion is reached. We refer to [1] and references therein for more details.

3. Transformation to Box Constraints

In this section, we state a variable transformation so that problem (1) with the conditions (2)-(4) is transformed to a quadratic minimization problem with unilateral box constraints.

For this purpose, we define the column index $\rho(j) \in \{\alpha_j, \dots, \beta_j - 1\}$ for a row index $j \in \mathbb{Z}_m$ of B, so that

$$|B_{j,\rho(j)}| = \max\{|B_{ji}| \mid i \in \{\alpha_j, \dots, \beta_j - 1\}\}.$$

We additionally assume that $B_{j,\rho(j)} \neq 0$. This is no restriction, since rows with only zero components can be neglected. The index $\rho(j)$ can be interpreted as the pivot index of the row with index j. For a column index $i \in \mathbb{Z}_n$, we define the number

$$\gamma_i := \begin{cases} j, & \exists j \in \mathbb{Z}_m : \ \alpha_j \leq i < \beta_j \\ -1, & \text{else,} \end{cases}$$

which indicates the row index with a non-zero component (if it exists). Obviously, there holds $\gamma_{\rho(j)} = j$. Furthermore, we define the vectors $\kappa, \sigma \in \mathbb{R}^n$ by

$$\sigma_i := \begin{cases} (1 - B_{\gamma_i,i})/B_{\gamma_i,i}, & \gamma_i \geq 0, \ i = \rho(\gamma_i) \\ -B_{\gamma_i,i}/B_{\gamma_i,\rho(\gamma_i)}, & \gamma_i \geq 0, \ i \neq \rho(\gamma_i) \\ 0, & \text{else,} \end{cases}$$

$$\kappa_i := \begin{cases} B_{\gamma_i,i} - 1, & \gamma_i \geq 0, \ i = \rho(\gamma_i) \\ B_{\gamma_i,i}, & \gamma_i \geq 0, \ i \neq \rho(\gamma_i) \\ 0, & \text{else} \end{cases}$$

with $i \in \mathbb{Z}_n$. For a vector $\xi \in \mathbb{R}^n$, we define the matrix $M^\xi \in \mathbb{R}^{n \times n}$ by

$$M^\xi_{ki} := \begin{cases} \xi_i, & \gamma_i \geq 0, \ k = \rho(\gamma_i) \\ 0, & \text{else} \end{cases}$$

with $i, k \in \mathbb{Z}_n$.

Lemma 1. *The matrices $I + M^\sigma$ and $I + M^\kappa$ are non singular, and*

$$(I + M^\kappa)^{-1} = I + M^\sigma.$$

Proof. First, we note $((I + M^\kappa)(I + M^\sigma))_{ki} = \delta_{ki} + M^\kappa_{ki} + M^\sigma_{ki} + (M^\kappa M^\sigma)_{ki}$ with

$$(M^\kappa M^\sigma)_{ki} = \sum_{r=1}^{n} M^\kappa_{kr} M^\sigma_{ri} = \begin{cases} \kappa_r \sigma_i, & \gamma_i \geq 0,\ k = \rho(\gamma_r),\ r = \rho(\gamma_i) \\ 0, & \text{else.} \end{cases}$$

For $i, k, r \in \mathbb{Z}_n$ with $\gamma_i \geq 0$, $k = \rho(\gamma_r)$ and $r = \rho(\gamma_i)$, we obtain $k = \rho(\gamma_{\rho(\gamma_i)}) = \rho(\gamma_i)$. Furthermore, there holds $\kappa_{\rho(\gamma_i)} = B_{\gamma_{\rho(\gamma_i)},\rho(\gamma_i)} - 1 = B_{\gamma_i,\rho(\gamma_i)} - 1$. Hence,

$$(M^\kappa M^\sigma)_{ki} = \begin{cases} (B_{\gamma_i,\rho(\gamma_i)} - 1)\sigma_i, & \gamma_i \geq 0,\ k = \rho(\gamma_i) \\ 0, & \text{else.} \end{cases}$$

and

$$((I + M^\kappa)(I + M^\sigma))_{ki} = \delta_{ki} + \begin{cases} \kappa_i + B_{\gamma_i,\rho(\gamma_i)}\sigma_i, & \gamma_i \geq 0,\ k = \rho(\gamma_i) \\ 0, & \text{else.} \end{cases}$$

For $k, i \in \mathbb{Z}_n$ with $\gamma_i = -1$ we immediately obtain that $((I+M^\kappa)(I+M^\sigma))_{ki} = \delta_{ki}$.
Let $i \in \mathbb{Z}_n$ with $\gamma_i \geq 0$ and $i \neq \rho(\gamma_i)$, then $((I+M^\kappa)(I+M^\sigma))_{ii} = 1$.
For $k \in \mathbb{Z}_n \backslash \{i\}$ with $k \neq \rho(\gamma_i)$, we obtain $((I+M^\kappa)(I+M^\sigma))_{ki} = 0$.
For $k = \rho(\gamma_i)$, we get

$$\begin{aligned} ((I + M^\kappa)(I + M^\sigma))_{ki} &= \kappa_i + B_{\gamma_i,\rho(\gamma_i)}\sigma_i \\ &= B_{\gamma_i,i} - B_{\gamma_i,\rho(\gamma_i)} B_{\gamma_i,i}/B_{\gamma_i,\rho(\gamma_i)} = 0. \end{aligned}$$

Let $i \in \mathbb{Z}_n$ with $\gamma_i \geq 0$ and $i = \rho(\gamma_i)$, then

$$\begin{aligned} ((I + M^\kappa)(I + M^\sigma))_{ii} &= 1 + \kappa_i + B_{\gamma_i,\rho(\gamma_i)}\sigma_i = 1 + \kappa_i + B_{\gamma_i,i}\sigma_i \\ &= 1 + (B_{\gamma_i,i} - 1) + B_{\gamma_i,i}(1 - B_{\gamma_i,i})/B_{\gamma_i,i} = 1. \end{aligned}$$

If $k \in \mathbb{Z}_n \backslash \{i\}$, then $k \neq \rho(\gamma_i)$, and therefore $((I+M^\kappa)(I+M^\sigma))_{ki} = 0$.
Summing up all these cases, we obtain $((I + M^\kappa)(I + M^\sigma))_{ki} = \delta_{ki}$ and the proof is complete. □

For $i \in \mathbb{Z}_n$, we set

$$\hat{K}_i := \begin{cases} (-\infty, g_{\gamma_i}], & \gamma_i \geq 0,\ i = \rho(\gamma_i) \\ (-\infty, \infty), & \text{else} \end{cases} \tag{7}$$

and $\hat{K} := \hat{K}_0 \times \hat{K}_1 \times \cdots \times \hat{K}_{n-1}$. Furthermore, let

$$\begin{aligned}\hat{E}(\hat{y}) &:= \frac{1}{2}\hat{y}^\top \hat{A}\hat{y} - \hat{y}^\top \hat{L}\\ \hat{A} &:= (I + M^\sigma)^\top A(I + M^\sigma)\\ \hat{L} &:= (I + M^\sigma)^\top L.\end{aligned}$$

Theorem 2. *There exists a unique $\hat{x} \in \hat{K}$, so that*

$$\hat{E}(\hat{x}) = \min_{\hat{y}\in\hat{K}} \hat{E}(\hat{y}).$$

Furthermore, $x = (I + M^\sigma)\hat{x}$.
Proof. The matrix $\hat{A}$ is positive definite and the existence of a unique $\hat{x} \in \hat{K}$ is guaranteed.
For $y \in K$ and $i \in \mathbb{Z}_n$ with $\gamma_i \geq 0$ and $i = \rho(\gamma_i)$, we have

$$\begin{aligned}((I + M^\kappa)y)_i &= y_i + \sum_{r=\alpha_{\gamma_i}}^{\beta_{\gamma_i}-1} \kappa_r y_r\\ &= y_i + (B_{\gamma_i,i} - 1)y_i + \sum_{r=\alpha_{\gamma_i}, r\neq i}^{\beta_{\gamma_i}-1} B_{\gamma_i,r}y_r\\ &= \sum_{r=\alpha_{\gamma_i}}^{\beta_{\gamma_i}-1} B_{\gamma_i,r}y_r \leq g_{\gamma_i}.\end{aligned}$$

Therefore, we obtain $K' := \{(I + M^\kappa)y \mid y \in K\} \subset \hat{K}$.
Let $\hat{y} \in \hat{K}$ and $y := (I + M^\sigma)\hat{y}$. Furthermore, let $\tilde{y} := M^\sigma\hat{y}$. Then, we obtain

$$\tilde{y}_i = \begin{cases} \sum_{r=\alpha_{\gamma_i}}^{\beta_{\gamma_i}-1} \sigma_r \hat{y}_r, & i = \rho(\gamma_i)\\ 0, & \text{else}\end{cases}$$

and

$$
\begin{aligned}
(By)_j &= (B(I+M^\sigma)\hat{y})_j = (B\hat{y})_j + (B\tilde{y})_j \\
&= (B\hat{y})_j + B_{j,\rho(j)} \sum_{r=\alpha_j}^{\beta_j-1} \sigma_r \hat{y}_r \\
&= (B\hat{y})_j + B_{j,\rho(j)} \sigma_{\rho(j)} \hat{y}_{\rho(j)} + B_{j,\rho(j)} \sum_{r=\alpha_j,\, r\neq\rho(j)}^{\beta_j-1} \sigma_r \hat{y}_r \\
&= (B\hat{y})_j + B_{j,\rho(j)} (1 - B_{\gamma_{\rho(j)},\rho(j)})/B_{\gamma_{\rho(j)},\rho(j)} \hat{y}_{\rho(j)} \\
&\qquad - \sum_{r=\alpha_j,\, r\neq\rho(j)}^{\beta_j-1} B_{j,r}\hat{y}_r \\
&= (B\hat{y})_j + (1 - B_{j,\rho(j)})\hat{y}_{\rho(j)} - \sum_{r=\alpha_j,\, r\neq\rho(j)}^{\beta_j-1} B_{j,r}\hat{y}_r = \hat{y}_{\rho(j)} \le g_j.
\end{aligned}
$$

Thus, we have $y \in K$. Lemma 1 yields $\hat{y} = (I + M^\kappa)y \in K'$, and hence

$$\hat{K} = K'. \tag{8}$$

Because of (8), there exists $\tilde{x} \in K$, so that $\hat{x} = (I + M^\kappa)\tilde{x}$. This yields

$$
\begin{aligned}
E(\tilde{x}) &= E((I+M^\sigma)\hat{x}) \\
&= \frac{1}{2}((I+M^\sigma)\hat{x})^\top A(I+M^\sigma)\hat{x} - ((I+M^\sigma)\hat{x})^\top L \\
&= \hat{E}(\hat{x}) = \min_{\hat{y}\in\hat{K}} \hat{E}(\hat{y}) = \min_{y\in K} \hat{E}((I+M^\kappa)y) \\
&= \frac{1}{2}((I+M^\kappa)y)^\top (I+M^\sigma)^\top A(I+M^\sigma)(I+M^\kappa)y \\
&\qquad - ((I+M^\sigma)y)^\top (I+M^\sigma)^\top L \\
&= \min_{y\in K} E(y).
\end{aligned}
$$

Since x is unique, it follows $\tilde{x} = x$ and, therefore, $x = (I + M^\sigma)\hat{x}$. □

4. The PSOR-Procedure for Sparse Matrices

With regard to Theorem 2, the solution of (1) with the conditions (2)-(4) is simply obtained by using the projective SOR-procedures as introduced in Section 2 with $\hat{A}$, $\hat{L}$ and $\hat{K}$ instead of A, L and

K. However, keeping in mind that A and M^σ typically have sparsity structures, it is not suggestive to expand the matrix product $\hat{A} = (I + M^\sigma)^\top A(I + M^\sigma)$. Therefore, the direct use of these procedures would be inappropriate.

In the following, we present a procedure which only needs the matrix-vector product of the matrix A at the most twice. We call this procedure SPSOR (Sparse PSOR). The computational effort is not significantly higher as in the case of box constraints. Only, an additional auxiliary vector $z \in \mathbb{R}^m$ is needed.

For an integer `i` and real valued arrays `x` and `z`, the basic routine reads

$$
\begin{array}{l}
\mathtt{SPSOR}[\mathtt{i},\mathtt{x},\mathtt{z}]: \\
\quad \mathtt{if}\ \gamma_\mathtt{i} = -1:\ \mathtt{x}_\mathtt{i} \leftarrow \mathtt{x}_\mathtt{i} + \epsilon_\mathtt{i}(\mathtt{L}_\mathtt{i} - \mathtt{A}_{\mathtt{i};\cdot}\mathtt{x}) \\
\quad \mathtt{else}\ : \\
\quad\quad \mathtt{if}\ \mathtt{i} \neq \rho(\gamma_\mathtt{i}): \\
\quad\quad\quad \mathtt{a} \leftarrow \epsilon_\mathtt{i}(\mathtt{L}_\mathtt{i} - \mathtt{A}_{\mathtt{i},\cdot}\mathtt{x} + \sigma_\mathtt{i}(\mathtt{L}_{\rho(\gamma_\mathtt{i})} - \mathtt{A}_{\rho(\gamma_\mathtt{i}),\cdot}\mathtt{x})); \\
\quad\quad\quad \mathtt{x}_\mathtt{i} \leftarrow \mathtt{x}_\mathtt{i} + \mathtt{a}; \\
\quad\quad\quad \mathtt{x}_{\rho(\gamma_\mathtt{i})} \leftarrow \mathtt{x}_{\rho(\gamma_\mathtt{i})} + \sigma_\mathtt{i}\mathtt{a}; \\
\quad\quad\quad \mathtt{z}_{\gamma_\mathtt{i}} \leftarrow \mathtt{z}_{\gamma_\mathtt{i}} + \sigma_\mathtt{i}\mathtt{a} \\
\quad\quad \mathtt{else}: \\
\quad\quad\quad \mathtt{b} \leftarrow \mathtt{x}_\mathtt{i} - \mathtt{z}_{\gamma_\mathtt{i}}; \\
\quad\quad\quad \mathtt{c} \leftarrow \mathtt{b} + \epsilon_\mathtt{i}(\mathtt{L}_\mathtt{i} - \mathtt{A}_{\mathtt{i},\cdot}\mathtt{x}); \\
\quad\quad\quad \mathtt{if}\ \mathtt{c} > \mathtt{g}_\mathtt{i}:\ \mathtt{c} \leftarrow \mathtt{g}_\mathtt{i}; \\
\quad\quad\quad \mathtt{z}_{\gamma_\mathtt{i}} \leftarrow \mathtt{z}_{\gamma_\mathtt{i}} + \sigma_\mathtt{i}(\mathtt{c} - \mathtt{b}); \\
\quad\quad\quad \mathtt{x}_\mathtt{i} \leftarrow \mathtt{c} + \mathtt{z}_{\gamma_\mathtt{i}}.
\end{array}
$$

Here, $\epsilon \in \mathbb{R}^n$ is defined as

$$
\epsilon_i := \begin{cases} \omega A_{ii}^{-1}, & \gamma_i = -1 \\ \omega(A_{ii} + 2\sigma_i A_{\rho(\gamma_i),i} + (\sigma_i)^2 A_{\rho(\gamma_i),\rho(\gamma_i)})^{-1}, & \gamma_i \geq 0,\ i \neq \rho(\gamma_i) \\ \omega((1+\sigma_i)A_{ii})^{-1}, & \gamma_i \geq 0,\ i = \rho(\gamma_i) \end{cases}
$$

with the relaxation parameter $0 < \omega < 2$.
The compact form of routine `SPSOR[i,x,z]` is adapted for the implementation. In the routine, three cases are considered: The case $\gamma_i = -1$ means that, x_i is not constrained. The case $i \neq \rho(\gamma_i)$ implies that x_i is constrained and the index i is not the pivot index of the row γ_i. The case $i = \rho(\gamma_i)$ denotes that x_i is constrained and the

index i is the pivot index. For more details, see the proof of Theorem 2.

The routine $\texttt{SPSOR}[\texttt{i},\texttt{x},\texttt{z}]$ can be interpreted as mappings $S_i^* : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $T_i^* : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ where $S_i^*(x,z)$ is equal to $\texttt{x}$ and $T_i^*(x,z)$ is equal to $\texttt{z}$ after calling this routine. Passing $\texttt{i}$ from $\texttt{0}$ to $\texttt{n}-\texttt{1}$ calling $\texttt{SPSOR}[\texttt{i},\texttt{x},\texttt{z}]$, we obtain the routine $\texttt{SPSOR}[\texttt{x},\texttt{z}]$ which is executed until a stopping criterion is reached.

The routine $\texttt{SPSOR}[\texttt{x},\texttt{z}]$ corresponds to the composition of the mappings S_i^* and T_i^*

$$(S^*,T^*) := (S_{n-1}^*,T_{n-1}^*) \circ (S_{n-2}^*,T_{n-2}^*) \circ \ldots \circ (S_1^*,T_1^*) \circ (S_0^*,T_0^*).$$

Let the sequence $\{x^\ell, z^\ell\}_{\ell\in\mathbb{N}}$ be defined by

$$(x^{\ell+1}, z^{\ell+1}) := (S^*,T^*)(x^\ell, z^\ell) \tag{9}$$

with $x^0 \in \mathbb{R}^n$ and $z^0 := -\mathcal{M}^\kappa(x^0)$. The main result of this section is to show that the sequence $\{x^\ell\}_{\ell\in\mathbb{N}}$ converges to the solution $x \in K$.

For these purposes, let $\mathcal{M}^\xi : \mathbb{R}^n \to \mathbb{R}^m$ be defined by

$$(\mathcal{M}^\xi(x))_j := \sum_{i=\alpha_j}^{\beta_j-1} \xi_i x_i$$

for $\xi \in \mathbb{R}^n$ and the mappings $\hat{S}_i$ and $\hat{S}$ be given as the mappings S and S_i of Section 2, but with $\hat{A}$, $\hat{L}$ and $\hat{K}$ instead of A, L and K.

Lemma 2. *Let $x \in \mathbb{R}^n$. Then,*

$$-M^\kappa x = M^\sigma(I + M^\kappa)x, \quad -\mathcal{M}^\kappa(x) = \mathcal{M}^\sigma((I + M^\kappa)x).$$

Proof. $M^\sigma(I+M^\kappa)x = (I+M^\sigma)(I+M^\kappa)x - (I+M^\kappa)x = x-(I+M^\kappa)x = -M^\kappa x$. The second assertion follows by the first assertion and the definition of $\mathcal{M}^\xi$. □

Lemma 3. *Let $i \in \mathbb{Z}_n$, and $\hat{\delta} \in \mathbb{R}^n$ with*

$$\hat{\delta}_i = \begin{cases} \epsilon_i(1+\sigma_i)^{-1} & ,\gamma_i \geq 0,\ i = \rho(\gamma_i) \\ \epsilon_i, & \textit{else}, \end{cases}$$

then $\omega \hat{A}_{ii}^{-1} = \hat{\delta}_i$.

Proof. If $\gamma_i = -1$, then $\hat{A}_{ii} = A_{ii}$. If $\gamma_i \geq 0$ and $i \neq \rho(\gamma_i)$, we have $\hat{A}_{ii} = A_{ii} + 2((M^\sigma)^\top A)_{ii} + ((M^\sigma)^\top A M^\sigma)_{ii} = A_{ii} + 2\sigma_i A_{\rho(\gamma_i),i} + (\sigma_i)^2 A_{\rho(\gamma_i),\rho(\gamma_i)}$. Finally, if $\gamma_i \geq 0$ and $i = \rho(\gamma_i)$, we obtain $\hat{A}_{ii} = A_{ii} + 2(M^\sigma A)_{ii} + ((M^\sigma)^\top A M^\sigma)_{ii} = A_{ii} + 2\sigma_i A_{ii} + (\sigma_i)^2 A_{ii} = (1+\sigma_i)^2 A_{ii}$. Therefore, in all three cases we obtain $\omega \hat{A}_{ii}^{-1} = \hat{\delta}_i$. □

Lemma 4. *Let $i \in \mathbb{Z}_n$, $x \in \mathbb{R}^n$, then*

$$(S_i^*, T_i^*)(x, -\mathcal{M}^\kappa(x)) \\ = ((I + M^\sigma)\hat{S}_i((I + M^\kappa)x), \mathcal{M}^\sigma(\hat{S}_i((I + M^\kappa)x))).$$

Proof. Let $\hat{x} := (I + M^\kappa)x$ and $z := -\mathcal{M}^\kappa(x) = \mathcal{M}^\sigma(\hat{x})$. From Lemma 3 we have $(\hat{S}_i(\hat{x}))_i = \hat{P}_i(\hat{x}_i + \hat{\delta}_i(\hat{L}_i - \hat{A}_{i,\cdot}\hat{x}))$ with

$$\hat{P}_i(s) := \begin{cases} g_i, & \gamma_i \geq 0,\ i = \rho(\gamma_i),\ s > g_i \\ s, & \text{else.} \end{cases}$$

If $\gamma_i = -1$, then, for $k \neq i$, it holds

$$\hat{x}_k = (\hat{S}_i(\hat{x}))_k, \qquad (M^\sigma \hat{x})_k = (M^\sigma \hat{S}_i(\hat{x}))_k. \tag{10}$$

Thus, we have

$$(S_i^*(x, -\mathcal{M}^\kappa(x)))_k = x_k = ((I + M^\sigma)\hat{x})_k \\ = ((I + M^\sigma)\hat{S}_i(\hat{x}))_k = ((I + M^\sigma)\hat{S}_i((I + M^\kappa)x))_k. \tag{11}$$

Furthermore, there is $\hat{x}_i = x_i$ and $\hat{L}_i = L_i$. Since $i \neq \rho(\gamma_r)$ for all $r \in \mathbb{Z}_n$ with $\gamma_r \geq 0$, we obtain $((I + M^\sigma)^\top)_{ir} = \delta_{ir}$ and

$$\hat{A}_{i,\cdot}\hat{x} = ((I + M^\sigma)^\top A(I + M^\sigma)(I + M^\kappa)x)_i \\ = ((I + M^\sigma)^\top Ax)_i = A_{i,\cdot}x.$$

Thus, we have

$$(S_i^*(x, -\mathcal{M}^\kappa(x)))_i = x_i + \epsilon_i(L_i - A_{i,\cdot}x) = \hat{x}_i + \epsilon_i(\hat{L}_i - \hat{A}_{i,\cdot}\hat{x}) \\ = (\hat{S}_i(\hat{x}))_i = ((I + M^\sigma)\hat{S}_i((I + M^\kappa)x))_i.$$

Since $\mathcal{M}^\sigma(\hat{S}_i(\hat{x})) = \mathcal{M}^\sigma(\hat{x})$, we have

$$T_i^*(x, -\mathcal{M}^\kappa(x)) = T_i^*(x, z) = z = \mathcal{M}^\sigma(\hat{x}) = \mathcal{M}^\sigma(\hat{S}_i((I + M^\kappa)x)).$$

If $\gamma_i \geq 0$ and $i \neq \rho(\gamma_i)$, then, for $k \notin \{i, \rho(\gamma_i)\}$, we also have (10) and, therefore, (11). Furthermore, there holds $\hat{x}_i = x_i$, $\hat{L}_i = ((I + M^\sigma)^\top L)_i = L_i + \sigma_i L_{\rho(\gamma_i)}$ and

$$\hat{A}_{i,\cdot}\hat{x} = ((I + M^\sigma)^\top A(I + M^\sigma))_{i,\cdot}(I + M^\kappa)x \\ = (A + (M^\sigma)^\top A)_{i,\cdot}x = A_{i,\cdot}x + \sigma_i A_{\rho(\gamma_i),\cdot}x.$$

With $a := \epsilon_i(L_i - A_{i,\cdot}x + \sigma_i(L_{\rho(\gamma_i)} - A_{\rho(\gamma_i),\cdot}x)) = \epsilon_i(\hat{L}_i - \hat{A}_{i,\cdot}\hat{x})$, we get

$$\begin{aligned}(S_i^*(x, -\mathcal{M}^\kappa(x)))_i = x_i + a = \hat{x}_i + \epsilon_i(\hat{L}_i - \hat{A}_{i,\cdot}\hat{x}) = (\hat{S}_i(\hat{x}))_i \\ = ((I + M^\sigma)\hat{S}_i((I + M^\kappa)x))_i.\end{aligned}$$

Since

$$\begin{aligned}\mathcal{M}^\sigma(\hat{x})_{\gamma_i} + \sigma_i a &= \sigma_i(\hat{x}_i + a) + \sum_{k=\alpha_{\gamma_i}, k\neq i}^{\beta_{\gamma_i}-1} \sigma_k \hat{x}_k \\ &= \sigma_i(\hat{S}_i(\hat{x}))_i + \sum_{k=\alpha_{\gamma_i}, k\neq i}^{\beta_{\gamma_i}-1} \sigma_k \hat{x}_k = \mathcal{M}^\sigma(\hat{S}_i(\hat{x}))_{\gamma_i}, \quad (12)\end{aligned}$$

we obtain

$$\begin{aligned}(S_i^*(x, -\mathcal{M}^\kappa(x)))_{\rho(\gamma_i)} = x_{\rho(\gamma_i)} + \sigma_i a = \hat{x}_{\rho(\gamma_i)} + (\mathcal{M}^\sigma(\hat{x}))_{\gamma_i} + \sigma_i a \\ = \hat{x}_{\rho(\gamma_i)} + (\mathcal{M}^\sigma(\hat{S}_i(\hat{x})))_{\gamma_i} = ((I + M^\sigma)\hat{S}_i(\hat{x}))_{\rho(\gamma_i)} \\ = ((I + M^\sigma)\hat{S}_i((I + M^\kappa)x))_{\rho(\gamma_i)}.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}(T_i^*(x, -\mathcal{M}^\kappa(x)))_j = (T_i^*(x, z))_j = z_j = (\mathcal{M}^\sigma(\hat{x}))_j \\ = (\mathcal{M}^\sigma((I + M^\kappa)x))_j \quad (13)\end{aligned}$$

for $j \neq \gamma_i$ and, by using (12),

$$\begin{aligned}(T_i^*(x, -\mathcal{M}^\kappa(x)))_{\gamma_i} = (T_i^*(x, z))_{\gamma_i} = z_{\gamma_i} + \sigma_i a = \\ (\mathcal{M}^\sigma(\hat{x}))_{\gamma_i} + \sigma_i a = (\mathcal{M}^\sigma(\hat{S}_i(\hat{x}))_{\gamma_i} = (\mathcal{M}^\sigma(\hat{S}_i((I + M^\kappa)x))_{\gamma_i}.\end{aligned}$$

If $\gamma_i \geq 0$ and $i = \rho(\gamma_i)$, then, for $k \neq i$, we also obtain (10) and, therefore, (11) holds. Furthermore, there is $\hat{x}_i = x_i + (M^\kappa x)_i = x_i + (\mathcal{M}^\kappa(x))_{\gamma_i} = x_i - z_{\gamma_i}$, $\hat{L}_i = ((I + M^\sigma)^\top L)_i = L_i + \sigma_i L_i = (1 + \sigma_i)L_i$ and

$$\begin{aligned}\hat{A}_{i,\cdot}\hat{x} = ((I + M^\sigma)^\top A(I + M^\sigma))_{i,\cdot}(I + M^\kappa)x \\ = (A + (M^\sigma)^\top A)_{i,\cdot}x = A_{i,\cdot}x + \sigma_i A_{\rho(\gamma_i),\cdot}x = (1 + \sigma_i)A_{i,\cdot}x.\end{aligned}$$

For $j \neq \gamma_i$, we have (13). Furthermore, there holds

$$\begin{aligned}(\mathcal{M}^\sigma(\hat{x}))_{\gamma_i} + \sigma_i((\hat{S}_i(\hat{x})_i - \hat{x}_i) \\ = \sigma_i \hat{S}_i(\hat{x})_i + \sum_{k=\alpha_{\gamma_i}, k\neq i}^{\beta_{\gamma_i}-1} \sigma_i \hat{x}_i = (\mathcal{M}^\sigma(\hat{S}_i(\hat{x})))_{\gamma_i}.\end{aligned}$$

With $b := x_i - z_{\gamma_i} = \hat{x}_i$ and $c := \hat{P}_i(b + \epsilon_i(L_i - A_{i,\cdot}x)) = (\hat{S}_i(\hat{x}))_i$, we get

$$\begin{aligned}(T_i^*(x, -\mathcal{M}^\kappa(x)))_{\gamma_i} &= (T_i^*(x, z))_{\gamma_i}\\ &= z_{\gamma_i} + \sigma_i(c - b) = (\mathcal{M}^\sigma(\hat{x}))_{\gamma_i} + \sigma_i((\hat{S}_i(\hat{x})_i - \hat{x}_i)\\ &= (\mathcal{M}^\sigma(\hat{S}_i(\hat{x})))_{\gamma_i} = (\mathcal{M}^\sigma(\hat{S}_i((I + M^\kappa)x)))_{\gamma_i}\end{aligned}$$

and

$$\begin{aligned}(S_i^*(x), -\mathcal{M}^\kappa(x))_i &= c + (T_i^*(x, -\mathcal{M}^\kappa(x)))_{\gamma_i}\\ &= (\hat{S}_i(\hat{x}))_i + (\mathcal{M}^\sigma(\hat{S}_i(\hat{x})))_{\gamma_i} = ((I + M^\sigma)\hat{S}_i(\hat{x}))_i\\ &= ((I + M^\sigma)\hat{S}_i((I + M^\kappa)x))_i.\end{aligned}$$

□

Lemma 5. *Let $x \in \mathbb{R}^n$, then*

$$\begin{aligned}&(S^*, T^*)(x, -\mathcal{M}^\kappa(x))\\ &\qquad = ((I + M^\sigma)\hat{S}((I + M^\kappa)x), \mathcal{M}^\sigma(\hat{S}((I + M^\kappa)x))).\end{aligned}$$

Proof. Let $(S^{*,k}, T^{*,k}) := (S_k^*, T_k^*) \circ \ldots \circ (S_0^*, T_0^*)$ and $\hat{S}^k := \hat{S}_k \circ \ldots \circ \hat{S}_0$ for $k \in \mathbb{Z}_n$. Lemma 4 yields $(S^{*,0}, T^{*,0})(x, -\mathcal{M}^\kappa(x)) = ((I + M^\sigma)\hat{S}^0((I + M^\kappa)x), \mathcal{M}^\sigma(\hat{S}^0((I + M^\kappa)x))$. By induction, it follows from Lemma 2 and Lemma 4 for $k > 0$, that

$$\begin{aligned}&(S^{*,k}, T^{*,k})(x, -\mathcal{M}^\kappa(x))\\ &= ((S_k^*, T_k^*) \circ (S^{*,k-1}, T^{*,k-1}))(x, -\mathcal{M}^\kappa(x))\\ &= (S_k^*, T_k^*)((I + M^\sigma)\hat{S}^{k-1}((I + M^\kappa)x), \mathcal{M}^\sigma(\hat{S}^{k-1}((I + M^\kappa)x))\\ &= (S_k^*, T_k^*)((I + M^\sigma)\hat{S}^{k-1}((I + M^\kappa)x),\\ &\qquad - \mathcal{M}^\kappa((I + M^\sigma)\hat{S}^{k-1}((I + M^\kappa)x)))\\ &= ((I + M^\sigma)\hat{S}_k((I + M^\kappa)(I + M^\sigma)\hat{S}^{k-1}((I + M^\kappa)x)),\\ &\qquad \mathcal{M}^\sigma(\hat{S}_k((I + M^\kappa)(I + M^\sigma)\hat{S}^{k-1}((I + M^\kappa)x)))\\ &= ((I + M^\sigma)\hat{S}_k(\hat{S}^{k-1}((I + M^\kappa)x)), \mathcal{M}^\sigma(\hat{S}_k(\hat{S}^{k-1}((I + M^\kappa)x)))\\ &= ((I + M^\sigma)\hat{S}^k((I + M^\kappa)x)), \mathcal{M}^\sigma(\hat{S}^k((I + M^\kappa)x))).\end{aligned}$$

Since $(S^*, T^*) = (S^{*,n-1}, T^{*,n-1})$ and $\hat{S} = \hat{S}^{n-1}$, the proof is complete. □

Lemma 6. *Let $\{x^\ell, z^\ell\}_{\ell\in\mathbb{N}}$ be given as in* (9)*.*
Then, $z^{\ell+1} = -\mathcal{M}^\kappa(x^{\ell+1})$.
Proof. By induction, it follows from Lemma 2 and Lemma 5, that

$$\begin{aligned} z^{\ell+1} &= T^*(x^\ell, z^\ell) = T^*(x^\ell, -\mathcal{M}^\kappa(x^\ell)) \\ &= \mathcal{M}^\sigma(\hat{S}((I+M^\kappa)x^\ell)) = \mathcal{M}^\sigma((I+M^\kappa)(I+M^\sigma)\hat{S}((I+M^\kappa)x^\ell)) \\ &= \mathcal{M}^\sigma((I+M^\kappa)S^*(x^\ell, -\mathcal{M}^\kappa(x^\ell)) = \mathcal{M}^\sigma((I+M^\kappa)S^*(x^\ell, z^\ell)) \\ &= \mathcal{M}^\sigma((I+M^\kappa)x^{\ell+1}) = -\mathcal{M}^\kappa(x^{\ell+1}). \end{aligned}$$

□

Theorem 3. *Let $\{x^\ell, z^\ell\}_{\ell\in\mathbb{N}}$ be given as in* (9)*. For any sequence $\{\hat{x}^\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{R}^n$, we have*

$$\hat{x}^\ell = (I+M^\kappa)x^\ell \quad \Leftrightarrow \quad \hat{x}^{\ell+1} = \hat{S}(\hat{x}^\ell),\ \hat{x}^0 = (I+M^\kappa)x^0.$$

Proof. Let $\hat{x}^\ell = (I+M^\kappa)x^\ell$ be fulfilled, then we obtain from Lemma 5 and Lemma 6, that

$$\begin{aligned} \hat{x}^{\ell+1} &= (I+M^\kappa)x^{\ell+1} = (I+M^\kappa)S^*(x^\ell, z^\ell) \\ &= (I+M^\kappa)S^*(x^\ell, -\mathcal{M}^\kappa(x^\ell)) = (I+M^\kappa)(I+M^\sigma)\hat{S}((I+M^\kappa)x^\ell) \\ &= \hat{S}(\hat{x}^\ell). \end{aligned}$$

Let $\hat{x}^{\ell+1} = \hat{S}(\hat{x}^\ell)$ be fulfilled, then it holds by induction

$$\begin{aligned} \hat{x}^{\ell+1} &= \hat{S}(\hat{x}^\ell) = (I+M^\kappa)(I+M^\sigma)\hat{S}((I+M^\kappa)x^\ell) \\ &= (I+M^\kappa)S^*(x^\ell, -\mathcal{M}^\kappa(x^\ell)) = (I+M^\kappa)S^*(x^\ell, z^\ell) \\ &= (I+M^\kappa)x^{\ell+1}. \end{aligned}$$

□

Corollary 1. *Let $\{x^\ell, z^\ell\}_{\ell\in\mathbb{N}}$ be given as in* (9)*. For the solution $x \in K$ of* (1)*, there holds*

$$\lim_{\ell\to\infty} x^\ell = x.$$

Proof. Let $\hat{x}^\ell := (I+M^\kappa)x^\ell$. From Theorem 3 we obtain $\hat{x}^{\ell+1} = \hat{S}(\hat{x}^\ell)$. With $\hat{E}(\hat{x}) = \min_{\hat{y}\in\hat{K}} \hat{E}(\hat{y})$, Theorem 1 yields $\lim_{\ell\to\infty} \hat{x}^\ell = \hat{x} \in \hat{K}$. From Theorem 2, it follows, that

$$x = (I+M^\sigma)\hat{x} = (I+M^\sigma)\lim_{\ell\to\infty} \hat{x}^\ell = \lim_{\ell\to\infty}(I+M^\sigma)\hat{x}^\ell = \lim_{\ell\to\infty} x^\ell. \quad (14)$$

□

In the same way as in Section 2, by passing $\mathtt{i}$ from $\mathtt{0}$ to $\mathtt{n}-\mathtt{1}$ and then from $\mathtt{n}-\mathtt{2}$ to $\mathtt{0}$ calling $\mathtt{SPSOR}[\mathtt{i},\mathtt{x},\mathtt{z}]$, we obtain a further routine $\mathtt{SPSSOR}[\mathtt{x},\mathtt{z}]$ (<u>S</u>parse <u>PSSOR</u>) and an accelerated variant which we call ASPSSOR (<u>A</u>ccelerated <u>SPSSOR</u>):

$$
\begin{array}{l}
\mathtt{ASPSSOR}[\mathtt{x},\mathtt{y},\mathtt{z}]: \\
\quad \mathtt{r} \leftarrow \mathtt{x}-\mathtt{y}; \\
\quad \mathtt{y} \leftarrow \mathtt{x}; \\
\quad \mathtt{SPSSOR}[\mathtt{x},\mathtt{z}]; \\
\quad \mathtt{s} \leftarrow \mathtt{x}-\mathtt{y}; \\
\quad \mathtt{MIN}[\mathtt{A},\mathtt{x},\mathtt{r},\mathtt{s}]; \\
\quad \mathtt{j}=\mathtt{0},\ldots,\mathtt{m}: \\
\qquad \mathtt{z_j} = -\sum_{r=\alpha_j}^{\beta_j-1} \kappa_r x_r; \\
\qquad \mathtt{if}\ \mathtt{x}_{\rho(\mathtt{j})} - \mathtt{z_j} > \mathtt{g_j}: \\
\qquad\quad \mathtt{z_j} \leftarrow \mathtt{z_j} + \sigma_{\rho(\mathtt{j})}(\mathtt{g_j} - \mathtt{x}_{\rho(\mathtt{j})} + \mathtt{z_j}); \\
\qquad\quad \mathtt{x}_{\rho(\mathtt{j})} \leftarrow \mathtt{g_j} + \mathtt{z_j}.
\end{array}
$$

It remains to show, that the ASPSSOR-procedure corresponds to the APSSOR-procedure with $\hat{A}$, $\hat{L}$ and $\hat{K}$ instead of A, L and K. For these purposes, we introduce the mappings $Q^*:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $R^*:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^m$ and $\hat{Q}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n$ where $Q^*(x,y,z)$ is equal to $\mathtt{x}$ and $R^*(x,y,z)$ is equal to $\mathtt{z}$ after calling `ASPSSOR[x,y,z]`, and $\hat{Q}(x,y)$ is equal to $\mathtt{x}$ after calling `APSSOR[x,y]` (with $\hat{A}$, $\hat{L}$ and $\hat{K}$ instead of A, L and K). Furthermore, we set

$$(\mathfrak{S}^*,\mathfrak{T}^*) := (S_0^*,T_0^*)\ldots\circ(S_{n-2}^*,T_{n-2}^*)\circ(S_{n-1}^*,T_{n-1}^*)\circ\ldots\circ(S_0^*,T_0^*)$$

and

$$\hat{\mathfrak{S}} := \hat{S}_0\circ\ldots\circ\hat{S}_{n-2}\circ\hat{S}_{n-1}\circ\ldots\circ\circ\hat{S}_0.$$

Obviously, the mappings $(\mathfrak{S}^*,\mathfrak{T}^*)$ and $\hat{\mathfrak{S}}$ correspond to the routines `SPSSOR[x,z]` and `PSSOR[x,z]`.

Lemma 7. *Let $x\in\mathbb{R}^n$, then it holds*

$$
\begin{aligned}
&(\mathfrak{S}^*,\mathfrak{T}^*)(x,-\mathcal{M}^\kappa(x)) \\
&\qquad = ((I+M^\sigma)\hat{\mathfrak{S}}((I+M^\kappa)x),\mathcal{M}^\sigma(\hat{\mathfrak{S}}((I+M^\kappa)x))).
\end{aligned}
$$

Proof. By applying the same arguments as in Lemma 5, we obtain the assertion by induction. □

Lemma 8. *For $x, r, s \in \mathbb{R}^n$, there holds*

$$\mathfrak{M}(A, x, r, s) = (I + M^\sigma)\mathfrak{M}(\hat{A}, (I + M^\kappa)x, (I + M^\kappa)r, (I + M^\kappa)s).$$

Proof. Let $\hat{x} := (I + M^\kappa)x$, $\hat{r} := (I + M^\kappa)r$ and $\hat{s} := (I + M^\kappa)s$. Since there holds (see (6))

$$\hat{E}(\mathfrak{M}(\hat{A}, \hat{x}, \hat{r}, \hat{s})) = \min_{\hat{y} \in \hat{x} + \operatorname{span}\{\hat{r}, \hat{s}\}} \hat{E}(\hat{y}),$$

we obtain

$$\begin{aligned}
E(\mathfrak{M}(A, x, r, s)) &= \min_{y \in x + \operatorname{span}\{r, s\}} E(y) = \min_{\substack{y \in (I+M^\sigma)\hat{x} \\ +\operatorname{span}\{(I+M^\sigma)\hat{r}, (I+M^\sigma)\hat{s}\}}} E(y) \\
&= \min_{\substack{\hat{y} \in (I+M^\kappa)((I+M^\sigma)\hat{x} \\ +\operatorname{span}\{(I+M^\sigma)\hat{r}, (I+M^\sigma)\hat{s}\})}} E((I + M^\sigma)\hat{y}) \\
&= \min_{\hat{y} \in \hat{x} + \operatorname{span}\{\hat{r}, \hat{s}\}} E((I + M^\sigma)\hat{y}) \\
&= \min_{\hat{y} \in \hat{x} + \operatorname{span}\{\hat{r}, \hat{s}\}} \hat{E}(\hat{y}) = \hat{E}(\mathfrak{M}(\hat{A}, \hat{x}, \hat{r}, \hat{s})) \\
&= E((I + M^\sigma)\mathfrak{M}(\hat{A}, \hat{x}, \hat{r}, \hat{s})).
\end{aligned}$$

The assertion follows from the uniqueness. $\square$

Lemma 9. *For $x, y \in \mathbb{R}^n$, there holds*

$$\begin{aligned}
Q^*(x, y, -\mathcal{M}^\kappa(x)) &= (I + M^\sigma)\hat{Q}((I + M^\kappa)x, (I + M^\kappa)y), \\
R^*(x, y, -\mathcal{M}^\kappa(x)) &= \mathcal{M}^\sigma(\hat{Q}((I + M^\kappa)x, (I + M^\kappa)y)).
\end{aligned}$$

Proof. Let $\hat{x} := (I + M^\kappa)x$, $\hat{y} := (I + M^\kappa)y$ and $z := -\mathcal{M}^\kappa(x) = \mathcal{M}^\sigma(\hat{x})$. Then, Lemma 1, Theorem 2 and Lemma 7 yield $r := x - y = (I + M^\sigma)(\hat{x} - \hat{y}) =: (I + M^\sigma)\hat{r}$ and

$$s := \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)) - x = (I + M^\sigma)(\hat{\mathfrak{S}}(\hat{x}) - \hat{x}) =: (I + M^\sigma)\hat{s}.$$

Let $j \in \mathbb{Z}_m$ with $x_{\rho(j)} - z_j \leq g_j$. Then we obtain from Lemma 2, Lemma 7 and Lemma 8

$$\begin{aligned}
(R^*(x, y, -\mathcal{M}^\kappa(x)))_j &= -\sum_{r=\alpha_j}^{\beta_j - 1} \kappa_r (\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s))_r \\
&= (-\mathcal{M}^\kappa(\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s)))_j \\
&= (\mathcal{M}^\sigma((I + M^\kappa)\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s)))_j \\
&= (\mathcal{M}^\sigma((I + M^\kappa)(I + M^\sigma)\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s})))_j \\
&= (\mathcal{M}^\sigma(\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s})))_j
\end{aligned}$$

Furthermore, there is $\hat{x}_{\rho(j)} = ((I + M^\kappa)x)_{\rho(j)} = x_{\rho(j)} - z_j \le g_j$ and, therefore, $(\hat{Q}(\hat{x}, \hat{y}))_k = (\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s}))_k$ for $k = \alpha_j, \ldots, \beta_j - 1$. Hence,

$$
\begin{aligned}
(\mathcal{M}^\sigma(\hat{Q}((I + M^\kappa)x, (I + M^\kappa)y)))_j &= (\mathcal{M}^\sigma(\hat{Q}(\hat{x}, \hat{y})))_j \\
&= (\mathcal{M}^\sigma(\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s})))_j \qquad (15)
\end{aligned}
$$

and

$$
(R^*(x, y, -\mathcal{M}^\kappa(x)))_j = (\mathcal{M}^\sigma(\hat{Q}((I + M^\kappa)x, (I + M^\kappa)y)))_j.
$$

Moreover, we obtain from (15) and Lemma 8

$$
\begin{aligned}
(Q^*(x, y, -\mathcal{M}^\kappa(x)))_{\rho(j)} &= (\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s))_{\rho(j)} \\
&= ((I + M^\sigma)\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s}))_{\rho(j)} \\
&= (\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s}))_{\rho(j)} + (\mathcal{M}^\sigma(\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s})))_j \\
&= (\hat{Q}(\hat{x}, \hat{y}))_{\rho(j)} + (\mathcal{M}^\sigma(\hat{Q}((\hat{x}, \hat{y})))_j \\
&= ((I + M^\sigma)\hat{Q}((I + M^\kappa)x, (I + M^\kappa)y))_{\rho(j)}.
\end{aligned}
$$

Let $j \in \mathbb{Z}_m$ and $x_{\rho(j)} - z_j > g_j$, then $(\hat{Q}(\hat{x}, \hat{y}))_{\rho(j)} = g_j$ and we get

$$
\begin{aligned}
&(R^*(x, y, -\mathcal{M}^\kappa(x)))_j \\
&= (-\mathcal{M}^\kappa(\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s)))_j \\
&\qquad + \sigma_{\rho(j)}(g_j - (\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s))_{\rho(j)} \\
&\qquad + (-\mathcal{M}^\kappa(\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s)))_j) \\
&= (\mathcal{M}^\sigma((I + M^\kappa)\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s)))_j \\
&\qquad + \sigma_{\rho(j)}(g_j - ((I + M^\kappa)\mathfrak{M}(A, \mathfrak{S}^*(x, -\mathcal{M}^\kappa(x)), r, s))_{\rho(j)}) \\
&= (\mathcal{M}^\sigma(\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s})))_j + \sigma_{\rho(j)}(g_j - (\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s}))_{\rho(j)}) \\
&= \sigma_{\rho(j)} g_j + \sum_{k=\alpha_j, k \ne \rho(j)}^{\beta_j - 1} \sigma_k (\mathfrak{M}(\hat{A}, \hat{\mathfrak{S}}(\hat{x}), \hat{r}, \hat{s}))_k \\
&= \sigma_{\rho(j)}(\hat{Q}(\hat{x}, \hat{y}))_{\rho(j)} + \sum_{k=\alpha_j, k \ne \rho(j)}^{\beta_j - 1} \sigma_k (\hat{Q}(\hat{x}, \hat{y}))_k \\
&= (\mathcal{M}^\sigma(\hat{Q}(\hat{x}, \hat{y})))_j = (\mathcal{M}^\sigma(\hat{Q}((I + M^\kappa)x, (I + M^\kappa)y)))_j.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
(Q^*(x,y,-\mathcal{M}^\kappa(x)))_{\rho(j)} &= g_j + (R^*(x,y,-\mathcal{M}^\kappa(x)))_j \\
&= \hat{Q}((I+M^\kappa)x,(I+M^\kappa)y)_{\rho(j)} \\
&\quad + (\mathcal{M}^\sigma(\hat{Q}((I+M^\kappa)x,(I+M^\kappa)y)))_j \\
&= ((I+M^\sigma)\hat{Q}((I+M^\kappa)x,(I+M^\kappa)y))_{\rho(j)}.
\end{aligned}
$$

Finnally, let $i \in \mathbb{Z}_n$ with $i \notin \{\rho(j) \mid j \in \mathbb{Z}_m\}$. Then, it is $(I+M^\sigma)_{ik} = \delta_{ik}$ and we obtain

$$
\begin{aligned}
(Q^*(x,y,-\mathcal{M}^\kappa(x)))_i &= (\mathfrak{M}(A,\mathfrak{S}^*(x,-\mathcal{M}^\kappa(x)),r,s))_i \\
&= ((I+M^\sigma)\mathfrak{M}(\hat{A},\hat{\mathfrak{S}}(\hat{x}),\hat{r},\hat{s}))_i = (\mathfrak{M}(\hat{A},\hat{\mathfrak{S}}(\hat{x}),\hat{r},\hat{s}))_i \\
&= (\hat{Q}(\hat{x},\hat{y}))_i = ((I+M^\sigma)\hat{Q}(\hat{x},\hat{y}))_i \\
&= ((I+M^\sigma)\hat{Q}((I+M^\kappa)x,(I+M^\kappa)y))_i.
\end{aligned}
$$

□

Now, we can state a similar result as given in Theorem 3. We introduce the sequence $\{(x^\ell,z^\ell)\}_{\ell\in\mathbb{N}}$ which is defined by

$$
(x^{\ell+1},z^{\ell+1}) := (Q^*,R^*)(x^\ell,x^{\ell-1},z^\ell) \tag{16}
$$

with $x^0,x^1 \in \mathbb{R}^n$, $x^0 \neq x^1$ and $z^1 := -\mathcal{M}^\kappa(x^1)$.

Lemma 10. *Let $\{x^\ell,z^\ell\}_{\ell\in\mathbb{N}}$ be given as in* (16)*. Then, it holds $z^{\ell+1} = -\mathcal{M}^\kappa(x^{\ell+1})$.*

Proof. By induction, it follows from Lemma 9

$$
\begin{aligned}
z^{\ell+1} &= R^*(x^\ell,x^{\ell-1},z^\ell) = R^*(x^\ell,x^{\ell-1},-\mathcal{M}^\kappa(x^\ell)) \\
&= \mathcal{M}^\sigma(\hat{Q}((I+M^\kappa)x^\ell,(I+M^\kappa)x^{\ell-1})) \\
&= \mathcal{M}^\sigma((I+M^\kappa)(I+M^\sigma)\hat{Q}((I+M^\kappa)x^\ell,(I+M^\kappa)x^{\ell-1})) \\
&= \mathcal{M}^\sigma((I+M^\kappa)Q^*(x^\ell,x^{\ell-1},-\mathcal{M}^\kappa(x^\ell))) \\
&= \mathcal{M}^\sigma((I+M^\kappa)Q^*(x^\ell,x^{\ell-1},z^\ell)) \\
&= \mathcal{M}^\sigma((I+M^\kappa)x^{\ell+1}) = -\mathcal{M}^\kappa(x^{\ell+1}).
\end{aligned}
$$

□

Theorem 4. *Let $\{x^\ell,z^\ell\}_{\ell\in\mathbb{N}}$ be given as in* (16)*. For any sequence $\{\hat{x}^\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{R}^n$, we have*

$$
\begin{aligned}
&\hat{x}^\ell = (I+M^\kappa)x^\ell \quad \Leftrightarrow \\
&\qquad \hat{x}^{\ell+1} = \hat{Q}(\hat{x}^\ell,\hat{x}^{\ell-1}),\ \hat{x}^0 = (I+M^\kappa)x^0,\ \hat{x}^1 = (I+M^\kappa)x^1.
\end{aligned}
$$

Proof. Let $\hat{x}^\ell = (I + M^\kappa)x^\ell$ be fulfilled. From Lemma 9 and Lemma 10 we obtain

$$\begin{aligned}\hat{x}^{\ell+1} &= (I + M^\kappa)x^{\ell+1} = (I + M^\kappa)Q^*(x^\ell, x^{\ell-1}, z^\ell)\\ &= (I + M^\kappa)Q^*(x^\ell, x^{\ell-1}, -\mathcal{M}^\kappa(x^{\ell+1}))\\ &= (I + M^\kappa)(I + M^\sigma)\hat{Q}((I + M^\kappa)x^\ell, (I + M^\kappa)x^{\ell-1})\\ &= \hat{Q}(\hat{x}^\ell, \hat{x}^{\ell-1}).\end{aligned}$$

Let $\hat{x}^{\ell+1} = \hat{Q}(\hat{x}^\ell, \hat{x}^{\ell-1}))$ be fulfilled. Then, we obtain by induction

$$\begin{aligned}\hat{x}^{\ell+1} &= \hat{Q}(\hat{x}^\ell, \hat{x}^{\ell-1})) = (I + M^\kappa)(I + M^\sigma)\hat{Q}(\hat{x}^\ell, \hat{x}^{\ell-1}))\\ &= (I + M^\kappa)(I + M^\sigma)\hat{Q}((I + M^\kappa)x^\ell, (I + M^\kappa)x^{\ell-1}))\\ &= (I + M^\kappa)Q^*(x^\ell, x^{\ell-1}, -\mathcal{M}^\kappa(x^\ell)) = (I + M^\kappa)Q^*(x^\ell, x^{\ell-1}, z^\ell)\\ &= (I + M^\kappa)x^{\ell+1}.\end{aligned}$$

□

Corollary 2. *Let* $\{x^\ell, z^\ell\}_{\ell\in\mathbb{N}}$ *be given as in* (16)*. Furthermore, assume that the sequence* $\{\hat{x}^\ell\}_{\ell\in\mathbb{N}}$ *defined by* $\hat{x}^{\ell+1} := \hat{Q}(\hat{x}^\ell, \hat{x}^{\ell-1})$ *with* $\hat{x}^0 = (I + M^\kappa)x^0$ *and* $\hat{x}^1 = (I + M^\kappa)x^1$ *converges to* $\hat{x} \in \hat{K}$ *with* $\hat{E}(\hat{x}) = \min_{\hat{y}\in\hat{K}} \hat{E}(\hat{y})$*. For the solution* $x \in K$ *of* (1)*, there holds*

$$\lim_{\ell\to\infty} x^\ell = x.$$

Proof. From Theorem 4 we obtain $\hat{x}^\ell = (I + M^\kappa)x^\ell$. The assertion is obtained by the same arguments as in (14). □

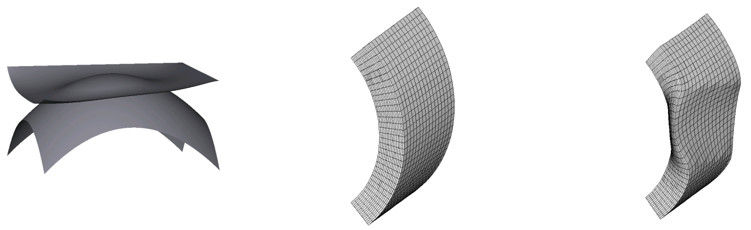
5. Numerical results

In this section, we discuss some numerical results in the context of contact problems which are given by obstacle problems and by linear elastic Signorini problems. We refer to [5], [6], [7] and [9] for more details concerning such contact problems.

Let $\Omega \subset \mathbb{R}^k$ be an open, connected and bounded set with the boundary $\Gamma := \Gamma_0 \cup \Gamma_1 := \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and a closed boundary part Γ_0. We consider the energy minimization problem

$$\min_{v\in\tilde{K}} \tilde{E}(v)$$

with a closed and convex set $\tilde{K} \subset H^1(\Omega, \Gamma_0)^t := \{v \in H^1(\Omega)^t \mid \gamma(v) = 0 \text{ on } \Gamma_0\}$. Here, γ denotes the trace operator related to Γ. For



a) b) c)

Fig. 1. Contact problems: **a)** membrane in contact (obstacle problem), **b)** undeformed body, **c)** body in contact (Signorini problem)

obstacle problems, we have $t = 1$ and the energy functional $\tilde{E}(v) := \frac{1}{2}(\nabla v, \nabla v)_0 - (f, v)_0 - (q, \gamma(v))_{0,\Gamma_1}$ with $f \in L^2(\Omega)$ and $q \in L^2(\Gamma_1)$. The set $\tilde{K}$ (admissable displacements in z-direction) is given by $\tilde{K} := \{v \in H^1(\Omega, \Gamma_0) \mid v \geq \psi_0\}$ with $\psi_0 : \Omega \to \mathbb{R}$ describing an obstacle in the interior of Ω. Figure 1a shows a membrane restricted by an obstacle defined on Ω. Here, we choose $\Omega := (-1, 1)^2$, $\Gamma_1 := (-1, 1) \times \{-1\} \cup \{-1\} \times (-1, 1)$, $f := -1$ and $q(x_0, x_1) := -\frac{1}{4}x_1^3$. Furthermore, we set $\psi_0(x_0, x_1) := -\frac{1}{2}(x_0^2 + x_1^2)$.

For (linear elastic) Signorini problems, we have $t \in \{2, 3\}$, $\tilde{E}(v) := \frac{1}{2}(\sigma(v), \epsilon(v))_0 - (f, v)_0 - (q, \gamma(v))_{0,\Gamma_1}$ and $\tilde{K} := \{v \in H^1(\Omega, \Gamma_0)^t \mid \delta_n(v) \leq \psi_1\}$ with a volume load $f \in L^2(\Omega)^t$, a surface load $q \in L^2(\Gamma_1)^t$ and a gap function $\psi_1 : \Gamma_1 \to \mathbb{R}$. The operators σ and ϵ denote the usual stress and strain tensors in linear elasticity. The expression $\delta_n(v)$ represents the outer normal contribution of a displacement field v.
Figures 1a and 1b show a contact problem from mechanical engineering modelled by a linear elastic Signorini problem (cf. [11]).

Choosing finite element methods with bilinear or trilinear trial functions on a grid with (inner) nodes $V_0, V_1, \ldots, V_{r-1} \in \mathbb{R}^k$, $k \in \{2, 3\}$, we obtain A and L as the stiffness matrix and the load vector, respectively.
One obtains (unilateral) box constraints in the case of obstacle problems. The PSOR-procedures as introduced in Section 2 can be used by setting $v_i := \psi_0(V_i)$ and $w_i := \infty$. In the case of Signorini problems, we have to deal with the conditions (2)-(4) and we apply the SPSOR-procedures as presented in Section 4. The matrix B and the vector $g \in \mathbb{R}^m$ are defined as follows: Let $W_0, W_1, \ldots, W_{s-1} \in \mathbb{R}^3$ be the grid nodes on Γ_1 and let $N_0, N_1, \ldots N_{s-1} \in \mathbb{R}^3$ be the outer normal vectors in $W_0, W_1, \ldots, W_{s-1}$. Then, we have $n = 3r$, $m = s$,

Table 1. Iterations of the PSOR-procedure.

n	$E(x^\ell)$	$\|ref - E(x^\ell)\|$	tol	$iter$
4	-0.18316	0.05595	1.0E-2	6
16	-0.26241	0.02329	1.0E-1	5
64	-0.24365	0.00452	1.0E-2	8
256	-0.24044	0.00131	1.0E-2	24
1024	-0.23946	3.35E-4	1.0E-3	128
4096	-0.23920	8.22E-5	1.0E-4	649
16384	-0.23914	1.93E-5	1.0E-5	3163
65536	-0.23912	3.81E-6	1.0E-6	14964

Table 2. Iterations of the APSSOR-procedure.

n	$E(x^\ell)$	$\|ref - E(x^\ell)\|$	tol	$iter$
4	-0.18316	0.05595	1.0E-1	2
16	-0.26253	0.02341	1.0E-1	3
64	-0.24366	0.00453	1.0E-2	6
256	-0.24049	0.00137	1.0E-2	9
1024	-0.23945	3.33E-4	1.0E-2	17
4096	-0.23920	8.23E-5	1.0E-3	53
16384	-0.23914	1.92E-5	1.0E-3	103
65536	-0.23912	3.81E-6	1.0E-4	279

Table 3. Iterations of the SQOPT-procedure.

n	$E(x^\ell)$	$\|ref - E(x^\ell)\|$	MOT	fkt
4	-0.18316	0.05595	1.0E-1	9
16	-0.26266	0.02353	1.0E-1	39
64	-0.24366	0.00453	1.0E-2	138
256	-0.24051	0.00138	1.0E-2	559
1024	-0.23946	3.39E-4	1.0E-2	2206
4096	-0.23920	8.24E-5	1.0E-3	8812

and

$$B_{j,3i+\kappa} := \begin{cases} (N_j)_\kappa, & W_j = V_i \\ 0, & \text{else}, \end{cases} \qquad \kappa = 0, 1, 2. \tag{17}$$

Furthermore, we obtain $g_j := \psi_1(W_j)$. The displacement in V_i is given by $(v_{3i}, v_{3i+1}, v_{3i+2})$.

In Tables 1 and 2, the number of iterations of the PSOR- and APSSOR-procudure is shown for an obstacle problem which is configurated as introduced above (cf. Figure 1). For each global refinement-

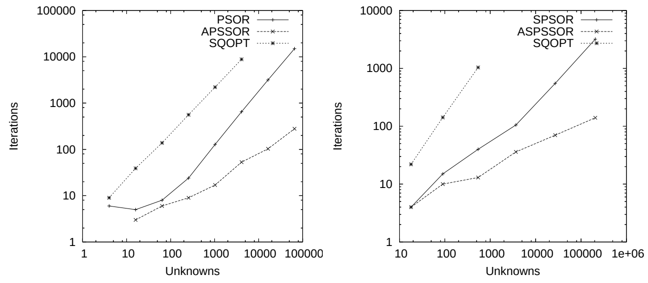


Fig. 2. Comparison of iteration numbers: **a)** obstacle problem, **b)** Signorini problem.

step, there are *iter* iterations needed, so that the value of $|x^{\ell+1} - x^{\ell}|$ reaches the given tolerance *tol*. The value of *tol* is chosen as the lowest power of 10^{-1}, so that the difference between a reference value for $E(x)$ and $E(x^{\ell})$ is stabilized (the first two non-zero digits do not change). In this example, the reference value is set to $ref = -0.23912486\ldots$ which is calculated by 7 refinement steps (or with $n = 262144$ unknowns) and a tolerance of $tol = 10^{-7}$. This proceeding guarantees, that the error caused by the procedures is less than the discretization error of the finite element method, but in the same range.
We see, that the APSSOR-procedure needs substantially less iterations than the PSOR-procedure. Hereby, we take into account, that the APSSOR-procedure needs twice the number of iterations of the PSOR-procedure and at the least 3 additional matrix-vector multiplications (`Ax`, `As`, `Ar` in the routine `MIN[A,x,r,s]`).
For comparison, we consider the standard quadratic optimization program SQOPT [3]. SQOPT is developed for convex, quadratic minimization problems with linear side conditions. Table 3 shows the number of function evaluations *fkt* of SQOPT for the obstacle problem. The tolerance parameter MOT corresponds to the SQOPT-parameter `MINOR OPTIMALITY TOLERANCE` (p.29 in [3]). The amount of one function evaluation in the SQOPT-algorithm roughly corresponds to one PSOR-step.

In Tables 4 and 5, the number of iterations *iter* and the number of function evaluations are tabulated for the linear elastic Signorini problem. The reference value is $ref = .10155913972\ldots$ determined by 6 refinement-steps ($n = 1609920$ unknowns) and a tolerance $tol = 10^{-7}$.
As we can see, the ASPSSOR-procedure needs the smallest iteration

Table 4. Iterations of the SPSOR-procedure.

n	$E(x^\ell)$	$\lvert ref - E(x^\ell)\rvert$	tol	$iter$
18	0.08950	0.01205	1.0E-1	4
90	0.07403	0.02752	1.0E-2	15
540	0.09211	0.00944	1.0E-3	40
3672	0.09965	0.00190	1.0E-3	105
26928	0.10132	2.30E-4	1.0E-4	549
205920	0.10157	1.22E-5	1.0E-5	3194

Table 5. Iterations of the ASPSSOR-procedure.

n	$E(x^\ell)$	$\lvert ref - E(x^\ell)\rvert$	tol	$iter$
18	0.08896	0.01259	1.0E-1	4
90	0.07396	0.02758	1.0E-2	10
540	0.09212	0.00943	1.0E-2	13
3672	0.09965	0.00190	1.0E-3	36
26928	0.10132	2.30E-4	1.0E-3	70
205920	0.10157	1.22E-5	1.0E-3	140

Table 6. Iterations of the SQOPT-procedure.

n	$E(x^\ell)$	$\lvert ref - E(x^\ell)\rvert$	MOT	fkt
18	0.08881	0.01274	1.0E-2	22
90	0.07396	0.02759	1.0E-3	142
540	0.09211	0.00944	1.0E-4	1038

number of all procedures. In Tables 3 and 6, the performance of SQOPT is shown only for a small number of unknowns. SQOPT stops without a result if the number of unknown is too large. In Figure 2a and 2b, the iteration numbers are depicted for the obstacle problem and the Signorini problem. In both cases, the slopes are different, where the APSSOR- and ASPSSOR-procedure provide the smallest slopes. Hence, the APSSOR- and ASPSSOR-procedure seem to be the procedures of the first choice.

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