Constraints Coefficients in hp-FEM

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Abstract Continuity requirements on irregular meshes enforce a proper constraint of the degrees of freedom that correspond to hanging nodes, edges or faces. This is achieved by using so-called constraints coefficients which are obtained from the appropriate coupling of shape functions.
In this note, a general framework for determining the constraints coefficients of tensor product shape functions is presented and its application to shape functions using integrated Legendre or Gauss-Lobatto polynomials. The constraints coefficients in the one-dimensional case are determined via recurrence relations. The constraints coefficients in the multi-dimensional case are obtained as products of these coefficients. The coefficients are available for arbitrary patterns of subdivisions.

1 Introduction

Local refinement processes arising from grid adaption are typically realized either by remeshing or by local refinements of grid elements. In the latter case so-called hanging nodes, edges or faces are unavoidable which result from refining a grid element without the refinement of neighboring elements. Applying conform finite element schemes, one has to ensure the finite element solution to be continuous. If no further local refinements (with possibly complex refinement patterns) are performed to eleminate grid irregularities, one has to constraint the degrees of freedom associated to hanging nodes, edges or faces. This can be done, e.g., by using Lagrange multipliers or static condensation or by incorporating the constraints in the iterative scheme that is used to determine the approximative solution. In all cases, a representation of shape functions in terms of transformed shape functions is needed. Such a representation is given by the so-called constraints coefficients.

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In a very general manner, constraints coefficients are defined as follows: Let P_q be a space of polynomials of degree $q \in \mathbb{N}$ on $\mathbb{R}^k$, $k \in \mathbb{N}$, and $\Upsilon : \mathbb{R}^k \to \mathbb{R}^k$ be an affine linear and bijective mapping. Furthermore, let $\xi = \{\xi_i\}_{0\leq i<n} \subset P_q$ be a linear independent set of polynomials. The numbers $\alpha_{ij} \in \mathbb{R}$ with $\xi_i \circ \Upsilon = \sum_{j=0}^{n-1} \alpha_{ij}\xi_j$ are called *constraints coefficients* of ξ for the mapping Υ.
In [3] constraints coefficients of the shape functions

$$\xi_0(x) := \frac{1}{2}(1-x), \quad \xi_1(x) := \frac{1}{2}(1+x), \quad \xi_i(x) := \begin{cases} x^i - 1, & i = 2,4,6,\ldots,q \\ x^i - x, & i = 3,5,7,\ldots,q \end{cases} \tag{1}$$

are determined. Since the functionals $\varphi_0(v) := v(-1)$, $\varphi_1(v) := v(1)$, $\varphi_j(v) := 1/j!d^jv/dx^j(0)$, $j = 2,\ldots,q$ fulfill the duality relation $\varphi_j(\xi_i) = \delta_{ij}$ (where δ_{ij} is the Kronecker delta), one simply obtains $\alpha_{ij} = \varphi_j(\xi_i \circ \Upsilon)$.
In [2] constraints coefficients of the Lagrange shape functions

$$\xi_0(x) := 1-x, \quad \xi_1(x) := x, \quad \xi_i := \frac{x(1-x)}{x_i(1-x_i)} \prod_{\ell=2;\ell\neq i}^{n-1} \frac{x-x_\ell}{x_i-x_\ell}, \quad i = 2,\ldots,q$$

are specified with $x_\ell \in (0,1)$, $\ell = 2,\ldots n-1$. The functionals $\varphi_0(v) := v(0)$, $\varphi_1(v) := v(1)$, $\varphi_j(v) := v(x_j)$, $j = 2,\ldots,n-1$, fulfill the duality relation $\varphi_j(\xi_i) = \delta_{ij}$ only for $i = 2,\ldots,n-1$. We get $\alpha_{i0} = (\xi_i \circ \Upsilon)(0)$ and $\alpha_{i1} = (\xi_i \circ \Upsilon)(1)$ for $i = 0,\ldots,n-1$ and $\alpha_{0j} = \alpha_{1j} = 0$ for $j = 2,\ldots,n-1$. Since $\varphi_j(\xi_i \circ \Upsilon) = \alpha_{i0}\varphi_j(\xi_0) + \alpha_{i1}\varphi_j(\xi_1) + \alpha_{ij}$, the remaining coefficients are determined by $\alpha_{ij} = (\xi_i \circ \Upsilon)(x_j) - \alpha_{i0}(1-x_j) - \alpha_{i1}x_j$.
A widely used family of shape functions are shape functions using integrated Legendre or Gauss-Lobatto polynomials ([7], [8], [9]). These polynomials belong to the family of so-called Gegenbauer polynomials $\{G_i^\rho\}_{i\in\mathbb{N}_0}$ which are defined by

$$(i+1)G_{i+1}^\rho(x) = 2(i+\rho)xG_i^\rho(x) - (i+2\rho-1)G_{i-1}^\rho(x) \tag{2}$$

with $\rho \in \mathbb{R}$, $G_0^\rho(x) := 1$ and $G_1^\rho(x) := 2\rho x$. Theoretical results about equivalent definitions of Gegenbauer polynomials and their special properties can be found, e.g., in [10]. With $\rho := -1/2$, we obtain integrated Legendre ($\beta_i := 1$) and Gauss-Lobatto ($\beta_i := \sqrt{(2i-1)/2}$) shape functions

$$\xi_0(x) := \frac{1}{2}(1-x),\ \xi_1(x) := \frac{1}{2}(1+x),\ \xi_i(x) := \beta_i G_i^{-1/2}(x), \quad i = 2,\ldots,q. \tag{3}$$

Because of the orthogonality relation of the Gegenbauer polynomials (cf. [10]), the functionals $\varphi_0(v) := v(-1)$, $\varphi_1(v) := v(1)$, $\varphi_j(v) := \mu_j \int_{-1}^{1}(1-x^2)^{-1}\xi_j(x)v(x)\,dx$ with $\mu_j := j(j-1)(2j-1)/(2\beta_j^2)$, $j = 2,\ldots,n-1$ fullfill the duality relation $\varphi_j(\xi_i) = \delta_{ij}$ for $i = 2,\ldots,n-1$ and $j = 0,\ldots,n-1$. Similar to the Lagrange shape functions, we obtain $\alpha_{i0} = (\xi_i \circ \Upsilon)(-1)$ and $\alpha_{i1} = (\xi_i \circ \Upsilon)(1)$ for $i = 0,\ldots,n-1$ and $\alpha_{0j} = \alpha_{1j} = 0$ for $j = 2,\ldots,n-1$. Since $\varphi_j(\xi_0) = (-1)^j(2j-1)/(2\beta_j^2)$ and $\varphi_j(\xi_1) = (2j-1)/(2\beta_j^2)$, the remaining coefficients are determined by $\alpha_{ij} = \varphi_j(\xi_i \circ \Upsilon) - (2j-1)/(2\beta_j^2)(\alpha_{i0}(-1)^j + \alpha_{i1})$.

In this note, we present a general framework for constraints coefficients of tensor product polynomials. Furthermore, we present an explicit formula of the constraints coefficients of integrated Legendre and Gauss-Lobatto shape functions without the integral representation given by φ_j. The formula is derived by the use of the recurrence relation (2). At the end of this note, the application of constraints coefficients to irregular grids is briefly discussed. Other areas of applications are hp-multigrid schemes (cf. [4], [5]) or grid transfer operations in timedependent problems.

2 Tensor Product Shape Functions

The space of polynomials in one variable of degree q is defined as $S^q := \{v : \mathbb{R} \to \mathbb{R} \mid v(x) = \sum_{0\le i\le q} c_i x^i,\ c_i \in \mathbb{R}\}$, the corresponding tensor product space is denoted by

$$S_k^q := \otimes_{i=0}^{k-1} S^q := \left\{ v : \mathbb{R}^k \to \mathbb{R} \mid v(x_0,\dots,x_{k-1}) = \prod_{i=0}^{k-1} v_i(x_i),\ v_0,\dots,v_{k-1} \in S^q \right\}.$$

Let $\hat{\xi} := \{\hat{\xi}_i\}_{0\le i<m}$ be a subset of S^q and L be an n times k matrix with entries in $\{0,\dots,m-1\}$. Then, we define $\Pi(\hat{\xi},L) := \left\{\prod_{r=0}^{k-1}\hat{\xi}_{L_{ir}}(x_r)\right\}_{0\le i<n} \subset S_k^q$.
For $\Upsilon(x) := \mathrm{diag}(a)x + b$ with $a,b \in \mathbb{R}^k$, it is easy to determine the constraints coefficients of $\Pi(\hat{\xi},L)$: Let $\hat{\alpha}_{ij}(a_r,b_r) \in \mathbb{R}$ be the constraints coefficients of $\hat{\xi}$ for $\Upsilon_r(x_r) := a_r x_r + b_r$. Furthermore, let $\mathscr{L} := \{(L_{i,0},\dots,L_{i,k-1}) \mid 0 \le i < n\}$.

Theorem 1. *Assume that $\Pi(\hat{\xi},L)$ is linear independent and there holds*

$$l \in \{0,\dots,m-1\}^k \backslash \mathscr{L} \Rightarrow \forall 0 \le i < n:\ \exists 0 \le r < k:\ \hat{\alpha}_{L_{ir},l_r} = 0. \tag{4}$$

Then, the constraints coefficients of $\Pi(\hat{\xi},L)$ for Υ are $\alpha_{ij} = \prod_{r=0}^{k-1}\hat{\alpha}_{L_{ir},L_{jr}}(a_r,b_r)$.
Proof: Let $x \in \mathbb{R}^k$. Because of (4), we obtain

$$\begin{aligned}
\Pi(\hat{\xi},L)_i(\Upsilon(x)) &= \prod_{r=0}^{k-1}\hat{\xi}_{L_{ir}}(a_r x_r + b_r) = \prod_{r=0}^{k-1}\sum_{l=0}^{m-1}\hat{\alpha}_{L_{ir},l}(a_r,b_r)\hat{\xi}_l(x_r) \\
&= \sum_{l_0=0}^{m-1}\cdots\sum_{l_{k-1}=0}^{m-1}\left(\prod_{r=0}^{k-1}\hat{\alpha}_{L_{ir},l_r}(a_r,b_r)\right)\left(\prod_{r=0}^{k-1}\hat{\xi}_{l_r}(x_r)\right) \\
&= \sum_{l\in\mathscr{L}}\left(\prod_{r=0}^{k-1}\hat{\alpha}_{L_{ir},l_r}(a_r,b_r)\right)\left(\prod_{r=0}^{k-1}\hat{\xi}_{l_r}(x_r)\right) = \sum_{j=0}^{n-1}\left(\prod_{r=0}^{k-1}\hat{\alpha}_{L_{ir},L_{jr}}(a_r,b_r)\right)\Pi(\hat{\xi},L)_j(x).
\end{aligned}$$

Since $\Pi(\hat{\xi},L)$ is assumed to be linear independent, the proof is completed. □

Finite element shape functions are basis polynomials that are defined on a reference element (unit square, cube or simplex). They constitute the global basis functions on the grid elements. In conform approaches shape functions are usually partitioned

into nodal modes, edge modes, face modes and inner modes. Nodal modes have the value 1 in exactly one vertex and vanish on the remaining vertices. Edge modes are different from zero on exactly one edge and vanish on the remaining edges and on all non-adjacent faces and all nodes. Face modes are different from zero on exactly one face and vanish on the remaining faces and on all edges and nodes. Inner modes vanish on all nodes, edges and faces, they are only different from zero in the interior. Using the notation $\Pi(\hat{\xi},L)$, the separation is established by splitting the matrix L into submatrices $L^\top := (L^0\ L^1\ \cdots\ L^k)^\top$. The submatrix L^0 generates the nodal modes, L^1 generates the edges modes and so on.
Let $\hat{\xi} = \hat{\xi}^q$ be shape functions in S^q which are partitioned into the nodal modes $\hat{\xi}_0$, $\hat{\xi}_1$ and inner modes $\hat{\xi}_i$, $2 \le i \le q$. With $\alpha(i,j) := i(i+1)/2 + j$, a proper definition of L in the two-dimensional case is, e.g.,

$$\begin{aligned}
&(L^0)^\top := \begin{pmatrix} 0\ 1\ 1\ 0 \\ 0\ 0\ 1\ 1 \end{pmatrix}^\top, \quad L^1_{i,1} := L^1_{3(q-1)+i,0} := 0, \quad L^1_{q-1+i,0} := L^1_{2(q-1)+i,1} := 1, \\
&L^1_{i,0} := L^1_{q-1+i,1} := L^1_{2(q-1)+i,0} := L^1_{3(q-1)+i,1} := i+2, \quad i = 0,\ldots,q-2, \\
&L^2_{\alpha(i,j),0} := j+2, \quad L^2_{\alpha(i,j),1} := i-j+2, \quad i = 0,\ldots,q-4+\tau,\ j = 0,\ldots,i.
\end{aligned} \tag{5}$$

This definition leads to the set of shape functions $\xi = \Pi(\hat{\xi},L)$:

$$\begin{aligned}
&\xi_0(x_0,x_1) := \hat{\xi}_0(x_0)\hat{\xi}_0(x_1), \quad \xi_1(x_0,x_1) := \hat{\xi}_1(x_0)\hat{\xi}_0(x_1), \\
&\xi_2(x_0,x_1) := \hat{\xi}_1(x_0)\hat{\xi}_1(x_1), \quad \xi_3(x_0,x_1) := \hat{\xi}_0(x_0)\hat{\xi}_1(x_1), \\
&\xi_{4+i}(x_0,x_1) := \hat{\xi}_{i+2}(x_0)\hat{\xi}_0(x_1), \quad \xi_{4+q-1+i}(x_0,x_1) := \hat{\xi}_1(x_0)\hat{\xi}_{i+2}(x_1), \\
&\xi_{4+2(q-1)+i}(x_0,x_1) := \hat{\xi}_{i+2}(x_0)\hat{\xi}_1(x_1), \quad \xi^q_{4+3(q-1)+i}(x_0,x_1) := \hat{\xi}_0(x_0)\hat{\xi}_{i+2}(x_1), \\
&\xi_{4q+\alpha(i,j)}(x_0,x_1) := \hat{\xi}_{j+2}(x_0)\hat{\xi}_{i-j+2}(x_1).
\end{aligned}$$

For $\tau = 2$ the set $\Pi(\hat{\xi},L)$ is a basis of S^q_2. Assuming that $\hat{\xi}$ is hierarchical (which means that $\hat{\xi}^{\tilde{q}}_i = \hat{\xi}^q_i$ for $0 \le i \le \tilde{q}$ and $\tilde{q} \le q$), the set $\Pi(\hat{\xi},L)$ has some important properties: For $\tau = 0$, we obtain a reduced set of shape functions (also known as Serendipity shape functions) with the same order of approximation (cf., e.g., p.175 in [1], [7]). Furthermore, the special definition of L implies that the edge modes (edge by edge) and the inner modes are hierarchical as well. This property can be exploited, e.g., for the efficient management of different polynomial degree distributions of neigboring grid elements. One simply omits the edge modes with polynomial degree $p_0 > p_1$, where p_1 is the polynomial degree in the neighboring element. The shape functions $\Pi(\hat{\xi},L)$ with integrated Legendre or Gauss-Lobatto shape functions $\hat{\xi}$ corresponds to the shape functions as proposed in [7] and [9] for *hp*-finite element methods. The use of the recurrence relation (2) admits a stable and fast evaluation of the shape functions and their derivatives. Derivatives of arbitrary order can be easily derived by the relation $\partial^\nu G^\rho_i = 2^\nu(\rho)_\nu G^{\rho+\nu}_{i-\nu}$ with $i,\nu \in \mathbb{N}_0$ and $(\rho)_\nu := \prod_{j=0}^{\nu-1}(\rho+j)$.

3 Constraints Coefficients of Integrated Legendre and Gauss-Lobatto Shape Functions

As a result of Theorem 1, it is sufficient to consider the one-dimensional case to determine the constraints coefficients in the multi-dimensional case.

Theorem 2. *Let $\hat{\xi}$ be a set of hierarchical shape functions and L be defined as in* (5)*. Then, the assumption* (4) *is fulfilled for $\tau \in \{0,2\}$.*
Proof. The assumption (4) is obviously fulfilled for $\tau = 2$. Let $q \geq 2$, $\tau = 0$ and $l \in \{0,\dots,q\}^2 \backslash \mathscr{L}$, then $l = (j+2, i-j+2)$ with $i \in \{\max\{q-3,0\}, q-2\}$ and $0 \leq j \leq i$. For the nodal mode ($\kappa = 0$) with index $0 \leq s < 4$ or for the edge mode ($\kappa = 1$) with index $0 \leq s < 4(q-1)$, we obtain $\deg(\hat{\xi}_{L^{\kappa}_{sr}}) = 1$ for at least one $r \in \{0,1\}$. Since $\min\{\deg(\hat{\xi}_{j+2}), \deg(\hat{\xi}_{i-j+2})\} \geq 2$, we have $\hat{\alpha}_{L^{\kappa}_{sr}, l_r} = 0$. For $q \geq 4$, the polynomial degree of the inner mode with index $0 \leq s < (q-3)(q-2)/2$ is bounded by $q-2 < \max\{j+2, i-j+2\} = \max\{\deg(\hat{\xi}_{j+2}), \deg(\hat{\xi}_{i-j+2})\}$. Therefore, there exists $r \in \{0,1\}$ such that $\hat{\alpha}_{L^{2}_{sr}, l_r} = 0$. $\square$

Theorem 3. *Let $\Upsilon(x) = ax + b$ with $a, b \in \mathbb{R}$ and $i \geq 2$. For integrated Legendre shape functions* (3)*, there holds:*

$$
\begin{aligned}
&\alpha_{00} = \frac{1+a-b}{2}, \quad \alpha_{10} = \frac{1-a+b}{2}, \quad \alpha_{20} = \frac{1-(a-b)^2}{2},\\
&\alpha_{i+1,0} = (b-a)\frac{2i-1}{i+1}\alpha_{i,0} - \frac{i-2}{i+1}\alpha_{i-1,0},\\
&\alpha_{01} = \frac{1-a-b}{2}, \quad \alpha_{11} = \frac{1+a+b}{2}, \quad \alpha_{21} = \frac{1-(a+b)^2}{2},\\
&\alpha_{i+1,1} = (a+b)\frac{2i-1}{i+1}\alpha_{i,1} - \frac{i-2}{i+1}\alpha_{i-1,1},\\
&\alpha_{22} = a^2, \quad \alpha_{i+1,2} = \frac{2i-1}{i+1}\left(\frac{a}{5}\alpha_{i,3} + b\alpha_{i,2} + a(\alpha_{i,0} - \alpha_{i,1})\right) - \frac{i-2}{i+1}\alpha_{i-1,2},\\
&\alpha_{i+1,j} = \frac{2i-1}{i+1}\left(a\frac{j}{2j-3}\alpha_{i,j-1} + a\frac{j-1}{2j+1}\alpha_{i,j+1} + b\alpha_{i,j}\right) - \frac{i-2}{i+1}\alpha_{i-1,j},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 3,\dots,i-1,\\
&\alpha_{i+1,i} = \frac{2i-1}{i+1}\left(a\frac{i}{2i-3}\alpha_{i,i-1} + b\alpha_{ii}\right), \quad i > 2,\\
&\alpha_{i+1,i+1} = a\alpha_{ii}, \quad \alpha_{i,j} = 0, \quad j > i.
\end{aligned}
$$

Proof. By comparing the coefficients in $\xi_i(ax+b) = \alpha_{i0}\xi_0(x) + \alpha_{i1}\xi_1(x)$, $i = 0,1,2$, we obtain α_{00}, α_{01}, α_{10}, α_{11}, α_{20}, α_{21} and α_{22}. From equation (2) we have:

$$x\xi_j(x) = (2j-1)^{-1}((j+1)\xi_{j+1}(x) + (j-2)\xi_{j-1}(x)), \quad j = 2,3,\dots.$$

Furthermore, we have

$$x\xi_0(x) = \frac{1}{2}x - \frac{1}{2}x^2 = -\frac{1}{2}(1-x) + \frac{1}{2}(1-x^2) = -\xi_0(x) + \xi_2(x),$$
$$x\xi_1(x) = \frac{1}{2}x + \frac{1}{2}x^2 = \frac{1}{2}(1+x) - \frac{1}{2}(1-x^2) = \xi_1(x) - \xi_2(x).$$

This yields

$$\begin{aligned}
&(i+1)\xi_{i+1}(ax+b)\\
&= (2i-1)(ax+b)\xi_i(ax+b) - (i-2)\xi_{i-1}(ax+b)\\
&= b(2i-1)\sum_{j=0}^{i}\alpha_{ij}\xi_j(x) + a(2i-1)x\sum_{j=0}^{i}\alpha_{ij}\xi_j(x) - (i-2)\sum_{j=0}^{i-1}\alpha_{i-1,j}\xi_j(x)\\
&= b(2i-1)\sum_{j=0}^{i}\alpha_{ij}\xi_j(x) + a(2i-1)\sum_{j=2}^{i}\alpha_{ij}\left(\frac{j+1}{2j-1}\xi_{j+1}(x) + \frac{j-2}{2j-1}\xi_{j-1}(x)\right)\\
&\quad + a(2i-1)\left(\alpha_{i,0}(-\xi_0(x)+\xi_2(x)) + \alpha_{i,1}(\xi_1(x)-\xi_2(x))\right)\\
&\quad - (i-2)\sum_{j=0}^{i-1}\alpha_{i-1,j}\xi_j(x)\\
&= a(i+1)\alpha_{ii}\xi_{i+1}(x) + \left(a(2i-1)\frac{i}{2i-3}\alpha_{i,i-1} + b(2i-1)\alpha_{ii}\right)\xi_i(x)\\
&\quad + a(2i-1)\sum_{j=3}^{i-1}\alpha_{i,j-1}\frac{j}{2j-3}\xi_j(x) + a(2i-1)\sum_{j=2}^{i-1}\alpha_{i,j+1}\frac{j-1}{2j+1}\xi_j(x)\\
&\quad + b(2i-1)\sum_{j=0}^{i-1}\alpha_{ij}\xi_j(x) - (i-2)\sum_{j=0}^{i-1}\alpha_{i-1,j}\xi_j(x) + a(2i-1)\left(\alpha_{i,0}-\alpha_{i,1}\right)\xi_2(x)\\
&\quad + a(2i-1)\alpha_{i,1}\xi_1(x) - a(2i-1)\alpha_{i,0}\xi_0(x)\\
&= a(i+1)\alpha_{ii}\xi_{i+1}(x) + \left(a(2i-1)\frac{i}{2i-3}\alpha_{i.i-1} + b(2i-1)\alpha_{ii}\right)\xi_i(x)\\
&\quad + \sum_{j=3}^{i-1}\Bigg(a(2i-1)\frac{j}{2j-3}\alpha_{i,j-1} + a(2i-1)\frac{j-1}{2j+1}\alpha_{i,j+1} + b(2i-1)\alpha_{ij}\\
&\qquad\qquad - (i-2)\alpha_{i-1,j}\Bigg)\xi_j(x)\\
&\quad + \left(a(2i-1)\frac{1}{5}\alpha_{i,3} + b(2i-1)\alpha_{i,2} - (i-2)\alpha_{i-1,2} + a(2i-1)\left(\alpha_{i,0}-\alpha_{i,1}\right)\right)\xi_2(x)\\
&\quad + \left(b(2i-1)\alpha_{i,1} - (i-2)\alpha_{i-1,1} + a(2i-1)\alpha_{i,1}\right)\xi_1(x)\\
&\quad + \left(b(2i-1)\alpha_{i,0} - (i-2)\alpha_{i-1,0} - a(2i-1)\alpha_{i,0}\right)\xi_0(x)
\end{aligned}$$

Division by $i+1$ completes the proof. □

It is easy to see, that the constraints coefficients of Gauss-Lobatto shape functions are $\sqrt{(2i-1)/(2j-1)}\alpha_{ij}$, $i,j \geq 2$. Furthermore, Theorem 3 can be extended to the case of Gegenbauer polynomials or general Jacobi polynomials.

4 Application to Hanging Nodes

Let $\mathscr{T}$ be a subdivision of $\Omega \subset \mathbb{R}^k$ consisting of quadrangles ($k = 2$) or hexahedrons ($k = 3$) and let $\Psi_T : [-1,1]^k \to T \in \mathscr{T}$ be a bijective and sufficiently smooth mapping. In conform finite element methods, the space of admissable functions is defined as $S^p(\mathscr{T}) := \{v \in C^0(\Omega) \mid \forall T \in \mathscr{T} : v_{|T} \circ \Psi_T \in S_k^{p_T}\}$ with the degree distribution $p = \{p_T\}_{T\in\mathscr{T}}$, $p_T \leq q$. By using so-called connectivity matrices $\pi_T \in \mathbb{R}^{\ell\times n_k}$, a basis $\{\phi_r\}_{0\leq r<\ell}$ of $S^p(\mathscr{T})$ is constructed via

$$\phi_{r|T} := \sum_{s=0}^{n_k-1} \pi_{T,rs}\hat{\phi}_{T,s}$$

with $\hat{\phi}_{T,s} := \Pi(\hat{\xi},L)_s \circ \Psi_T^{-1}$, $0 \leq s < n_k$, where n_k is the number of shape functions. In particular, the stiffness matrix K and the load vector b are assembled via $K := \sum_{T\in\mathscr{T}} \pi_T K_T \pi_T^\top$ and $b := \sum_{T\in\mathscr{T}} \pi_T b_T$ with local stiffness matrices $K_T \in \mathbb{R}^{n_k\times n_k}$ and local load vectors $b_T \in \mathbb{R}^{n_k}$.

In the presence of hanging nodes, the definition of π_T is the crucial point. The entries are ± 1 (or 0), if the associated shape functions are related to a non-hanging node, edge or face. Otherwise, the entries are given by the constraints coefficients as introduced in the previous sections. Figure 1a shows a typical situation in 3D which is obtained by refining the neighbored grid element of the left hexahedron (denoted by T_L), for example by dividing it into eight small hexahedrons. One of them (denoted by T_R) is examplarly depicted on the right hand side of T_L. The entries of the connectivity matrix of T_L related to the nodes v_0 and v_1, to the edges e_0, e_1, e_2 and to the face f are defined as follows. The entries related to v_0 and e_0 are given by the constraints coefficients α_{ij} of the one-dimensional case: Let $\phi_{\hat{r}}$ be a basis function of $\{\phi_r\}_{0\leq r<\ell}$, that belongs to V_0, V_1 or E. Furthermore, let $\{\hat{\phi}_{T_L,s}\}_{s\in\mathscr{S}_L}$ be the polynomials of $\{\hat{\phi}_{T_L,s}\}_{0\leq s<n_3}$, that belong to V_0, V_1 and E, and let $\{\hat{\phi}_{T_R,s}\}_{s\in\mathscr{S}_R}$ be the polynomials of $\{\hat{\phi}_{T_R,s}\}_{0\leq s<n_3}$, that belong to V_0, v_0 and e_0. Since V_0, V_1 and E are non-hanging, it holds

$$\pm\hat{\phi}_{T_L,\hat{s}|e_0} = \phi_{\hat{r}|e_0} = \sum_{s\in\mathscr{S}_R} \pi_{T,\hat{r}s}\hat{\phi}_{T_R,s|e_0}$$

with $\hat{s} \in \mathscr{S}_L$. Provided that E is subdivided into two subedges with proportions of division z and $1-z$, $z \in (0,1)$, and e_0 is its first subedge, we define a mapping Υ by $\Upsilon(x) := zx + z - 1$ which maps $[-1,(2-z)/z]$ onto $[-1,1]$. If e_0 is the second subedge of E, we set $\Upsilon(x) := (1-z)x + z$ which maps $[(z+1)/(z-1),1]$ onto $[-1,1]$. Due to the tensor structure of $\Pi(\hat{\xi},L)$, there exist bijective mappings $\Delta_L : \{0,\ldots,n_1-1\} \to \mathscr{S}_L$, $\Delta_R : \{0,\ldots,n_1-1\} \to \mathscr{S}_R$, and $\Psi_{e_0} : [-1,1] \to e_0$, such that $\hat{\phi}_{T_L,\hat{s}|e_0} \circ \Psi_{e_0} = \hat{\xi}_{\Delta_L^{-1}(\hat{s})} \circ \Upsilon_{|[-1,1]}$ and $\hat{\phi}_{T_R,\Delta_R(j)|e_0} \circ \Psi_{e_0} = \hat{\xi}_j$, $0 \leq j < n_1$. Therefore, we obtain

$$\pm\hat{\xi}_{\Delta_L^{-1}(\hat{s})} \circ \Upsilon = \sum_{j=0}^{n_1-1} \pi_{T_R,\hat{r},\Delta_R(j)}\hat{\xi}_j$$

and, finally, $\pi_{T_R,\hat{r},\Delta_R(j)} = \pm\alpha_{\Delta_L^{-1}(\hat{s}),j}$.
By analogy, the entries related to v_1, e_1, e_2 and f are the constraints coefficients of the two-dimensional case. We consider the polynomials of $\{\hat{\phi}_{T_L,s}\}_{0\leq s<n_3}$, that belong to F and its nodes and edges, restricted to F and those of $\{\hat{\phi}_{T_R,s}\}_{0\leq s<n_3}$, that belong to v_1, e_1, e_2 and f, restricted to f. For more details, see [6].

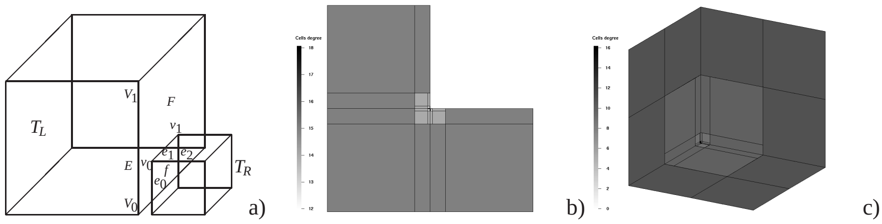

Fig. 1 **a**: Local refinement in 3D. **b-c**: *hp*-adaptive grids with unsymmetric divisions.

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