Mixed FEM of higher-order for a frictional contact problem

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This paper presents mixed finite element methods of higher-order for an idealized frictional contact problem in linear elasticity. The approach relies on a saddle point formulation where the frictional contact condition is captured by a Lagrange multiplier. The convergence of the mixed scheme is proven and some a priori estimates for the h- and p-method are derived. Furthermore, a posteriori error estimates are presented which rely on the estimation of the discretization error of an auxiliary problem and some further terms capturing the error in the friction and complementary conditions. Numerical results confirm the applicability of the a posteriori error estimates within h- and hp-adaptive schemes.

1 Introduction

The aim of this note is to derive a mixed finite element method of higher-order for an idealized frictional contact problem in linear elasticity. We consider a model problem with Tresca friction which captures the main characteristics of many frictional contact problems. The discretization approach is based on mixed finite elements for contact problems introduced by Haslinger et al. in [1]. This approach was originally developed for low-order finite elements and is based on a saddle point formulation. In this paper, we extend it to a higher-order discretization. The introduced Lagrange multiplier is defined on the contact boundary and enforces the frictional condition via box constraints in some Gauss quadrature points.

The stability of the mixed scheme is guaranteed by a discrete inf-sup condition. The proof of its convergence is based on the verification of some approximation properties and intensively utilizes the definition of the box constraints in Gauss points enabling to allow for higher-order interpolation, cf. [2].

The mixed method also allows for a posteriori error estimates and their application within h- and hp-adaptive schemes. The error control relies on the estimation of the discretization error of an auxiliary problem and some further terms which capture the error in the friction and complementary conditions.

Higher-order discretizations based on mixed schemes for frictional contact problems are rarely studied in literature. We refer to [3] for some estimates using a mortar approach. The definition of the contact conditions in some quadrature points only is already suggested in this work. To derive similar results for Signorini's problem with geometrical contact conditions, we refer to [4, 5].

2 Mixed variational formulation of an idealized frictional problem

Let $\Omega \subset \mathbb{R}^k$, $k \in \mathbb{N}$, be a domain with a polygonal boundary $\Gamma := \partial\Omega$. Moreover, let $\Gamma_D \subset \Gamma$ be closed with positive measure and let $\Gamma_C \subset \Gamma\backslash\Gamma_D$ with $\overline{\Gamma}_C \subsetneq \Gamma\backslash\Gamma_D$ and $\Gamma_N := \Gamma\backslash(\Gamma_D \cup \overline{\Gamma}_C)$. $L^2(\Omega)$, $H^1(\Omega)$ and $H^{1/2}(\Gamma_C)$ denote the usual Sobolev spaces and $H^1_D(\Omega) := \{v \in H^1(\Omega) \mid \gamma(v) = 0 \text{ on } \Gamma_D\}$ with the trace operator γ. The space $H^{-1/2}(\Gamma_C)$ denotes the topological dual space of $H^{1/2}(\Gamma_C)$ with the norms $\|\cdot\|_{-1/2,\Gamma_C}$ and $\|\cdot\|_{1/2,\Gamma_C}$, respectively. Let $(\cdot,\cdot)_{0,\omega}$, $(\cdot,\cdot)_{0,\Gamma'}$ be the usual L^2-scalar products on $\omega \subset \Omega$ and $\Gamma' \subset \Gamma$, respectively. We define $\|v\|^2_{0,\omega} := (v,v)_{0,\omega}$ and omit the subscript ω whenever $\omega = \Omega$. Moreover, $\|\cdot\|^2_1$ is the usual H^1-norm. We denote the gradient operator in the weak sense by ∇ and the Laplace operator by Δ. For functions in $L^2(\Omega)$ or $L^2(\Gamma_C)$, the inequality symbols $\geq$ and $\leq$ are defined by means of "almost everywhere".

An idealized frictional problem is to find a function $u \in H^1_D(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \Gamma_N, \quad |\partial_n u| \leq s \text{ with } \left\{ \begin{array}{ll} |\partial_n u| & < s \Rightarrow u = 0, \\ \partial_n u & = s \Rightarrow u \geq 0, \\ -\partial_n u & = s \Rightarrow u \leq 0 \end{array} \right\} \text{ on } \Gamma_C \tag{1}$$

for $f \in L^2(\Omega)$ and $s \in L^\infty(\Gamma_C)$, $s > 0$, [1, 6]. It is well-known, that $u \in H^1_D(\Omega)$ is a solution if and only if the variational inequality

$$(\nabla u, \nabla(v-u))_0 + (s, |\gamma(v)| - |\gamma(u)|)_{0,\Gamma_C} \geq (f, v-u)_0 \tag{2}$$

holds for all $v \in H^1_D(\Omega)$ which is equivalent to the minimization problem $(E+j)(u) = \min_{v\in H^1_D}(\Omega)(E+j)(v)$ with $j(v) := (s, |\gamma_C(v)|)_{0,\Gamma_C}$ and the trace opterator $\gamma_C := \gamma_{|\Gamma_C}$, [6]. Since j is strictly convex, continuous and coercive,

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the unique existence of a minimizer u is guaranteed. With $\Lambda := \{\mu \in L^2(\Gamma_C) \mid |\mu| \leq 1\}$ it is easy to see that $j(v) = \sup_{\mu\in\Lambda}(\mu, s\gamma_C(v))_{0,\Gamma_C}$ and, therefore, $(E+j)(u) = \inf_{v\in H^1_D(\Omega)} \sup_{\mu\in\Lambda} \mathcal{L}(v,\mu)$ with the Lagrange functional $\mathcal{L}(v,\mu) := E(v) + (\mu, s\gamma_C(v))_{0,\Gamma_C}$ on $H^1_D(\Omega) \times \Lambda$. Thus, u is a minimizer of $E + j$ if $(u, \lambda) \in H^1_D(\Omega) \times \Lambda$ is a saddle point of $\mathcal{L}$. The existence of a saddle point is guaranteed by the boundedness of Λ, [7, Prop IV.2.3 and Remark IV.2.1]. Due to the stationarity conditions, the pair $(u, \lambda) \in H^1_D(\Omega) \times \Lambda$ is equivalently characterized by the mixed variational formulation,

$$\begin{aligned}&(\nabla u, \nabla v)_0 = (f, v)_0 - (\lambda, s\gamma_C(v))_{0,\Gamma_C},\\ &(\mu - \lambda, s\gamma_C(u))_{0,\Gamma_C} \leq 0\end{aligned} \tag{3}$$

for all $v \in H^1_D(\Omega)$ and $\mu \in \Lambda$. Since $H^{1/2}(\Gamma_C)$ is dense in $L^2(\Gamma_C)$, we conclude from (3) that the Lagrange multiplier is unique. It is easy to see that $-s\lambda = \partial_n u$, cf. [1]

To define a higher-order finite element discretization based on quadrangles or hexahedrons, let $\mathcal{T}_h$ and $\mathcal{E}_H$ be finite element meshes of Ω and Γ_C with mesh sizes h and H, respectively. Moreover, let $\Psi_T : [-1,1]^k \to T \in \mathcal{T}_h$ and $\Phi_E : [-1,1]^{k-1} \to E \in \mathcal{E}_H$ be bijective transformations and let $p_T, q_E \in \mathbb{N}$ be degree distributions on $\mathcal{T}_h$ and $\mathcal{E}_H$, respectively. Using the polynomial tensor product space S^q_k of order q on the reference element $[-1,1]^k$, we define

$$\begin{aligned}S^p_h &:= \left\{v \in H^1_D(\Omega) \mid \forall T \in \mathcal{T}_h : v_{|T} \circ \Psi_T \in S^{p_T}_k\right\},\\ M^q_H &:= \left\{\mu \in L^2(\Gamma_C) \mid \forall E \in \mathcal{E}_H : \mu_{|E} \circ \Phi_E \in S^{q_E}_{k-1}\right\}.\end{aligned}$$

Moreover, we set

$$\Lambda^q_H := \{\mu^q_H \in M^q_H \mid \forall E \in \mathcal{E}_H, \forall x \in \mathcal{C}_E : |\mu^q_H(\Phi_E(x))| \leq 1\},$$

where $\mathcal{C}_E$ denotes a finite set of nodes. The discrete saddle point problem of the idealized frictional problem is then to find a pair $(u^p_h, \lambda^q_H) \in S^p_h \times \Lambda^q_H$ such that

$$\begin{aligned}&(\nabla u^p_h, \nabla v^q_h)_0 = (f, v^p_h)_0 - (\lambda^q_H, s\gamma_C(v^p_h))_{0,\Gamma_C},\\ &(\mu^q_H - \lambda^q_H, s\gamma_C(u^p_h))_{0,\Gamma_C} \leq 0\end{aligned} \tag{4}$$

for all $v^p_h \in S^p_h$ and $\mu^q_H \in \Lambda^q_H$. Due to the boundedness of Λ^q_H, the existence of a discrete saddle point is guaranteed by the same arguments as in the non-discretized case. The first component of the discrete saddle point is unique. The second component is unique if, for instance, a discrete inf-sup condition is fulfilled, i.e., there exists an $\alpha > 0$ such that

$$\alpha\|\mu^q_H\|_{\tau,\Gamma_C} \leq \sup_{v^p_h\in S^p_h,\, \|v^p_h\|_1=1} (\mu^q_H, s\gamma_C(v^p_h))_{0,\Gamma_C},\ \tau \leq 0. \tag{5}$$

Remark 2.1 Indeed, it is shown in [8] that (5) is even uniformly fulfilled if $hH^{-1}\max\{1,q\}^2 p^{-1}$ is sufficiently small for constant p and q as well as $\tau = -1/2$.

3 A priori estimates

Let Ω be a subset of $\mathbb{R}^2$ and the polynomial degree distributions p and q be constant. The convergence of the mixed method can be stated without any regularity assumptions using some standard techniques of convex analysis. Only, the coercivity and the approximation properties of S^q_h and M^q_H are used. In the following, a sequence $\{v^p_h\}$ with $v^p_h \in \mathcal{S}^p_h$ converges to $v \in H^1_D(\Omega)$ if $v^p_h \to v$ as $h \to 0$ for a fixed p or as $p \to \infty$ for a fixed h. Similarly, the convergence of a sequence $\{\mu^q_H\}$ with $\mu^q_H \in M^q_H$ is defined. Moreover, we omit $h, H \to 0$ and $p, q \to \infty$ using the usual lim-notation. It is shown in [1, Theorem 1.1.5.3] that the sequence $\{u^p_h\}$ converges strongly to u and the sequence of Lagrange multipliers $\{\lambda^q_H\}$ converges weakly to λ, if (i) for all $v \in H^1_D(\Omega)$, there exists a sequence $\{v^p_h\}$ with $v^p_h \in \mathcal{S}^p_h$ which converges strongly to v, (ii) for all $\mu \in \Lambda$ there exists a sequence $\{\mu^q_H\}$ with $\mu^q_H \in M^q_H$ which converges strongly to μ, and (iii) for all sequences $\{\mu^q_H\}$ with $\mu^q_H \in \Lambda^q_H$ converging weakly to $\mu \in L^2(\Gamma_C)$, there holds $\mu \in \Lambda$.

Condition (i) is fulfilled due to the approximation properties of S^p_h. In order to show that Conditions (ii) and (iii) are also fulfilled, let $\mathcal{C}_E$, $E \in \mathcal{E}_H$, be the $q_E + 1$ Gauss points. Note that polynomials $\hat{P}$ of order $2q_E + 1$ are exactly integrated by the quadrature rule defined by $\mathcal{C}_E$, i.e, with some weights $\alpha_{\hat{x}} \geq 0$, there holds $\int_{-1}^{1} \hat{P}(\hat{x})\, d\hat{x} = \sum_{\hat{x}\in\mathcal{C}_E} \alpha_{\hat{x}} \hat{P}(\hat{x})$. For polynomials P on $E \in \mathcal{E}_H$ we have $\int_E P(x)\, dx = \sum_{\hat{x}\in\mathcal{C}_E} \beta_{\hat{x}} P(\Phi_E(\hat{x}))$ with $\beta_{\hat{x}} := \alpha_{\hat{x}} |\det \nabla\Phi_E(\hat{x})^\top \nabla\Phi_E(\hat{x})| \geq 0$, $\hat{x} \in \mathcal{C}_E$. Using the set of points $\mathcal{C}_E$ on each $E \in \mathcal{E}_H$, we define the standard interpolation operator I^q_H which maps continuous functions into M^q_H. There holds

$$\|v - I^q_H(v)\|_{0,\Gamma_C} \lesssim H^{\min(q+1,\theta)}/q^\theta \|v\|_{\theta,\Gamma^1_C}$$

for $v \in H^\theta(\Gamma_C)$ with $\theta > 1/2$, cf. [2, Theorem 5.6]. Here, $\lesssim$ abbreviates $\leq$ up to a positive constant which is independent of the mesh and the polynomial degree.

Lemma 3.1 *Let $\{\mu_H^q\}$ with $\mu_H^q \in M_H^q$ be a bounded sequence in $L^2(\Gamma_C)$ and $v \in H^\theta(\Gamma_C)$ with $\theta > 1/2$. It holds*

$$|(\mu_H^q, v - I_H^q(v))_0| \lesssim H^{\min(q+1,\theta)} q^\theta \|v\|_{\theta,\Gamma_C}.$$

P r o o f. From Cauchy's inequality as well as the interpolation estimate, we have

$$|(\mu_H^q, v - I_H^q(v))_{0,\Gamma_C}| \le \|\mu_H^q\|_{0,\Gamma_C} \|v - I_H^q(v)\|_{0,\Gamma_C} \lesssim H^{\min(q+1,\theta)}/q^\theta \|v\|_{\theta,\Gamma_C} \|\mu_H^q\|_{0,\Gamma_C}.$$

As the sequence $\{\mu_H^q\}$ is assumed to be bounded in $L^2(\Gamma_C)$, we obtain the assertion. □

Theorem 3.2 *$\{u_h^p\}$ converges strongly to u and $\{\lambda_H^q\}$ converges weakly to λ.*

P r o o f. Let $\mu \in \Lambda$ and $\epsilon > 0$. Due to the density of continuous functions in $L^2(\Gamma_C)$, there exists a continuous function μ_ϵ on Γ_C with $\|\mu - \mu_\epsilon\|_{0,\Gamma_C^1} \le \epsilon$. Define $\bar{\mu}_\epsilon := \max\{\min\{\mu_\epsilon, 1\}, -1\}$, then $\bar{\mu}_\epsilon \in H^1(\Gamma_C)$ and $\|\mu - \bar{\mu}_\epsilon\|_{0,\Gamma_C} \le \|\mu - \mu_\epsilon\|_{0,\Gamma_C} \le \epsilon$. For a fixed q there exists an H so that $\|\bar{\mu}_\epsilon - I_H^q(\bar{\mu}_\epsilon)\|_{0,\Gamma_C} \le \epsilon$. Define $\mu_H^q := I_H^q(\bar{\mu}_\epsilon) \in \Lambda_H^q$, then $\|\mu - \mu_H^q\|_{0,\Gamma_C} \le \|\mu - \mu_\epsilon\|_{0,\Gamma_C^1} + \|\bar{\mu}_\epsilon - \mu_H^q\|_{0,\Gamma_C} \le 2\epsilon$, which gives us Condition (ii). The same holds for a fixed H.

Obvioulsy, for $q \in \{0,1\}$ we have $\Lambda_H^q \subset \Lambda$, so that Condition (iii) is immediately given. To show Condition (iii) for $q \ge 2$, let the sequence $\{\mu_H^q\}$ with $\mu_H^q \in M_H^q$ converge weakly to $\mu \in L^2(\Gamma_C)$ and $v \in H^\theta(\Gamma_C)$ with $\theta > 1/2$ and $v \ge 0$. From Lemma 3.1 we obtain

$$\begin{aligned}(\mu \pm 1, \mp v)_{0,\Gamma_C} &= \lim(\mu_H^q \pm 1, \mp v)_{0,\Gamma_C} = \lim(\mu_H^q \pm 1, \mp(v - I_H^q(v))_{0,\Gamma_C} \mp (\mu_H^q \pm 1, I_H^q(v))_{0,\Gamma_C} \\ &= \mp \lim(\mu_H^q \pm 1, I_H^q(v))_{0,\Gamma_C} = \mp \lim \sum_{E \in \mathcal{E}_H} \sum_{\hat{x} \in \mathcal{C}_E} \beta_{\hat{x}} (\mu_H^q(\Phi_E(\hat{x}) \pm 1) v(\Phi_E(\hat{x})) \le 0.\end{aligned}$$

Thus, we obtain $\mu \in \Lambda$ using the density of $H^\theta(\Gamma_C)$ in $L^2(\Gamma_C)$. □

To derive convergence rates, we apply well-known a priori estimates as, for instance, introduced in [1]. Assume condition (5) to be uniformly fulfilled. Then,

$$\begin{aligned}&\|u - u_h^p\|_1^2 \lesssim \|u - v_h^p\|_1^2 + \|\lambda - \mu_H^q\|_{\tau,\Gamma_C}^2 + (\lambda - \mu_H^q, s\gamma_C(u))_{0,\Gamma_C} + (\lambda_H^q - \mu, s\gamma_C(u))_{0,\Gamma_C}, \\ &\|\lambda - \lambda_H^q\|_{\tau,\Gamma_C} \lesssim \|u - u_h^p\|_1 + \|\lambda - \mu_H^q\|_{\tau,\Gamma_C}\end{aligned} \tag{6}$$

for all $v_h^p \in \mathcal{S}_h^p$, $\mu_H^q \in \Lambda_H^q$ and $\mu \in \Lambda$.

Theorem 3.3 *Let $u \in H^{1+\kappa}(\Omega)$, $\lambda \in H^\theta(\Gamma_C)$ and $s\gamma_C(u) \in H^{\tilde{\kappa}}(\Gamma_C)$ with $\theta, \kappa, \tilde{\kappa} > 1/2$. Assume the inf-sup condition (5) to be uniformly fulfilled and let the set of points of Γ_C, in which λ changes from negative to positive, be finite. Furthermore, let H be sufficiently small. Then, there holds*

$$\|u - u_h^p\|_1 + \|\lambda - \lambda_H^q\|_{\tau,\Gamma_C} \lesssim H^{\min(q+1,\theta)/2}/q^{\theta/2} + H^{\min(q+1,\tilde{\kappa})/2}/q^{\tilde{\kappa}/2} + h^{\min(p,\kappa)}/p^\kappa.$$

P r o o f. There holds $I_H^q(\lambda) \in \Lambda_H^q$ and, therefore, $(\lambda - I_H^q(\lambda), \gamma_C(u))_{0,\Gamma_C} \lesssim \|\lambda - I_H^q(\lambda)\|_0 \lesssim H^{\min(q+1,\theta)}/q^\theta$. We conclude from Theorem 3.2 that the sequence $\{\lambda_H^q\}$ converges weakly and is, therefore, bounded in $L^2(\Gamma_C)$. Let $\mathcal{E}_H^* := \{E \in \mathcal{E}_H \mid \forall x \in E: \ \lambda(x) \in (-1,1)\}$ and $\mathcal{E}_H^\pm := \{E \in \mathcal{E}_H \backslash \mathcal{E}_H^* \mid \forall x \in E: \ \pm\lambda(x) \ge 0\}$. From (1), we conclude $u(x) = 0$ for $x \in E \in \mathcal{E}_H^*$ and $\pm u(x) \le 0$ and, therefore, $(\lambda_H^q(x) \pm 1)u(x) \le 0$ for $x \in E \in \mathcal{E}_H^\pm$. Due to the continuity of λ and the assumption on its change of sign, we have $\mathcal{E}_H = \mathcal{E}_H^* \cup \mathcal{E}_H^+ \cup \mathcal{E}_H^-$ for a sufficiently small mesh size H. Define $\delta(x) := \pm 1$ if $x \in E \in \mathcal{E}_H^\pm$ and $\delta := 0$ if $x \in E \in \mathcal{E}_H^*$. Then, $\delta \in \Lambda$ and we obtain from Lemma 3.1

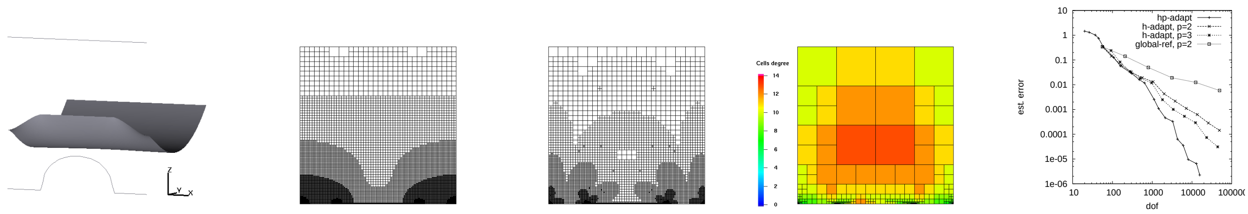
$$\begin{aligned}(\lambda_H^q - \delta, s\gamma_C(u))_{0,\Gamma_C} &\le |(\lambda_H^q - \delta, \gamma_C(u) - I_H^q(s\gamma_C(u)))_{0,\Gamma_C}| + (\lambda_H^q - \delta, I_H^q(s\gamma_C(u)))_{0,\Gamma_C} \\ &\lesssim H^{\min(q+1,\tilde{\kappa})}/q^{\tilde{\kappa}} + \sum_{E \in \mathcal{E}_H^+ \cup \mathcal{E}_H^-} \sum_{\hat{x} \in \mathcal{C}_E} \beta_{\hat{x}} \underbrace{(\lambda_H^q(\Phi_E(\hat{x})) - \delta)(su)(\Phi_E(\hat{x}))}_{\le 0}.\end{aligned}$$

The assertion follows from (6) using some well-known approximation estimates for the hp-method, cf. [2]. □

4 A posteriori estimates

The aim of this section is to derive a reliable a posteriori error estimate for $\|u - u_h\|_1$. The basic idea is to consider the following auxiliary problem: Find $u_\star \in H_D^1(\Omega)$ such that

$$(\nabla u_\star, \nabla v) = (f, v) - (\lambda_H^q, \gamma_C(v))_{0,\Gamma_C} \tag{7}$$



(a) (b) (c) (d) (e)

Fig. 1 **(a)** Solution of the idealized frictional contact problem, **(b)-(d)** adaptive meshes ($p = 2$, $p = 3$, hp), **(e)** convergence rates.

for all $v \in H^1_D(\Omega)$. Obviously, the solution $u_\star$ of (7) exists and is unique. Moreover, u^p_h is the finite element solution of (7). For all $\mu \in \Lambda$, there holds

$$\begin{aligned}\|u - u^p_h\|_1^2 &\lesssim (\nabla(u-u_\star), \nabla(u-u^p_h))_0 + \|u_\star - u^p_h\|_1 \|u-u^p_h\|_1 \\ &\lesssim (\lambda^q_H - \lambda, s\gamma_C(u-u^p_h))_{0,\Gamma_C} + \epsilon^{-1}\|u_\star - u^p_h\|_1^2 + \epsilon\|u-u^p_h\|_1^2 \\ &= (\lambda^q_H - \mu, s\gamma_C(u-u^p_h))_{0,\Gamma_C} + (\mu - \lambda, s\gamma_C(u-u^p_h))_{0,\Gamma_C} + \epsilon^{-1}\|u_\star - u^p_h\|_1^2 + \epsilon\|u-u^p_h\|_1^2 \\ &\lesssim \|\lambda^q_H - \mu\|_{0,\Gamma_C}\|u-u^p_h\|_1 + j(u^p_h) - (\mu, s\gamma_C(u^p_h))_{0,\Gamma_C} + \epsilon^{-1}\|u_\star - u^p_h\|_1^2 + \epsilon\|u-u^p_h\|_1^2,\end{aligned}$$

where we use Young's inequality $2ab \le \epsilon a^2 + (\epsilon)^{-1}b^2$ for $a, b, \epsilon > 0$. Applying a suitable ϵ yields

$$\|u - u^p_h\|_1^2 \lesssim \|u_\star - u^p_h\|_1^2 + \|\lambda^q_H - \mu\|^2_{0,\Gamma_C} + j(u^p_h) - (\mu, s\gamma_C(u^p_h))_{0,\Gamma_C}.$$

Using an error estimator $\eta_\star$ to estimate $\|u_\star - u^p_h\|_1$ and setting, for instance, $\bar{\lambda}^q_H(x) := \max\{\min\{\lambda^q_H(x), 1\}, -1\}$ we finally get the error estimation

$$\|u - u^p_h\|_1^2 \lesssim \eta_\star^2 + \|\lambda^q_H - \bar{\lambda}^q_H\|^2_{0,\Gamma_C} + (s, |u^p_h| - \bar{\lambda}^q_H\gamma_C(u^p_h))_{0,\Gamma_C}.$$

Remark 4.1 In principle, each error estimator known from the literature of variational equations can be used to define $\eta_\star$. We refer to [9] for an overview of h-adaptive methods. For hp-adaptivity, we need an error estimator which takes the degree distribution p into account. Such an estimator can be found in [10].

5 Numerical results

In our numerical experiments, we study the idealized frictional contact problem with $\Omega := (-1,1)^2$, $\Gamma_D := (-1,1) \times \{1\}$, $\Gamma_C := (-1,1) \times \{-1\}$, $f := -1$ and $s(x,y) = 2(1-x^2)$. In Figure 1(a), the finite element solution u^p_h is depicted. In addition, the function 1 and the discrete Lagrangian multiplier $-\lambda^q_H$ are sketched in. We observe, that the conditions as stated in (1) are approximatively fulfilled. In Figure 1(b),(c), h-adaptive meshes for $p = 2$ and $p = 3$ are shown. We find local refinements towards both ends of the contact zone and within the contact zone. In Figure 1(d), we observe the typical geometrical refinement patterns of an hp-adaptive mesh. Here, we use the hp-strategy as introduced in [5]. Some corners of the domain and the ends of the contact zone are resolved by h-refinements with small polynomial degrees ($p = 1$ or $p = 2$), whereas, away from the corners and the contact zone, the polynomial degree increases. We obtain almost exponential convergence rates for hp-adaptivity and optimal algebraic convergence rates for h-adaptivity, see Figure 1(e).

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