Identification and properties of Real Harmonizable Fractional Lévy Motions.

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Abstract

In this article the class of Real Harmonizable Fractional Lévy Motions is introduced. It is shown that RHFLM share many properties with Fractional Brownian Motion. These fields are locally asymptotically self-similar with a constant index H, and have Hölderian paths. Moreover the identification of H for RHFLM can be performed with the so-called generalized variation method. Besides the Fractional Brownian Motion this class contains non-Gaussian fields that are asymptotically self-similar at infinity with a Real Harmonizable Fractional Stable Motion of index $\tilde{H}$ as tangent field. This last property should be useful to model phenomena with multiscale behavior.

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1 Introduction

Fractional Brownian Motion (in short FBM) $B_H(t)$ of fractional index H, introduced in (Mandelbrot and Ness 1968), provides a very powerful model in applied mathematics. FBM is the only centered Gaussian self-similar process with stationary increments (in short sssi) and with index H. Let us recall the definition of self-similarity. A field $X(x), x \in \mathbb{R}^d$ is self-similar with index H if, for all $\lambda > 0$:

$$(X(\lambda x))_{x\in\mathbb{R}^d} \overset{(d)}{=} \lambda^H (X(x))_{x\in\mathbb{R}^d} ,$$

where $\overset{(d)}{=}$ stands for equality in distribution. It is a global property and for some applications one needs a local version of this notion. A field $X(x)$ is locally asymptotically self-similar (in short lass) at point x with index H if

$$\lim_{\lambda\to 0+} \left(\frac{X(x+\lambda u) - X(x)}{\lambda^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (T(u))_{u\in\mathbb{R}^d} ,$$

where the non-degenerate field $(T(u))_{u\in\mathbb{R}^d}$ is called the tangent field at point x. This definition has been introduced in (Benassi, Jaffard and Roux 1997, Peltier and Levy-Vehel 1996) and various examples of lass Gaussian fields have been studied : Filtered White Noises (Benassi, Cohen, Istas and Jaffard 1998b), Multifractional Brownian where the fractional index H is replaced by a fractional function $H(x)$ (Benassi et al. 1997, Peltier and Levy-Vehel 1996), and for irregular fractional functions the Generalized Brownian Motion (Ayache and Véhel 1999) and the Step Fractional Brownian Motion (Benassi, Bertrand, Cohen and Istas 2000). These fields share many properties with the FBM. For instance the sample paths are locally almost surely $C^{H'}$ for every $H' < H(x)$. In these models the identification of the fractional function is a central problem. It has have been performed with semi-parametric estimators based on generalized quadratic variations (Istas and Lang 1997, Benassi, Cohen and Istas 1998a, Benassi et al. 2000).

In this paper we propose a class of second order fields called Real Harmonizable Fractional Lévy Motion (in short RHFLM), including non-Gaussian fields and the FBM, that have stationary increments, that are lass with index H and with tangent FBM, that have sample paths locally almost surely $C^{H'}$ for every $H' < H$. These fields are obtained by integrating fractionally a Lévy Measure $L(d\xi)$ that have moments of every order :

$$X_H(x) = \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} L(d\xi) . \tag{1}$$

When the Lévy Measure $L(d\xi)$ is a Brownian measure $W(d\xi)$, (1) yields the harmonizable representation of the FBM (Samorodnitsky and Taqqu 1994). Let us also recall that the Real Harmonizable Fractional Stable Motion (in short RHFSM) has a representation similar to (1) where $L(d\xi)$ is a complex isotropic stable $\alpha-$symmetric measure $M_\alpha(d\xi)$. Nevertheless Real Harmonizable Fractional Stable Motions are not RHFLM since the variance of their increments are infinite. Actually the complex isotropic stable $\alpha-$ symmetric measure can be truncated to yield a RHFLM that has an asymptotic self-similarity at infinity with an index $\tilde{H}$ different from H :

$$\lim_{\lambda \to +\infty} \left(\frac{X(\lambda u)}{\lambda^{\tilde{H}}} \right)_{u \in \mathbb{R}^d} \stackrel{(d)}{=} (T(u))_{u \in \mathbb{R}^d} ,$$

where the non-degenerate field $(T(u))_{u\in\mathbb{R}^d}$ is called the asymptotic field. Roughly speaking this "truncated" Real Harmonizable Fractional Stable Motion can be seen as a bridge between FBM and Real Harmonizable Fractional Stable Motions. We think that this class of identifiable models that exhibit a very different behavior at low scale when ($\lambda \to 0^+$, lass) and at large scale ($\lambda \to +\infty$), where discrete structures (related to the "big jumps" of $M_\alpha(d\xi)$) appears, should be useful for applications (see (Hermann and Roux 1990) for instance). This phenomenon of different asymptotic self similarities at low and large scales has already been encountered in (Benassi and Deguy 1999) but in (Benassi and Deguy 1999) the asymptotic fields are both FBM which is less remarkable.

From the statistical point of view, the fractional index of a RHFLM can be estimated with the method of generalized quadratic variations which cannot be applied to RHFSM (See (Abry, Delbeke and Flandrin 2000a, Abry, Pesquet-Popescu and Taqqu 2000b) for other methods of estimation in the case of RHFSM.)

In section two the construction of RHFLM is performed. The asymptotic self-similarity properties and the regularity of the sample paths of such fields are studied in section 3 and section 4 is devoted to the identification of the fractional index.

2 Construction of non-Gaussian Fractional Fields

In this part the RHFLMs are obtained and we precisely describe the construction of the Lévy measure $L(d\xi)$. For sake of simplicity, we work first with a Lévy measure (written $M(d\xi)$) without Brownian component. Since it is desirable that the parameter H can be identified with generalized quadratic variations we impose that the field X_H have moments of every order $\mathbb{E}(|X_H(x)|^p) < +\infty \quad \forall x \in \mathbb{R}^d$. Actually if $M(d\xi)$ is a symmetric α-stable measure the Real Harmonizable Fractional Stable Motion X_H (cf. (Samorodnitsky and Taqqu 1994)) is self-similar but $X_H(x)$ fails to have moments of every order. Hence we consider a non vanishing Lévy measure $M(d\xi)$ represented by a Poisson random measure $N(d\xi, dz)$ in the sense of section 3.12 of (Samorodnitsky and Taqqu 1994) but with a control measure that has moments of every order. More precisely let $N(d\xi, dz)$ be a Poisson measure on $\mathbb{R}^d \times \mathbb{C}$ for which the mean measure $n(d\xi, dz) = \mathbb{E}N(d\xi, dz) = d\xi\nu(dz)$ satisfies

$$\forall p \geq 2, \quad \int_{\mathbb{C}} |z|^p \nu(dz) < +\infty. \tag{2}$$

Let us recall the basic properties of the compensated Poisson measure

$$\tilde{N} = N - n.$$

For every function $\varphi : \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$ such that $\varphi \in L^2(\mathbb{R}^d \times \mathbb{C})$ the stochastic integral

$$\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z) \tilde{N}(d\xi, dz)$$

is defined as the limit in $L^2(\Omega)$ of

$$\int_{\mathbb{R}^d \times \mathbb{C}} \varphi_k(\xi, z) \tilde{N}(d\xi, dz)$$

where φ_k is a simple function of the form $\sum_{i \in I} a_i \mathbf{1}_{A_i}$. The set I is finite and

$$\int_{\mathbb{R}^d \times \mathbb{C}} \sum_{i \in I} a_i \mathbf{1}_{A_i} \tilde{N}(d\xi, dz) \stackrel{def}{=} \sum_{i \in I} a_i \tilde{N}(A_i)$$

where the Poisson random variables have intensity $n(A_i)$ and are independent if the sets A_i are disjoints. It is classical to have the isometry property :

$$\mathbb{E}\left| \int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z) \tilde{N}(d\xi, dz) \right|^2 = \int_{\mathbb{R}^d \times \mathbb{C}} |\varphi(\xi, z)|^2 n(d\xi, dz). \tag{3}$$

Furthermore if φ is real valued so is $\int \varphi d\tilde{N}$, and if $\Re(z)$ denotes the real part of a complex z then $\Re(\int \varphi d\tilde{N}) = \int \Re(\varphi) d\tilde{N}$, the same property is true for the imaginary part $\Im$ of stochastic integrals.

Then the characteristic function of the stochastic integral is $\forall u,\ v \in \mathbb{R}$:

$$\mathbb{E}\exp\left(i(u\int \Re(\varphi)d\tilde{N} + v\int \Im(\varphi)d\tilde{N})\right) =$$
$$\exp[\int_{\mathbb{R}^d\times\mathbb{C}}[\exp(i(u\Re(\varphi)+v\Im(\varphi)))$$
$$-1-i(u\Re(\varphi)+v\Im(\varphi))]d\xi\nu(dz)] \quad (4)$$

where the integral in the right hand term of the previous equation is convergent since

$$|\exp(ix)-1-ix| \leq C|x|^2 \quad \forall x \in \mathbb{R}. \quad (5)$$

Let us remark that the Poisson measure N has to be defined on $\mathbb{R}^d\times\mathbb{C}$ to have the field X_H real valued as in the case of the Real Harmonizable Fractional Stable Motion. More precisely we consider the compensated Poisson measure $\tilde{N} = N - n$ and

$$\int_{\mathbb{R}^d} f(\xi)M(d\xi) \stackrel{def}{=} \int_{\mathbb{R}^d\times\mathbb{C}}[f(\xi)z + f(-\xi)\bar{z}]\tilde{N}(d\xi,dz) \quad (6)$$

for every function $f:\mathbb{R}^d\to\mathbb{C}$ where $f\in L^2(\mathbb{R}^d)$. Then if

$$\forall\xi\in\mathbb{R}^d,\quad f(-\xi)=\overline{f(\xi)} \quad (7)$$

the following integral is a real number since

$$\begin{aligned}\int_{\mathbb{R}^d} f(\xi)M(d\xi) &= \int_{\mathbb{R}^d\times\mathbb{C}} 2\Re(f(\xi)z)\tilde{N}(d\xi,dz) &(8)\\ &= 2\Re\left(\int_{\mathbb{R}^d\times\mathbb{C}} f(\xi)z\tilde{N}(d\xi,dz)\right). &(9)\end{aligned}$$

Hence X_H is a real valued field since the integrand

$$\frac{e^{-ix\cdot\xi}-1}{\|\xi\|^{\frac{d}{2}+H}} \quad (10)$$

satisfies the symmetry (7).

Classically the field X_H has stationary increments if the control measure $\nu(dz)$ is assumed to be rotationally invariant. Let P be the map $P(\rho\exp(i\theta)) = (\theta,\rho)\in[0,2\pi)\times\mathbb{R}^+_*$. From now on the measure satisfies the following property :

$$P(\nu(dz)) = d\theta\nu_\rho(d\rho) \quad (11)$$

where $d\theta$ is the uniform measure on $[0, 2\pi)$. When f satisfies (7) :

$$\int f(\xi) \exp(i\theta\xi) M(d\xi) \stackrel{(d)}{=} \int f(\xi) M(d\xi),$$

where $\stackrel{(d)}{=}$ means that the random variables have the same distributions, is a consequence of (4). Moreover stochastic integral have symmetric distributions :

$$-\int f(\xi) M(d\xi) \stackrel{(d)}{=} \int f(\xi) M(d\xi).$$

Hence under the assumption (11) X_H is a field with stationary increments which considerably simplifies the following work. Moreover an isometry property for the Lévy measure $M(d\xi)$ when f satisfies (7) :

$$\mathbb{E}|\int_{\mathbb{R}^d} f(\xi) M(d\xi)|^2 = 4\pi \|f\|^2_{L^2(\mathbb{R}^d)} \int_0^{+\infty} \rho^2 \nu_\rho(d\rho) \tag{12}$$

holds, it is a consequence of (3).

To study the field X_H moment of order $2p$, $\mathbb{E}((X_H(s) - X_H(t))^{2p})$ is computed with the help of the corresponding L^{2p} norms of the deterministic integrand (10). Actually the characteristic function (4) allows to compute every moments of the stochastic integrals $\int f(\xi) M(d\xi)$.

Proposition 2.1 *If $f \in \bigcap_{q=1}^p L^{2q}(\mathbb{R}^d)$ and f satisfies (7) then $\int f(\xi) M(d\xi)$ is in $L^{2p}(\Omega)$ and*

$$\mathbb{E}((\int f(\xi) M(d\xi))^{2p}) = \sum_{n=1}^{p} (2\pi)^n \sum_{P_n} \prod_{q=1}^{n} \frac{(2m_q)! \|f\|_{2m_q}^{2m_q} \int_0^{+\infty} \rho^{2m_q} \nu_\rho(d\rho)}{(m_q!)}, \tag{13}$$

where $\sum_{P_n}$ stands for the sum over the set of partitions P_n of $\{1, \ldots, 2p\}$ in n subsets K_q such that the cardinality of K_q is $2m_q$ with $m_q \geq 1$ and where $\|f\|_{2m_q}$ is the $L^{2m_q}(\mathbb{R}^d)$ norm of f.

Proof :

An expansion in power series of both sides of (4) yields the result.

We now introduce the Real Harmonizable Fractional Lévy Motion.

Definition 2.2 *Let us call a real valued field which admits a harmonizable representation*

$$X_H(x) = \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} L(d\xi),$$

where $L(d\xi) = aM(d\xi) + bW(d\xi)$ is the sum of a Lévy measure $aM(d\xi)$ and of an independent $bW(d\xi)$ Wiener measure, a Real Harmonizable Fractional Lévy Motion (in short RHFLM). We suppose that $M(d\xi)$ satisfies the finite moment assumption (2), and the rotational invariance (11).

With this definition X_H is the sum of the process $\int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} bW(d\xi)$ which is a Fractional Brownian Motion and of $\int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} aM(d\xi)$, both processes being independent. In particular the FBM is a RHFLM obtained for $L(d\xi) = W(d\xi)$.

3 Some properties of Real Harmonizable Fractional Lévy Motions

In this section we investigate some properties of self-similarity and regularity types that the RHFLM shares with the Fractional Brownian Motion. In the first part of this section we prove that two asymptotic self-similarity properties are true for the RHFLM. Since no trivial invariance property is assumed for the Lévy measure, it is the best result we can hope in that direction. In the second part we see that almost surely the paths of the RHFLM are Hölder-continuous with a pointwise Hölder exponent H. We suppose for sake of simplicity in this part that $L(d\xi) = M(d\xi)$.

3.1 Asymptotic self-similarity

Since we know the characteristic function of stochastic integrals of the measure $M(d\xi)$ we can prove the local self-similarity of the RHFLMs. Actually it is a consequence of the scaling property of the integrand (10) and of a central limit theorem for the stochastic measure $M(d\xi)$.

Proposition 3.1 *The Real Harmonizable Fractional Lévy Motion is locally self-similar with parameter H in the sense that for every fixed $x \in \mathbb{R}^d$:*

$$\lim_{\epsilon \to 0^+} \left(\frac{X_H(x+\epsilon u) - X_H(x)}{\epsilon^H} \right)_{u \in \mathbb{R}^d} \stackrel{(d)}{=} \left(4\pi \int_0^{+\infty} \rho^2 \nu_\rho(d\rho) \right)^{1/2} (B_H(u))_{u \in \mathbb{R}^d}, \quad (14)$$

where the convergence is a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets. The limit is the distribution of a FBM.

Proof : The convergence of the finite dimensional margins is proved first. Since the RHFLM has stationary increments we only have to prove the convergence for $x = 0$. Let us consider the multivariate function :

$$g_{u,v,H}(\epsilon, \xi, z) = i2\Re \left(z \sum_{k=1}^n v_k \frac{e^{-i\epsilon u_k \cdot \xi} - 1}{\epsilon^H \|\xi\|^{\frac{d}{2}+H}} \right) \quad (15)$$

where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are in $\mathbb{R}^n$. Then

$$\mathbb{E} \exp \left(i \sum_{k=1}^n v_k \frac{X_H(\epsilon u_k)}{\epsilon^H} \right) = \exp \left(\int_{\mathbb{R}^d \times \mathbb{C}} [\exp(g_{u,v,H}(\epsilon, \xi, z)) - 1 - g_{u,v,H}(\epsilon, \xi, z)] d\xi d\nu(z) \right). \quad (16)$$

The change of variable $\lambda = \epsilon\xi$ is applied to the integral of the previous right hand term to get :

$$\int_{\mathbb{R}^d \times \mathbb{C}} [\exp(\epsilon^{d/2} g_{u,v,H}(1, \lambda, z)) - 1 - \epsilon^{d/2} g_{u,v,H}(1, \lambda, z)] \frac{d\lambda}{\epsilon^d} d\nu(z). \quad (17)$$

Then as $\epsilon \to 0^+$ a dominated convergence argument yields that

$$\lim_{\epsilon \to 0^+} \mathbb{E} \exp \left(i \sum_{k=1}^n v_k \frac{X_H(\epsilon u_k)}{\epsilon^H} \right) = \exp \left(\int_{\mathbb{R}^d \times \mathbb{C}} g_{u,v,H}^2(1, \lambda, z) d\lambda d\nu(z) \right). \quad (18)$$

Moreover (11) allows to express the logarithm of the previous limit as :

$$4\pi \int_0^{+\infty} \rho^2 \nu_\rho(d\rho) \int_{\mathbb{R}^d} \frac{|\sum_{k=1}^n v_k e^{-iu_k\cdot\lambda} - 1|^2}{\|\lambda\|^{2d+H}} d\lambda \tag{19}$$

and this last integral is the variance of $\sum_{k=1}^n v_k B_H(u_k)$ which concludes the proof of the convergence of finite dimensional margins.

Let us proceed to the proof that the distributions are tight. We need to estimate

$$\mathbb{E}(X_H(x) - X_H(y))^{2p}$$

for p big enough. Unfortunately when $H > 1 - d/2$ these moments are not finite because of the asymptotic of the integrand :

$$g_0(x,\xi) = \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} \tag{20}$$

when $\|\xi\| \to 0$. In the case $H > 1 - d/2$ we thus apply a transformation to the integrand g_0 to analyze in two different ways its behavior at both ends of the spectrum.

Let us first consider the easy case : $H < 1 - d/2$. Then $g_0(x,.) \in L^{2q}(\mathbb{R}^d) \quad \forall q \in \mathbb{N}^*$ and

$$\|g_0(x,.) - g_0(y,.)\|^{2q}_{L^{2q}(\mathbb{R}^d)} = \|x-y\|^{2Hq+d(q-1)} \|g_0(e_1,.)\|^{2q}_{L^{2q}(\mathbb{R}^d)}$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. Because of (13) we know that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} = \sum_{n=1}^{p} D(n) \|x-y\|^{2Hp+d(p-n)}$$

for some nonnegative constants $D(n)$. Hence there exists $C < +\infty$ such that :

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} \le C\|x-y\|^{2Hp}. \tag{21}$$

Hence if $H < 1 - d/2$

$$\mathbb{E}\left(\frac{(X_H(x+\epsilon u) - X_H(x+\epsilon v))^{2p}}{\epsilon^{2Hp}} \right) \le \|u-v\|^{2Hp}$$

and one can take $p > 1/2H$ to show the tightness.

When $H > 1 - d/2$ let us take K an integer such that $K \geq 1 + d/2$;

$$P_K(t) = \sum_{k=1}^{K} \frac{t^k}{k!},$$

and φ an even C^1-function such that $\varphi(t) = 1$ when $|t| \leq 1/2$ and $\varphi(t) = 0$ when $|t| > 1$. Then

$$g_K(x,\xi) = \frac{e^{-ix\cdot\xi} - 1 - P_K(-ix\cdot\xi)\varphi(\|x\|\|\xi\|)}{\|\xi\|^{\frac{d}{2}+H}}$$

is in $L^{2q}(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$ and $q \in \mathbb{N}^*$.

The field X_H is then split into two fields $X_H = X_H^+ + X_H^-$ where

$$X_H^+(x) = \int g_K(x,\xi) M(d\xi) \tag{22}$$

and

$$X_H^-(x) = \int \frac{P_K(-ix\cdot\xi)}{\|\xi\|^{\frac{d}{2}+H}} \varphi(\|x\|\|\xi\|) M(d\xi). \tag{23}$$

A method similar to the one used for X_H when $H < 1 - d/2$ is applied to X_H^+ and we check that X_H^- has almost surely C^1 paths.

Let us start by the remark

$$\|g_K(x,.)\|_{L^{2q}(\mathbb{R}^d)}^{2q} = \|x\|^{2Hq+d(q-1)} \|g_K(e_1,.)\|_{L^{2q}(\mathbb{R}^d)}^{2q}.$$

As in the easy case we have to estimate when $\epsilon \to 0^+$

$$I_\epsilon = \int_{\mathbb{R}^d} |g_K(x,\xi) - g_K(x+\epsilon u,\xi)|^{2q} d\xi.$$

Let us split this integral into

$$I_\epsilon^+ = \int_{\epsilon\|\xi\|\geq 1} |g_K(x,\xi) - g_K(x+\epsilon u,\xi)|^{2q} d\xi$$

and

$$I_\epsilon^- = \int_{\epsilon\|\xi\|< 1} |g_K(x,\xi) - g_K(x+\epsilon u,\xi)|^{2q} d\xi$$

as $I_\epsilon = I_\epsilon^+ + I_\epsilon^-$. Actually :

$$g_K(x,\xi) - g_K(y,\xi) = g_0(x-y,\xi)$$

on $\{\epsilon\|\xi\| \geq 1\}$ for ϵ small enough, and we get by the change of variable $\lambda = \epsilon\xi$

$$I_\epsilon^+ = \epsilon^{2Hq+d(q-1)} \int_{\|\lambda\|\geq 1} \frac{|e^{-ie_1\cdot\lambda} - 1|^{2q}}{\|\lambda\|^{2Hq+dq}} d\lambda. \tag{24}$$

Then a Taylor expansion is applied to I_ϵ^-

$$I_\epsilon^- = \int_{\|\xi\|<\frac{1}{\epsilon}} |dg_K(\theta(x,\epsilon u,\xi),\xi).\epsilon u|^{2q} d\xi$$

where $dg_K(\theta(x,\epsilon u,\xi),\xi)$ is the differential of the map $g_K(.,\xi)$ and $\theta(x,\epsilon,\xi)$ is a point in the segment $(x, x+\epsilon u)$. Note that

$$\int_{\|\xi\|<C} \|dg_K(\theta(x,\epsilon u,\xi),\xi)\|^{2q} d\xi < +\infty$$

for every fixed C and that

$$\|dg_K(\theta(x,\epsilon u,\xi),\xi)\|^{2q} = O(\|\xi\|^{2q(1-\frac{d}{2}-H)}) \quad \text{when} \quad \|\xi\| \to +\infty,$$

hence

$$\left(\int_{\|\xi\|<\frac{1}{\epsilon}} \|dg_K(\theta(x,\epsilon u,\xi),\xi)\|^{2q} d\xi\right) \epsilon^{2q} = 0(\epsilon^{2Hq+d(q-1)}) \quad \text{when} \quad \epsilon \to 0^+$$

and

$$|I_\epsilon^-| \leq C\epsilon^{2Hq+d(q-1)} \tag{25}$$

when $\epsilon \to 0^+$. Because of (24) and (25) there exists a positive constant C such that

$$\int_{\mathbb{R}^d} |g_K(x,\xi) - g_K(y,\xi)|^{2q} d\xi \leq C\|x-y\|^{2Hq+d(q-1)}$$

and consequently

$$\mathbb{E}(X_H^+(x) - X_H^+(y))^{2p} \leq C\|x-y\|^{2Hp}. \tag{26}$$

when $\|x\| \leq 1$, $\|y\| \leq 1$, which yields that the distributions of

$$\left(\frac{X_H^+(x+\epsilon.) - X_H^+(x)}{\epsilon^H}\right)_{\epsilon>0}$$

are tight. To conclude let us write X_H^- as

$$\int_{\epsilon\|\xi\|\leq 1/2} \frac{P_K(-ix\cdot\xi)}{\|\xi\|^{\frac{d}{2}+H}} M(d\xi) + \int_{1/2\leq\epsilon\|\xi\|\leq 1} \frac{P_K(-ix\cdot\xi)}{\|\xi\|^{\frac{d}{2}+H}} \varphi(i\|x\|\|\xi\|) M(d\xi).$$

The first integral of the previous line is actually a polynomial in the variables $(x_1, \dots, x_d)$ with coefficients that are random variables, hence it has almost surely C^1 paths. Let us remark that the integrand of the second integral is bounded with compact support in $\mathbb{R}^d \times \mathbb{C}$ and is C^1 in the variable x, so is the integral which yields that X_H^- is almost surely C^1. Then it is clear that

$$\lim_{\epsilon\to 0^+} \left(\frac{X_H^-(x+\epsilon u) - X_H^-(x)}{\epsilon^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} 0$$

which concludes the proof.

We now want to exhibit an example of RHFLM that has asymptotic self-similarity properties when the increment is taken on large scales. Actually if the control measure $\nu_\rho(d\rho)$ is $\frac{d\rho}{|\rho|^{1+\alpha}}\mathbf{1}(|\rho| < 1)$ where $0 < \alpha < 2$, we show that at large scales the RHFLM is asymptotically self-similar with parameter $0 < \tilde{H} < 1$ such that $\tilde{H} + \frac{d}{\alpha} = H + \frac{d}{2}$. Moreover the limit field is a RHFSM with parameter $\tilde{H}$. This shows that at large scales the behavior of RHFLM can be very far from the Gaussian model even if the RHFLM is a field that have moments of order 2. The RHFLM with control measure $\frac{d\rho}{|\rho|^{1+\alpha}}\mathbf{1}(|\rho| < 1)$ can be viewed roughly speaking as in between a RHFSM at large scales and a FBM at low scales. Let us now state precisely the asymptotic self-similarity.

Proposition 3.2 *Let us assume that $\tilde{H}$ defined by $\tilde{H} + \frac{d}{\alpha} = H + \frac{d}{2}$ is such that $0 < \tilde{H} < 1$. The Real Harmonizable Fractional Lévy Motion with control measure $\nu_\rho(d\rho)$:*

$$\frac{d\rho}{|\rho|^{1+\alpha}}\mathbf{1}(|\rho| < 1)$$

is asymptotically self-similar with parameter $\tilde{H}$. For every fixed $x \in \mathbb{R}^d$

$$\lim_{R\to+\infty} \left(\frac{X_H(Ru)}{R^{\tilde{H}}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(Y_{\tilde{H}}(u)\right)_{u\in\mathbb{R}^d} \tag{27}$$

where the limit is in distribution for all finite dimensional margins of the fields and the limit is a Real Harmonizable Fractional Stable Motion that has a representation :

$$Y_{\tilde{H}}(x) = 2\Re \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{\alpha}+\tilde{H}}} M_\alpha(d\xi), \qquad (28)$$

where $M_\alpha(d\xi)$ is complex isotropic stable $\alpha-$symmetric random measure.

Proof :

As in the previous proposition we consider a multivariate function :

$$g_{u,v,H}(R,\xi,z) = i2\Re\left(z\sum_{k=1}^n v_k \frac{e^{-iRu_k\cdot\xi} - 1}{R^{\tilde{H}}\|\xi\|^{\frac{d}{2}+H}}\right) \qquad (29)$$

where u and v are in $\mathbb{R}^n$. And

$$\mathbb{E}\exp\left(i\sum_{k=1}^n v_k \frac{X_H(Ru_k)}{R^{\tilde{H}}}\right) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{C}} [\exp(g_{u,v,H}(R,\xi,z)) - 1 - g_{u,v,H}(R,\xi,z)] d\xi d\nu(z)\right). \qquad (30)$$

Then the change of variable $\lambda = \frac{\xi}{R}$ is applied and $\tilde{H}$ has been chosen such that integral in the previous equation is now :

$$\int_{\mathbb{R}^d\times[0,2\pi]\times\mathbb{R}_*^+} [\exp(g_{u,v,H}(1,\lambda,R^{d/\alpha}\rho e^{i\theta})) - 1 - g_{u,v,H}(1,\lambda,R^{d/\alpha}\rho e^{i\theta})] \mathbf{1}(|\rho|<1) R^d d\lambda d\theta \frac{d\rho}{|\rho|^{1+\alpha}}. \qquad (31)$$

Let us set $r = R^{d/\alpha}\rho$ the integral becomes

$$I(R) = \int_{\mathbb{R}^d\times[0,2\pi]\times\mathbb{R}} [\exp(g_{u,v,H}(1,\lambda,re^{i\theta})) - 1 - g_{u,v,H}(1,\lambda,re^{i\theta})] \mathbf{1}(|r|<R^{d/\alpha}) d\lambda d\theta \frac{dr}{2|r|^{1+\alpha}}. \qquad (32)$$

Let us recall that

$$-C(\alpha)|x|^\alpha = \int_{\mathbb{R}} [e^{ixr} - 1 - ixr\mathbf{1}(|r| \le R^{d/\alpha})] \frac{dr}{2|r|^{1+\alpha}} \qquad (33)$$

for every $R > 0$, where $C(\alpha) = \int_0^{+\infty}(1-\cos(r))\frac{dr}{2r^{1+\alpha}}$. Let us write

$$J_R = \int_{\mathbb{R}} [e^{ixr} - 1 - ixr]\mathbf{1}(|r| \le R^{d/\alpha})\frac{dr}{2|r|^{1+\alpha}}$$

then

$$\begin{aligned}\lim_{R\to+\infty}(J_R + C(\alpha)|x|^\alpha) &= \lim_{R\to+\infty}\int_{\mathbb{R}}[e^{ixr}-1]\mathbf{1}(|r| > R^{d/\alpha})\frac{dr}{2|r|^{1+\alpha}} \\ &= 0\end{aligned}$$

hence :

$$\lim_{R\to+\infty} I(R) = -C(\alpha)\int_{\mathbb{R}^d\times[0,2\pi]}\left|2\Re\left(e^{i\theta}\sum_{k=1}^n v_k \frac{e^{-iu_k\lambda}-1}{\|\lambda\|^{\frac{d}{\alpha}+\tilde{H}}}\right)\right|^\alpha d\lambda d\theta \quad (34)$$

which is also :

$$-C(\alpha)\int_0^{2\pi}|2\cos(\theta)|^\alpha d\theta \int_{\mathbb{R}^d}\left|\sum_{k=1}^n v_k \frac{e^{-iu_k\lambda}-1}{\|\lambda\|^{\frac{d}{\alpha}+\tilde{H}}}\right|^\alpha d\lambda. \quad (35)$$

Since this last expression is the logarithm of

$$\mathbb{E}\exp\left(i\sum_{k=1}^n v_k Y_{\tilde{H}}(u_k)\right),$$

the proof is complete.

3.2 Regularity of the sample paths of the RHFLM

The Kolmogorov Theorem (see for instance (Karatzas and Shreve 1988)) and Proposition 2.1 show that H can be considered roughly speaking as the Hölder exponent of the sample paths of the RHFLMs. Let us recall the definition of the pointwise exponent $H_f(x)$ of a deterministic function f at point x by

$$H_f(x) = \sup\{H', \quad \lim_{\epsilon\to 0}\frac{f(x+\epsilon)-f(x)}{\|\epsilon\|^{H'}} = 0\} \quad (36)$$

then the regularity of the sample paths is described by the following proposition.

Proposition 3.3 *For every $H' < H$ there exists a continuous modification of the RHFLM such that :*

$$\mathbb{P}\left[\omega; \sup_{0<\|x-y\|<\epsilon(\omega),\|x\|\leq 1,\|y\|\leq 1} \left(\frac{X_H(x)-X_H(y)}{\|x-y\|^{H'}}\right) \leq \delta\right] = 1 \tag{37}$$

where $\epsilon(\omega)$ is an almost surely positive random variable and $\delta > 0$. Moreover at every point x the pointwise exponent $H_{X_H}(x)$ of the RHFLM X_H is almost surely equal to H.

Proof :

In the first part of the proof we will use the estimation of the moments

$$\mathbb{E}(X_H(x) - X_H(y))^{2p}$$

performed in the proof of Proposition 3.1 and Kolmogorov Theorem. When $H < 1 - d/2$ we already know by (21) that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} \leq C\|x-y\|^{2Hp}$$

when $\|x\| \leq 1, \|y\| \leq 1$ and Kolmogorov Theorem yields (37) for every $H' < H$. When $H > 1 - d/2$ we recall that X_H has been split into

$$X_H = X_H^+ + X_H^-$$

where X_H^+ and X_H^- are defined in (22) and (23). Furthermore we know that X_H^+ is H'-Hölder-continuous for every $H' < H$ by Kolmogorov Theorem and the inequality (26), and that X_H^- has almost surely C^1 sample paths which concludes the proof of (37).

Because of (37) at every point x the Hölder exponent satisfies $H(x) \geq H$. To show $H(x) \leq H$ let us use the local self-similarity (14). Actually if $H' > H$ we can deduce from (14) that

$$\lim_{\epsilon \to 0^+} \frac{\epsilon^{H'}}{|X_H(x+\epsilon) - X_H(x)|} \stackrel{(d)}{=} 0,$$

which is also a convergence in probability. Hence we can find a sequence $(\epsilon_n)_{n\in\mathbb{N}} \to 0^+$ such that :

$$\lim_{n\to+\infty} \frac{|X_H(x+\epsilon_n) - X_H(x)|}{\epsilon_n^{H'}} = +\infty \quad \text{almost surely.}$$

This argument concludes the proof of Proposition 3.3.

4 Identification of the fractional index

Let W be a Brownian measure on $L^2(\mathbb{R}^d)$ and M a Lévy measure satisfying the assumptions of Definition 2.2. Assume that M and W are independent. Define then the following Lévy measure:

$$L \;=\; \sigma W + M \;.$$

Recall that X_H defined by:

$$X_H(x) = \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} L(d\xi) \;.$$

is the sum of a Fractional Brownian Motion and a Real Harmonizable Fractional Lévy Motion, both motions being independent.

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbf{N}^d$ and $n \in \mathbf{N}^\star$, define:

$$\begin{aligned} \frac{\mathbf{k}}{n} &= \left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) \;, \\ X_H\left(\frac{\mathbf{k}}{n}\right) &= X_H\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) \;. \end{aligned}$$

The aim of this section is to perform the identification of the fractional index H in a semi-parametric setup from discrete observations of field X_H on $[0,1]^d$: the variance σ and the control measure $\nu(dz)$ of M are therefore unknown. Field X_H is observed at times $\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right)$, $0 \le k_i \le n$, $i = 1, \ldots, d$.

Let $(a_\ell), \ell = 0, \ldots, K$ be a real valued sequence such that:

$$\sum_{\ell=0}^{K} a_\ell = 0 \;, \;\; \sum_{\ell=0}^{K} \ell a_\ell = 0 \;. \tag{38}$$

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbf{N}^d$ define:

$$a_{\mathbf{k}} \;=\; a_{k_1} \ldots a_{k_d} \;.$$

Define the increments of field X_H associated with the sequence a:

$$\begin{aligned} \Delta X_{\mathbf{p}} &= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} X_H\left(\frac{\mathbf{k}+\mathbf{p}}{n}\right) \\ &= \sum_{k_1,\ldots,k_d=0}^{K} a_{k_1} \ldots a_{k_d} X_H\left(\frac{k_1+p_1}{n}, \ldots, \frac{k_d+p_d}{n}\right) \;, \end{aligned}$$

one can take for instance $K = 2$, $a_0 = 1$, $a_1 = -2$, $a_2 = 1$. Define the quadratic variations associated with sequence a:

$$Q_n = \frac{1}{(n-K)^d} \sum_{\mathbf{p}=\mathbf{0}}^{\mathbf{n}-\mathbf{K}} (\Delta X_{\mathbf{p}})^2 .$$

The estimator of fractional index H is:

$$\widetilde{H}_n = \frac{1}{2} \log_2 \frac{Q_{n/2}}{Q_n} .$$

As pointed out in the introduction, this estimator is the estimator used by (Istas and Lang 1997) to estimate the fractional index of a Fractional Brownian Motion.

Theorem 4.1 *As $n \to +\infty$, one has:*

$$\widetilde{H}_n \overset{(\mathbf{P})}{\to} H ,$$

where $\overset{(\mathbf{P})}{\to}$ means a convergence in probability. Moreover, there exists a constant $C > 0$ such that, as $n \to +\infty$, one has:

$$n^d \mathbf{E}(\widetilde{H}_n - H)^2 \to C .$$

Proof :
First define the following constants A and B:

$$A = \sigma^2 + 4\pi \int_{\mathbb{R}+} \rho^2 \nu_\rho(d\rho)$$
$$B = 4\pi \int_{\mathbb{R}+} \rho^4 \nu_\rho(d\rho) .$$

Define the following two functional spaces:

$$\mathcal{F}_2 = \{f \in L^2(\mathbb{R}^d),\ f(-\xi) = \overline{f(\xi)}, \forall \xi \in \mathbb{R}^d\} ,$$
$$\mathcal{F}_4 = \{f \in L^2(\mathbb{R}^d) \cap L^4(\mathbb{R}^d),\ f(-\xi) = \overline{f(\xi)}, \forall \xi \in \mathbb{R}^d\} .$$

According to Proposition 2.1, we then have:

- For all $f_1, f_2 \in \mathcal{F}_2$:

$$\mathbb{E} \int f_1(\xi) L(d\xi) \int f_2(\xi) L(d\xi) = A \int f_1(\xi) f_2(-\xi) d\xi . \quad (39)$$

- For all $f_1, f_2, f_3, f_4 \in \mathcal{F}_4$:

$$
\begin{aligned}
\mathbb{E}\prod_{i=1}^{4}\int f_i(\xi)L(d\xi) \quad = \quad & A^2\left(\int f_1(\xi)f_2(-\xi)d\xi \times \int f_3(\xi)f_4(-\xi)d\xi\right. \\
& + \int f_1(\xi)f_3(-\xi)d\xi \times \int f_2(\xi)f_4(-\xi)d\xi \\
& \left. + \int f_1(\xi)f_4(-\xi)d\xi \times \int f_2(\xi)f_3(-\xi)d\xi\right) \\
& + B\ \left(\int f_1(\xi)f_2(-\xi)f_3(\xi)f_4(-\xi)d\xi\right. \\
& + \int f_1(\xi)f_2(\xi)f_3(-\xi)f_4(-\xi)d\xi \\
& \left. + \int f_1(\xi)f_2(-\xi)f_3(-\xi)f_4(\xi)d\xi\right) . \qquad (40)
\end{aligned}
$$

Define now:

$$V_n \quad = \quad n^{2H}Q_n \ .$$

Expectation of V_n
We deduce from (39):

$$\mathbb{E}(\Delta X_{\mathbf{p}})^2 \quad = \quad A\int_{\mathbb{R}^d}\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\frac{\mathbf{k}}{n}\cdot\xi}\right|^2}{||\xi||^{d+2H}}d\xi \ .$$

The change of variables $\lambda = \dfrac{\xi}{n}$ leads to:

$$\mathbb{E}(\Delta X_{\mathbf{p}})^2 \quad = \quad An^{-2H}\int_{\mathbb{R}^d}\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}}d\lambda \ ,$$

and therefore

$$\mathbb{E}V_n \quad = \quad A\int_{\mathbb{R}^d}\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}}d\lambda \ .$$

Variance of V_n
We deduce from (40):

$$\mathbb{E}\left[(\Delta X_{\mathbf{p}})^2(\Delta X_{\mathbf{p}'})^2\right] \quad = \quad T_1 + T_2 + T_3 + T_4 \ ,$$

with

$$\begin{aligned}
T_1 &= A^2\left(\int_{\mathbb{R}^d}\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{-i\frac{\mathbf{k}}{n}\cdot\xi}\right|^2}{||\xi||^{d+2H}}d\xi\right)^2 ,\\
T_2 &= 2A^2\left(\int_{\mathbb{R}^d} e^{-i\frac{\mathbf{p}-\mathbf{p}'}{n}\cdot\xi}\ \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{-i\frac{\mathbf{k}}{n}\cdot\xi}\right|^2}{||\xi||^{d+2H}}d\xi\right)^2 ,\\
T_3 &= B\int_{\mathbb{R}^d}\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\frac{\mathbf{k}}{n}\cdot\xi}\right|^4}{||\xi||^{2d+4H}}d\xi ,\\
T_4 &= B\int_{\mathbb{R}^d} e^{-i\frac{2\mathbf{p}-2\mathbf{p}'}{n}\cdot\xi}\ \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\frac{\mathbf{k}}{n}\cdot\xi}\right|^4}{||\xi||^{2d+4H}}d\xi
\end{aligned}$$

So that:

$$\mathrm{var} V_n = G_n + NG_n ,$$

with

$$\begin{aligned}
G_n &= 2A^2\frac{n^{4H}}{(n-K)^{2d}}\sum_{\mathbf{p},\mathbf{p}'=0}^{\mathbf{n}-\mathbf{K}}\left(\int_{\mathbb{R}^d} e^{-i\frac{\mathbf{p}-\mathbf{p}'}{n}\cdot\xi}\ \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\frac{\mathbf{k}}{n}\cdot\xi}\right|^2}{||\xi||^{d+2H}}d\xi\right)^2 ,\\
NG_n &= Bn^{4H}\int_{\mathbb{R}^d}\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\frac{\mathbf{k}}{n}\cdot\xi}\right|^4}{||\xi||^{2d+4H}}d\xi\\
&\quad +B\frac{n^{4H}}{(n-K)^{2d}}\sum_{\mathbf{p},\mathbf{p}'=0}^{\mathbf{n}-\mathbf{K}}\int_{\mathbb{R}^d} e^{-i\frac{2\mathbf{p}-2\mathbf{p}'}{n}\cdot\xi}\ \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\frac{\mathbf{k}}{n}\cdot\xi}\right|^4}{||\xi||^{2d+4H}}d\xi
\end{aligned}$$

We first study the part G_n of the variance.

The change of variables $\lambda = \dfrac{\xi}{n}$ leads to:

$$\begin{gathered}
\int_{\mathbb{R}^d} e^{-i\frac{\mathbf{p}-\mathbf{p}'}{n}\cdot\xi}\ \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{-i\frac{\mathbf{k}}{n}\cdot\xi}\right|^2}{||\xi||^{d+2H}}d\xi = \\
n^{-2H}\int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda}\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}}e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}}d\lambda .
\end{gathered} \tag{41}$$

Define the operator $\mathbf{D} = \prod_{j=1}^{d} \frac{\partial}{\partial x_i}$. Let us suppose that $\forall j, \quad p_j \neq p'_j$, integrating by parts leads to :

$$\int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}} d\lambda$$

$$= i^d \prod_{j=1}^{d} \frac{1}{(p_j - p'_j)} \int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \mathbf{D} \left[\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}} \right] d\lambda \ . \qquad (42)$$

The conditions $\sum_{\ell=0}^{K} a_\ell = 0$, $\sum_{\ell=0}^{K} \ell a_\ell = 0$ ensures the convergence of the integral.

Since there exists a constant C_1 such that, as $n \to +\infty$,

$$\left(\frac{1}{n} \sum_{m,m'=0, m\neq m'}^{n-K} \frac{1}{(m-m')^2} \right) \to C_1 \ ,$$

as $n \to +\infty$,

$$n^d G_n \quad \to \quad C_2 \ .$$

We know study the part NG_n of the variance. Using the change of variables $\lambda = \frac{\xi}{n}$, one obtains, for T_3:

$$\frac{1}{n^d} \int_{\mathbb{R}^d} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\lambda}\right|^4}{||\lambda||^{2d+4H}} d\lambda \ .$$

It remains to study the part of NG_n depending of T_4. This part can be written as:

$$\sum_{\mathbf{p},\mathbf{p}'=\mathbf{0}}^{\mathbf{n}-\mathbf{K}} \int_{\mathbb{R}^d} e^{2i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\lambda}\right|^4}{||\lambda||^{2d+4H}} d\lambda \ .$$

Using as previously integration by parts, one proves that this part is negligible with respect to the previous parts.

To sum up, we have proved that there exists $C > 0$ such that:

$$n^d \mathrm{var} V_n \quad \to \quad C \ ,$$

and Theorem 4.1 is proved.

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