Lévy Processes, Pseudo-differential Operators and Dirichlet Forms in the Heisenberg Group

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Abstract

N.Jacob and his colleagues have recently made many interesting investigations of Markov processes in Euclidean space where the infinitesimal generator of the associated semigroup is a pseudo-differential operator in the Kohn-Nirenberg sense. We wish to extend this programme to the Heisenberg group where we can utilise the Weyl calculus to build pseudo-differential operators and we begin by considering Levy processes. We obtain the general form of symbol for infinitesimal generators. We then investigate a natural sub-class of group-valued processes whose components are a Lévy process in phase space and the associated Lévy area process on the real line. These are applied to give a new probabilistic proof for Mehler's formula.

In addition, we describe some properties of the generator in its Schrödinger representation and of the associated Dirichlet form.

1 Introduction

Let $X = (X(t), t \geq 0)$ be a Markov process whose state space is a Lie group G, then we obtain a Markov semigroup $(T(t), t \geq 0)$ on the Banach space $C_b(G)$ of bounded, continuous functions on G, through the prescription

$$T(t)(f)(\sigma) = \mathbf{E}(f(X(t))|X(0) = \sigma)$$

for each $f \in C_b(G), \sigma \in G$, i.e. $(T(t), t \geq 0)$ is a strongly continuous one-parameter contraction semigroup which is positivity preserving and conservative i.e. each $T(t)1 = 1$. In the case where G is $\mathbf{R}^n$, there has recently been much interesting work by N.Jacob and his collaborators in constructing and investigating such processes where the infinitesimal generator $\mathcal{A}$ of the semigroup is a pseudo-differential operator (in the Kohn-Nirenberg sense). More specifically we have, for sufficiently regular functions f in the domain of $\mathcal{A}$ and for each $x \in \mathbf{R}^n$,

$$(\mathcal{A}f)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R^n}} e^{ix.\xi} q(x, \xi) \hat{f}(\xi) d\xi \tag{1}$$

where $\hat{f}$ is the Fourier transform and in a classical paper by Courrège [8], it was shown that a generator of a Feller process has such a representation whenever the test functions are in its domain. The function q is the symbol of the operator $\mathcal{A}$ and a key feature of this approach is to extract information about the probabilistic behaviour of X from the properties of q. The prototype for such studies is the case where X is a Lévy process (i.e. a process with stationary and independent increments). In this case, q is a function of ξ alone and is precisely the characteristic exponent of the process as determined by the celebrated Lévy-Khinchine formula. We refer readers to the monograph [18] and the recent survey article [17] for more details.

In this paper, we are motivated by the prospect of extending this analysis to more general Lie groups. Of course, a major obstacle here is the lack of a suitable Fourier transform, so we need to restrict our endeavours to a context where such objects exist. One possibility is to take the group to be semisimple with finite centre. Here we can employ the spherical transform based on Harish-Chandra's theory of spherical functions. For recent investigations of Lévy processes in this context - see [4]. We take a different approach in this article and consider the case where G is the Heisenberg group $\mathbf{H}^n$. This is interesting for a number of reasons. It's manifold structure is that of a $(2n+1)$-dimensional Euclidean space and from a physical point of view it is useful to think of this as an extension of an even-dimensional phase space comprising n position and n momentum co-ordinates. In particular, the ordinary Fourier transform can now be utilised. As a group it is step 2 nilpotent and so noncommutativity enters quite mildly. Through the Schrödinger representation, this group plays a leading role in quantum mechanics, but it also figures prominently in an number of other branches of mathematics, particularly harmonic analysis (see [14]).

There is a rich theory of pseudo-differential operators in the Heisenberg group based on the so-called the Weyl functional calculus and by analogy with (1) we might aim to study Markov processes whose generators can be so represented as

$$(\mathcal{A}_\pi f)(x) = (2\pi)^{-n} \int_{\mathbf{R^{2n}}} \sigma(\frac{1}{2}(x+y), \xi) e^{i(x-y)\cdot\xi} f(y) dy d\xi \tag{2}$$

We should emphasise that this calculus works through the Schrödinger representation π and so we deal not with the original semigroup generator $\mathcal{A}$, but its image $\mathcal{A}_\pi$ which can be thought of as its "quantisation". Indeed, operators of the form (2) were originally introduced by Hermann Weyl as a means of associating quantum mechanical observables to phase space functions q (see [28], Chapter 4, section D.14). For a more recent and extensive mathematical account of these operators see Chapter 2 of [9].

Our main focus in this paper is on the operators $\mathcal{A}_\pi$, and we seek to understand these from both an analytic and probabilistic viewpoint within the context of Lévy processes in $\mathbf{H}^n$. Indeed, these processes are more complicated than their counterparts in Euclidean space and it is clearly important to learn as much as possible about them before progressing to greater generality. The paper is organised as follows. In section 1, we briefly review the relationship between the Lévy-Khintchine formula and infinitesimal generators in Euclidean space. Section 2 summarises results we need about Lévy processes in Lie groups while section 3 is a primer on the Heisenberg group. Our main results can be found in sections 4 and 5. In section 4, we give the Weyl symbol for operators $\mathcal{A}_\pi$ associated to general Lévy processes in $\mathbf{H}^n$. We then introduce a class of such processes which are called phase-dominated as $\mathcal{A}_\pi$ only acts non-trivially on phase-space co-ordinates. We show that any such process can be written in the form $\rho = (A, X_Q, X_P)$ where $X = (X_Q, X_P)$ is a Lévy process in $\mathbf{R}^{2n}$ and A is the associated stochastic area process. In the case where X is a standard Brownian motion, this leads to a new probabilistic proof of Mehler's formula for the symbol of the semigroup generated by $\mathcal{A}_\pi$. In section 5, we investigate analytic properties of the operator $\mathcal{A}_\pi$. In particular, we show that the smooth functions of compact support form a core for this operator. We also study the associated Dirichlet form, in the symmetric case, and show that $\mathcal{A}_\pi$ also generates a sub-Markov semigroup in the L^2-space.

For related work to this, readers might consult the monograph [21] which mainly studies limit theorems for Brownian motion in $\mathbf{H}^n$, including the central limit theorem and the law of the iterated logarithm. Stable processes in $\mathbf{H}^n$ are investigated in [11], see also [19]. Pap [22] has studied Lévy processes in general nilpotent Lie groups and has obtained a related result to that of our theorem 3 below. In [23], he has made an extensive study of Gaussian measures on $\mathbf{H}^1$. Hoh [13] has recently investigated Markov processes on manifolds whose generators are pseudo-differential operators in the Kohn-Nirenberg sense in local co-ordinates while Baldus [5, 6] has studied a very general framework for Markov processes in manifolds, employing a calculus of pseudo-differential operators due to Hörmander.

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Notation We will use Einstein summation convention throughout and will sometimes be quite cavalier about writing matrix entries c^{ij} as c_{ij}. We emphasise that no metric tensor is involved in this purely notational raising and lowering of indices.

$|.|$ will always denote the Euclidean norm in $\mathbf{R}^d$.

We denote $D = (D_1, \ldots, D_n)$ where each $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index so each $\alpha_j \in \mathbf{N} \cup \{0\}$, we define

$$D^\alpha = \frac{1}{i^{|\alpha|}}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We call n the length of the multi-index and note that multi-indices of the same length can be added component-wise.

We will sometimes find it notationally convenient to write first order partial derivatives $\frac{\partial}{\partial x_j}$ simply as ∂_j.

Throughout this paper all stochastic processes will be defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.

All function spaces will consist of complex-valued functions unless otherwise indicated. In particular, $C_c^\infty(\mathbf{R}^n)$ comprises the smooth functions of compact support on $\mathbf{R}^n$ and $S(\mathbf{R}^n)$ is the Schwartz space of rapidly decreasing functions on $\mathbf{R}^n$ so that $f \in S(\mathbf{R}^n)$ if f is smooth and $\sup_{x \in \mathbf{R}^n} |x|^\beta |D^\alpha f(x)| < \infty$ for all $\beta \in \mathbf{N} \cup \{0\}$ and all multi-indices α.

2 Review of the Euclidean Case

Let $Y = (Y(t), t \geq 0)$ be a Lévy process in $\mathbf{R}^n$ then we have the Lévy-Khintchine formula

$$\mathbf{E}(e^{iu.Y(t)}) = e^{t\varphi(u)}$$

for all $u \in \mathbf{R}^n, t \geq 0$ where

$$\varphi(u) = im.u - \frac{1}{2}u.au + \int_{\mathbf{R}^n - \{0\}} \left(e^{iu.y} - 1 - i\frac{u.y}{1+|y|^2} \right) \nu(dy) \tag{3}$$

and $m \in \mathbf{R}^n$, a is a non-negative symmetric $n \times n$ matrix and ν is a Lévy measure on $\mathbf{R}^n - \{0\}$ i.e. $\int_{\mathbf{R}^n - \{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty$ (see e.g. [26].

Let $C_0(\mathbf{R}^n)$ denote the Banach space (when equipped with the supremum norm) of continuous functions on $\mathbf{R}^n$ which vanish at infinity. We obtain a Feller semigroup $(T(t), t \geq 0)$ on $C_0(\mathbf{R}^n)$ by the prescription

$$(T(t)f)(x) = \mathbf{E}(f(x + Y(t)))$$

for all $f \in C_0(\mathbf{R}^n), x \in \mathbf{R}^n, t \geq 0$. We denote the infinitesimal generator of the semigroup as $\mathcal{A}$. The following result is classical but we give a short proof for completeness.

Proposition 1 *1. $\mathcal{A}$ is a pseudo-differential operator of the form*

$$\mathcal{A} = \varphi(D)$$

2. For all $f \in S(\mathbf{R}^n), x \in \mathbf{R}^n$,

$$\begin{aligned} \mathcal{A}(f)(x) &= i(m.D)f(x) - \frac{1}{2}a^{ij}D_iD_jf(x) + \\ &+ \int_{\mathbf{R}^n - \{0\}} \left(f(x+y) - f(x) - i\frac{(y.D)f(x)}{1+|y|^2} \right) \nu(dy) \end{aligned}$$

Proof If $f \in S(\mathbf{R}^n)$ we denote its Fourier transform by $\hat{f}$ so that for each $u \in \mathbf{R}^n, \hat{f}(u) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-iu.x} f(x)dx$. We note that the mapping $f \to \hat{f}$ is a continuous linear bijection from $S(\mathbf{R}^n)$ to itself and Fourier inversion yields

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x} \hat{f}(u)du$$

then by Fubini's theorem and the Lévy-Khintchine formula we find that for all $x \in \mathbf{R}^n, t \geq 0$,

$$\begin{aligned} (T(t)f)(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x} \mathbf{E}(e^{i(u,Y(t))}) \hat{f}(u)du \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x} e^{t\varphi(u)} \hat{f}(u)du \end{aligned}$$

Hence by dominated convergence we have $S(\mathbf{R}^n) \subset \mathrm{Dom}(\mathcal{A})$ and

$$\mathcal{A}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{iu.x} \varphi(u) \hat{f}(u)du$$

so that (1) is established.(2) then follows by standard use of the Fourier transform. $\square$

Of course we can go much further in characterising the domain of $\mathcal{A}$. It is shown in Sato [26], pp. 208-11 that the twice continuously differentiable functions in $C_0(\mathbf{R}^n)$ are in the domain and that $C_c^\infty(\mathbf{R}^n)$ is a core. If we extend the semigroup to act in $L^2(\mathbf{R}^n)$, the entire domain forms a non-isotropic Sobolev space whose structure is determined by the symbol φ (see e.g. [18] p.49-50).

3 Lévy Processes in Lie Groups

Let G be a Lie group with Lie algebra $\mathbf{g}$. A Lévy process in G is a G-valued stochastic process $\rho = (\rho(t), t \geq 0)$ which satisfies the following

1. ρ has stationary and independent left increments, where the increment between s and t with $s \leq t$ is $\rho(s)^{-1}\rho(t)$.
2. $\rho(0) = e$ (a.s.)
3. ρ is stochastically continuous i.e.

$$\lim_{s \to t} P(\rho(s)^{-1}\rho(t) \in A) = 0$$

for all $A \in \mathcal{B}(G)$ with $e \notin \bar{A}$

Let $C_0(G)$ be the Banach space (with respect to the supremum norm) of functions on G which vanish at infinity. Just as in the Euclidean case, we obtain a Feller semigroup $(T(t), t \geq 0)$ on $C_0(G)$ by the prescription

$$T(t)f(\tau) = \mathbf{E}(f(\tau\rho(t)))$$

for each $t \geq 0, \tau \in G, f \in C_0(G)$ and its infinitesimal generator will be denoted as $\mathcal{L}$.

We fix a basis $\{Z_1, \ldots, Z_n\}$ for $\mathbf{g}$ and define a dense subspace $C_2(G)$ of $C_0(G)$ as follows:-

$$C_2(G) = \{f \in C_0(G); Z_i^L(f) \in C_0(G) \text{ and } Z_i^L Z_j^L(f) \in C_0(G) \text{ for all } 1 \leq i, j \leq n\}$$

where Z^L denotes the left invariant vector field associated to $Z \in \mathbf{g}$ by differential left translation.

In [16], Hunt proved that there exist functions $y_i \in C_2(G), 1 \leq i \leq n$ so that each

$$y_i(e) = 0 \text{ and } Z_i^L y_j(e) = \delta_{ij}$$

and a map $h \in \text{Dom}(\mathcal{L})$ which is such that

1. $h > 0$ on $G - \{e\}$.
2. There exists a compact neighborhood of the identity V such that for all $\tau \in V$,

$$h(\tau) = \sum_{i=1}^n y_i(\tau)^2$$

Any such function is called a *Hunt function* in G.

A positive measure ν defined on $\mathcal{B}(G-\{e\})$ is called a *Lévy measure* whenever

$$\int_{G-\{e\}} h(\sigma)\nu(d\sigma) < \infty$$

for some Hunt function h.

We are now ready to state the main result of [16].

Theorem 1 (Hunt's theorem) *Let ρ be a Lévy process in G with infinitesimal generator $\mathcal{L}$ then*

1. $C_2(G) \subseteq Dom(\mathcal{L})$
2. *For each $\tau \in G, f \in C_2(G)$*

$$\mathcal{L}(f)(\tau) = b^i Z_i^L f(\tau) + c^{ij} Z_i^L Z_j^L f(\tau) + \int_{G-\{e\}} (f(\tau\sigma) - f(\tau) - y^i(\sigma) Z_i^L f(\tau))\nu(d\sigma) \tag{4}$$

where $b = (b^1, \ldots b^n) \in \mathbf{R}^n, c = (c^{ij})$ is a non-negative-definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

Furthermore, any linear operator with a representation as in (4) is the restriction to $C_2(G)$ of the infinitesimal generator of a unique (up to modification) Lévy process.

Several obscure features of Hunt's paper were later clarified by Ramaswami in [25] and then incorporated into the seminal treatise of [12]. For a survey of this and related ideas see [2].

Now let h be a complex, separable Hilbert space and $U(h)$ be the group of all unitary operators in h. Let $\pi : G \to U(h)$ be a strongly continuous, unitary representation of G in h and let $C^\infty(\pi) = \{\psi \in h; g \to \pi(g)\psi \text{ is } C^\infty\}$ be the dense linear space of *smooth vectors* for π in h. We define a strongly continuous contraction semigroup $(\mathcal{T}_t^\pi, t \geq 0)$ of linear operators on h by

$$\mathcal{T}_t^\pi = \mathbf{E}(\pi(\rho(t)))$$

Alternatively if we fix $\psi_1, \psi_2 \in h$ and define $f \in C_b(G)$ by $f(\sigma) =< \psi_1, \pi(\sigma)\psi_2 >$ where $\sigma \in G$ then we have

$$(T(t)f)(e) =< \psi_1, \mathcal{T}_t^\pi \psi_2 >$$

Let $\mathcal{L}^\pi$ denote the infinitesimal generator of this semigroup. It follows from the arguments of [3], that $C^\infty(\pi) \subseteq \text{Dom}(\mathcal{L}^\pi)$ and for all $\psi \in C^\infty(\pi)$ we have

$$\begin{aligned} \mathcal{L}^\pi \psi &= b^i d\pi(Z_i)\psi + c^{ij} d\pi(Z_i) d\pi(Z_j)\psi + \\ &\quad + \int_{G-\{e\}} (\pi(\sigma) - I - y^i(\sigma) d\pi(Z_i))\psi\nu(d\sigma) \end{aligned} \tag{5}$$

4 The Heisenberg Group

This section is based on Chapter 2 of [27] (see also the monograph [9]). The *Heisenberg group* $\mathbf{H}^n$ is a Lie group with underlying manifold $\mathbf{R}^{n+1}$ equipped with the composition law

$$(a_1, q_1, p_1)(a_2, q_2, p_2) = (a_1 + a_2 + \frac{1}{2}(p_1.q_2 - q_1.p_2), q_1 + q_2, p_1 + p_2)$$

where each $a_i \in \mathbf{R}, q_i, p_i \in \mathbf{R}^n (i = 1, 2)$.

A basis for the Lie algebra of left-invariant vector fields is $\{T, L_1, \ldots, L_n, M_1, \ldots, M_n\}$ where for $1 \leq j \leq n$,

$$T = \frac{\partial}{\partial t}, L_j = \frac{\partial}{\partial q_j} + \frac{1}{2} p_j \frac{\partial}{\partial t}, M_j = \frac{\partial}{\partial p_j} - \frac{1}{2} q_j \frac{\partial}{\partial t}$$

and we have the commutation relations

$$[L_j, L_k] = [M_j, M_k] = [M_j, T] = [L_j, T] = 0, [M_j, L_k] = \delta_{jk} T$$

for $1 \leq j, k \leq n$ so that $\mathbf{H}^n$ is step-2 nilpotent.

By the Stone-von Neumann uniqueness theorem, all irreducible representations of $\mathbf{H}^n$ are either one-dimensional or are unitarily equivalent to the *Schrödinger representation* in $L^2(\mathbf{R}^n)$ which is given by

$$\pi(a, q, p) = e^{i(aI + q.X + p.D)}$$

where $X = (X_1, \ldots, X_n)$ and each $X_i u(x) = x_i u(x)$ for $u \in S(\mathbf{R}^n)$. In fact the linear operator $p.D + q.X$ is essentially self-adjoint on $S(\mathbf{R}^n)$. A basis for the representation of the Lie algebra is

$$d\pi(T) = iI, d\pi(L_j) = iX_j, d\pi(M_j) = iD_j$$

for $1 \leq j \leq n$, where I is the identity operator.

The Schrödinger representation allows us to define an interesting class of pseudo-differential operators using the *Weyl functional calculus.* Indeed let $\sigma \in S'(\mathbf{R}^{2n})$ then we may define

$$\sigma(X, D) = (2\pi)^{-n} \int_{\mathbf{R^{2n}}} \hat{\sigma}(q, p) e^{i(q.X + p.D)} dq dp$$

and $\sigma(X, D)$ is a continuous linear operator from $S(\mathbf{R}^n)$ to $S'(\mathbf{R}^n)$.

Moreover, we have the following useful alternative form

$$(\sigma(X, D)f)(x) = (2\pi)^{-n} \int_{\mathbf{R^{2n}}} \sigma(\frac{1}{2}(x + y), \xi) e^{i(x-y).\xi} f(y) dy d\xi \tag{6}$$

for each $f \in S(\mathbf{R}^n), x \in \mathbf{R}^n$.

In particular we have the classical symbol class - if $\sigma \in C^\infty(\mathbf{R}^{2n})$ and for all multi-indices, α, β we can find $C_{\alpha,\beta} \geq 0, K \in \mathbf{R}$ and $\delta < \frac{1}{2}$ such that

$$|D_x^\alpha D_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|^2)^{K + \delta(|\alpha| + |\beta|)}$$

then $\sigma(X, D)$ maps $S(\mathbf{R}^n)$ continuously into itself and extends to a continuous operator on $S'(\mathbf{R}^n)$.

Two straightforward examples which we will find useful below are:

1. If $\sigma(x,\xi) = e^{i(q.x+p.\xi)}$ then $\sigma(X,D) = \pi(0,q,p)$
2. If $\sigma(x,\xi) = |x|^2 + |\xi|^2$ then $\sigma(X,D) = H_{osc} = -\Delta + |X|^2$

The sub-Laplacian H_{osc} is of course the well-known harmonic oscillator Hamiltonian of elementary quantum mechanics.

5 Lévy Processes in the Heisenberg Group

Let ρ be a Lévy process in $\mathbf{H}^n$ and consider the unitary operator valued process $\pi(\rho)$ where π is the Schrödinger representation. We compute the form of the generator (5) on the domain $C^\infty(\pi) = S(\mathbf{R}^n)$. We will find it simplifies matters if we write the vector $b = (b_0, b^1, b^2)$, where $b_0 \in \mathbf{R}$ and $b^i \in \mathbf{R}^n, i = 1,2$. We also write the non-negative definite matrix $C = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{01} & C_1 & E \\ c_{02} & E & C_2 \end{pmatrix}$ where for $i = 1,2, c_{00} \geq 0, c_{0i} \in \mathbf{R}^n$ and C_i, E are symmetric $n \times n$ matrices

Proposition 2

$$\begin{aligned} \mathcal{L}^\pi &= i\left(b_0 + \sum_{j=1}^n E_{jj}\right) I + (ib_j^1 - 2c_j^{01})X^j + (ib_j^2 - 2c_j^{02})D^j \qquad (7) \\ &- c_{00}I - c_{jk}^1 X^j X^k - c_{jk}^2 D^j D^k - 2E_{jk}X^j D^k \\ &+ \int_{\mathbf{R}^{2n+1}-\{0\}} \left(e^{i(aI+q.X+p.D)} - I - \frac{i(aI+q.X+p.D)}{1+|a|^2+|q|^2+|p|^2}\right)\nu(da,dq,dp) \end{aligned}$$

Proof This follows directly from (5). The most interesting part is the quadratic term which we write as

$$c^{ij} d\pi(Z_i) d\pi(Z_j) = (iI \ iX \ iD)\begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{01} & C_1 & E \\ c_{02} & E & C_2 \end{pmatrix}\begin{pmatrix} iI \\ iX \\ iD \end{pmatrix}$$

Using the fact that E is symmetric we obtain for each $\psi \in S(\mathbf{R}^n)$

$$\begin{aligned} (E^{jk}X_jD_k + E^{kj}D_kX_j)\psi &= E^{jk}(X_jD_k + D_kX_j)\psi \\ &= 2E^{jk}X_jD_k\psi - i\sum_{j=1}^n E_{jj}\psi \end{aligned}$$

□

The following is a fairly straightforward computation using the Weyl calculus

Proposition 3 *$\mathcal{L}^\pi$ is a pseudo-differential operator with symbol*

$$\begin{aligned} \sigma_{\mathcal{L}^\pi}(x,\xi) &= i\left(b_0 + \sum_{j=1}^n E_{jj}\right) + (ib_j^1 - 2c_j^{01})x^j + (ib_j^2 - 2c_j^{02})\xi^j \\ &- c_{00} - c_{jk}^1 x^j x^k - c_{jk}^2 \xi^j \xi^k - 2E_{jk}x^j\xi^k \\ &+ \int_{\mathbf{R}^{2n+1}-\{0\}} \left(e^{i(a+q.x+p.\xi)} - 1 - \frac{i(a+q.x+p.\xi)}{1+|a|^2+|q|^2+|p|^2}\right)\nu(da,dq,dp) \end{aligned}$$

where $x, \xi \in \mathbf{R}^n$.

Now let each of the vectors $c_{0i} = 0$ for $i = 1, 2$ and define a Lévy process Y in $\mathbf{R}^{2n+1}$ with characteristics $(m, 2C, \nu)$ where C and ν are as above and $m_0 = b_0 + \sum_{j=1}^{n} E_{jj}, m_j = b_j^1, m_{j+n} = b_j^2$ for $1 \leq j \leq n$, then we can as in Proposition 1 associate a symbol φ_Y to the generator of Y via the Kohn-Nirenberg calculus and the Lévy-Khintchine formula.

Using propositions 1 and 2 we have the following correspondence between the processes ρ and Y at the level of their symbols.

Theorem 2

$$\varphi_Y(1, u) = \sigma_{\mathcal{L}^\pi}(x, \xi)$$

where each $u = (x, \xi) \in \mathbf{R}^{2n}$

It is interesting to contemplate theorem 2 from the following viewpoint. Suppose that we were only given the symbol as data from which we wish to construct a process, then we can build either the generator of a Lévy process Y in $\mathbf{R}^{2n+1}$ via the Kohn-Nirenberg calculus or of ρ on $\mathbf{H}^n$ by means of the Weyl calculus. If we know the diffusion matrix C a priori, we could argue that the non-commutativity of $\mathbf{H}^n$ can be detected in either context by the presence of its off-diagonal elements in the drift. We now introduce a class of Lévy processes in $\mathbf{H}^n$ for which this method of detecting non-commutativity no longer works.

We say that ρ is *phase-dominated* if $E = 0$, each $c_{0i} = 0 (i = 1, 2), b_0 = c_{00} = 0$ and ν has support in $(\{0\} \times \mathbf{R}^{2n}) - \{0\}$ (which we can identify with $\mathbf{R}^{2n} - \{0\}$).

We employ the phrase "phase-dominated" to indicate that the characteristics of ρ only operate on 'phase-space co-ordinates'.

We investigate phase-dominated processes more extensively. To this end we introduce a Lévy process $\Upsilon = (\Upsilon(t), t \geq 0)$ in $\mathbf{R}^{2n}$. It has the following Lévy-Itô decomposition (see e.g. [26], Chapter 4).

$$\Upsilon^j(t) = \alpha^j t + \tau_i^j B^i(t) + \int_0^{t+} \int_{|x|<1} x^j \tilde{N}(ds, dx) + \int_0^{t+} \int_{|x|\geq 1} x^j N(ds, dx)$$

for each $1 \leq j \leq 2n, t \geq 0$, where $\alpha = (\alpha^1, \ldots, \alpha^{2n}) \in \mathbf{R}^{2n}, (\tau_i^j)$ is a $2n \times r$ matrix, $B = (B^1, \ldots, B^r)$ is standard Brownian motion in $\mathbf{R}^r, N$ is a Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R}^{2n} - \{0\})$ which is independent of B and has intensity measure η and $\tilde{N}$ is the associated compensator so $\tilde{N}(t, A) = N(t, A) - t\eta(A)$ for each $t \geq 0, A \in \mathcal{B}(\mathbf{R}^{2n} - \{0\})$. We will find it convenient to define

$$\Upsilon_Q(t) = (\Upsilon^1(t), \ldots, \Upsilon^n(t)), \quad \Upsilon_P(t) = (\Upsilon^{n+1}(t), \ldots, \Upsilon^{2n}(t))$$

and we introduce the Lévy stochastic area of Υ_Q and Υ_P to be the process $(A(t), t \geq 0)$ comprising Itô stochastic integrals which is defined by

$$\begin{aligned} A(t) &= \frac{1}{2} \sum_{j=1}^{n} \int_0^t (\Upsilon_P^j(s) d\Upsilon_Q^j(s) - \Upsilon_Q^j(s) d\Upsilon_P^j(s)) \\ &= \frac{1}{2} \sum_{j=1}^{n} \int_0^t (\Upsilon^{j+n}(s) d\Upsilon^j(s) - \Upsilon^j(s) d\Upsilon^{j+n}(s)) \end{aligned}$$

We say that the Lévy process Υ *diffuses phasewise* if $\tau = \begin{pmatrix} \tau_Q & 0 \\ 0 & \tau_P \end{pmatrix}$ where τ_Q and τ_P are $(n \times r_i)$ matrices for $i = 1, 2$ with $r_1 + r_2 = r$. We write $\gamma_Q = \tau_Q \tau_Q^T$ and $\gamma_P = \tau_P \tau_P^T$

Theorem 3 *A Lévy process $\rho = (\rho(t), t \geq 0)$ in $\mathbf{H}^n$ is phase-dominated if and only if there exists a Lévy process Υ in phase space $\mathbf{R}^{2n}$ which diffuses phasewise for which*

$$\rho(t) = (A(t), \Upsilon_Q(t), \Upsilon_P(t)) \quad a.s. \tag{8}$$

for each $t \geq 0$. The infinitesimal generator of ρ is given by

$$\begin{aligned}(\mathcal{L}f)(\sigma) &= \alpha^j(L_j f)(\sigma) + \alpha^{j+n}(M_j f)(\sigma) + \frac{1}{2}\gamma_Q^{ij}(L_i L_j f)(\sigma) + \frac{1}{2}\gamma_P^{ij}(M_i M_j f)(\sigma) \\ &+ \int_{\mathbf{R}^{2n}-\{0\}}(f(\sigma.(0, x_Q, x_P)) - f(\sigma) - [x^j(L_j f)(\sigma) - x^{j+n}(M_j f)(\sigma)]1_{|x|<1})\eta(dx)\end{aligned} \tag{9}$$

for each $f \in C^2(\mathbf{H}^n), \sigma \in \mathbf{H}^n$, where . is the composition law in $\mathbf{H}^n, x_Q = (x^1, \ldots, x^n)$ and $x_P = (x^{n+1}, \ldots, x^{2n})$.

Proof To ease the notation, we will take $n = 1$. There is also no great loss of generality in taking $\alpha^1 = \alpha^2 = 0, \tau_1^1 = \tau_2^2 = 1$ and assuming that η has support in $\{|x| \in \mathbf{R}^{2n}, |x| < 1, x \neq 0\}$. We then have the following

$$\Upsilon_Q(t) = B^1(t) + \int_0^{t+}\int_{|x|<1} x^1 \tilde{N}(ds, dx), \quad \Upsilon_P(t) = B^2(t) + \int_0^{t+}\int_{|x|<1} x^2 \tilde{N}(ds, dx)$$

Now assume that ρ is defined as in (8).

By Itô's formula for semimartingales with jumps (see e.g. [24], Chapter 2) we obtain for each $f \in C^2(\mathbf{R}^3)$,

$$\begin{aligned}f(\rho(t)) &= f(0) + \int_0^t (\partial_a f)(\rho(s-))dA(s) + \int_0^t (\partial_q f)(\rho(s-))d\Upsilon_Q(s) \\ &+ \int_0^t (\partial_p f)(\rho(s-))d\Upsilon_P(s) + \int_0^t (\partial_a \partial_q f)(\rho(s-))d[[A, \Upsilon_Q]]_c(s) \\ &+ \int_0^t (\partial_a \partial_p f)(\rho(s-))d[[A, \Upsilon_P]]_c(s) + \int_0^t (\partial_q \partial_p f)(\rho(s-))d[[\Upsilon_Q, \Upsilon_P]]_c(s) \\ &+ \frac{1}{2}\int_0^t (\partial_a^2 f)(\rho(s-))d[[A, A]]_c(s) + \frac{1}{2}\int_0^t (\partial_q^2 f)(\rho(s-))d[[\Upsilon_Q, \Upsilon_Q]]_c(s) \\ &+ \frac{1}{2}\int_0^t (\partial_p^2 f)(\rho(s-))d[[\Upsilon_P, \Upsilon_P]]_c(s) + \sum_{0 \leq s \leq t}[f(\rho(s-) + \Delta\rho(s)) - f(\rho(s-)) \\ &- (\partial_a f)(\rho(s-))\Delta A(s) - (\partial_q f)(\rho(s-))\Delta\Upsilon_Q(s) - (\partial_p f)(\rho(s-))\Delta\Upsilon_P(s)]\end{aligned}$$

where $[[.,.]]_c$ denotes the continuous part of quadratic variation (see e.g. [24] p.62). The following are then easily deduced from the quadratic variation of Brownian motion.

$$d[[\Upsilon_Q, \Upsilon_P]]_c(t) = 0, \quad d[[\Upsilon_Q, \Upsilon_Q]]_c(t) = d[[\Upsilon_P, \Upsilon_P]]_c(t) = dt$$

$$d[[\Upsilon_Q, A]]_c(t) = \frac{1}{2}\Upsilon_P(t)dt, \quad d[[\Upsilon_P, A]]_c(t) = -\frac{1}{2}\Upsilon_Q(t)dt$$

$$d[[A, A]]_c(t) = \frac{1}{4}((\Upsilon_Q(t))^2 + (\Upsilon_P(t))^2)$$

We can now simplify the expression for $f(\rho(t))$ by incorporating these results, the Lévy-Itô decompositions of Υ_Q and Υ_P and by introducing the Lie algebra basis $T = \partial_a, L = \partial_q + \frac{1}{2}pT$ and $M = \partial_p - \frac{1}{2}qT$. We then obtain

$$\begin{aligned}
f(\rho(t)) &= f(0) + \int_0^t (Lf)(\rho(s-))dB^1(s) + \int_0^t (Mf)(\rho(s-))dB^2(s) \\
&+ \frac{1}{2}\int_0^t (L^2 f)(\rho(s-))ds + \frac{1}{2}\int_0^t (M^2 f)(\rho(s-))ds \\
&+ \int_0^{t+}\int_{|x|<1} \{f(A(s-) + \frac{1}{2}(\Upsilon_P(s-)x^1 - \Upsilon_Q(s-)x^2), \Upsilon_Q(s-) + x^1, \Upsilon_P(s-) + x^2) \\
&- f(\rho(s-))\}\tilde{N}(ds, dx) \\
&+ \int_0^{t+}\int_{|x|<1} \{f(A(s-) + \frac{1}{2}(\Upsilon_P(s-)x^1 - \Upsilon_Q(s-)x^2), \Upsilon_Q(s-) + x^1, \Upsilon_P(s-) + x^2) \\
&- f(\rho(s-)) - \frac{1}{2}(\Upsilon_P(s-)x^1 - \Upsilon_Q(s-)x^2)(\partial_a f)(\rho(s-)) - x^1(\partial_q f)(\rho(s-)) \\
&- x^2(\partial_p f)(\rho(s-))\}ds\eta(dx) \\
&= f(0) + \int_0^t (Lf)(\rho(s-))dB^1(s) + \int_0^t (Mf)(\rho(s-))dB^2(s) \\
&+ \int_0^{t+}\int_{|x|<1} [f(\rho(s-).x) - f(\rho(s-))]\tilde{N}(ds, dx) + \int_0^t \mathcal{L}(f)(\rho(s-))ds
\end{aligned}$$

where for each $\sigma \in \mathbf{H}^1$,

$$\begin{aligned}
\mathcal{L}(f)(\sigma) &= \frac{1}{2}(L^2 f)(\sigma) + \frac{1}{2}(M^2 f)(\sigma) \\
&+ \int_{|x|<1} (f(\sigma.x) - f(\sigma) - x^1(Lf)(\sigma) - x^2(Mf)(\sigma))\eta(dx)
\end{aligned}$$

Sufficiency now follows immediately from theorem 3.1 of [1]. For necessity, let ρ be an arbitrary phase-dominated Lévy process in $\mathbf{H}^n$, then it has an infinitesimal generator of the form (9) and hence we can define a Lévy process in $\mathbf{R}^{2n}$ with characteristics (α, τ, η) which diffuses phasewise. The required result then follows by reversing the steps in the argument given above and appealing to the uniqueness of solutions to stochastic differential equations of this type (see e.g. [1], [22]). □

Note 1. For readers with a background in stochastic differential equations, we remark that since the exponential map from $\mathbf{R}^{2n+1}$ to $\mathbf{H}^n$ is surjective, the equation for $f \circ \rho$ which appears in the proof of theorem 3 is equivalent to

$$d\rho(t) = \sigma^i_j Z^j(\rho(t-)) \diamond d\Upsilon_i(t) \tag{10}$$

with initial condition $\rho(0) = e$ (a.s.), where $\diamond$ denotes the Marcus canonical form (see the appendix to [2] and references therein). Here $\sigma^i_j = \text{diag}(0, 1, \ldots, 1), Z^i(i = 0, 1, \ldots, 2n)$ are the usual Lie algebra generators and $\Upsilon = (\Upsilon_0, \Upsilon_Q, \Upsilon_P)$ is a Lévy process on $\mathbf{R}^{2n+1}$ (note the component Υ_0 is included simply for "accounting purposes" and plays no role in determining the process ρ).

2. Using different methods to ours, Pap [22] has shown that an arbitrary Lévy process in $\mathbf{H}^1$ can be written as $\rho(t) = (\Upsilon_0(t) + A(t), \Upsilon_Q(t), \Upsilon_P(t))$ for each $t \geq 0$,

where Υ and A are as above. The generalisation to $\mathbf{H}^n$ is clear (albeit notationally messy) using either the approach given above or that of [22].

Example (Phase-dominated Brownian motion)

Let $\rho(t) = (2A(t), \sqrt{2}B_Q(t), \sqrt{2}B_P(t))$ where $B = (B_0, B_Q, B_P)$ is a standard Brownian motion in $\mathbf{R}^{2n+1}$ and $2A(t) = \sum_{j=1}^n \int_0^t (B_P^j(s)dB_Q^j(s) - B_Q^j(s)dB_P^j(s)$, for each $t \geq 0$. In this case, the $\diamond$ in equation (10) is the Stratonovitch differential and $\Upsilon = B$. We note that $\mathcal{L}_\pi = -H_{osc}$ which has symbol $-(|x|^2 + |\xi|^2)$. This example also appears in [29] as a special case of a prescription for solutions of general SDEs driven by Brownian motion on nilpotent Lie groups. A related process is studied in [15].

Even in the case of phase-dominated Lévy processes, we can detect the non-commutativity of the group by examining the passage from generator to semigroup. We begin by recalling the situation in $\mathbf{R}^n$. If Y is a Lévy process then its generator is a pseudo-differential operator with symbol φ and the semigroup $(T(t), t \geq 0)$ again consists of pseudo-differential operators with the symbol of each $T(t)$ being $e^{t\varphi}$, indeed this was established in the proof of Proposition 1.

Now consider the case of $\mathbf{H}^n$ and take ρ to be phase-dominated Brownian motion so that $\mathcal{L}_\pi$ has symbol $-(|x|^2 + |\xi|^2)$. Here the symbol of the semigroup $(\mathcal{T}_t^\pi, t \geq 0)$ is not a straightforward exponential, as we will now demonstrate. The following result is very well-known and three different proofs can be found in [27] pp 68-76. As far as we know, the probabilistic proof that we present here is new.

Theorem 4

$$\sigma_{\mathcal{T}_t^\pi}(x, \xi) = (\cosh(t))^{-n} e^{-\tanh(t)(|x|^2 + |\xi|^2)}$$

Proof Suppose that $n = 1$ and observe that for each $t \geq 0$, by theorem 3,

$$\pi(\rho(t)) = e^{i(2A(t) + \sqrt{2}B^1(t)X + \sqrt{2}B^2(t)D)}$$

and so

$$\mathcal{T}_t^\pi = \int_\Omega e^{i(2A(t)(\omega) + \sqrt{2}B^1(t)(\omega)X + \sqrt{2}B^2(t)(\omega)D)} dP(\omega)$$

By a straightforward application of Fubini's theorem in (6), we see that

$$\sigma_{\mathcal{T}_t^\pi}(x, \xi) = \int_\Omega e^{i(2A(t)(\omega) + \sqrt{2}B^1(t)(\omega)x + \sqrt{2}B^2(t)(\omega)\xi)} dP(\omega)$$

Now we use a classical formula for the Lévy area (see [30], equation (2.7), p.19)

$$\mathbf{E}(\exp(i2A(t))|B^1(t) = a, B^2(t) = b) = \left(\frac{t}{\sinh(t)}\right) \exp\left(\frac{(a^2 + b^2)(1 - t\coth(t))}{2t}\right)$$

Hence, on carrying out a standard Gaussian integral, we find

$$\begin{aligned}
\sigma_{\mathcal{T}_t^\pi}(x, \xi) &= \int_{\mathbf{R^2}} e^{i(\sqrt{2}ax + \sqrt{2}b\xi)} \mathbf{E}(\exp(i2A(t))|B^1(t) = a, B^2(t) = b) \frac{e^{\frac{-(a^2+b^2)}{2t}}}{2\pi t} dadb \\
&= \frac{1}{\sinh(t)} \int_{\mathbf{R^2}} e^{i(\sqrt{2}ax + \sqrt{2}b\xi)} \exp\left(\frac{-(a^2 + b^2)\coth(t)}{2}\right) \frac{dadb}{2\pi} \\
&= \frac{1}{\cosh(t)} \exp\left(-\tanh(t)(x^2 + \xi^2)\right).
\end{aligned}$$

The result for $n > 1$ follows by a straightforward independence argument.

$\square$.

Note In related work to this, Pap [23] has recently found a probabilistic interpretation of the formula for the fundamental solution of $\frac{\partial u}{\partial t} = H_{osc}u$. As is shown in [27], this is closely related to Mehler's formula.

6 Properties of the Operator $\mathcal{L}^\pi$

In this section we will make frequent use of the projection-valued measure $P_{a,q,p}$ in $L^2(\mathbf{R}^n)$ associated to the spectral decomposition of the self-adjoint operator $\overline{aI + q.X + p.D}$, where $(a, q, p) \in \mathbf{R}^{2n+1}$.

We know that $\mathcal{L}^\pi$ is a densely defined, closed linear operator in $L^2(\mathbf{R}^n)$. Our first task is to show that C_c^∞ is a core for $\mathcal{L}^\pi$. The key is the following technical result.

Proposition 4 *There exists $C > 0$ such that for all $f \in C_c^\infty(\mathbf{R}^n)$,*

$$||\mathcal{L}^\pi f|| \leq C\left[||f|| + \sum_{j=1}^{n}(||X_j f|| + ||D_j f||) + \sum_{j,k=1}^{n}(||X^j X^k f|| + ||X^j D^k f|| + ||D^j D^k f||)\right] \tag{11}$$

Proof Using the defining property of the Lévy measure ν, we can rewrite $\mathcal{L}^\pi$ (as given by (7)) as

$$\mathcal{L}^\pi = \mathcal{L}_1^\pi + \mathcal{L}_2^\pi + \mathcal{L}_3^\pi$$

where for some $\alpha, \beta_j, \gamma_j \in \mathbf{C}, 1 \leq j \leq n$,

$$\mathcal{L}_1^\pi = \alpha I + \beta_j X^j + \gamma_j D^j - c_{00}I - c_{jk}^1 X^j X^k - c_{jk}^2 D^j D^k - 2E_{jk}X^j D^k,$$

$$\mathcal{L}_2^\pi = \int_B (e^{i(aI+q.X+p.D)} - I - i(aI + q.X + p.D))\nu(da, dq, dp),$$

$$\mathcal{L}_3^\pi = \int_{B^c} (e^{i(aI+q.X+p.D)} - I)\nu(da, dq, dp),$$

and where $B = \{y \in \mathbf{R}^{2n+1}, ||y|| \leq 1\}$. Now for each $f \in C_c^\infty$, we find that

$$\begin{aligned} ||\mathcal{L}_3^\pi f|| &\leq \int_{B^c} ||(e^{i(aI+q.X+p.D)} - I)f||\nu(da, dq, dp) \\ &\leq 2\nu(B^c)||f|| \qquad \ldots(i) \end{aligned}$$

By the spectral theorem and Taylor's theorem, we see that for each $(a, q, p) \in \mathbf{R}^{2n+1}$,

$$\begin{aligned} ||[e^{i(aI+q.X+p.D)} - I - i(aI + q.X + p.D)]f||^2 &= \int_{\mathbf{R}^{2n+1}} |e^{i\lambda} - 1 - i\lambda|^2 ||P_{a,q,p}(d\lambda)f||^2 \\ &\leq \frac{1}{4}\int_{\mathbf{R}^{2n+1}} |\lambda|^4 ||P_{a,q,p}(d\lambda)f||^2 \\ &= \frac{1}{4}||(aI + q.X + p.D)^2 f||^2 \end{aligned}$$

We thus obtain

$$\begin{aligned}
||\mathcal{L}_2^\pi f|| &\leq \frac{1}{2}\int_B ||(aI + q.X + p.D)^2 f||\nu(da, dq, dp) \\
&\leq C_1 \int_B (|a|^2 + |q|^2 + |p|^2)\Bigg[||f|| + \sum_{j=1}^n (||X_j f|| + ||D_j f||) + \\
&+ \sum_{j,k=1}^n (||X^j X^k f|| + ||X^j D^k f|| + ||D^j D^k f||)\Bigg]\nu(da, dq, dp) \\
&\leq C_2 \Bigg[||f|| + \sum_{j=1}^n (||X_j f|| + ||D_j f||) + \sum_{j,k=1}^n (||X^j X^k f|| + ||X^j D^k f|| + ||D^j D^k f||)\Bigg] \ldots (ii)
\end{aligned}$$

where $C_1, C_2 > 0$.

The required result now follows on combining (i), (ii) and the expression for $||\mathcal{L}_1^\pi f||$.□

Theorem 5 *$C_c^\infty(\mathbf{R}^n)$ is a core for $\mathcal{L}^\pi$.*

Proof Let $f \in \text{Dom}(\mathcal{L}^\pi)$, then we can find $(f_n, n \in \mathbf{N})$ in $C_c^\infty(\mathbf{R}^n)$ such that $f = \lim_{n\to\infty} f_n$. By Proposition 4, we deduce that $\lim_{m,n\to\infty} |||\mathcal{L}^\pi(f_n - f_m)|| = 0$, hence the sequence $(\mathcal{L}^\pi f_n, n \in \mathbf{N})$ is Cauchy and so convergent to some $g \in L^2(\mathbf{R}^n)$. But $\mathcal{L}^\pi$ is closed, hence $g = \mathcal{L}^\pi f$ and the result is established. □

In the following we will find it convenient to denote the restriction of $\mathcal{L}^\pi$ to $C_c^\infty(\mathbf{R}^n)$ by $\mathcal{L}_0^\pi$.

By (7) we see that $\mathcal{L}_0^\pi$ is symmetric if and only if the following conditions hold:-

- $b_0 = b_j^i = 0 (i = 1, 2, j = 1, \ldots n), E = 0$
- ν is a symmetric measure i.e. $\nu(A) = \nu(-A)$ for all $A \in \mathcal{B}(\mathbf{R}^{2n+1})$.

If we make the further assumption that each $c_{0i} = 0 (i = 1, 2)$, then $-\mathcal{L}_0^\pi$ is also a positive operator in the sense that $< \psi, (-\mathcal{L}_0^\pi)\psi > \geq 0$ for all $\psi \in S(\mathbf{R}^n)$ as the following result shows:-

Proposition 5 *$-\mathcal{L}_0^\pi$ is a positive symmetric operator where*

$$\begin{aligned}
-\mathcal{L}_0^\pi &= c_{00} I + c_{jk}^1 X^j X^k + c_{jk}^2 D^j D^k \\
&+ \int_{\mathbf{R}^{2n+1} - \{0\}} (I - \cos(aI + q.X + p.D))\nu(da, dq, dp)
\end{aligned} \tag{12}$$

Proof Since C is a non-negative definite symmetric matrix, it is easily verified that the first line of (12) is a positive symmetric operator. We write

$$M_\pi = \int_{\mathbf{R}^{2n+1} - \{0\}} (1 - \cos(aI + q.X + p.D))\nu(da, dq, dp)$$

By the spectral theorem, for each $\psi \in S(\mathbf{R}^n)$ we have

$$< \psi, M_\pi \psi > = \int_{\mathbf{R}^{2n+1} - \{0\}} \int_{\mathbf{R}} (1 - \cos(\lambda))||P_{a,q,p}(d\lambda)\psi||^2 \nu(da, dq, dp) \geq 0$$

and the result follows. □

From now on we will restrict the action of the positive symmetric operator $-\mathcal{L}_0^\pi$ and it closure $-\mathcal{L}^\pi$ to the real Hilbert space $L^2(\mathbf{R}^n, \mathbf{R})$, where the domain of the former operator comprises the real-valued smooth functions of compact support, $C_c^\infty(\mathbf{R}^n, \mathbf{R})$.

Now since $-\mathcal{L}_0^\pi$ is positive symmetric we can define a positive quadratic form $\mathcal{E}_\pi$ with domain $C_c^\infty(\mathbf{R}^n, \mathbf{R})$ by the prescription

$$\mathcal{E}_\pi(\psi) = < (-\mathcal{L}_0^\pi)^{\frac{1}{2}}\psi, (-\mathcal{L}_0^\pi)^{\frac{1}{2}}\psi >$$

for each $\psi \in C_c^\infty(\mathbf{R}^n, \mathbf{R})$, then $\mathcal{E}_\pi$ is closable with closure $\overline{\mathcal{E}_\pi}$ and there exists a positive self-adjoint operator A^π which is an extension of $-\mathcal{L}_0^\pi$ such that $\mathrm{Dom}((A^\pi)^{\frac{1}{2}}) = \mathrm{Dom}(\overline{\mathcal{E}_\pi})$ and $\overline{\mathcal{E}_\pi}(\psi) = < (A^\pi)^{\frac{1}{2}}\psi, (A^\pi)^{\frac{1}{2}}\psi >$ for all $\psi \in \mathrm{Dom}((A^\pi)^{\frac{1}{2}})$. A^π is called the *Friedrichs extension* of $-\mathcal{L}_0^\pi$ (see e.g [7], p.4). It follows directly from Theorem 5 that $A^\pi = -\mathcal{L}^\pi$ and hence $(\mathcal{T}_t^\pi, t \geq 0)$ is a symmetric semigroup in $L^2(\mathbf{R}^n, \mathbf{R})$.

Theorem 6 *$\overline{\mathcal{E}_\pi}$ is a symmetric Dirichlet form.*

Proof We must show that for all $\psi \in \mathrm{Dom}(\overline{\mathcal{E}_\pi})$ then $(\psi \wedge 0) \vee 1 \in \mathrm{Dom}(\overline{\mathcal{E}_\pi})$ and

$$\overline{\mathcal{E}_\pi}((\psi \wedge 0) \vee 1) \leq \overline{\mathcal{E}_\pi}(\psi)$$

Using the terminology of Proposition 5, we first show that if $\psi \in \mathrm{Dom}(\overline{M_\pi})$ then $(\psi \wedge 0) \vee 1 \in \mathrm{Dom}(\overline{M_\pi})$,

$$\begin{aligned}
||\overline{M_\pi}((\psi \wedge 0) \vee 1)||^2 &= \int_{\mathbf{R}^{2n+1}-\{0\}} \int_{\mathbf{R}} |1 - \cos(\lambda)|^2 ||P_{a,q,p}(d\lambda)((\psi \wedge 0) \vee 1)||^2 \nu(da, dq, dp) \\
&\leq \int_{\mathbf{R}^{2n+1}-\{0\}} \int_{\mathbf{R}} |1 - \cos(\lambda)|^2 ||P_{a,q,p}(d\lambda)\psi||^2 \nu(da, dq, dp) \\
&= ||\overline{M_\pi}(\psi)||^2 < \infty
\end{aligned}$$

Now for $\psi \in \mathrm{Dom}(\overline{M_\pi})$, we may use a similar argument to obtain

$$\begin{aligned}
& < (\psi \wedge 0) \vee 1, \overline{M_\pi}((\psi \wedge 0) \vee 1)) > \\
= & \int_{\mathbf{R}^{2n+1}-\{0\}} < (\psi \wedge 0) \vee 1, (I - \cos(aI + q.X + p.D))(\psi \wedge 0) \vee 1 > \nu(da, dq, dp) \\
= & \int_{\mathbf{R}^{2n+1}-\{0\}} \int_{\mathbf{R}} (1 - \cos(\lambda)) ||P_{a,q,p}(d\lambda)((\psi \wedge 0) \vee 1)||^2 \nu(da, dq, dp) \\
\leq & \int_{\mathbf{R}^{2n+1}-\{0\}} \int_{\mathbf{R}} (1 - \cos(\lambda)) ||P_{a,q,p}(d\lambda)\psi||^2 \nu(da, dq, dp) \\
= & < \psi, \overline{M_\pi}\psi >
\end{aligned}$$

The result follows on combining this with standard facts about diffusions (see e.g. [10]). □

By general properties of Dirichlet forms (see e.g. [10], [7]), $\mathcal{L}^\pi$ generates a symmetric sub-Markov semigroup in $L^2(\mathbf{R}^n, \mathbf{R})$ so that whenever $0 \leq f \leq 1$ (a.e.), we have $0 \leq \mathcal{T}_t^\pi f \leq 1$ (a.e.).

Note that when $-\mathcal{L}^\pi$ is positive and symmetric, the symbol of $\mathcal{L}^\pi$ takes the form

$$\sigma_{\mathcal{L}^\pi}(x, \xi) = -c_{00} - c_{jk}^1 x^j x^k - c_{jk}^2 \xi^j \xi^k + \int_{\mathbf{R}^{2n+1}-\{0\}} (\cos(a + q.x + p.\xi) - 1)\nu(da, dq, dp)$$

and following a calculation in [11], p.109, we can write the associated Dirichlet form as

$$\overline{\mathcal{E}_{\pi}}(\psi) = -(2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \sigma_{\mathcal{L}^{\pi}}(x,\xi)\Gamma(\psi)(x,\xi)dxd\xi$$

where $\Gamma(\psi)(x,\xi) = \int_{\mathbf{R}^n} e^{-iy.\xi}\psi(x+\frac{1}{2}y)\psi(x-\frac{1}{2}y)dy$

By Theorem 5, $\overline{\mathcal{E}}$ is regular and by the construction of Chapter 7 in [10], we can assert the existence of a symmetric (with respect to Lebesgue measure) Hunt process with state space $\mathbf{R}^n$ which is unique up to exceptional sets and whose transition semigroup is a quasi-continuous version of $(\mathcal{T}_t^{\pi}, t \geq 0)$. It is an interesting problem to relate this equivalence class of processes more directly to ρ. We note that similar results can be obtained for non-symmetric Dirichlet forms, provided the sector condition of [20] holds.

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