ASYMPTOTIC AND INCREASING PROPAGATION OF CHAOS EXPANSIONS FOR GENEALOGICAL PARTICLE MODELS

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Abstract. This article is concerned with the propagation-of-chaos properties of genetic type particle models. This class of models arises in a variety of scientific disciplines including theoretical physics, macromolecular biology, engineering sciences, and more particularly in advanced signal processing. From the pure mathematical point of view, these interacting particle systems can be regarded as a mean field particle interpretation of a class of Feynman-Kac measures on path spaces.

In the present article, we design an original integration theory of propagation-of-chaos based on the fluctuation analysis of a class of interacting particle random fields. We provide analytic functional representations of the asymptotic distributions of finite particle blocks, yielding what seems to be the first result of this kind for this class of interacting particle systems. These asymptotic expansions are expressed in terms of the limiting Feynman-Kac semigroups and a class of interacting jump operators. These results provide both sharp estimates of the negligible bias introduced by the interaction mechanisms, and central limit theorems for nondegenerate U-statistics and von Mises statistics associated with genealogical tree models. Applications to nonlinear filtering problems and interacting Markov chain Monte Carlo algorithms are discussed.

Key words. Interacting particle systems, historical and genealogical tree models, propagation-of-chaos, central limit theorems, Gaussian fields.

AMS subject classifications.

1. Introduction. In this article, we present an original fluctuation analysis of the propagation of chaos properties of a class of genetic type interacting particle systems. This subject has various natural links to statistical physics, macromolecular biology, engineering sciences, and more particularly, to rare event estimation, nonlinear filtering, global optimization, and sequential Monte Carlo theory.

The objects on which these particle models are sought vary considerably from one application area to another. But, from the strict mathematical and physical point of view, the genetic models discussed in this study can be interpreted as mean field particle approximations of an abstract and general class of Feynman-Kac path-measures. From the point of view of statistics and engineering sciences, these models can also be interpreted as methods to sample from complex distributions on path spaces. In observing this connection, we mention that the propagation of chaos properties of interacting processes allows us to measure the statistical bias, as well as the degree of independence between the sampled particles. For a detailed review of these model application areas, and precise asymptotic theory of these particle schemes, we refer the reader to the recent research books of the first and second authors [2, 7].

The study of the propagations of chaos properties for discrete and continuous time genetic models was begun in [5, 6], and it has been further developed in [2]. These three studies have been influenced by articles of G. Ben Arous and O.I. Zeitouni [1], C. Graham and S. Méléard [8], and S. Méléard [11]. In the articles [1, 5], increasing propagation of chaos properties are discussed for McKean-Vlasov diffusions, as well as for genetic type particle models with sufficiently regular mutation transitions. We note that the change of reference measure technique, developed in [5], relies entirely

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on some regularity conditions on the mutation transition, which are not satisfied for path-particle models. As a result, they do not apply to genealogical particle tree models. In the articles [8, 11], the authors present strong propagation of chaos results for the N-particle approximating model associated with a class of generalized Boltzmann equations. Using general interacting graphs and precise coupling techniques, they show that the order of convergence for the total variation distance between the law of the q-first particles and the limiting distribution on a compact interval $[0, t]$ is $q^2\, c(t)/N$. The connections between spatially homogeneous Boltzmann equations and continuous time Feynman-Kac formulae are described in some detail in [2, 6]. Loosely speaking, the microscopic colliding particle interpretations of the generalized Boltzmann equations introduced by S. Méléard [11] can also be interpreted as selective interacting jump models. In the articles [2, 5, 6], the authors have also obtained, in the context of genetic models, the same increasing propagation of chaos property using an original and alternative Feynman-Kac semigroup approach.

The increasing propagation of chaos property developed in the above series of articles leads us inevitably to question the sharpness of the order q^2/N. Furthermore, in most of these studies, the remainder control constant $c(t)$ increases exponentially fast to infinity, as the time parameter increases. As a consequence, these estimates cannot really be used in practice to quantify the degree of independence, and the performance of the particle interpretation models.

The increasing propagation of chaos properties discussed in this paper show that the order q^2/N is sharp. Additionally, we derive precise asymptotic expansions of the law of finite particle blocks with respect to the size of the system. Our analysis does not rely on any kind of regularity conditions on the mutation transitions. As a consequence, it applies to path space genealogical tree models. In contrast to traditional studies on this theme, our approach also describes the precise fluctuations associated with these asymptotic weak expansions. From the viewpoint of statistics, these fluctuations can be interpreted as central limit theorems for nondegenerate U-statistics. These results extend the original central limit theorem for U-statistics due to Hoeffding [9, 10] for independent and indentically distributed random variables to mean field and genealogical processes. Moreover, we provide a semigroup technique to estimate the first order operator of these asymptotic expansions. For sufficiently regular models, we show that the asymptotic remainder constant $c(t)$ is uniformly bounded, with respect to the time parameter.

This article is organized as follows:

In a preliminary section, Subsection 1.1, we provide a mathematical description of the Feynman-Kac models and their probabilistic particle interpretations.

In Subsection 1.2, we describe our main results. We start with a brief reminder of the propagation of chaos properties of interacting particle systems. Then we develop an asymptotic propagation of chaos estimate for polynomial tensor product test functions. This first result applies to a general class of particle McKean interpretation models on abstract measurable state spaces. We already mentioned that this result also provides a sharp estimate of the bias introduced by the particle interaction mechanisms. In the second part, we extend this propagation of chaos property to the context of simple genetic models on locally compact and separable metric spaces. We provide a first order asymptotic expansion of the distribution of finite particle blocks.

In Subsection 1.3, we illustrate this abstract class of models with two concrete situations arising in applied probability and engineering sciences. In the first, we discuss a class of interacting Markov chain Monte Carlo algorithms to sample from complex high dimensional distributions and to sample paths of Markov chains restricted to their terminal values. These particle models, also known as Sequential Monte Carlo samplers, were recently introduced by the first two authors [3]. Their applications to statistics, and global optimization were further developed in a joint work with the third author [4]. The second model application area discussed in this section is concerned with nonlinear filtering problems. This class of Feynman-Kac type models are currently used in advanced signal processing and Bayesian analysis. In this context, the corresponding genetic particle approximation models are also known as particle filters. In both situations, it is essential to estimate the bias induced by the interaction mechanisms, as well as the degree of independence between the particles. The asymptotic propagation of chaos expansions developed in this article provide precise answers to these two fundamental questions.

In Section 2 we have collected a series of results on both the weak convergence of random fields, and on the combinatorial transport properties of q-tensor product particle measures. Section 3 presents the proof of the main results of this article. In the final section, Section 4, we present an original semigroup contraction technique to control the first order operator of the asymptotic propagation of chaos expansion uniformly in time.

1.1. Description of the models. Let $(E_n, \mathcal{E}_n)_{n\geq 0}$ be a collection of measurable state spaces, and denote by $\mathcal{B}_b(E_n)$ the Banach space of all bounded and measurable functions f on E_n, equipped with the uniform norm $\|f\| = \sup_{x_n\in E_n} |f(x_n)|$. We also consider a collection of potential functions G_n on the state spaces E_n, a distribution η_0 on E_0, and a collection of Markov transitions $M_n(x_{n-1}, dx_n)$ from E_{n-1} into E_n. To simplify the presentation, and avoid unnecessary technical discussion, we shall suppose that the potential functions are chosen such that $\sup_{(x_n,x'_n)\in E_n^2} G_n(x_n)/G_n(x'_n) < \infty$. We associate the Feynman-Kac measures defined for any $f_n \in \mathcal{B}_b(E_n)$ by the formulae

$$\eta_n(f_n) = \gamma_n(f_n)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f_n) = \mathbb{E}[f_n(X_n) \textstyle\prod_{0\leq k<n} G_k(X_k)] \tag{1.1}$$

to the pair potentials/transitions (G_n, M_n). In (1.1), $(X_n)_{n\geq 0}$ represents a Markov chain with initial distribution η_0, and elementary transitions M_n. This abstract formulation is particularly useful for describing Markov motions on path spaces. For instance, X_n may represent the historical process

$$X_n = (X'_0, \ldots, X'_n) \in E_n = E'_{[0,n]} = E'_0 \times \ldots \times E'_n \tag{1.2}$$

associated with an auxiliary Markov chain X'_n which takes values in some measurable state spaces E'_n. As we shall see, this apparently innocent observation is particularly useful for modelling and analyzing genealogical evolution processes. By the Markov property and the multiplicative structure of (1.1), it is easily checked that the flow $(\eta_n)_{n\geq 0}$ satisfies the following equation

$$\eta_{n+1} = \Psi_n(\eta_n)M_{n+1} =_{\text{def.}} \int_{E_n} \Psi_n(\eta_n)(dx_n)\ M_{n+1}(x_n, .) \tag{1.3}$$

where the Boltzmann-Gibbs transformation Ψ_n is defined by

$$\Psi_n(\eta_n)(dx_n) = \frac{1}{\eta_n(G_n)}\ G_n(x_n)\ \eta_n(dx_n).$$

The particle approximation of the flow (1.3) depends on the choice of the McKean interpretation model. These probabilistic interpretations consist of a chosen collection of Markov transitions K_{n+1,η_n}, indexed by the set of probability measures η_n on E_n, and satisfying the compatibility condition

$$\Psi_n(\eta_n)M_{n+1} = \eta_n K_{n+1,\eta_n} (= \int \eta_n(dx_n) K_{n+1,\eta_n}(x_n,.)).$$

These collections are not unique, we can choose, for instance, $K_{n+1,\eta_n} = S_{n,\eta_n} M_{n+1}$, where $S_{n,\eta_n}(x_n, dy_n)$ is the updating Markov transition on E_n defined by

$$S_{n,\eta_n}(x_n, dy_n) = \epsilon_n G_n(x_n)\ \delta_{x_n}(dy_n) + (1 - \epsilon_n G_n(x_n))\ \Psi_n(\eta_n)(dy_n) \tag{1.4}$$

In the above formula, ϵ_n represents any (possibly null) constant such that $\|\epsilon_n G_n\| \leq 1$. The corresponding nonlinear equation

$$\eta_{n+1} = \eta_n K_{n+1,\eta_n}$$

can be interpreted as the evolution of the law of the states of a canonical Markov chain X_n whose elementary transitions K_{n+1,η_n} depend on the law of the current state. That is, we have that

$$\overline{\mathbb{P}}(X_{n+1} \in dx_{n+1} \mid X_n = x_n) = K_{n+1,\eta_n}(x_n, dx_{n+1}) \quad \text{with} \quad \overline{\mathbb{P}} \circ X_n^{-1} = \eta_n \tag{1.5}$$

The law $\overline{\mathbb{P}}_n$ of the random canonical path $(X_p)_{0\leq p\leq n}$ under the McKean measure $\overline{\mathbb{P}}$ is simply defined by

$$\overline{\mathbb{P}}_n(d(x_0,\ldots,x_n)) = \eta_0(dx_0)\ K_{1,\eta_0}(x_0,dx_1)\ \ldots\ K_{n,\eta_{n-1}}(x_{n-1},dx_n)$$

The mean field particle model associated with such McKean model is an E_n^N-valued Markov chain $\xi_n = (\xi_n^i)_{1\leq N}$ with elementary transitions defined in symbolic form by

$$\mathbb{P}(\xi_n \in d(x_n^1,\ldots,x_n^N) \mid \xi_{n-1}^i) = \prod_{i=1}^N K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n^i) \quad \text{with } \eta_{n-1}^N =_{\text{def.}} \frac{1}{N}\sum_{i=1}^N \delta_{\xi_{n-1}^i} \tag{1.6}$$

In the above displayed formula, $d(x_n^1,\ldots,x_n^N)$ represents an infinitesimal neighborhood of the point $(x_n^1,\ldots,x_n^N) \in E_n^N$. The initial configuration $\xi_0 = (\xi_0^i)_{1\leq N}$ consists of N independent and identically distributed random variables with distribution η_0.

By the definition of the McKean transitions we find that (1.6) is the overlapping of simple selection/mutation genetic transitions. The selection stage consists of N randomly evolving path-particles $\xi_{n-1}^i \rightsquigarrow \widehat{\xi}_{n-1}^i$ according to the update transition $S_{n,m(\xi_{n-1}^i)}(\xi_{n-1}^i, .)$. In other words, with probability $\epsilon_{n-1}G_{n-1}(\xi_{n-1}^i)$, we set $\widehat{\xi}_{n-1}^i = \xi_{n-1}^i$; otherwise, the particle jumps to a new location, randomly drawn from the discrete distribution $\Psi_{n-1}(\eta_{n-1}^N)$. During the mutation stage, each of the selected particles $\widehat{\xi}_{n-1}^i \rightsquigarrow \xi_n^i$ evolves according to the Markov transition M_n.

If we consider the historical process $X_n = (X_0', \ldots, X_n')$ introduced in (1.2), then the above mean field model consists of N path particles evolving according to the same selection/mutation transitions. It is immediately clear that the resulting particle model can be interpreted as the evolution of a genealogical tree model. Also notice that for $\epsilon = 0$, the particle interpretation model reduces to a simple mutation/selection genetic model.

It is, of course, out of the scope of this article to present a full asymptotic analysis of these genealogical particle models. We refer the interested reader to the recent research monograph [2], and the references therein. For instance, it is well known that the occupation measures of the ancestral lines converge to the desired Feynman-Kac measures. That is, we have with various precision estimates, and as N tends to infinity, the weak convergence result $\lim_{N\to\infty} \eta_n^N = \eta_n$.

1.2. Statement of main results. Several propagation of chaos estimates have been recently obtained which ensure that the particles ξ_n^i are asymptotically independent and identically distributed with common distribution η_n. The weakest form of this property can be stated as follows. We say that the particle model $\xi_n = (\xi_n^i)_{1\leq i\leq N}$ is weakly chaotic with respect to the measure η_n if we have

$$\lim_{N\to\infty} \mathbb{E}\left(\prod_{i=1}^{q} f_n^{(i)}(\xi_n^i)\right) = \prod_{i=1}^{q} \eta_n(f_n^{(i)}) \tag{1.7}$$

for any finite block size $q \leq N$, any time horizon, and any sequence of functions $(f_n^{(i)})_{1\leq i\leq q} \in \mathcal{B}_b(E_n^q)$. This propagation of chaos property is known to be satisfied for the McKean interpretation models defined in (1.4). Note that if $q = 1$ then (1.7) simply says that the law of a single particle is asymptotically unbiased. In other words, by the exchangeability property of the particle model we have that

$$\mathbb{E}(f_n^{(1)}(\xi_n^1)) = \mathbb{E}(\eta_n^N(f_n^{(1)})) \to \eta_n(f_n^{(1)}) \quad \text{as} \quad N \to \infty. \tag{1.8}$$

If we take $\epsilon = 0$ in (1.4), the particle model reduces to a simple genetic model. In this context, we also have the increasing propagation of chaos estimate

$$\|\mathrm{Law}(\xi_n^1, \ldots, \xi_n^q) - \eta_n^{\otimes q}\|_{\mathrm{tv}} \leq c(n)\; q^2/N \tag{1.9}$$

for some finite constant, which only depends on the time parameter n and where $\|\,.\,\|_{\mathrm{tv}}$ denotes the total variation norm on the set of bounded measures.

The main object of this article is to provide an asymptotic functional expansion of these two convergence results with respect to the precision parameter N. To simplify the presentation, we shall often use the following slight abuse of notation; for a given Markov transition K from a measurable space $(E, \mathcal{E})$ into another space $(F, \mathcal{F})$, and for any pair of functions $f, g \in \mathcal{B}_b(F)$, we write

$$\begin{aligned} K[(f - K(f))(g - K(g))](x) &=_{\mathrm{def.}} K[(f - K(f)(x))(g - K(g)(x))](x) \\ &= K(fg)(x) - K(f)(x)K(g)(x) \end{aligned}$$

Our asymptotic expansions are expressed in terms of a collection of independent and centered Gaussian fields W_n on the Banach function spaces $\mathcal{B}_b(E_n)$; with for any $f_n, g_n \in \mathcal{B}_b(E_n)$,

$$\mathbb{E}(W_n(f_n)W_n(g_n)) = \eta_{n-1}K_{n,\eta_{n-1}}([f_n - K_{n,\eta_{n-1}}(f_n)][g_n - K_{n,\eta_{n-1}}(g_n)]) \tag{1.10}$$

For $n = 0$ we use the convention $K_{0,\eta_{-1}} = \eta_0$. The Gaussian fields W_n represent the asymptotic fluctuations of the local sampling errors associated with the mean field particle approximation sampling steps. To describe precisely our first main result, we let $Q_{p,n}$, with $0 \leq p \leq n$, be the Feynman-Kac semi-group associated with the flow $\gamma_n = \gamma_p Q_{p,n}$. For $p = n$, we use the convention that $Q_{n,n} = Id$. Using the Markov property, it is not difficult to check that $Q_{p,n}$ has the following functional representation

$$Q_{p,n}(f_n)(x_p) = \mathbb{E}\left[f_n(X_n)\; \textstyle\prod_{p\leq k<n} G_k(X_k) \mid X_p = x_p\right]$$

for any test function $f_n \in \mathcal{B}_b(E_n)$, and any state $x_p \in E_p$. For $p = n$ we use the convention $\prod_{\emptyset} = 1$ and $Q_{p,n} = Id$. We also denote by $R_{p,n}$ the renormalized semigroup from E_p into E_n given by

$$R_{p,n}(f_n) = \frac{Q_{p,n}(f_n)}{\eta_p(Q_{p,n}(1))} = \frac{\gamma_p(1)}{\gamma_n(1)}\; Q_{p,n}(f_n) \tag{1.11}$$

THEOREM 1.1. *For any time horizon $n \geq 0$, any block size parameter $0 < q \leq N$, and every sequence of functions $f_n^{(k)} \in \mathcal{B}_b(E_n)$, with $\eta_n(f_n^{(k)}) = 1$, and $1 \leq k \leq q$, we have*

$$\begin{aligned}
&\lim_{N\to\infty} N\; \mathbb{E}\left[\prod_{k=1}^{q} \eta_n^N(f_n^{(k)}) - 1\right] \\
&= - \textstyle\sum_{i=1}^{q} \sum_{p=0}^{n-1} \gamma_p(1)^2\; \mathbb{E}\left(W_p(Q_{p,n}1)\; W_p(Q_{p,n}[f_n^{(i)} - 1])\right) \\
&\qquad\qquad + \textstyle\sum_{1\leq i<j\leq q} \quad \sum_{p=0}^{n} \mathbb{E}\left(W_p(R_{p,n}(f_n^{(i)} - 1))W_p(R_{p,n}(f_n^{(j)} - 1))\right)
\end{aligned} \tag{1.12}$$

In addition, we have the weak asymptotic propagation of chaos estimate

$$\lim_{N\to\infty} N \quad \left[\mathbb{E}\left(\prod_{k=1}^{q} f_n^{(k)}(\xi_n^k)\right) - \mathbb{E}\left(\prod_{k=1}^{q} \eta_n^N(f_n^{(k)})\right)\right] = - \sum_{1\leq i<j\leq q} (\eta_n(f_n^{(i)} f_n^{(j)}) - 1) \tag{1.13}$$

The asymptotic expansions described in Theorem 1.1 provide sharp estimates of the weak propagation of chaos and unbiased properties stated in (1.7) and (1.8). For instance, if we take $q = 1$ in (1.12), and consider the simple genetic interpretation model associated with the choice of $\epsilon_n = 0$ in (1.4), then, for any $f_n \in \mathcal{B}_b(E_n)$, we have the asymptotic unbiased expansion

$$\lim_{N\to\infty}\; N\; \mathbb{E}(\eta_n^N(f_n) - \eta_n(f_n)) = -\sum_{p=0}^{n-1} \eta_p\left[R_{p,n}(1)\; R_{p,n}(f_n - \eta_n(f_n))\right]$$

Theorem 1.1 provides weak asymptotic expansions of the q-tensor particle measures $(\eta_n^N)^{\otimes q}$ on tensor product test functions $F_n = (f_n^{(1)} \otimes \ldots \otimes f_n^{(q)})$. From the point of view of statistics, $(\eta_n^N)^{\otimes q}(F_n)$ can also be interpreted as a sequence of U-statistics for interacting processes. The fluctuations associated with these mathematical objects are discussed in Section 3.1.

Our next objective is to further extend the propagation of chaos analysis of the simple genetic model to test functions which are not necessarily tensor product functions. For this purpose, we introduce the canonical projection operators $(p_n^i)_{1\leq i\leq q}$ and the collection of selection jump operators $(C_{i,j})_{1\leq i<j\leq q}$ on $\mathcal{B}_b(E_n^q)$ defined for any $F_n \in \mathcal{B}_b(E_n^q)$ by the formulae

$$\begin{aligned}
p_n^i(F_n)(x_n^1, \ldots, x_n^q) &= \int_{E_n} \eta_n(dx_n^i)\; F_n(x_n^1, \ldots, x_n^i, \ldots, x_n^q) \\
C_{i,j}(F_n)(x_n^1, \ldots, x_n^q) &= F_n(\theta_{i,j}(x_n^1, \ldots, x_n^q))
\end{aligned}$$

with the change of coordinate mapping $\theta_{i,j}$ from E_n^q into itself, defined by

$$\theta_{i,j}(x_n^1,\ldots,x_n^q)^j = x_n^i$$

and $\theta_{i,j}(x_n^1,\ldots,x^q)^k = x_n^k$ for $k \neq j$ otherwise; i.e. the j^{th} component x_n^j of $(x_n^1,\ldots,x_n^q)$ is set equal to x_n^i whereas the others are not modified. We also associate with a given function $f_n \in \mathcal{B}_b(E_n)$ the q-empirical function $\overline{f}_n^q \in \mathcal{B}_b(E_n^q)$ defined by

$$\overline{f}_n^q(x_n^1,\ldots,x_n^q) = \frac{1}{q}\sum_{i=1}^q f_n(x_n^i).$$

Finally, we let $R_{p,n}^{\otimes q}$ denote the q-tensor product semigroup associated with $R_{p,n}$ and defined for any $F_n \in \mathcal{B}_b(E_n^q)$ and $(x_p^1,\ldots,x_p^q) \in E_p^q$ by

$$R_{p,n}^{\otimes q}(F_n)(x_p^1,\ldots,x_p^q) = \int_{E_n^q} R_{p,n}(x_p^1,dx_n^1)\ldots R_{p,n}(x_p^q,dx_n^q)\ F_n(x_n^1,\ldots,x_n^q).$$

We are now in position to state the second main result of this article.

Theorem 1.2. *Let η_n^N be the particle measures associated with the simple genetic model. We assume that the state spaces E_n are locally compact and separable metric spaces. For any time horizon $n \geq 0$, any block size parameter $0 < q \leq N$, and every function $F_n \in \mathcal{C}_b(E_n^q)$, we have*

$$\lim_{N\to\infty} N\ \mathbb{E}[(\eta_n^N)^{\otimes q}(F_n) - \eta_n^{\otimes q}(F_n)] = \mathcal{L}_n^{(q)}(F_n)$$

where the bounded linear operator $\mathcal{L}_n^{(q)}$ on $\mathcal{C}_b(E_n^q)$ is given by

$$\begin{aligned}\mathcal{L}_n^{(q)}(F_n) = -q\ &\sum_{p=0}^{n-1}\eta_p^{\otimes q}\left[\overline{R}_{p,n}^q(1)\ R_{p,n}^{\otimes q}(F_n - \eta_n^{\otimes q}(F_n))\right]\\ &+\sum_{p=0}^{n}\sum_{1\leq i<j\leq q}\eta_p^{\otimes q}C_{i,j}R_{p,n}^{\otimes q}(Id-p_n^i)(Id-p_n^j)(F_n)\end{aligned}\qquad(1.14)$$

In addition, we have the weak asymptotic propagation-of-chaos estimate

$$\lim_{N\to\infty} N\ \left[\mathbb{E}(F_n(\xi_n^1,\ldots,\xi_n^q)) - \mathbb{E}((\eta_n^N)^{\otimes q}(F_n))\right] = -\sum_{1\leq i<j\leq q}\eta_n^{\otimes q}[C_{i,j}-Id](F_n)\qquad(1.15)$$

This theorem readily provides an asymptotic first order expansion of the distribution $\mathbb{P}_n^{(q,N)}$ of the first q particles $(\xi_n^1,\ldots,\xi_n^q)$. More precisely, combining (1.14) and (1.15), we find that

$$\mathbb{P}_n^{(q,N)} = \eta_n^{\otimes q} + N^{-1}\ \mathcal{M}_n^{(q)}\ +\ \mathcal{R}_n^{(q,N)}\qquad(1.16)$$

where a remainder linear operator $\mathcal{R}_n^{(q,N)}$ on $\mathcal{C}_b(E_n^q)$ is such that

$$\lim_{N\to\infty} N\mathbb{E}(\mathcal{R}_n^{(q,N)}(F_n)) = 0$$

and the bounded linear operator $\mathcal{M}_n^{(q)}$ on $\mathcal{C}_b(E_n^q)$ is defined for any $F_n \in \mathcal{C}_b(E_n^q)$ by

$$\mathcal{M}_n^{(q)}(F_n) = \mathcal{L}_n^{(q)}(F_n) - \sum_{1 \le i < j \le q} \eta_n^{\otimes q}[C_{i,j} - Id](F_n).$$

Under appropriate regularity conditions on the Markov transitions M_n, we shall also prove the uniform estimate

$$\sup_{n \ge 0} \sup_{F_n : \|F_n\| \le 1} |\mathcal{M}_n^{(q)}(F_n)| \le \text{Cte} \ \times \ q^2$$

Although this article is restricted to discrete generation models, the fluctuation and combinatorial analyses developed here are well suited to the analysis of continuous time Feynman-Kac models and their interacting particle interpretation. For an account of these continuous models we refer the reader to [2, 6], and the references therein. The reader will find that the integration of propagation of chaos presented in Section 3 relies entirely on a precise analysis of the combinatorial properties of tensor product particle measures, and on the fluctuation of the particle approximation measures around their limiting values. Such observations suggest that these techniques may apply to any interacting processes as soon as we have some information about their fluctuations.

Finally, we mention that the weak asymptotic expansions presented in (3) lead to the conjecture that strong versions, with respect to the total variation distance, exist with a first order term given by the norm of the operator $\mathcal{M}_n^{(q)}$.

1.3. Some model application areas. One of the most exciting developments in applied probability and engineering sciences are those centered around the recently established connection between branching and interacting particle systems, and nonlinear filtering and Bayesian methodology. For an overview of the theoretical and the applied sides of this subject, we refer the interested reader to [2, 7], and references therein.

To motivate the abstract mathematical models discussed in the present article, we have chosen to illustrate their impact on two separate application model areas.

The first one is concerned with the analysis of Markov chains with fixed terminal values. This rather recent subject has also been stimulated by the need to find efficient simulation techniques to sample from complex distributions (see [3, 4]). As we shall see in Subsection 1.3.1, for judicious choices of potential functions, the Feynman-Kac flow introduced in (1.1) can be interpreted as a nonlinear and stationary Metropolis-Hasting type model. In this context, the corresponding particle interpretation can be seen as a genealogical tree based simulation method.

In Subsection 1.3.2, we introduce nonlinear filtering applications. This rapidly developing area is concerned with estimating the conditional distribution of a given Markov chain with respect to some observation sequence.

1.3.1. Interacting Metropolis models. One recurrent problem in various scientific disciplines is to obtain an efficient simulation method to produce random samples of a given sequence of distributions π_n defined on some measurable state spaces $(F_n, \mathcal{F}_n)$. The prototype of these target measures is given by the annealed Boltzmann-Gibbs measures on some common homogeneous space $F_n = F$. These measures are defined by the formula

$$\pi_n(dy_n) = \frac{e^{-\beta_n V(y_n)}}{\mu(e^{-\beta_n V})} \ \mu(dy_n) \tag{1.17}$$

The parameter β_n represents an inverse cooling schedule. The reference measure μ, and the energy function V are chosen such that $\mu(e^{-\beta V}) \in (0, \infty)$, for all $\beta > 0$. More generally, another interesting problem is to sample a backward canonical Markov path sequence

$$(Y_0, Y_1, \ldots, Y_n) \in (F_n \times F_{n-1} \times \ldots \times F_0)$$

of length n, starting in F_n and with initial distribution π_n, and evolving randomly from F_{k+1} into F_k according to a given Markov transition L_k. More formally, $\mathbb{P}^L_{n,\pi_n}$ is given by the formula

$$\mathbb{P}^L_{n,\pi_n}(d(y_n, y_{n-1} \ldots, y_0)) = \pi_n(dy_n) L_{n-1}(y_n, dy_{n-1}) \ldots L_0(y_1, y_0)$$

As noticed in [3], these two problems have a common and natural Feynman-Kac formulation. To describe these interpretations, we first consider an auxiliary sequence of Markov transitions $M_{n+1}(y_n, dy_{n+1})$ from $(F_n, \mathcal{F}_n)$ into $(F_{n+1}, \mathcal{F}_{n+1})$. We denote by $(\pi_n \times M_{n+1})_1$ and $(\pi_{n+1} \times L_n)_2$, the distributions on the transition space $(F_n \times F_{n+1})$ defined by

$$\begin{aligned} d(\pi_n \times M_{n+1})_1(y_n, y_{n+1}) &= \pi_n(dy_n)\ M_{n+1}(y_n, dy_{n+1}) \\ d(\pi_{n+1} \times L_n)_2(y_n, y_{n+1}) &= \pi_{n+1}(dy_{n+1})\ L_n(y_{n+1}, dy_n) \end{aligned}$$

We further assume that the mathematical objects (π_n, M_n, L_n) are chosen so that these measures are absolutely continuous with respect to each other, and the corresponding Radon-Nykodim derivatives

$$G_n = \frac{d(\pi_{n+1} \times L_n)_2}{d(\pi_n \times M_{n+1})_1}$$

are bounded positive functions on $(F_n \times F_{n+1})$. In this notation, it is immediate to check the following time reversal formula

$$\begin{aligned} &\mathbb{E}^L_{n,\pi_n}\left(f_n(Y_n, Y_{n-1}, \ldots, Y_0)\right) \\ &= \mathbb{E}^M_{\pi_0}\left(f_n(Y_0, Y_1, \ldots, Y_n)\ \textstyle\prod_{0 \le p < n} G_p(Y_p, Y_{p+1})\right) \end{aligned} \tag{1.18}$$

for any test function $f_n \in \mathcal{B}_b(F_0 \times \ldots \times F_n)$. Here, $\mathbb{E}^M_{\pi_0}$ is the expectation operator with respect to the distribution $\mathbb{P}^M_{\pi_0}$ of a forward canonical Markov path sequence

$$(Y_0, Y_1, \ldots, Y_n) \in (F_0 \times F_1 \times \ldots \times F_n)$$

starting with an initial distribution π_0, and evolving from F_k into F_{k+1}, according to the Markov transitions M_{k+1}. Arguing as in [3], we also prove the Feynman-Kac functional representation

$$\begin{aligned} &\mathbb{E}^L_{n,\pi_n}(f_n(Y_n, Y_{n-1}, \ldots, Y_0) \mid Y_n = y_0) \\ &\propto \mathbb{E}^M_{y_0}\left(f_n(Y_0, Y_1, \ldots, Y_n)\ \textstyle\prod_{0 \le p < n} G_p(Y_p, Y_{p+1})\right) \end{aligned} \tag{1.19}$$

where $\mathbb{E}^M_{y_0}$ is the expectation operator with respect to the distribution $\mathbb{P}^M_{y_0}$ of a forward canonical Markov path sequence starting at $Y_0 = y_0$. The genealogical tree interpretation of the Feynman-Kac model given in (1.19) clearly yields an elegant backward particle simulation method for sampling Markov paths which are restricted to having particular terminal values.

These particle approximation models can also be interpreted as a sequence of interacting Metropolis type algorithms. We illustrate this observation in the context of the target Boltzmann-Gibbs measures introduced in (1.17). In this case, we notice that, the n-th marginal of the Feynman-Kac path measure introduced in (1.18) coincides with the Boltzmann-Gibbs measure at temperature β_n; that is, we have

$$\mathbb{E}^M_{\pi_0}\left(\varphi_n(Y_n) \prod_{0\leq p<n} G_p(Y_p, Y_{p+1})\right) = \pi_n(\varphi_n) \tag{1.20}$$

for any $\varphi_n \in \mathcal{B}_b(E_n)$. In contrast to traditional non-interacting Metropolis models, we emphasize that the McKean interpretation (1.5) of the flow (1.20) consists of a nonlinear Markov chain with distribution π_n, at each time n. Furthermore, the corresponding particle interpretation clearly behaves as an interacting Metropolis model. It is also essential to notice that the usual Metropolis rejection mechanism has been replaced by a selective interacting jump transition. As noticed in [4]; in practice, one judicious choice of Markov mutation transition is to take a Markov chain Monte Carlo kernel M_n such that $\pi_n M_n = \pi_n$. In this case, the interaction potential functions are given by

$$G_n(y_n, y_{n+1}) = \exp\left[-(\beta_{n+1} - \beta_n)V(y_n)\right]$$

as soon as $L_n(y_{n+1}, dy_n) = \pi_{n+1}(dy_n)\ \frac{dM_{n+1}(y_n, \bullet)}{d\pi_{n+1}}(y_{n+1})$. When M_n represents the elementary transition of a simulated annealing model at temperature β_n, the resulting particle model behaves as an interacting simulated annealing algorithm. At low temperature, the potential function is close to one, and thus the particles are more likely not to interact.

1.3.2. Nonlinear filtering problems. The filtering problem is to estimate a stochastic signal process that we cannot observe directly. More precisely, the signal is partially observed by some noisy sensors. The noise may come from the model uncertainties, or from inherent perturbations, such as thermal noise in electronic devices.

The signal/observation pair sequence $(X_n, Y_n)_{n\geq 0}$ is defined as a Markov chain which takes values in some product of measurable spaces $(E_n \times F_n)_{n\geq 0}$. We further assume that the initial distribution ν_0 and the Markov transitions P_n of (X_n, Y_n) have the form

$$\begin{aligned}\nu_0(d(x_0, y_0)) &= g_0(x_0, y_0)\ \eta_0(dx_0)\ q_0(dy_0)\\ P_n((x_{n-1}, y_{n-1}), d(x_n, y_n)) &= g_n(x_n, y_n)\ M_n(x_{n-1}, dx_n)\ q_n(dy_n)\end{aligned}$$

where g_n are strictly positive functions on $(E_n \times F_n)$ and q_n is a sequence of measures on F_n. The initial distribution η_0 of the signal X_n, and its Markov transitions M_n from E_{n-1} into E_n, are assumed known. A version of the conditional distributions of the signal states given their noisy observations is expressed in terms of Feynman-Kac formulae of the same type as the ones discussed. More precisely, let G_n be the non homogeneous function on E_n defined for any $x_n \in E_n$ by

$$G_n(x_n) = g_n(x_n, y_n). \tag{1.21}$$

Note that G_n depends on the observation value y_n at time n. In this notation, the conditional distribution of the signal X_n, given the sequence of observations from the origin up to time n, has the Feynman-Kac functional representation

$$\mathbb{E}(f_n(X_0,\ldots,X_n) \mid Y_0=y_0,\ldots,Y_n=y_n) \propto \mathbb{E}(f_n(X_0,\ldots,X_n)\prod_{0\leq p\leq n} G_p(X_p)).$$

In this context, the corresponding genealogical tree based models can be interpreted as an adaptative and stochastic grid approximation. Note that the selection transition is dictated by the likelihood function (1.21); i.e. the current observation delivered by the sensors. Therefore, the resulting birth and death genetic mechanism gives more reproductive opportunities to particles evolving in state space regions with high conditional probability mass. This filtering problem arises in numerous scientific disciplines, including financial mathematics, robotics, telecommunications and tracking; see for instance [2, 7], and references therein.

2. Preliminary results. This section is decomposed into two separate parts. In the first subsection, Subsection 2.1, we provide a summary of results on the fluctuations of a class of random fields associated with the particle occupation measures η_n^N introduced in (1.6). The definitions and results come essentially from Chapter 9 in the monograph [2]. We shall simplify and further extend this study to random field product models. This approach is central in our method of analysing the sharp asymptotic propagation of chaos properties. In Subsection 2.2, we provide a transport equation relating a pair of q-tensor product particle measures. We also mention that weaker versions of this identity are sometimes used in nonparameteric statistics to relate U-statistics to von Mises statistics.

2.1. Fluctuation analysis. The fluctuation analysis of the particle measures η_n^N around their limiting values η_n is essentially based on the asymptotic analysis of the local sampling errors associated with the particle approximation sampling steps. These local errors are expressed in terms of the random fields W_n^N, given for any $f_n\in\mathcal{B}_b(E_n)$ by the formula

$$W_n^N(f_n)=\sqrt{N}\;[\eta_n^N-\Psi_{n-1}(\eta_{n-1}^N)M_n](f_n)=\frac{1}{\sqrt{N}}\sum_{i=1}^N[f_n(\xi_n^i)-K_{n,\eta_{n-1}^N}(f_n)(\xi_{n-1}^i)].$$

The next central limit theorem for random fields is pivotal. Its complete proof can be found in [2].

Theorem 2.1. *For any fixed time horizon $n\geq 1$, the sequence $(W_p^N)_{1\leq p\leq n}$ converges in law, as N tends to infinity, to a sequence of n independent, Gaussian and centered random fields $(W_p)_{1\leq p\leq n}$; with, for any $f_p,g_p\in\mathcal{B}_b(E_p)$, and $1\leq p\leq n$,*

$$\mathbb{E}(W_p(f_p)W_p(g_p))=\eta_{p-1}K_{p,\eta_{p-1}}([f_p-K_{p,\eta_{p-1}}(f_p)][g_p-K_{p,\eta_{p-1}}(g_p)]) \tag{2.1}$$

For the McKean selection transition (1.4), with $\epsilon_n=0$, we find that

$$K_{p,\eta_{p-1}}(x_{p-1},.)=\Psi_{p-1}(\eta_{p-1})M_p.$$

In this case, the correlation term (2.1) takes the form

$$\mathbb{E}(W_p(f_p)W_p(g_p))=\eta_p([f_p-\eta_p(f_p)][g_p-\eta_p(g_p)]).$$

More generally, for any choice of $\epsilon_n \geq 0$, we have

$$\begin{aligned}\mathbb{E}(W_p(f_p)W_p(g_p)) &= \eta_p([f_p - \eta_p(f_p)][g_p - \eta_p(g_p)]) \\ &\quad - \eta_{p-1}((\epsilon_{p-1}G_{p-1})^2[M_p(f_p) - \eta_p(f_p)][M_p(g_p) - \eta_p(g_p)]) \\ &= \eta_p([f_p - \eta_p(f_p)][g_p - \eta_p(g_p)]) \\ &\quad - \epsilon_{p-1}^2 \eta_{p-1}(Q_{p-1,p}[f_p - \eta_p(f_p)]Q_{p-1,p}[g_p - \eta_p(g_p)]).\end{aligned}$$

These two observations show that local variances induced by sampling errors are reduced for the McKean transitions associated with the choice $\epsilon_n > 0$.

These multivariate fluctuations also yield, for any finite collection of functions $(f_n^{(i)})_{1\leq i\leq d} \in \mathcal{B}_b(E_n)^d$, with $d \geq 1$

$$(W_n^N(f_n^1), \ldots, W_n^N(f_n^d)) \longrightarrow_{N\to\infty} (W_n(f_{p_1}^1), \ldots, W_n(f_n^d))$$

where $(W_n(f_n^1), \ldots, W_n(f_n^d))$ is a d-dimensional centered Gaussian random variable with a $(d \times d)$-covariance matrix $\Sigma(f)$ whose (i,j)-elements $\Sigma(f_n^i, f_n^j)$ are given by

$$\Sigma(f_n^i, f_n^j) = \eta_{n-1}K_{n,\eta_{n-1}}([f_n^i - K_{n,\eta_{n-1}}(f_n^i)][f_n^j - K_{n,\eta_{n-1}}(f_n^j)]).$$

We also observe that the unnormalized distributions (γ_n) can be expressed in terms of the normalized measures with the product formula $\gamma_{n+1}(1) = \gamma_n(G_n) = \eta_n(G_n)\ \gamma_n(1)$. This readily implies that, for any $f_n \in \mathcal{B}_b(F_n)$,

$$\gamma_n(f_n) = \eta_n(f_n) \prod_{0\leq p<n} \eta_p(G_p). \tag{2.2}$$

Mimicking (2.2), the *unbiased* particle approximation measures γ_n^N of the unnormalized model γ_n are defined as

$$\gamma_n^N(f_n) = \eta_n^N(f_n) \prod_{0\leq p<n} \eta_p^N(G_p).$$

To explain what we have in mind when making these definitions, we now consider the elementary decomposition

$$\gamma_n^N - \gamma_n = \sum_{p=0}^n [\gamma_p^N Q_{p,n} - \gamma_{p-1}^N Q_{p-1,n}]$$

For $p = 0$, we take the convention $\eta_{-1}^N Q_{-1,n} = \gamma_n$. The next important thing to note is that

$$\gamma_{p-1}^N Q_{p-1,p} = \gamma_{p-1}^N(G_{p-1})\Psi_{p-1}(\eta_{p-1}^N)M_p \quad \text{and} \quad \gamma_{p-1}^N(G_{p-1}) = \gamma_p^N(1)$$

This decomposition now readily implies that

$$W_n^{\gamma,N}(f_n) =_{\text{def.}} \sqrt{N}[\gamma_n^N - \gamma_n](f_n) = \sum_{p=0}^n \gamma_p^N(1)\ W_p^N(Q_{p,n}f_n) \tag{2.3}$$

Since the random variable $\gamma_p^N(1)$ only depends on the flow $(\eta_k^N)_{0\leq k<p}$, it is easy to check that γ_n^N is an unbiased estimate of γ_n; in the sense that $\mathbb{E}(\gamma_n^N(f_n)) = \gamma_n(f_n)$, for any $f_n \in \mathcal{B}_b(E_n)$. To take the final step, we recall that the random sequence

$(\gamma_p^N(1))_{1\leq p\leq n}$ converges in law, as N tends to infinity, to the deterministic sequence $(\gamma_p(1))_{1\leq p\leq n}$ (see, for instance, [2]). A simple application of Slutsky's Lemma, implies that the random fields $W_n^{\gamma,N}$ converge in law, as N tends to infinity, to the Gaussian random fields W_n^γ defined for any $f_n\in\mathcal{B}_b(E_n)$ by

$$W_n^\gamma(f_n)=\sum_{p=0}^n \gamma_p(1)\ W_p(Q_{p,n}f_n) \tag{2.4}$$

In much the same way, the sequence of random fields

$$\begin{aligned} W_n^{\eta,N}(f_n) &=_{\text{def.}}\sqrt{N}\ [\eta_n^N-\eta_n](f_n)\\ &=\gamma_n^N(1)^{-1}\ \times\ W_n^{\gamma,N}(f_n-\eta_n(f_n)) \end{aligned} \tag{2.5}$$

converge in law, as N tends to infinity, to the Gaussian random fields W_n^η defined for any $f_n\in\mathcal{B}_b(E_n)$ by

$$W_n^\eta(f_n)=\sum_{p=1}^n(\gamma_p(1)/\gamma_n(1))\ W_p\left(Q_{p,n}(f_n-\eta_n(f_n))\right)$$

Our final objective is to analyze the asymptotic behavior of random fields of product sequences. These properties are one of the stepping stones in the integration analysis of propagation of chaos used in the further developement of Section 3.

We let $Poly(E_n^q)$ be the set of all linear combinations $F_n=\sum_{k\geq0}\lambda_k\ F_n^k$ of polynomial q-tensor product functions

$$F_n^k=f_n^{(k,1)}\otimes\ldots\otimes f_n^{(k,q)},\quad\text{with}\quad (f_n^{(k,i)})_{1\leq i\leq q}\in\mathcal{B}_b(E_n)^q$$

where $\sum_{k\geq0}|\lambda_k|\ \|F_k\|<\infty$. We are now ready to prove the following proposition.

Proposition 2.2. *For any time horizon $n\geq1$, any particle block size parameter $1\leq q\leq N$ and any sequence $(\nu^i)_{1\leq i\leq d}\in\{\gamma,\eta\}$, the sequence of random fields $(W_n^{\nu^1,N}\otimes\ldots\otimes W_n^{\nu^q,N})$ on $Poly(E_n^q)$ converges in law, as N tends to infinity, to the Gaussian random field $(W_n^{\nu^1}\otimes\ldots\otimes W_n^{\nu^q})$. In addition, we have for any $F_n\in Poly(E_n^q)$*

$$\lim_{N\to\infty}\mathbb{E}\left((W_n^{\nu^1,N}\otimes\ldots\otimes W_n^{\nu^q,N})(F_n)\right)=\mathbb{E}\left((W_n^{\nu^1}\otimes\ldots\otimes W_n^{\nu^q})(F_n)\right)$$

Proof:
We recall from [2] that for $\nu\in\{\gamma,\eta\}$, for any $f_n\in\mathcal{B}_b(E_n)$, and $p\geq1$, we have the $\mathbb{L}_p$-mean error estimates

$$\sup_{N\geq1}\mathbb{E}(|W_n^{\nu,N}(f_n)|^p)^{1/p}\leq c_p(n)\ \|f_n\| \tag{2.6}$$

for some finite constant $c_p(n)$, which depends only on the pair of parameters (p,n), and with the random fields $(W_n^{\gamma,N},W_n^{\eta,N})$ defined in (2.3) and (2.5). By the Borel-Cantelli lemma this property ensures that the random sequence of pairs $(\gamma_n^N(f_n),\eta_n^N(f_n))$ converges almost surely to $(\gamma_n(f_n),\eta_n(f_n))$, as N tends to infinity.

Using Hölder's inequality, we can also prove that any polynomial function of a finite number of terms $W_n^{\nu,N}(f_n)$, with $\nu\in\{\gamma,\eta\}$ and $f_n\in\mathcal{B}_b(E_n)$, forms a uniformly integrable collection of random variables (indexed by the size and precision parameter $N\geq1$). This property, combined with the continuous mapping theorem, and Skorohod embedding theorem, ends the proof of the proposition. ∎

2.2. Combinatorial properties of tensor product measures. This section is concerned with a precise combinatorial analysis of the particle measures on E_n^q defined by the formulae

$$(\eta_n^N)^{\otimes q} = \frac{1}{N^q} \sum_{\beta \in \langle N \rangle^{\langle q \rangle}} \delta_{(\xi_n^{\beta(1)}, \ldots,\ \xi_n^{\beta(q)})} \quad \text{and} \quad (\eta_n^N)^{\odot q} = \frac{1}{(N)_q} \sum_{\beta \in < q,N >} \delta_{(\xi_n^{\beta(1)}, \ldots,\ \xi_n^{\beta(q)})} \tag{2.7}$$

where $\langle N \rangle^{\langle q \rangle}$ represents the set of all mappings from $\langle q \rangle = \{1, \ldots, q\}$ into $\langle N \rangle$, and $\langle q, N \rangle$ the subset of all $(N)_q = N!/(N-q)!$ one-to-one mappings from the set $\langle q \rangle$ into $\langle N \rangle$. More precisely, the aim of this short section is to express $(\eta_n^N)^{\otimes q}$ as a linear transport of the measure $(\eta_n^N)^{\odot q}$, with respect to some Markov transition. For this purpose, we let $\mathcal{P}(q,p)$ be the set of all partitions of the set $\langle q \rangle$ into p blocks, equipped with the order relation on the subsets A, B of $\langle q \rangle$ given by

$$A \leq B \Longleftrightarrow \inf\{i \ : \ i \in A\} \leq \inf\{i \ : \ i \in B\}.$$

Finally, we associate with a given partition $\pi = (\pi_i)_{1 \leq i \leq p} \in \mathcal{P}(q,p)$, of p increasing blocks, the mapping $\beta_\pi \in < q, N >$ defined by $\beta_\pi = \sum_{i=1}^p \beta(i)\ 1_{\pi_i}$. In this notation, we have for any $F_n \in \mathcal{B}_b(E_n^q)$

$$(\eta_n^N)^{\otimes q}(F_n) = \frac{1}{N^q} \sum_{p=1}^q \sum_{\pi \in \mathcal{P}(q,p)} \sum_{\beta \in < p,N >} F_n(\xi_n^{\beta_\pi(1)}, \ldots, \xi_n^{\beta_\pi(q)}) \tag{2.8}$$

This formula is reasonably well known (for a complete proof we refer the reader to [2], section 8.6, on page 267). The simplest nontrivial example often used in statistics is the 2-tensor product measure. In this case, we have

$$\begin{aligned} &\mathcal{P}(2,1) = \{\pi\} \quad \text{with} \quad \pi = \{\pi_1, \pi_2\} \quad \text{and} \quad \pi_1 = \{1\} \leq \pi_2 = \{2\} \\ &\mathcal{P}(2,2) = \{\pi\} \quad \text{with} \quad \pi = \pi_1 = \{1,2\} \end{aligned}$$

Thus, equation (2.8) for $q = 2$, takes the simplest form

$$(\eta_n^N)^{\otimes 2}(F_n) = \frac{1}{N^2} \sum_{\beta \in <1,N>} F_n(\xi_n^{\beta(1)}, \xi_n^{\beta(1)}) + \frac{1}{N^2} \sum_{\beta \in <2,N>} F_n(\xi_n^{\beta(1)}, \xi_n^{\beta(2)}).$$

In a more general set-up, we observe that the q-th term in the r.h.s. of (2.8) is given by

$$\frac{1}{N^q} \sum_{\beta \in <q,N>} F_n(\xi_n^{\beta(1)}, \ldots, \xi_n^{\beta(q)})$$

For $p = (q-1)$, the set $\mathcal{P}(q, q-1)$ consists of all $q(q-1)/2$ partitions π with one block of two elements, and $(q-2)$ blocks of one element. For instance, for $\pi = \{\pi_1, \ldots, \pi_{q-1}\}$, with

$$\pi_1 = \{1,2\} \leq \pi_2 = \{3\} \leq \ldots \leq \pi_{q-1} = \{q\}$$

we find that

$$\sum_{\beta \in <q-1,N>} F_n(\xi_n^{\beta_\pi(1)}, \ldots, \xi_n^{\beta_\pi(q)}) = \sum_{\beta \in <q-1,N>} F_n(\xi_n^{\beta(1)}, \xi_n^{\beta(1)}, \xi_n^{\beta(2)} \ldots, \xi_n^{\beta(q-1)}).$$

To go one step further in our discussion, we observe that, for any $p \leq q$, we have

$$F_n(\xi_n^{\beta_\pi(1)}, \ldots, \xi_n^{\beta_\pi(q)}) = C_\pi^{p,q}(F_n)(\xi_n^{\beta(1)}, \ldots, \xi_n^{\beta(p)}),$$

with the Markov operator $C_\pi^{p,q}$ from E_n^p into E_n^q defined by

$$C_\pi^{p,q}(F_n)(x^1, \ldots, x^p) = F_n(\sum_{i=1}^p x^i \ 1_{\pi_i}(1), \ldots, \sum_{i=1}^p x^i \ 1_{\pi_i}(q)).$$

We extend $C_\pi^{p,q}$ to a Markov operator $C_\pi^{(p,q)}$ from E_n^q into itself by setting

$$C_\pi^{(p,q)}(F_n)(x^1, \ldots, x^q) = C_\pi^{p,q}(F_n)(x^1, \ldots, x^p).$$

The operator $C_\pi^{(p,q)}$ has the following interpretation; let π be a partition of q elements into p increasing blocks $\pi_1, \ldots, \pi_p$ of respective sizes $b_1, \ldots, b_p$. Sampling a configuration according to $C_\pi^{(p,q)}((x^1, \ldots, x^q), \boldsymbol{.})$ consists of duplicating the individuals x^i, b_i times, where $1 \leq i \leq p$. In this sense, $C_\pi^{(p,q)}$ can also be interpreted as a selection jump operator. By construction, an elementary calculation provides the following identities

$$\frac{1}{(N)_p} \sum_{\beta \in <p,N>} F_n(\xi_n^{\beta_\pi(1)}, \ldots, \xi_n^{\beta_\pi(q)}) = (\eta_n^N)^{\odot p} C_\pi^{p,q} = (\eta_n^N)^{\odot q} C_\pi^{(p,q)}.$$

Using (2.8), its straightforward to prove the following proposition.

Proposition 2.3. *For any block size parameter $1 \leq q \leq N$, we have the decomposition*

$$(\eta_n^N)^{\otimes q} = \frac{1}{N^q} \sum_{p=1}^q \ S(q,p) \ (N)_p \ (\eta_n^N)^{\odot q} C^{(p,q)}, \tag{2.9}$$

with $S(q,p)$ is the Stirling number of the second kind, and the Markov selection-jump type operator $C^{(p,q)}$ from E_n^q into itself is given by*

$$C^{(p,q)} = \frac{1}{S(q,p)} \sum_{\pi \in \mathcal{P}(q,p)} C_\pi^{(p,q)}.$$

3. Asymptotic propagation of chaos properties. The main objective of this section is to prove the theorems stated in Section 1.2.

In Subsection 3.1 we provide a weak asymptotic expansion of the distribution $\mathbb{P}_n^{(q,N)}$ of the first q particles $(\xi_n^i)_{1 \leq i \leq q}$ on tensor product functions. We analyze a general class of McKean selection models (1.4). The study is decomposed into two parts. First we examine the bias of the tensor product particle measures $(\eta_n^N)^{\otimes q}$ using the random field fluctuation analysis presented in Subsection 2.1. Then we transfer this result to $\mathbb{P}_n^{(q,N)}$ with the help of the combinatorial transport equation developed in Subsection 2.2.

In Subsection 3.3, we extend the weak asymptotic expansions derived in Subsection 3.1 to the class of continuous and bounded functions on product spaces. This

*$S(q,p)$ corresponds to the number of partitions of q elements in p blocks.

analysis is restricted to a simple genetic particle model evolving on locally compact and separable metric spaces. We recall that these genetic models correspond to the McKean selection transition (1.4), with $\epsilon_n = 0$. Our strategy consists of analyzing the tensor product bias, and the asymptotic first order quantities as linear functional operators. This integration of a propagation of chaos interpretation, combined with a density argument allows us to derive an asymptotic expansion $\mathbb{P}_n^{(q,N)}$.

3.1. General McKean particle models. As we mentioned above, Proposition 2.2 is pivotal in the analysis of the bias of the path-particle models. To illustrate our approach, we present an elementary consequence of Proposition 2.2. We first rewrite (2.5) as follows

$$\begin{aligned}
&W_n^{\eta,N}(f_n)\\
&= \gamma_n(1)^{-1}\ W_n^{\gamma,N}\left(f_n - \eta_n(f_n)\right) + \left(\gamma_n^N(1)^{-1} - \gamma_n(1)^{-1}\right)\ \times\ \mathcal{W}_n^{\gamma,N}\left(\left(f_n - \eta_n(f_n)\right)\right)\\
&= \gamma_n(1)^{-1}\ W_n^{\gamma,N}\left(f_n - \eta_n(f_n)\right) - \tfrac{1}{\sqrt{N}}\ [\gamma_n^N(1)\gamma_n(1)]^{-1}\ W_n^{\gamma,N}(f_n - \eta_n(f_n))W_n^{\gamma,N}(1).
\end{aligned}$$

This readily yields that

$$N\ \mathbb{E}(\eta_n^N(f_n) - \eta_n(f_n)) = -\mathbb{E}\left([\gamma_n^N(1)\gamma_n(1)]^{-1}\ W_n^{\gamma,N}(f_n - \eta_n(f_n))W_n^{\gamma,N}(1)\right).$$

Note that the random sequence $(1/(\gamma_n(1)\gamma_n^N(1)))_{N\geq 1}$ is uniformly bounded, and it converges in law to $\gamma_n(1)^{-2}$, as N tends to infinity. Now using Proposition 2.2, we find the sharp asymptotic bias estimate

$$\begin{aligned}
&\lim_{N\to\infty}\ N\ \mathbb{E}(\eta_n^N(f_n) - \eta_n(f_n))\\
&= -\gamma_n(1)^{-2}\ \mathbb{E}\left(W_n^{\gamma}(1)\ W_n^{\gamma}\left(f_n - \eta_n(f_n)\right)\right)\\
&= -\sum_{p=0}^{n}\left(\gamma_p(1)/\gamma_n(1)\right)^2 \mathbb{E}\left(W_p(Q_{p,n}(1))\ W_p(Q_{p,n}[f_n - \eta_n(f_n)])\right). \qquad (3.1)
\end{aligned}$$

The next technical proposition extends this result to the tensor empirical measures $(\eta_n^N)^{\otimes q}$.

PROPOSITION 3.1. *For any $n, q \geq 1$, and every sequence of functions $f_n^{(k)} \in \mathcal{B}_b(E_n)$, with $\eta_n(f_n^{(k)}) = 1$, and $1 \leq k \leq q$, we have*

$$\begin{aligned}
&\lim_{N\to\infty} N\ \mathbb{E}\left[\prod_{k=1}^{q}\eta_n^N(f_n^{(k)}) - 1\right]\\
&= -\sum_{i=1}^{q}\gamma_n(1)^{-2}\ \mathbb{E}\left(W_n^{\gamma}(1)\ W_n^{\gamma}\left(f_n^{(i)} - 1\right)\right) + \sum_{1\leq i<j\leq q}\ \mathbb{E}\left(W_n^{\eta}(f_n^{(i)})W_n^{\eta}(f_n^{(j)})\right).
\end{aligned}$$

Proof:
We use the decomposition

$$\prod_{i=1}^{q} a_i - 1 = \sum_{1\leq p\leq q}\ \sum_{1\leq j_1<\ldots<j_p\leq q}\ \prod_{l=1}^{p}(a_{j_l} - 1)$$

which is valid for any $q \geq 0$, and any collection of real numbers $(a_i)_{1\leq i\leq q}$. Using (2.6) and Hölder's inequality we readily find that

$$|\mathbb{E}[\prod_{p=1}^{q}(\eta_n^N(f_n^{(p)})-1)]| \leq \prod_{p=1}^{q}\mathbb{E}[|\eta_n^N(f_n^{(p)})-1|^q]^{1/q} \leq \frac{c_q(n)}{N^{q/2}}\ [\prod_{p=1}^{q}\|f_n^{(p)}\|]$$

for some finite constant $c_q(n)$, which depends only on the pair of parameters (q,n). If we take $a_i = \eta_n^N(f_n^{(i)})$ in the above decomposition, then we obtain

$$\prod_{p=1}^{q}\eta_n^N(f_n^{(p)}) - 1 = \sum_{1\leq p\leq q}\ \sum_{1\leq j_1<\ldots<j_p\leq q} N^{-p/2}\left[\prod_{l=1}^{p} W_n^{\eta,N}(f_n^{(j_l)})\right] \tag{3.2}$$

from which we conclude that

$$\mathbb{E}\left[\prod_{k=1}^{q}\eta_n^N(f_n^{(k)})-1\right] = \sum_{i=1}^{q}\ \mathbb{E}\left[(\eta_n^N(f_n^{(i)})-1\right]$$
$$+\sum_{1\leq i<j\leq q}\ \mathbb{E}\left[(\eta_n^N(f_n^{(i)})-1)(\eta_n^N(f_n^{(j)})-1)\right] + \mathrm{O}\left(\frac{1}{N\sqrt{N}}\right).$$

The end of the proof is now a straightforward consequence of (3.1) and Proposition 2.2. ■

3.2. Fluctuations of the particle tensor product measures. Before going into further details, it is important to make a couple of remarks. We first observe that formula (3.2) can be used to analyze the fluctuations of the particle tensor product measures $(\eta_n^N)^{\otimes q}$ on the set of test functions $Poly(E_n^q)$. Indeed, a simple argument shows that the random sequence $\sqrt{N}[\prod_{k=1}^{q}\eta_n^N(f_n^{(k)})-1]$ converges in law, as $N\to\infty$, to the centered Gaussian sum $\sum_{k=1}^{q}W_n^{\eta}(f_n^{(k)})$. More formally, if we set $F_n = (f_n^{(1)}\otimes\ldots\otimes f_n^{(q)})$, then we obtain the fluctuations of the U-statistics associated with an interacting particle model; that is, we have that

$$\lim_{N\to\infty}\sqrt{N}[(\eta_n^N)^{\otimes q} - \eta_n^{\otimes q}](F_n) = \sum_{k=1}^{q}(\eta_n^{\otimes(k-1)}\otimes W_n^{\eta}\otimes\eta_n^{\otimes(q-k)})(F_n). \tag{3.3}$$

By linearity arguments, we prove that the above convergence also holds for any $F_n \in Poly(E_n^q)$. Using (3.2) again, we have the second order fluctuations

$$\begin{aligned}&\lim_{N\to\infty} N\ \left\{(\eta_n^N)^{\otimes q} - \eta_n^{\otimes q} - \sum_{k=1}^{q}[\eta_n^{\otimes(k-1)}\otimes(\eta_n^N-\eta_n)\otimes\eta_n^{\otimes(q-k)}]\right\}(F_n)\\ &= \sum_{1\leq k<l\leq q}[\eta_n^{\otimes(k-1)}\otimes W_n^{\eta}\otimes\eta_n^{\otimes(l-k-1)}\otimes W_n^{\eta}\otimes\eta_n^{\otimes(q-l)}](F_n)\end{aligned} \tag{3.4}$$

for any $F_n \in Poly(E_n^q)$. These asymptotic results clearly enhance and strengthen the weak expansions in Proposition 3.1.

The statement of Proposition 3.1 corresponds to the first part of the Theorem 1.1. The end of the proof of the theorem results from the following proposition.

Proposition 3.2. *For any time horizon $n\geq 1$, any particle block size parameter $1\leq q\leq N$, and every sequence of functions $f_n^{(k)}\in\mathcal{B}_b(E_n)$, with $\eta_n(f_n^{(k)})=1$, and*

$1 \leq k \leq q$, *we have*

$$\lim_{N\to\infty} N \ \left[\mathbb{E}\left(f_n^{(1)}(\xi_n^1)\ldots f_n^{(q)}(\xi_n^q)\right)-1\right]$$

$$= -\sum_{i=1}^q \gamma_n(1)^{-2}\ \mathbb{E}\left(W_n^\gamma(1)\ W_n^\gamma\left(f_n^{(i)}-1\right)\right)$$

$$+\sum_{1\leq i<j\leq q}\left[\ \mathbb{E}\left(W_n^\eta(f_n^{(i)})W_n^\eta(f_n^{(j)})\right)+\left(1-\eta_n(f_n^{(i)}f_n^{(j)})\right)\right]$$

Proof:
After some elementary manipulations, and using (2.8), we prove that

$$\mathbb{E}\left((\eta_n^N)^{\otimes q}(f_n^{(1)}\otimes\ldots\otimes f_n^{(q)})\right)-1=\frac{(N)_q}{N^q}\ I_n^{(1,N)}(f)+\frac{(N)_{q-1}}{N^q}\ I_n^{(2,N)}(f)+\mathrm{O}\left(\frac{1}{N^2}\right) \tag{3.5}$$

with $I_n^{(1,N)}(f)=\left[\mathbb{E}(f_n^{(1)}(\xi_n^1)\ldots f_n^{(q)}(\xi_n^q))-1\right]$, and

$$I_n^{(2,N)}(f)$$

$$=\sum_{1\leq i<j\leq q}\left[\mathbb{E}\left((f_n^{(i)}f_n^{(j)})(\xi_n^i)\ [\prod_{1\leq k<j,\ k\neq i}f_n^{(k)}(\xi_n^k)]\ [\prod_{j\leq k<q}f_n^{(k+1)}(\xi_n^k)]\right)-1\right].$$

Meanwhile, by (1.9) we observe that $\lim_{N\to\infty} I_n^{(1,N)}(f)=0$, and since we have

$$N\left[1-\frac{(N)_q}{N^q}\right]\leq N\left[1-\left(1-\frac{q-1}{N}\right)^{q-1}\right)\leq (q-1)^2$$

we also find that $\lim_{N\to\infty} N\left(1-\frac{(N)_q}{N^q}\right)\ I_n^{(1,N)}(f)=0$. In much the same way, we have

$$\lim_{N\to\infty} I_n^{(2,N)}(f)=I_n^{(2)}(f)=_{\text{def.}}\sum_{1\leq i<j\leq N}\left[\eta_n(f_n^{(i)}f_n^{(j)})-1\right].$$

This clearly yields that $\lim_{N\to\infty}(N)_{q-1}N^{-(q-1)}\ I_n^{(2,N)}(f)=I_n^{(2)}(f)$. These estimates together with (3.5) imply that

$$\lim_{N\to\infty} N\ I_n^{(1,N)}(f)=-I_n^{(2)}(f)+\lim_{N\to\infty} N\ \mathbb{E}\left[\prod_{k=1}^q \eta_n^N(f_n^{(k)})-1\right].$$

The end of the proof is now a straightforward consequence of Proposition 3.1.

■

We end this section with a discussion of the fluctuations of $(\eta_n^N)^{\odot q}$. Using the combinatorial transport equation (2.9), we first observe that

$$[(\eta_n^N)^{\otimes q}-\eta_n^{\otimes q}]$$

$$=\frac{(N)_q}{N^q}\ [(\eta_n^N)^{\odot q}-\eta_n^{\otimes q}]+\frac{(N)_{q-1}}{N^q}\ S(q,q-1)\ [(\eta_n^N)^{\odot q}C^{q-1,q}-\eta_n^{\otimes q}]+\epsilon_n^{(N,q)}(F_n)$$

with a remainder term $\epsilon_n^{(N,q)}$, such that $\sup_{N\geq 1} N^2|\epsilon_n^{(N,q)}(F_n)| < \infty$. Using (3.3) and taking (3.4) into account, we can verify the following central limit theorem for nondegenerate von Mises type statistics.

THEOREM 3.3. *For any time horizon $n \geq 1$, any particle block size parameter $1 \leq q \leq N$, and for any $F_n \in Poly(E_n^q)$, we have the following convergence in law*

$$\lim_{N\to\infty} \sqrt{N}[(\eta_n^N)^{\odot q} - \eta_n^{\otimes q}](F_n) = \sum_{k=1}^{q} (\eta_n^{\otimes(j-1)} \otimes W_n^\eta \otimes \eta_n^{\otimes(q-j)})(F_n).$$

Finally, we also find that for the second order fluctuations

$$\lim_{N\to\infty} N \left\{(\eta_n^N)^{\odot q} - \eta_n^{\otimes q} - \sum_{k=1}^q [\eta_n^{\otimes(j-1)} \otimes (\eta_n^N - \eta_n) \otimes \eta_n^{\otimes(q-j)}]\right\}(F_n)$$

$$= -S(q,q-1)\ \eta_n^{\otimes q}[C^{q-1,q} - Id](F_n)$$

$$+ \sum_{1\leq k<l\leq q} [\eta_n^{\otimes(k-1)} \otimes W_n^\eta \otimes \eta_n^{\otimes(l-k-1)} \otimes W_n^\eta \otimes \eta_n^{\otimes(q-l)}](F_n).$$

3.3. Simple genetic particle models. In this section, we analyze the simple genetic particle model associated with the McKean interpretation (1.4), with $\epsilon_n = 0$. We also restrict our analysis to locally compact and separable metric spaces E_n.

DEFINITION 3.4. *Let $(\mathcal{L}_n^{(q,N)}, \mathcal{M}_n^{(q,N)})$ be the pair of linear operators on $\mathcal{B}_b(E_n^q)$ defined for any $F_n \in \mathcal{B}_b(E_n^q)$ by the formulae*

$$\mathcal{L}_n^N(F_n) = N\ \mathbb{E}[(\eta_n^N)^{\otimes q}(F_n) - \eta_n^{\otimes q}(F_n)]$$
$$\mathcal{M}_n^N(F_n) = N\ \mathbb{E}[(\eta_n^N)^{\odot q}(F_n) - \eta_n^{\otimes q}(F_n)]$$

where the pair of particle measures $((\eta_n^N)^{\otimes q}, (\eta_n^N)^{\odot q})$ on E_n^q are defined in (2.7).

By (2.9), we have that

$$\mathcal{L}_n^{(q,N)}(F_n)$$

$$= \frac{(N)_q}{N^q}\ \mathcal{M}_n^{(q,N)}(F_n) + \frac{1}{N^{q-1}} \sum_{p=1}^{q-1}\ S(q,p)\ \ (N)_p\ \mathbb{E}\left[(C^{(p,q)} - Id)F_n(\xi_n^1, \ldots, \xi_n^q)\right].$$

By the propagation of chaos estimates presented in (1.9), we have the bounds

$$\sup_{N\geq 1} \sup_{\|F_n\|\leq 1} \{|\mathcal{L}_n^{(q,N)}(F_n)| \vee |\mathcal{M}_n^{(q,N)}(F_n)|\} \leq c(n)\ (q^2 + \sum_{p=1}^{q-1} S(q,p)). \tag{3.6}$$

In other words, $(\mathcal{L}_n^{(q,N)})_{N\geq 1}$, and $(\mathcal{M}_n^{(q,N)})_{N\geq 1}$ are uniformly bounded sequences of linear operators on $\mathcal{B}_b(E_n^q)$. On the other hand, for polynomial functions of the form

$$F_n = f_n^{(1)} \otimes \ldots \otimes f_n^{(q)}$$

with $(f_n^{(k)})_{1\leq k\leq q} \in \mathcal{B}_b(E^q)$, we have, from Proposition 3.1,

$$\lim_{N\to\infty} N\ \mathbb{E}[(\eta_n^N)^{\otimes q}(F_n) - \eta_n^{\otimes q}(F_n)] = \mathcal{L}_n^{(q)}(F_n) =_{\text{def.}} \mathcal{L}_n^{(q,N),1}(F_n) + \mathcal{L}_n^{(q,N),2}(F_n) \tag{3.7}$$

where the pair of linear operators $(\mathcal{L}_n^{(q,N),1}, \mathcal{L}_n^{(q,N),2})$ are defined by

$$\mathcal{L}_n^{(q,N),1}(F_n) = -\sum_{i=1}^{q} \gamma_n(1)^{-2}\ \mathbb{E}\left[W_n^\gamma(1)\ W_n^\gamma\left(f_n^{(i)} - \eta_n(f_n^{(i)})\right)\right]\ [\prod_{1\leq j\leq q,\ j\neq i} \eta_n(f_n^{(j)})],$$

$$\mathcal{L}_n^{(q,N),2}(F_n) = \sum_{1\leq i<j\leq q} \mathbb{E}\left(W_n^\eta(f_n^{(i)}) W_n^\eta(f_n^{(j)})\right)\ [\prod_{1\leq k\leq q,\ k\notin\{i,j\}} \eta_n(f_n^{(k)})].$$

By the definition of the random field W_n^η given in (2.4), we find that

$$\begin{aligned}
&\gamma_n(1)^{-2}\ \mathbb{E}\left[W_n^\gamma(1)\ W_n^\gamma\left(f_n^{(i)} - \eta_n(f_n^{(i)})\right)\right] \\
&= \textstyle\sum_{p=0}^n \eta_p[R_{p,n}(1)\ R_{p,n}(f_n^{(i)} - \eta_n(f_n^{(i)}))] \\
&= \textstyle\sum_{p=0}^n \eta_p[R_{p,n}(1)\ R_{p,n}(f_n^{(i)})] - \eta_n(f_n^{(i)})\ \sum_{p=0}^n \eta_p[R_{p,n}(1)^2].
\end{aligned}$$

¿From these observations, we obtain the operator decomposition

$$\mathcal{L}_n^{(q,N),1}(F_n) = \sum_{p=0}^{n} \mathcal{L}_{p,n}^{(q,N),1}(F_n)$$

with the collection of linear operators $\mathcal{L}_{p,n}^{(q,N),1}$ given by

$$\begin{aligned}
\mathcal{L}_{p,n}^{(q,N),1}(F_n) = &-\sum_{i=1}^{q} \eta_p[R_{p,n}(1)\ R_{p,n}(f_n^{(i)})]\ [\prod_{1\leq j\leq q,\ j\neq i} \eta_p R_{p,n}(f_n^{(j)})] \\
&+ q\ \eta_p[R_{p,n}(1)^2]\ \eta_n^{\otimes q}(F_n).
\end{aligned}$$

By the definition of the renormalized semigroup $R_{p,n}$ given in (1.11), we recall that $\eta_p R_{p,n} = \eta_n$. This implies that

$$\mathcal{L}_{p,n}^{(q,N),1}(F_n) = -q\ \eta_p^{\otimes q}[\overline{R}_{p,n}^q(1)\ R_{p,n}^{\otimes q}(F_n)] + q\ \eta_p^{\otimes q}[\overline{R}_{p,n}^q(1)\ R_{p,n}^{\otimes q}(1)]\ \eta_n^{\otimes q}(F_n).$$

with the collection of functions $\overline{R}_{p,n}^q(1)$ on E_n^q given by

$$\overline{R}_{p,n}^q(1)(x_n^1, \ldots, x_n^q) = \frac{1}{q}\sum_{i=1}^{q} \overline{R}_{p,n}(1)(x_n^i).$$

The above arguments show that

$$\mathcal{L}_{p,n}^{(q,N),1}(F_n) = -q\ \eta_p^{\otimes q}\left[\overline{R}_{p,n}^q(1)\ R_{p,n}^{\otimes q}(F_n - \eta_n^{\otimes q}(F_n))\right].$$

In much the same way, we have the operator decomposition

$$\mathcal{L}_n^{(q,N),2}(F_n) = \sum_{p=0}^{n} \mathcal{L}_{p,n}^{(q,N),2}(F_n),$$

with the collection of linear operators $\mathcal{L}_{p,n}^{(q,N),2}$ given by

$$\mathcal{L}_{p,n}^{(q,N),2}(F_n) = \sum_{1\leq i<j\leq q} \eta_p \left[R_{p,n}(f_n^{(i)} - \eta_n(f_n^{(i)}))\ R_{p,n}(f_n^{(j)} - \eta_n(f_n^{(j)}))\right] \times \left[\prod_{1\leq k\leq q,\ k\notin\{i,j\}} \eta_n(f_n^{(k)})\right]$$

$$= \sum_{1\leq i<j\leq q} \eta_p^{\otimes q} C_{i,j} R_{p,n}^{\otimes q}(I - p_n^i)(I - p_n^j)(F_n).$$

Since $\mathcal{L}_n^{(q)}$ is a bounded linear operator, $(\mathcal{L}_n^{(q,N)})_{N\geq 1}$ is a sequence of uniformly bounded operators, and recalling that the set of linear combinations of polynomial functions is dense in $\mathcal{C}_b(E_n^q)$, we easily prove the first assertion of Theorem 1.2. First, an elementary calculation reveals

$$|\mathcal{L}_n^{(q)}(F_n)| \leq c(n)\ q^2\ \|F_n\| \tag{3.8}$$

for any $F_n \in \mathcal{C}_b(E_n^q)$, and for some finite constant $c(n) < \infty$, whose values do not depend on F_n. Let $(F_n^\epsilon)_{\epsilon\geq 0}$ be an ϵ-approximation of $F_n \in \mathcal{C}_b(E_n^q)$, by linear combinations of polynomial functions F_n^ϵ. If one uses (3.6) and (3.8), then we get the estimate

$$|\mathcal{L}_n^{(q,N)}(F_n) - \mathcal{L}_n^{(q)}(F_n)| \leq c(n)\ q^2\ \|F_n - F_n^\epsilon\| + |\mathcal{L}_n^{(q,N)}(F_n^\epsilon) - \mathcal{L}_n^{(q)}(F_n^\epsilon)|$$

Letting $N \to \infty$, and then $\epsilon \to 0$, we obtain the following proposition.

Proposition 3.5. *For any time horizon $n \geq 0$, any function $F_n \in \mathcal{C}_b(E_n^q)$, and any particle block size parameter $1 \leq q \leq N$, we have the convergence*

$$\lim_{N\to\infty} \mathcal{L}_n^{(q,N)}(F_n) = \mathcal{L}_n^{(q)}(F_n)$$

with the linear operator $\mathcal{L}_n^{(q)}$ defined by

$$\mathcal{L}_n^{(q)}(F_n) = -q \sum_{p=0}^{n-1} \eta_p^{\otimes q} \left[\overline{R}_{p,n}^q(1)\ R_{p,n}^{\otimes q}(F_n - \eta_n^{\otimes q}(F_n))\right] + \sum_{p=0}^{n} \sum_{1\leq i<j\leq q} \eta_p^{\otimes q} C_{i,j} R_{p,n}^{\otimes q}(Id - p_n^i)(Id - p_n^j)(F_n)$$

Arguing, as in the beginning of the proof of Proposition 3.2, we also observe that

$$\begin{aligned}
&\mathbb{E}[(\eta_n^N)^{\otimes q}(F_n) - \eta_n^{\otimes q}(F_n)] \\
&= \tfrac{(N)_q}{N^q}\ \mathbb{E}[(\eta_n^N)^{\odot q}(F_n) - \eta_n^{\otimes q}(F_n)] \\
&\qquad + \tfrac{(N)_{q-1}}{N^q}\ S(q, q-1)\ \mathbb{E}[(\eta_n^N)^{\odot q} C^{(q-1,q)}(F_n) - \eta_n^{\otimes q}(F_n)] + \mathrm{O}(N^{-2})
\end{aligned}$$

The propagation of chaos estimate presented in (1.9), combined with the exchangeability property of the particle configurations, implies that

$$N\ |\mathbb{E}((\eta_n^N)^{\odot q} C^{(p,q)}(F_n)) - \eta_n^{\otimes q} C^{(p,q)}(F_n)| \leq q^2\ c(n)\ \|F_n\|$$

for any $1 \leq p \leq q \leq N$, and any function $F_n \in \mathcal{B}_b(E_n^q)$. From these estimates we also deduce that

$$\mathcal{L}_n^{(q)}(F_n) = \mathcal{M}_n^{(q)}(F_n) + \frac{q(q-1)}{2}\ \eta_n^{\otimes q}[C^{(q-1,q)} - Id](F_n)$$

where the bounded linear operator $\mathcal{M}_n^{(q)}$ on $\mathcal{B}_b(E_n^q)$ is given by

$$\mathcal{M}_n^{(q)}(F_n) =_{\text{def.}} \lim_{N\to\infty} N\ \mathbb{E}[(\eta_n^N)^{\odot q}(F_n) - \eta_n^{\otimes q}(F_n)]$$

As an aside, using (1.9) again, we find that $|\mathcal{M}_n^{(q)}(F_n)| \leq c(n)\ q^2\ \|F_n\|$. In conclusion, for any $F_n \in \mathcal{C}_b(E_n^q)$, $n \geq 1$, and $1 \leq q \leq N$ we have proved that

$$\lim_{N\to\infty} N\ \mathbb{E}[F_n(\xi_n^1, \ldots, \xi_n^q) - \eta_n^{\otimes q}(F_n)] = \mathcal{M}_n^{(q)}(F_n)$$

where the bounded linear operator $\mathcal{M}_n$ on $\mathcal{C}_b(E_n^q)$ is given by

$$\mathcal{M}_n^{(q)}(F_n) = \mathcal{L}_n^{(q)}(F_n) - \sum_{1\leq i<j\leq q} \eta_n^{\otimes q}[C_{i,j} - Id](F_n)$$

where $\mathcal{L}_n^{(q)}$ is the operator defined in (3.7). This clearly ends the proof of Theorem 1.2.

4. First order uniform estimations. The asymptotic propagation of chaos expansions stated in Theorem 1.2 are expressed in terms of Feynman-Kac tensor product semigroups. Except in particular situations, such as finite state space models, these functional semigroups are rather complex, and difficult to solve analytically. The aim of this section is to estimate these quantities. More precisely, we design an original contraction semigroup technique for estimating the norm of the first order operators $\mathcal{M}_n^{(q)}$ introduced in (1.16).

Before proceeding, we recall the definition of the Dobrushin coefficient and provide some key properties. We recall that the total variation distance between two probability measures μ, ν on some measurable space $(E, \mathcal{E})$ can alternatively be defined by

$$\|\mu - \nu\|_{\text{tv}} = 2^{-1} \sup_{(A,B)\in\mathcal{E}^2} (\mu(A) - \nu(B)) = \sup\left\{|\mu(f) - \nu(f)|\ ;\ f \in \text{Osc}(E)\right\}$$

where $\text{Osc}(E)$ represents the set of measurable functions f on E such that

$$\text{osc}(f) =_{\text{def.}} \sup_{(x,y)\in E^2} |f(x) - f(y)| \leq 1.$$

The Dobrushin contraction coefficient $\beta(M)$ of a Markov operator M from E into another measurable space $(F, \mathcal{F})$ is the quantity defined by

$$\beta(M) = \sup_{(x,y)\in E^2} \|M(x,.) - M(y,.)\|_{.\text{tv}}$$

The coefficient $\beta(M)$ can also be seen as the largest constant satisfying one of the two inequalities for any bounded measurable function f, or for any probability measures μ, ν,

$$\|\mu M - \nu M\|_{\text{tv}}\ \leq\ \beta(M)\ \|\mu - \nu\|_{\text{tv}} \quad \text{and} \quad \text{osc}(M(f))\ \leq\ \beta(M)\ \text{osc}(f)$$

We end this brief reminder by recalling that $\|\mu^{\otimes q} - \nu^{\otimes q}\|_{\mathrm{tv}} \;\leq\; q\ \|\mu - \nu\|_{\mathrm{tv}}$, from which we readily find the rather crude estimate $\beta(M^{\otimes q}) \leq q\ \beta(M)$.

We are now in position to estimate the norm of the pair operators $(\mathcal{L}_n^{(q)}, \mathcal{M}_n^{(q)})$ introduced in (1.14) and (1.16). Let $P_{p,n}$ be the renormalized Feynman-Kac semigroup from E_p into E_n defined for any $f_n \in \mathcal{B}_b(E_n)$ by the formula

$$P_{p,n}(f_n) = Q_{p,n}(f_n)/Q_{p,n}(1).$$

Proposition 4.1. *For any time horizon $n \geq 1$, any particle block size parameter $1 \leq q \leq N$ and any function $F_n \in \mathcal{B}_b(E_n^q)$, with $\mathrm{osc}(F_n) \leq 1$, we have the estimates*

$$|\mathcal{M}_n^{(q)}(F_n)| \leq |\mathcal{L}_n^{(q)}(F_n)| + q(q-1)/2$$

and

$$|\mathcal{L}_n^{(q)}(F_n)| \leq \frac{3}{2}\ q^2\ \sum_{p=0}^{n} \beta(P_{p,n})\ \eta_p(R_{p,n}(1)^2)$$

Before discussing the details of the proof of the above result, we underline the impact of the above estimate on simple genetic models with sufficiently regular mutations. If we set

$$r_{p,n} = \sup_{(x_p,y_p)\in E_p^2} Q_{p,n}(1)(x_p)/Q_{p,n}(1)(y_p)$$

then, recalling that $\eta_p R_{p,n} = \eta_n$, we conclude that

$$|\mathcal{M}_n^{(q)}(F_n)| \leq \frac{3}{2}\ q^2\ \sum_{p=0}^{n} r_{p,n}\ \beta(P_{p,n}) + q(q-1)/2 \tag{4.1}$$

When the mutation transitions M_n satisfy a Doeblin type mixing condition, it is well known that the summation term in the r.h.s. of (4.1) is uniformly bounded with respect to the time parameter. For instance, let us assume that for some constants $\epsilon > 0$, and $r < \infty$, the following pair condition is met

$$M_{n+1}(x_n, .) \geq \epsilon\ M_{n+1}(y_n, .) \quad \text{and} \quad G_n(x_n) \leq r G_n(y_n)$$

for any time horizon $n \geq 0$, and for any pair of states $(x_n, y_n) \in E_n^2$. In this case, we have the rather crude estimates

$$r_{p,n} \leq r\ \epsilon^{-2} \quad \text{and} \quad \beta(P_{p,n}) \leq (1-\epsilon^2)^{(n-p)} \Rightarrow \sum_{p=0}^{n}\ r_{p,n}\ \beta(P_{p,n}) \leq r\ \epsilon^{-4}$$

from which we conclude that

$$|\mathcal{M}_n^{(q)}(F_n)| \leq \frac{3}{2}\ q^2\ r\ \epsilon^{-4} + q(q-1)/2$$

For more refined estimates we refer the reader to Chapter 4 in [2], and the references therein.

Now we come to the proof of the proposition.

Proof of Proposition 4.1:
Firstly, we observe that

$$R_{p,n}^{\otimes q}(F_n)/R_{p,n}^{\otimes q}(1) = P_{p,n}^{\otimes q}(F_n)$$

for any pair of test functions $(f_n, F_n) \in (\mathcal{B}_b(E_n) \times \mathcal{B}_b(E_n^q))$. After some elementary calculations, we find that for any $0 \leq p \leq n$ and $F_n \in \mathcal{B}_b(E_n^q)$, with $\mathrm{osc}(F_n) \leq 1$,

$$|\eta_p^{\otimes q}\left[\overline{R}_{p,n}^q(1)\ R_{p,n}^{\otimes q}(F_n - \eta_n^{\otimes q}(F_n))\right]| \leq\ q\ \eta_p(R_{p,n}(1)^2)\ \beta(P_{p,n})$$

To estimate the second term in the r.h.s. of (1.14), with some obvious abuse of notation, we observe

$$\begin{aligned}
&P_{p,n}^{\otimes q}(Id - p_n^i)(F_n)(x_p^1, \ldots, x_p^q)\\
&= \int [\prod_{1\leq k\leq q,\ k\neq i} P_{p,n}(x_p^k, dy_n^k)]\\
&\qquad \times \{\int P_{p,n}(x_p^i, dy_n^i) F_n(y_n^1, \ldots, y_n^q) - \int \eta_p P_{p,n}(dy_n'^i) F_n(y_n^1, \ldots, y_n'^i, \ldots, y_n^q))\}.
\end{aligned}$$

This yields that

$$\|P_{p,n}^{\otimes q}(Id - p_n^i)(F_n)\| \leq \beta(P_{p,n})\ \mathrm{osc}(F_n).$$

In much the same way, we find

$$\|P_{p,n}^{\otimes q}(Id - p_n^i)p_n^j(F_n)\| \leq \beta(P_{p,n})\ \mathrm{osc}(p_n^j F_n) \leq \beta(P_{p,n})\ \mathrm{osc}(F_n).$$

These two estimates readily imply that

$$\|P_{p,n}^{\otimes q}(Id - p_n^i)(Id - p_n^j)(F_n)\| \leq \beta(P_{p,n})\ \mathrm{osc}(F_n)$$

from which we conclude that

$$\begin{aligned}
|\eta_p^{\otimes q} C_{i,j} R_{p,n}^{\otimes q}(Id - p_n^i)(Id - p_n^j)(F_n)| &\leq \beta(P_{p,n})\ \eta_p^{\otimes q} C_{i,j} R_{p,n}^{\otimes q}(1)\\
&= \beta(P_{p,n})\ \eta_p(R_{p,n}(1)^2)
\end{aligned}$$

when $\mathrm{osc}(F_n) \leq 1$.

$$|\mathcal{L}_n^{(q)}(F_n)| \leq \frac{3}{2}\ q^2\ \sum_{p=0}^{n} \beta(P_{p,n})\ \eta_p(R_{p,n}(1)^2)$$

The remainder of the proof of the proposition is now clear. ■

REFERENCES

[1] G. Ben Arous, and O.I. Zeitouni, Increasing Propagation of Chaos for Mean Field Models, *Ann. Inst. H. Poincaré Probab. Statist.*, vol. 35, pp. 85–102, 1999.

[2] P. Del Moral, *Feynman-Kac Formulae: Genealogical and Interacting Particle Systems with Applications,* Springer-Verlag, Series: Probability and its Applications, Springer New York, 2004.

[3] P. Del Moral and A. Doucet. On a Class of Genealogical and Interacting Metropolis Models. J. Azéma, M. Emery, M. Ledoux, and M. Yor, editors, *Séminaire de Probabilités XXXVII*, Lecture Notes in Mathematics no. 1832, pp. 415–446. Springer-Verlag, Berlin, 2003.

[4] P. Del Moral, A. Doucet, and G.W. Peters. Sequential Monte Carlo Samplers, *Technical Report, Cambridge University*, CUED/F-INFENG/TR 443, Dec. 2002.

[5] P. Del Moral and L. Miclo, Genealogies and Increasing Propagations of Chaos for Feynman-Kac and Genetic Models, *The Annals of Applied Probability*, vol. 11, no. 4, pp. 1166-1198, 2001.

[6] P. Del Moral and L. Miclo, On the Strong Propagation of Chaos for Interacting Particle Approximations of Feynman-Kac Formulae, *Publications du Laboratoire de Statistique et Probabilités, no. 08-00, Université Paul Sabatier, France*, 2000.

[7] A. Doucet, N. de Freitas, and N. Gordon, editors. *Sequential Monte Carlo Methods in Pratice.* Statistics for Engineering and Information Science. Springer New York, 2001.

[8] C. Graham and S. Méléard, Stochastic Particle Approximations for Generalized Boltzmann Models and Convergence Estimates, *The Annals of Probability*, vol. 25, no. 1, pp. 115–132, 1997.

[9] W. Hoeffding, A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.*, 19, 293-325, (1948).

[10] W. Hoeffding, The strong law of large numbers for U-statistics. *Inst. Statist. Univ. of North Carolina, Mimeo Report,* No. 302 (1961).

[11] S. Méléard, Asymptotic Behaviour of Some Interacting Particle Systems: McKean-Vlasov and Boltzmann Models, *Probabilistic Models for Nonlinear Partial Differential Equations, Montecatini Terme, 1995*, D. Talay and L. Tubaro, editors, Springer-Verlag, Lecture Notes in Mathematics 1627, 1996.