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by

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Exponential decay laws in perturbation theory of threshold and embedded eigenvalues*

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Abstract

Exponential decay laws for the metastable states resulting from perturbation of unstable eigenvalues are discussed. Eigenvalues embedded in the continuum as well as threshold eigenvalues are considered. Stationary methods are used, i.e. the evolution group is written in terms of resolvent via Stone's formula and a partition technique (Schur-Livsic-Feschbach-Grushin formula) is used to localize the essential terms. No analytic continuation of the resolvent is required. The main result is about the threshold case: for Schrödinger operators in odd dimensions the leading term of the life-time in the perturbation strength, ε, is of order $\varepsilon^{2+\nu/2}$, where ν is an odd integer, $\nu \geq -1$. Examples covering all values of ν are given. For eigenvalues properly embedded in the continuum the results sharpen the previous ones.

1 Introduction

Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and E_0 a finitely degenerate eigenvalue of H: $HP_0 = E_0P_0, \quad \dim P_0 < \infty$. On $P_0\mathcal{H}$ the

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evolution is stationary:

$$P_0 e^{-itH} P_0 = e^{-itE_0} P_0. \tag{1.1}$$

The problem we consider is what happens with the evolution compressed to $P_0\mathcal{H}$, when a perturbation is added, i.e. H is replaced by

$$H_\varepsilon = H + \varepsilon W. \tag{1.2}$$

On heuristic grounds one expects that

$$P_0 e^{-itH_\varepsilon} P_0 = e^{-ith_\varepsilon} P_0 + \delta(\varepsilon, t), \tag{1.3}$$

where h_ε is a (dissipative) "effective hamiltonian" in $P_0\mathcal{H}$ and $\delta(\varepsilon, t)$ is an error term vanishing in the limit $\varepsilon \to 0$.

Among the questions to be answered are:

i. Find sufficient conditions for (1.3) to hold true. In particular one can ask whether there are interesting cases, in which such a simple description of the compressed dynamics does not exist (e.g. non exponential decay laws).

ii. Compute the effective hamiltonian h_ε.

iii. Estimate

$$\sup_{t>0} \|\delta(\varepsilon, t)\| = \delta(\varepsilon). \tag{1.4}$$

The above questions can be completely answered in the elementary case of a regular perturbation of discrete eigenvalues. Here the Kato-Rellich analytic perturbation theory gives

$$h_\varepsilon = h_\varepsilon^* = U_\varepsilon^* P_\varepsilon H_\varepsilon P_\varepsilon U_\varepsilon, \tag{1.5}$$

$$\delta(\varepsilon) \leq const.\varepsilon^2, \tag{1.6}$$

where P_ε is the perturbed projection, and U_ε is the Sz.-Nagy transformation matrix of the pair P_ε, P_0 (see e.g. [18]). Moreover, one can show that (1.6) is optimal, i.e. the power of ε in the error term cannot exceed 2. One remark is in order here. One can ask whether h_ε as given by (1.3) and (1.4) is unique. The answer is no, and one can easily see that if one takes $\tilde{h}_\varepsilon = W_\varepsilon^* h_\varepsilon W_\varepsilon$ with W_ε unitary, $[W_\varepsilon, P_0] = 0$, and $\|W_\varepsilon - 1\| \leq const.\varepsilon^2$, then $\tilde{h}_\varepsilon$ still satisfies (1.3) and (1.6). However, there is a uniqueness statement: the spectrum, $\sigma(h_\varepsilon)$, must coincide with the spectrum of H_ε emerging from E_0, i.e. h_ε is unique up to a unitary rotation.

Consider now the really interesting case, when E_0 is embedded (properly or at a threshold) in the continuous spectrum of H or/and W is singular with respect to H, as e.g. in the Stark effect. A fairly complete answer is known in the case of dilation analytic hamiltonians, if in addition one supposes that E_0 is not situated at a threshold. More precisely, using the analytic perturbation theory in the framework of Aguilar-Balslev-Combes dilation analytic hamiltonians, as developed by Simon [25], Hunziker [9] proved that (1.3) with the estimate (1.6) holds true. The important point here is that h_ε is no more self-adjoint, but only dissipative, which reflects the fact that generically under the effect of the perturbation the stationary state becomes metastable with (up to a uniform error of order ε^2) an exponential decay law. In the non-degenerate case (1.3) gives the rigorous foundation (control on error term included!) for the famous survival probability formula (here $h_\varepsilon = \lambda_\varepsilon P_0$)

$$|\langle \Psi_0, e^{-itH_\varepsilon}\Psi_0\rangle|^2 \sim e^{-2|\operatorname{Im}\lambda_\varepsilon|t}, \tag{1.7}$$

as given by the Dirac second order time dependent perturbation theory (Fermi Golden Rule).

The question we address is to what extent Hunziker's results can be generalized to:

i. A non-analytic (smooth) context.

ii. Threshold eigenvalues.

Our main interest is in the threshold eigenvalues case (however we shall give also results for properly embedded eigenvalues, extending and sharpening the existing ones). While for properly embedded eigenvalues one has a (generically) universal behavior as $\varepsilon \to 0$ of the decay rate constant, $\Gamma_\varepsilon \equiv 2|\operatorname{Im}\lambda_\varepsilon| \sim \varepsilon^2$, given by the "universal" Fermi Golden Rule, for threshold eigenvalues the situation is by far more complicated. As remarked by Baumgartner [2], even at the heuristic level the usual Fermi Golden Rule prescription to compute the decay rate constant does not work. The deep reason is that in the neighborhood of a threshold the resolvent (Greens function) has a complicated non universal structure. After all it is well known that quantum mechanics at threshold is a tricky business! It turns out that contrary to the properly embedded eigenvalue case, for threshold eigenvalues the behavior as $\varepsilon \to 0$ of Γ_ε is (generically) not universal; in the Schrödinger operators case it depends upon the dimension of the space, angular momentum, as well as upon the existence of threshold resonances.

Our main result [14] is that for threshold eigenvalues of Schrödinger operators in odd dimensions, the leading term of the decay rate constant in the perturbation strength, ε, is of order $\varepsilon^{2+\nu/2}$, where ν is an odd integer, $\nu \geq -1$. We give examples for all values of ν, for which we compute the leading term in Γ_ε, and give estimates for the error term.

There are basically two general approaches to derive (1.3). The first one, initiated by Soffer and Weinstein [26], consists in a direct study of the

Schrödinger evolution governed by H_ε:

$$i\partial_t\psi(t) = H_\varepsilon\psi(t) \tag{1.8}$$

for initial conditions localized in energy around E_0. The second one, initiated by Orth [24], is the stationary approach, which by use of the Stone formula reduces the computation of the l.h.s. of (1.3) to the computation of an integral over energies involving the compressed resolvent. In both methods, in order to isolate the significant contributions, one uses variants of projection techniques (appearing in the literature under various names as: Liapunov-Schmidt projection method, Schur complements, Livsic-Feschbach matrix, Grushin method, etc; for more comments and references see [14]).

We use the stationary approach. We refine it as to cover the threshold eigenvalues case (and also to sharpen the existing results for properly embedded eigenvalues) by adding two things:

i. Detailed asymptotic expansions near a threshold of the resolvent of Schrödinger operators in odd dimensions obtained in [10],[11],[22],[14],[12].

ii. A careful study of the integral appearing in the Stone formula, especially regarding the interval of energies giving the significant contributions.

Finally we would like to stress that we do not touch here the huge field related to resonances, from the spectral-scattering theory point of view. For further references we send the reader to [8],[23], as well as to the recent review [7].

2 The basic formula

The first step is to localize in energy. Thus we consider $P_0 e^{-itH_\varepsilon} g_\varepsilon(H_\varepsilon)P_0$, where $0 \leq g_\varepsilon(x) \leq 1$ is the (possibly smoothed) characteristic function of an interval in a neighborhood of E_0. The crucial point here is the following beautiful, elementary remark due to Hunziker [9]:

Proposition 1. *Suppose that for some* $h_\varepsilon : P_0\mathcal{H} \to P_0\mathcal{H}$,

$$\|P_0 e^{-itH_\varepsilon} g_\varepsilon(H_\varepsilon)P_0 - e^{-ith_\varepsilon}P_0\| \leq \delta(\varepsilon). \tag{2.1}$$

Then

$$\|P_0 e^{-itH_\varepsilon} P_0 - e^{-ith_\varepsilon}P_0\| \leq 2\delta(\varepsilon). \tag{2.2}$$

Then one can use the freedom of choice of $g_\varepsilon(x)$ to be able to compute h_ε, and to optimize the error estimate. We note that usually $g_\varepsilon(x)$ is chosen independent of ε. One of the key points of our approach is to make an appropriate ε-*dependent* choice of $g_\varepsilon(x)$. For example, in the case of perturbing threshold eigenvalues, it is crucial that $g_\varepsilon(x)$ is the characteristic function of an interval, which is "far" from the threshold, i.e. does not contain the unperturbed eigenvalue. In what follows we choose an interval $I_\varepsilon = (e_0(\varepsilon) - d(\varepsilon), e_0(\varepsilon) + d(\varepsilon))$, and take $g_\varepsilon(x) = \chi_{I_\varepsilon}(x)$ as the cut-off function. As already said the central point in our approach is to find the "right" location $e_0(\varepsilon)$, and the "right" size function $d(\varepsilon)$, such that energies in I_ε give the resonance behavior, and energies outside I_ε only contribute to the error term $\delta(\varepsilon, t)$.

A remark is in order here. By taking a smoothed out characteristic function one can obtain a refinement of (2.2) in the form

$$P_0 e^{-itH_\varepsilon} P_0 = (I + A(\varepsilon)) e^{-ith_\varepsilon} (I + A(\varepsilon)) + \delta(\varepsilon, t),$$

where $A(\varepsilon) = \mathcal{O}(\varepsilon^p)$ for some $p > 0$, and $\delta(\varepsilon, t)$ now exhibits decay in t for t large. However, our concern here is with error estimates uniform in time, so we take just the characteristic function as our cut-off function.

The next step is to write down a workable formula for the compressed evolution in (2.1). For this purpose we use the Stone formula to express the compressed evolution in terms of compressed resolvent, and then we use the Schur-Livsic-Feschbach-Grushin (SLFG) partition formula to express the compressed resolvent as an inverse. We briefly recall the SLFG formula (for details, further references, and historical remarks, we send the reader to [14]). Let $R_\varepsilon(z) = (H(\varepsilon) - z)^{-1}$, and let $R_{0,\varepsilon}(z)$ be the resolvent of $Q_0 H(\varepsilon) Q_0$, as an operator in $Q_0\mathcal{H}$. where

$$Q_0 = 1 - P_0. \tag{2.3}$$

Then we have in the decomposed space $\mathcal{H} = P_0\mathcal{H} \oplus Q_0\mathcal{H}$

$$R_\varepsilon(z) = \begin{bmatrix} R_{\text{eff}}(z) & -\varepsilon R_{\text{eff}}(z) P_0 W Q_0 R_{0,\varepsilon}(z) \\ -\varepsilon R_{0,\varepsilon}(z) Q_0 W P_0 R_{\text{eff}}(z) & R_{22} \end{bmatrix}, \tag{2.4}$$

with

$$R_{\text{eff}}(z) = \left(P_0 H(\varepsilon) P_0 - \varepsilon^2 P_0 W Q_0 R_{0,\varepsilon}(z) Q_0 W P_0 - z P_0\right)^{-1}.$$

We do not give the formula for R_{22}, since it is not needed here, see [14] for this formula.

More precisely, by using the Stone formula, the SLFG formula, and by rearranging the Neumann series for the perturbed resolvent, one arrives at the following basic formula for the compressed evolution [14]:

Proposition 2.

$$
\begin{aligned}
P_0 e^{-itH_\varepsilon)} g_\varepsilon(H_\varepsilon) P_0 &= \lim_{\eta\to 0} \frac{1}{\pi} \int dx\; e^{-itx} g_\varepsilon(x)\, \mathrm{Im}\, P_0 (H_\varepsilon - x - i\eta)^{-1} P_0 \\
&= \lim_{\eta\to 0} \frac{1}{\pi} \int dx\; e^{-itx} g_\varepsilon(x)\, \mathrm{Im}\, F(x+i\eta, \varepsilon)^{-1} \qquad (2.5)
\end{aligned}
$$

where using the notation

$$W = A^* D A, \quad D = D^* = D^{-1}, \tag{2.6}$$

$$G(z) = A Q_0 (H - z)^{-1} Q_0 A^*, \tag{2.7}$$

as an operator in $P_0 \mathcal{H}$*, the function* $F(z, \varepsilon)$ *is given by*

$$
\begin{aligned}
F(z,\varepsilon) = (E_0 - z) P_0 &+ \varepsilon P_0 W P_0 - \varepsilon^2 P_0 A^* D G(z) D A P_0 \\
&+ \varepsilon^3 P_0 A^* D G(z) [D + \varepsilon G(z)]^{-1} G(z) D A P_0. \qquad (2.8)
\end{aligned}
$$

The formulas (2.5) and (2.8) are the starting formulas of our approach, and at this point the hard work starts. What is needed is to show that on the interval I_ε, up to a controllable error, $F(x+i\eta,\varepsilon) = h_\varepsilon - x - i\eta$, so that one can isolate the resonant term and estimate the remainder. All that depends crucially on the smoothness properties of $F(z,\varepsilon)$. The main point of the formula (2.8) is that $F(z,\varepsilon)$ inherits the smoothness properties of $G(z)$. This allows, assuming appropriate conditions on $G(z)$, to prove "semi-abstract" results, and then apply them to various concrete cases, by checking these assumptions. In what follows the assumptions for the threshold case are modeled on Schrödinger operators in odd dimensions.

3 The results

3.1 Properly embedded eigenvalues

Let for $a > 0$

$$D_a(E_0) = \{ z \in \mathbf{C} \,|\, |z - E_0| < a, \mathrm{Im}\, z > 0 \}. \tag{3.1}$$

We denote by $C^{n,\theta}(D_a(E_0))$ the functions in $D_a(E_0)$ that are n times continuously norm-differentiable, with the n^{th} derivative satisfying a uniform Hölder condition in $D_a(E_0)$, of order θ, $0 \le \theta \le 1$. The main assumption in this subsection is that

$$G(z) \in C^{n,\theta}(D_a(E_0)). \tag{3.2}$$

Such conditions can be verified in an abstract setting, using the Mourre estimate and the multiple commutator technique, see e.g. [1], [3], [6], and references therein. Note this assumption implies that $G(z)$ has boundary values $G(x+i0)$, which are in $C^{n,\theta}((E_0-a, E_0+a))$. For Schrödinger operators the smoothness of $G(z)$ also follows, if the potential decays sufficiently fast at infinity.

We give first the result in the non-degenerate case [14].

Theorem 3. *Assume* $G(z) \in C^{n,\theta}(D_a(E_0))$. *Assume* $\dim P_0 = 1$ *and* $n+\theta > 0$. *Write* $F(x+i0,\varepsilon) = (R(x,\varepsilon) + iI(x,\varepsilon))P_0$. *Then for* ε *sufficiently small there exists a (unique for* $n+\theta \geq 1$*) solution to* $R(x,\varepsilon) = 0$ *in the interval* $(E_0 - a, E_0 + a)$, *denoted by* $x_0(\varepsilon)$. *Let* $\Gamma(\varepsilon) = I(x_0(\varepsilon),\varepsilon)$, *write*

$$\lambda_\varepsilon = x_0(\varepsilon) - i\Gamma(\varepsilon), \tag{3.3}$$

and let Ψ_0 *denote a normalized eigenfunction for eigenvalue* E_0 *of* H. *Then for* ε *sufficiently small, and for all* $t > 0$, *the following results hold true:*

(i) *Assume* $n = 0$, $0 < \theta < 1$, *and*

$$\Gamma(\varepsilon) \geq C\varepsilon^\gamma \quad \textit{with} \quad 2 \leq \gamma < \frac{2}{1-\theta}. \tag{3.4}$$

Then we have

$$|\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0\rangle - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \leq C\frac{1}{1-\theta}\varepsilon^\delta, \tag{3.5}$$

where

$$\delta = 2 - \gamma(1-\theta) > 0. \tag{3.6}$$

(ii) *For* $n + \theta \geq 1$ *we have*

$$|\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0\rangle - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \leq C \begin{cases} \varepsilon^2|\ln\varepsilon| & \textit{for } n = 0, \theta = 1, \\ \varepsilon^2 & \textit{for } n+\theta > 1. \end{cases} \tag{3.7}$$

The results in the theorem above sharpen and amplify similar results in [5, 4, 19, 20, 21, 26, 27]. Let us stress that in the high regularity case, i.e. $n + \theta \geq 1$, there is no lower bound condition for $\Gamma(\varepsilon)$. In particular, λ_ε can be an eigenvalue.

We turn now to the degenerate case. In the degenerate case the results are by far less complete. In particular, in order to prove (1.3) and (1.4), one has to impose a condition on the size of the imaginary part of $\operatorname{Im} F(E_0 + i0)$, namely the so-called Fermi Golden Rule condition (see (3.8) below). One can relax (3.8), if one imposes conditions on the spectrum of P_0WP_0, such that one can apply the methods and results from the non-degenerate case [24, 16]. Our main result [16] here sharpening the ones in [21, 28] is contained in

Theorem 4. *Assume $N \geq 2$ and $G(z) \in C^{n,\theta}(D_a(E_0))$ with $n+\theta \geq 2$. Assume there exists $\gamma > 0$ such that*

$$\operatorname{Im} P_0 A^* D G(E_0 + i0) D A P_0 \geq \gamma P_0. \tag{3.8}$$

Then there exists a function $\delta(\varepsilon, t)$ satisfying (1.4) *with $p = 2$, such that*

$$P_0 e^{-itH_\varepsilon} P_0 = e^{-ith_\varepsilon} P_0 + \delta(\varepsilon, t). \tag{3.9}$$

Here h_ε on $P_0\mathcal{H}$ is given by

$$\begin{aligned} h_\varepsilon = {} & E_0 P_0 + \varepsilon P_0 W P_0 - \varepsilon^2 P_0 W Q_0 (H - E_0 - i0)^{-1} Q_0 W P_0 \\ & - \varepsilon^3 \Big\{ P_0 W Q_0 (H - E_0 - i0)^{-1} Q_0 W Q_0 (H - E_0 - i0)^{-1} Q_0 W P_0 \\ & + \tfrac{1}{2} \Big[P_0 W P_0 W \frac{d}{dE} Q_0 (H - E - i0)^{-1} Q_0 \Big|_{E=E_0} W P_0 \\ & + P_0 W \frac{d}{dE} Q_0 (H - E - i0)^{-1} Q_0 \Big|_{E=E_0} W P_0 W P_0 \Big] \Big\}. \end{aligned} \tag{3.10}$$

3.2 Threshold eigenvalues

As already said in the Introduction, the usual methods to prove the smoothness of $G(z)$ do not work at thresholds, and actually it may not be smooth, or even blows up, in the neighborhood of the origin. The way out from this difficulty is to use the asymptotic expansion of $G(z)$ around the threshold (see [10, 11, 12, 22] and references therein). Let us stress that the asymptotic expansions of the resolvent around thresholds are not universal; e.g. in the Schrödinger case the type of expansions depend on dimension, and on the threshold spectral properties of the hamiltonian. The asymptotic expansion in the assumption below (see [14, Section 3]) is modeled after Schrödinger and Dirac operators in odd dimensions.

Assumption 5. (A1) *There exists $a > 0$, such that $(-a, 0) \subset \rho(H)$ (the resolvent set) and $[0, a] \subset \sigma_{\mathrm{ess}}(H)$.*

(A2) *Assume that zero is a non-degenerate eigenvalue of H: $H\Psi_0 = 0$, with $\|\Psi_0\| = 1$, and there are no other eigenvalues in $[0, a]$. Let $P_0 = |\Psi_0\rangle\langle\Psi_0|$ be the orthogonal projection onto the one-dimensional eigenspace.*

(A3) *Assume*

$$\langle \Psi_0, W\Psi_0 \rangle = b > 0. \tag{3.11}$$

(A4) *For* $\mathrm{Re}\,\kappa \geq 0$ *and* $z \in \mathbf{C} \setminus [0, \infty)$ *we let*

$$\kappa = -i\sqrt{z}, \quad z = -\kappa^2. \tag{3.12}$$

There exist $N \in \mathbf{N}$ *and* $\delta_0 > 0$*, such that for* $\kappa \in \{\kappa \in \mathbf{C} \,|\, 0 < |\kappa| < \delta_0, \mathrm{Re}\,\kappa \geq 0\}$ *we have*

$$G(z) = \sum_{j=-1}^{N} \widetilde{G}_j \kappa^j + \kappa^{N+1} \widetilde{G}_N(\kappa), \tag{3.13}$$

where

$$\widetilde{G}_j \quad \text{are bounded and self-adjoint}, \tag{3.14}$$

$$\widetilde{G}_{-1} \quad \text{is of finite rank and self-adjoint}, \tag{3.15}$$

$$\widetilde{G}_N(\kappa) \quad \text{is uniformly bounded in } \kappa. \tag{3.16}$$

From (3.13) *we get*

$$\langle \Psi_0, A^* D G(z) D A \Psi_0 \rangle = \sum_{j=-1}^{N} g_j \kappa^j + \kappa^{N+1} g_N(\kappa), \tag{3.17}$$

where

$$g_j = \langle \Psi_0, A^* D \widetilde{G}_j D A \Psi_0 \rangle, \tag{3.18}$$

$$g_N(\kappa) = \langle \Psi_0, A^* D \widetilde{G}_N(\kappa) D A \Psi_0 \rangle. \tag{3.19}$$

(A5) *There exists an odd integer,* $-1 \leq \nu \leq N$*, such that*

$$g_\nu \neq 0, \quad \widetilde{G}_j = 0 \quad \text{for } j = -1, 1, \ldots, \nu - 2. \tag{3.20}$$

The main (semi)-abstract result dealing with threshold case is as follows [14]:

Theorem 6. *Let* $x_0(\varepsilon)$, $\Gamma(\varepsilon)$ *be as in Theorem* 3*. Suppose* (A1)–(A5) *in Assumption* 5 *hold true. Then for sufficiently small* $\varepsilon > 0$ *we have*

$$|\langle \Psi_0, e^{-itH_\varepsilon} \Psi_0 \rangle - e^{-it(x_0(\varepsilon) - i\Gamma(\varepsilon))}| \leq C\varepsilon^{p(\nu)}. \tag{3.21}$$

Here $p(\nu) = \min\{2, (2+\nu)/2\}$*, and*

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+\nu/2}(1 + \mathcal{O}(\varepsilon)), \tag{3.22}$$

$$x_0(\varepsilon) = b\varepsilon(1 + \mathcal{O}(\varepsilon)). \tag{3.23}$$

4 A uniqueness result

The spectrum of the effective hamiltonian, h_ε, (λ_ε in the nondegenerate case) gives information about the "location" of resonances resulting from the perturbation of stationary states, as can be seen in the cases, when one can define the resonances as poles of the analytic continuation of the resolvent or the scattering matrix. Then a natural question is to ask, to what extent the effective hamiltonian h_ε as *defined* by (1.3) and (1.4) is unique. If h_ε has an asymptotic expansion as $\varepsilon \to 0$, the question is how many expansion coefficients are uniquely determined. The following result [3], [17] gives the answer to this problem.

Theorem 7. *I. Assume* $\operatorname{Rank} P_0 = 1$.

Assume that h_ε^1 and h_ε^2 both satisfy (1.3) *and* (1.4), *with the same value for p. Assume that for some $c_0 > 0$ and $q > 0$ we have*

$$-c_0\varepsilon^q P_0 \le \operatorname{Im} h_\varepsilon^1 \le 0 \quad \text{for } 0 \le \varepsilon < \varepsilon_0. \tag{4.1}$$

Then for ε_0 sufficiently small we have

$$\|h_\varepsilon^1 - h_\varepsilon^2\|_{\mathcal{B}(P_0\mathcal{H})} \le C\varepsilon^{p+q}, \quad 0 \le \varepsilon < \varepsilon_0. \tag{4.2}$$

II. Assume $1 \le \operatorname{Rank} P_0 < \infty$.

(i) *Assume that h_ε^1 and h_ε^2 both satisfy* (1.3) *and* (1.4), *with the same value for p. Assume that h_ε^1 satisfies*

$$h_\varepsilon^1 = E_0P_0 + \varepsilon h_1^1 + \varepsilon f^1(\varepsilon), \quad 0 \le \varepsilon < \varepsilon_0, \tag{4.3}$$

such that $h_1^1 = (h_1^1)^*$, $\operatorname{Im} f^1(\varepsilon) \le 0$, and $f^1(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Assume that h_ε^2 is a bounded family of operators on $P_0\mathcal{H}$. Then for ε_0 sufficiently small we have

$$\|h_\varepsilon^1 - h_\varepsilon^2\|_{\mathcal{B}(P_0\mathcal{H})} \le C\varepsilon^{p+1}, \quad 0 \le \varepsilon < \varepsilon_0. \tag{4.4}$$

(ii) *Assume that h_ε^1 and h_ε^2 both satisfy* (1.3) *and* (1.4), *with $p = 2$. Assume that h_ε^1 satisfies*

$$h_\varepsilon^1 = E_0P_0 + \varepsilon h_1 + \varepsilon^2 h_2 + o(\varepsilon^2), \quad 0 \le \varepsilon < \varepsilon_0, \tag{4.5}$$

such that $h_1 = h_1^*$ and $\operatorname{Im} h_\varepsilon^1 \le 0$. Assume that h_ε^2 is a bounded family of operators on $P_0\mathcal{H}$. Then there exists a family of invertible operators $U(\varepsilon)$ on $P_0\mathcal{H}$ with $U(\varepsilon) = P_0 + O(\varepsilon^2)$, such that for ε_0 sufficiently small we have

$$\|h_\varepsilon^1 - U(\varepsilon)^{-1}h_\varepsilon^2 U(\varepsilon)\|_{\mathcal{B}(P_0\mathcal{H})} \le C\varepsilon^4, \quad 0 \le \varepsilon < \varepsilon_0. \tag{4.6}$$

5 Examples

In this section, for all $\nu = -1, 1, 3, \ldots$, we give examples for which Assumption 5 holds true, and then Theorem 6 gives (1.3) with $|\operatorname{Im}\lambda_\varepsilon| \sim \varepsilon^{2+\frac{\nu}{2}}$. In each case we compute the leading term g_ν. As examples we consider one and two channel Schrödinger operators in three dimensions [14]. For more examples, see [14, 15, 16].

5.1 Example 1: one channel case, $\nu = -1$

In this case

$$H = -\Delta + V(\mathbf{x}), \tag{5.1}$$

$$(Wf)(\mathbf{x}) = W(\mathbf{x})f(\mathbf{x}), \tag{5.2}$$

in $L^2(\mathbf{R}^3)$, with V, W satisfying

$$\langle\cdot\rangle^\beta V \in L^\infty(\mathbf{R}^m), \tag{5.3}$$

$$\langle\cdot\rangle^\gamma W \in L^\infty(\mathbf{R}^m), \tag{5.4}$$

and β, γ are sufficiently large (see below). Here $E_0 = 0$. About H we suppose that it has a non-degenerate threshold eigenvalue

$$(-\Delta + V)\Psi_0 = 0, \quad \|\Psi_0\| = 1, \tag{5.5}$$

as well as a threshold resonance with canonical resonance function Ψ_c . We recall that H has a threshold resonance if there exist additional non-zero solutions to $(-\Delta + V)\Psi = 0$, in the space $L^{2,-s}(\mathbf{R}^3)$, $1/2 < s \le 3/2$. Among these solutions, one can choose a distinguished one, Ψ_c, called the *canonical zero resonance function*, and all the others can be written as $\Psi = \alpha\Psi_c + \tilde{\Psi}$ with $\alpha \neq 0$ and $\tilde{\Psi} \in L^2(\mathbf{R}^3)$ (for definition and further details see [14, Appendix A]). In the theorem below we take Ψ_0 to be real-valued.

Theorem 8. *Assume that V and W satisfy* (5.3) *and* (5.4) *with $\beta > 9$ and $\gamma > 5$, respectively. Assume that* (A1-3) *holds for $H = -\Delta + V$. Let*

$$X_j = \int_{\mathbf{R}^3} \Psi_0(\mathbf{x})V(\mathbf{x})x_j d\mathbf{x}, \quad j = 1, 2, 3. \tag{5.6}$$

Assume either that $X_j \neq 0$ for at least one j, or that $\langle\Psi_0, W\Psi_c\rangle \neq 0$. Then $\nu = -1$, and we have

$$g_{-1} = \frac{b^2}{12\pi}(X_1^2 + X_2^2 + X_3^2) + |\langle\Psi_0, W\Psi_c\rangle|^2. \tag{5.7}$$

If H does not have a resonance at the threshold, but still $X_j \neq 0$ for at least one j, then the second term in the right hand side of (5.7) *should be omitted, i.e.*

$$g_{-1} = \frac{b^2}{12\pi}(X_1^2 + X_2^2 + X_3^2). \tag{5.8}$$

The following example shows the significance of the conditions in the theorem. Take

$$V(\mathbf{x}) = \begin{cases} -V_0, & \text{if } |\mathbf{x}| \leq 1, \\ 0, & \text{if } |\mathbf{x}| > 1. \end{cases}$$

Here $V_0 > 0$ is a parameter. By adjusting this parameter, one can get a radial solution to $(-\Delta + V)\psi = 0$ for any angular momentum $\ell = 0, 1, \ldots$, which decays as $|\mathbf{x}|^{-\ell}$, as $|\mathbf{x}| \to \infty$. Thus for $\ell = 0$ we get a zero resonance. For $\ell = 1$ we get zero eigenvalues, such that at least one $X_j \neq 0$, see (5.6). For $\ell \geq 2$ all $X_j = 0$. For $\ell \geq 1$ the eigenvalue at zero is not simple. Examples with a simple zero eigenvalue can be obtained using only the radial part. Note that in order to get $\langle \Psi_0, W\Psi_c \rangle \neq 0$ one will have to take a non-radial perturbation W.

5.2 Example 2: two channel case, $\nu = -1, 1$

In the two channel case we consider examples of a non-degenerate bound state of zero energy in the "closed" channel decaying due to the interaction with a three dimensional Schrödinger operator in the open channel. Since only the bound state in the closed channel is relevant in the forthcoming discussion, we shall take $\mathbf{C}$ as the Hilbert space representing the closed channel, i.e. $\mathcal{H} = L^2(\mathbf{R}^3) \oplus \mathbf{C}$. As the unperturbed hamiltonian we take

$$H = \begin{bmatrix} -\Delta + V & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.9}$$

where V satisfies (5.3), and as the perturbation we take

$$W = \begin{bmatrix} W_{11} & |W_{12}\rangle\langle 1| \\ |1\rangle\langle W_{12}| & b \end{bmatrix}, \tag{5.10}$$

which is a shorthand for

$$W \begin{bmatrix} f(\mathbf{x}) \\ \xi \end{bmatrix} = \begin{bmatrix} W_{11}(\mathbf{x})f(\mathbf{x}) + W_{12}(\mathbf{x})\xi \\ \int \overline{W_{12}(\mathbf{x})} f(\mathbf{x}) + b\xi \end{bmatrix}. \tag{5.11}$$

Here we assume

$$\langle\cdot\rangle^{\gamma} W_{11} \in L^\infty(\mathbf{R}^m), \quad \langle\cdot\rangle^{\gamma/2} W_{12} \in L^\infty(\mathbf{R}^m), \tag{5.12}$$

and furthermore that W_{11} is real-valued. In order to satisfy (3.11) we assume $b > 0$ in (5.10).

Concerning the two channel case we have the following result.

Theorem 9. *Assume that V and W satisfy* (5.3) *and* (5.4) *with $\beta > 9$ and $\gamma > 5$, respectively.*

(i) *Assume that $-\Delta + V$ has neither a threshold resonance nor a threshold eigenvalue. Then $\nu \geq 1$, and we have*

$$g_1 = \frac{-1}{4\pi} |\langle W_{12}, (I + G_0^0 V)^{-1} 1 \rangle|^2. \tag{5.13}$$

where the integral kernel of G_0^0 is $\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$.

(ii) *Assume that $-\Delta + V$ has a threshold resonance, and no threshold eigenvalue. Let Ψ_c denote the canonical zero resonance function. Assume that $\langle W_{12}, \Psi_c \rangle \neq 0$. Then $\nu = -1$, and*

$$g_{-1} = |\langle W_{12}, \Psi_c \rangle|^2. \tag{5.14}$$

5.3 Example 3: two channel radial case, $\nu \geq 3$

Here we consider radial part of Schrödinger operator with spherical symmetric potentials for angular momentum $\ell = 1, 2, \ldots$

$$H_{0,\ell} = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}, \quad \ell = 1, 2, \ldots, \tag{5.15}$$

on the space $\mathcal{H} = L^2(\mathbf{R}_+)$ in the two channel set-up, where we now take the Hilbert space $\mathcal{H} = L^2(\mathbf{R}_+) \oplus \mathbf{C}$, and replace (5.9) by

$$H = \begin{bmatrix} H_{0,\ell} & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.16}$$

It will provide us with examples of resolvent expansions, where we can verify Assumption (A5) with $\nu \geq 3$ odd and arbitrarily large. Note that the cases $\nu = -1$ and $\nu = 1$ were covered in the preceding examples.

Theorem 10. *Consider the two channel case with H given by* (5.16)*. Assume that W given by* (5.10) *satisfies* (5.12) *with $\gamma > 2\ell + 5$. Assume that*

$$\langle W_{12}, r^{\ell+1} \rangle \neq 0.$$

Then we have $\nu = 2\ell + 1$ and

$$g_\nu = (-1)^{\ell+1} \left[\frac{\sqrt{\pi}}{2^{\ell+1}\Gamma(\ell + \frac{3}{2})} \right]^2 |\langle W_{12}, r^{\ell+1} \rangle|^2, \tag{5.17}$$

where Γ denotes the usual Gamma function.

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