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ON A TRANSFERENCE RESULT FOR MATRIX WEIGHTS

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ABSTRACT. We consider a periodic matrix weight W defined on $\mathbb{R}^d$ and taking values in the $N\times N$ positive-definite matrices. For such weights, we prove a transference results between multiplier operators on $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$, $1<p<\infty$, respectively. As an application, we prove that Bochner-Riesz summation at the critical index in $L_p(\mathbb{T}^d;W)$ converges if and only if W satisfies a matrix Muckenhoupt A_p-condition.

1. INTRODUCTION

A matrix weight is a locally integrable function $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ taking values in the set of positive definite Hermitian forms. The associated weighted space $L_p(\mathbb{R}^d;W)$, $1\le p<\infty$, is the set of measurable (vector-)functions $f:\mathbb{R}^d\to\mathbb{C}^N$ satisfying

$$\|f\|^p_{L_p(\mathbb{R}^d;W)} := \int_{\mathbb{R}^d} |W^{1/p}f|^p\,dx<\infty. \tag{1.1}$$

For periodic weights, i.e., $W:\mathbb{T}^d\to\mathbb{C}^{N\times N}$, we define the associated weighted space $L_p(\mathbb{T}^d;W)$, $1\le p<\infty$, as the set of measurable periodic (vector-)functions $f:\mathbb{T}^d\to\mathbb{C}^N$ satisfying

$$\|f\|^p_{L_p(\mathbb{T}^d;W)} := \int_{\mathbb{T}^d} |W^{1/p}f|^p\,dx<\infty. \tag{1.2}$$

In this paper, we study transference results for multiplier operators on $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$ for periodic weights W. By a multiplier operator on a weighted vector-valued space, we mean a scalar multiplier that acts coordinatewise. More precisely, for a scalar multiplier operator T on $\mathbb{R}^d$ (or $\mathbb{T}^d$), we lift T to an operator on functions f taking values in $\mathbb{C}^N$ by letting it act separately on each coordinate function,

$$(Tf)_j = Tf_j,\qquad j=1,2,\dots N. \tag{1.3}$$

It is well-known that in the scalar case, there is a close connection between bounded L_p multipliers on the line and on the torus, and it turns out that such results can be considered in the matrix weighted case as well. Transference can thus reduce the workload needed to prove L_p-boundedness for multipliers on e.g. the tours; one only needs to consider the corresponding multiplier on the line (or vice-versa).

Scalar transference results for scalar L_p-multipliers were first established by de Leeuw [4]. A systematic treatment of transference for multipliers and maximal multiplier operators was given by Coifman and Weiss [3]. More recent developments can be found in [1,2,8].

A number of authors have studied boundedness of multipliers on $L_p(\mathbb{R}^d;W)$. In their seminal papers [7,10], Treĭl′ and Volberg proved that the Hilbert transform is bounded if and only if the weight W belongs to an appropriate matrix Muckenhoupt A_p class. This result was extended

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by Goldberg [5] who proved that boundedness of standard multiplies on $L_p(\mathbb{R}^d;W)$ are closely related to the matrix Muckenhoupt A_p condition on the weight W. The transference results obtained here allow us to obtain similar conclusions for multiplier sequences on $L_p(\mathbb{T}^d;W)$.

This paper is organized as follows. Section 2 contains the main results on transference for multipliers between $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$. Our main application is presented in Section 3, where we consider multipliers on L_p spaces with weights satisfying a matrix Muckenhoupt A_p condition. In Section 4, we characterize boundedness of Bochner-Riesz summation on $L_p(\mathbb{T}^d;W)$ in terms of properties of the weight W in . It is proved in Section 4 that boundedness of this summation procedure at the critical index is essentially equivalent to the weight W belonging to the matrix Muckenhoupt class A_p.

2. Main transference results

This section contains our main result. We give results in two directions. In Proposition 2.4, we transfer boundedness for multipliers on $L_p(\mathbb{R}^d;W)$ to boundedness for discrete multipliers on $L_p(\mathbb{T}^d;W)$, while in Proposition 2.5 we transfer in the other direction from $L_p(\mathbb{T}^d;W)$ to $L_p(\mathbb{R}^d;W)$.

Before we state the transference results, we need to define the classes of bounded multipliers on $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$ that will be considered.

Definition 2.1. Let $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ be a matrix weight, and let $1\le p<\infty$. We denote by $\mathscr{M}_p(\mathbb{R}^d;W)$ the set of all bounded functions b on $\mathbb{R}^d$ such that the operator

$$T_b(f) := \big(b\hat{f}\big)^{\vee}$$

extends to a bounded operator on $L_p(\mathbb{R}^d;W)$. The norm $\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)}$ of an element $b\in\mathscr{M}_p(\mathbb{R}^d;W)$ is by definition the norm of the operator T_b on $L_p(\mathbb{R}^d;W)$.

Similarly, for $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ a periodic matrix weight, we denote by $\mathscr{M}_p(\mathbb{T}^d;W)$ the set of bounded sequences $\mathbf{a}=\{a_k\}_{k\in\mathbb{Z}^d}$ such that the operator

$$T_{\mathbf{a}}(f)(x) := \sum_{k\in\mathbb{Z}^d} a_k\hat{f}(k)e^{2\pi i k\cdot x}$$

extends to a bounded operator on $L_p(\mathbb{T}^d;W)$. The norm $\|\{a_k\}\|_{\mathscr{M}_p(\mathbb{R}^d;W)}$ of an element $\mathbf{a}\in\mathscr{M}_p(\mathbb{R}^d;W)$ is defined to be the norm of the operator $T_{\mathbf{a}}$ on $L_p(\mathbb{T}^d;W)$.

2.1. **Multipliers in $\mathscr{M}_p(\mathbb{R}^d;W)$.** We now focus on multipliers b in $\mathscr{M}_p(\mathbb{R}^d;W)$. The basic idea of transference is to sample b on $\mathbb{Z}^d$ and thereby obtain a multiplier in $\mathscr{M}_p(\mathbb{T}^d;W)$. For this to work, b must be well-behaved pointwise. A very useful notion in the theory of (scalar) transference is that of a regulated function. Let us recall the definition of a regulated function.

Definition 2.2. Let $t_0\in\mathbb{R}^d$. A bounded measurable function b on $\mathbb{R}^d$ is called regulated at the point t_0 if

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^d}\int_{|t|\le\varepsilon}\big(b(t_0-t)-b(t_0)\big)dt = 0.$$

The function b is called regulated if it is regulated at every point $t_0\in\mathbb{R}^d$.

We now turn to vector-valued multipliers. Our idea is to use scalar transference combined with a duality argument. The dual space of $L_p(D;W)$, for $1 < p < \infty$, and $D \in \{\mathbb{T}^d, \mathbb{R}^d\}$, can be identified with, $L_q(D;W^{-q/p})$, where q is the dual exponent to p given by $\frac{1}{p}+\frac{1}{q}=1$, see [10] for further details. The pairing of $L_p(D;W)$ and $L_p(D;W)^* = L_q(D;W^{-q/p})$ is given by the integral

$$\int_D \langle f(x), g(x)\rangle_{\ell_2(\mathbb{C}^N)}\, dx = \sum_{j=1}^N \int_D f_j(x)\overline{g_j(x)}\, dx. \tag{2.1}$$

The integrals on the right-hand side of (2.1) are ordinary scalar integrals, and in the proof of Proposition 2.4 below we use the following well-known lemma from (scalar) transference repeatedly.

Lemma 2.3 ([3, 6]). *Let T be the operator on $\mathbb{R}^d$ whose multiplier is $b(\xi)$, and let S be the operator on $\mathbb{T}^d$ whose multiplier is the sequence $\{b(m)\}_{m\in\mathbb{Z}^d}$. Assume that $b(\xi)$ is regulated at every point $m \in \mathbb{Z}^d$. Let $L_\varepsilon(x) = e^{-\pi\varepsilon|x|^2}$ for $x \in \mathbb{R}^d$ and $\varepsilon > 0$. For every pair of trigonometric polynomials P and Q on $\mathbb{R}^d$, and $\alpha, \beta > 0$ with $\alpha+\beta = 1$, we have the identity*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{d/2} \int_{\mathbb{R}^d} T(PL_{\varepsilon\alpha})(x)\overline{Q(x)L_{\varepsilon\beta}(x)}dx = \int_{\mathbb{T}^d} S(P)(x)\overline{Q(x)}dx. \tag{2.2}$$

We also need the following observation about weights $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ with $W \in L_{1,\text{loc}}$. Let $\alpha > 0$ and $1 \le p < \infty$. Since $W(x)$ is a nonnegative and self-adjoint matrix at each point x, we have uniformly in x,

$$\begin{aligned}
\|W^\alpha(x)\| = \|W^{\frac{2\alpha}{p}}(x)\|^{\frac{p}{2}} &\asymp [\operatorname{trace}(W^{\frac{2\alpha}{p}}(x))]^{\frac{p}{2}} \\
&= \Bigg(\sum_{j=1^N} |W^{\frac{\alpha}{p}}(x)e_j|^2\Bigg)^{p/2} \\
&\asymp \sum_{j=1}^N |W^{\frac{\alpha}{p}}(x)e_j|^p,
\end{aligned}$$

where $\{e_j\}_{j=1}^N$ is any orthonormal basis for $\mathbb{C}^N$. We apply the result for $(\alpha, p) := (1, p)$, and then for $(\alpha, p) := (1/p, 1)$ to conclude that $W \in L_{1,\text{loc}}$ implies that $W^{1/p} \in L_{1,\text{loc}}$ for $p \ge 1$.

With this notation in place, we can now state the first part of our main result.

Proposition 2.4. *Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a periodic matrix weight with $W, W^{-1} \in L_{1,loc}$, and suppose that b is a regulated function on $\mathbb{R}^d$ that lies in $\mathscr{M}_p(\mathbb{R}^d;W)$ for some $1 \le p < \infty$. Then $\{b(m)\}_{m\in\mathbb{Z}^d}$ is in $\mathscr{M}_p(\mathbb{T}^d;W)$. Moreover,*

$$\|\{b(m)\}\|_{\mathscr{M}_p(\mathbb{T}^d;W)} \le \|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)}.$$

Proof. The idea of proof is to use scalar transference together with the fact that the dual space to $L_p(\mathbb{T}^d;W)$ is $L_q(\mathbb{T}^d;W^{-q/p})$, with $\frac{1}{p}+\frac{1}{q}=1$, see [5]. Let $\mathscr{P}^{d,N}$ be the family

$$\left\{\mathbf{P}(x) = [P_1(x), \dots P_N(x)]^T\right\}$$

of vectors of trigonometric polynomials on $\mathbb{R}^d$. Take any $\mathbf{P} \in \mathscr{P}^{d,N}$. We now use (2.1) and Lemma 2.3 to calculate the norm of $S(\mathbf{P})$ in $L_p(\mathbb{T}^d;W)$. We notice that $W^{-1/p} \in L_{1,\text{loc}}$, so it follows that $\mathscr{P}^{d,N}$ is dense in $L_q(\mathbb{T}^d;W^{-q/p})$ which again implies that

$$\|S(\mathbf{P})\|_{L_p(\mathbb{T}^d;W)} = \sup_{\mathbf{Q}\in\mathscr{P}^{d,N}:\|\mathbf{Q}\|_{L_q(\mathbb{T}^d;W^{-q/p})}\le 1} \left| \int_{\mathbb{T}^d} \langle S(\mathbf{P})(x),\mathbf{Q}(x)\rangle_{\ell_2}\,dx \right|. \tag{2.3}$$

We now estimate the right hand side of (2.3). Define $L_\varepsilon(x) := e^{-\pi\varepsilon|x|^2}$ for $x\in\mathbb{R}^d$ and $\varepsilon>0$. Using the scalar transference result (2.2) of Lemma 2.3, we obtain

$$\begin{aligned}
\int_{\mathbb{T}^d} \langle S(\mathbf{P})(x),\mathbf{Q}(x)\rangle_{\ell_2}\,dx &= \sum_{i=1}^N \int_{\mathbb{T}^d} S(P_i)(x)\overline{Q_i(x)}\,dx \\
&= \sum_{i=1}^N \lim_{\varepsilon\to 0^+} \varepsilon^{d/2}\int_{\mathbb{R}^d} T(P_iL_{\varepsilon/p})(x)\overline{Q_i(x)L_{\varepsilon/q}(x)}\,dx \\
&= \lim_{\varepsilon\to 0^+} \varepsilon^{d/2}\int_{\mathbb{R}^d} \langle T(\mathbf{P}L_{\varepsilon/p})(x),\mathbf{Q}(x)L_{\varepsilon/q}(x)\rangle_{\ell_2}\,dx \\
&= \lim_{\varepsilon\to 0^+} \varepsilon^{d/2}\int_{\mathbb{R}^d} \langle W^{1/p}(x)T(\mathbf{P}L_{\varepsilon/p})(x),W^{-1/p}(x)\mathbf{Q}(x)L_{\varepsilon/q}(x)\rangle_{\ell_2}\,dx \\
&\le \limsup_{\varepsilon\to 0^+}\left(\int_{\mathbb{R}^d} \varepsilon^{d/2}L_\varepsilon(x)|W^{1/p}(x)T(\mathbf{P})(x)|^p\,dx\right)^{1/p} \\
&\qquad \times \limsup_{\varepsilon\to 0^+}\left(\int_{\mathbb{R}^d} \varepsilon^{d/2}L_\varepsilon(x)|W^{-1/p}(x)\mathbf{Q}(x)|^q\,dx\right)^{1/q} \\
&\le \|T\|_{L_p(\mathbb{R}^d;W)\to L_p(\mathbb{R}^d;W)} \limsup_{\varepsilon\to 0^+}\left(\int_{\mathbb{R}^d} \varepsilon^{d/2}L_\varepsilon(x)|W^{1/p}(x)\mathbf{P}(x)|^p\,dx\right)^{1/p} \\
&\qquad \times \limsup_{\varepsilon\to 0^+}\left(\int_{\mathbb{R}^d} \varepsilon^{d/2}L_\varepsilon(x)|W^{-1/p}(x)\mathbf{Q}(x)|^q\,dx\right)^{1/q} \\
&= \|T\|_{L_p(\mathbb{R}^d;W)\to L_p(\mathbb{R}^d;W)} \left(\int_{\mathbb{T}^d} |W^{1/p}(x)\mathbf{P}(x)|^p\,dx\right)^{1/p} \\
&\qquad \times \left(\int_{\mathbb{T}^d} |W^{-1/p}(x)\mathbf{Q}(x)|^q\,dx\right)^{1/q}.
\end{aligned} \tag{2.4}$$

In the last step, we have used that for any periodic function $f\in L_1(\mathbb{T}^d)$, using Poisson's summation formula,

$$\begin{aligned}
\varepsilon^{d/2}\int_{\mathbb{R}^d} f(x)L_\varepsilon(x)\,dx &= \varepsilon^{d/2}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{T}^d} f(x-k)e^{-\pi\varepsilon|x-k|^2}\,dx \\
&= \int_{\mathbb{T}^d} f(x)\varepsilon^{d/2}\sum_{k\in\mathbb{Z}^d} e^{-\pi\varepsilon|x-k|^2}\,dx \\
&= \int_{\mathbb{T}^d} f(x)\sum_{k\in\mathbb{Z}^d} e^{-\pi|k|^2/\varepsilon}e^{2\pi i x\cdot k}\,dx \\
&= \int_{\mathbb{T}^d} f(x)\,dx + E_\varepsilon,
\end{aligned}$$

where

$$|E_\varepsilon| \le \|f\|_{L_1(\mathbb{T}^d)} \sum_{k\ge 1} e^{-\pi|k|^2/\varepsilon} \to 0,$$

as $\varepsilon \to 0$. We can now complete the proof. Using the estimate (2.4) in (2.3), we immediately see that

$$\|S(\mathbf{P})\|_{L_p(\mathbb{T}^d;W)\to L_p(\mathbb{T}^d;W)} \le \|T\|_{L_p(\mathbb{R}^d;W)\to L_p(\mathbb{R}^d;W)} \|\mathbf{P}\|_{L_p(\mathbb{T}^d;W)}.$$

Moreover, S can be extended to a bounded operator on $L_p(\mathbb{T}^d;W)$ with the required norm estimate since $\mathscr{P}^{d,N}$ is dense in $L_p(\mathbb{T}^d;W)$. Therefore, $\|\{b(m)\}\|_{\mathscr{M}_p(\mathbb{T}^d;W)} \le \|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)}$. □

2.2. **Multipliers in $\mathscr{M}_p(\mathbb{T}^d;W)$.** We now turn to a converse result to Proposition 2.4. At a first glance, the statement of Proposition 2.5 below may appear unnatural since it requires information about dilated versions of the weight W. However, as will be demonstrated in Section 4, the most interesting class of weights is the Muckenhoupt class A_p, which is actually dilation invariant making the statement appear more natural.

Proposition 2.5. *Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a periodic matrix weight with $W, W^{-1} \in L_{1,loc}$, and let $1 \le p < \infty$. Suppose that b is a bounded continuous function on $\mathbb{R}^d$ with $\{b(m/M)\}_{m\in\mathbb{Z}^d} \in \mathscr{M}_p(\mathbb{R}^d;W(M\cdot))$ uniformly in $M \in \mathbb{N}$. Then b is in $\mathscr{M}_p(\mathbb{R}^d;W)$. Moreover,*

$$\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)} \le C_p := \sup_{M\in\mathbb{N}} \|\{b(m/M)\}_m\|_{\mathscr{M}_p(\mathbb{T}^d;W(M\cdot))}.$$

Proof. Let $\mathbf{F}(x) = [F_1(x),\dots,F_N(x)]^T$ and $\mathbf{G}(x) = [G_1(x),\dots,G_N(x)]^T$ be vectors of compactly supported smooth functions. There is an $M_0 \ge 1$ such that $M \ge M_0$ implies that $\mathbf{F}(Mx)$ and $\mathbf{G}(Mx)$ are supported in $[-1/2,1/2)^d$. Let $M \in \mathbb{N}$ with $M \ge M_0$, and define

$$\mathbf{F}_M(x) = \sum_{k\in\mathbb{Z}^d} \mathbf{F}(M(x-k)), \qquad \mathbf{G}_M(x) = \sum_{k\in\mathbb{Z}^d} \mathbf{G}(M(x-k)).$$

A straightforward calculation shows that the Fourier coefficients of $\mathbf{F}_M$ and $\mathbf{G}_M$ satisfy $\widehat{\mathbf{F}_M}(m) = M^{-d}\widehat{\mathbf{F}}(m/M)$ and $\widehat{\mathbf{G}_M}(m) = M^{-d}\widehat{\mathbf{G}}(m/M)$. We use these facts to obtain,

(2.5)

$$\begin{aligned}
&\left|\sum_{i=1}^N \sum_{m\in\mathbb{Z}^d} b(m/M)\hat{F}_i(m/M)\overline{\hat{G}_i(m/M)}\operatorname{Vol}\left(\left[\tfrac{m}{M},\tfrac{m+1}{M}\right]^d\right)\right| \\
&\qquad = \left| M^d \sum_{i=1}^N \sum_{m\in\mathbb{Z}^d} b(m/M)\widehat{F_{i,M}}(m)\overline{\widehat{G_{i,M}}(m)}\right| \\
&\qquad = \left| M^d \int_{\mathbb{T}^d} \sum_{i=1}^N \left(\sum_{m\in\mathbb{Z}^d} b(m/M)\widehat{F_{i,M}}(m)e^{2\pi i m\cdot x}\right)\overline{G_{i,M}(x)}\,dx\right|
\end{aligned}$$

$$
\begin{aligned}
&= M^d \int_{\mathbb{T}^d} \langle T_{\{b(m/M)\}} \mathbf{F}_M(x), \mathbf{G}_M(x) \rangle_{\ell_2}\, dx \\
&= M^d \int_{\mathbb{T}^d} \langle W^{1/p}(Mx) T_{\{b(m/M)\}} \mathbf{F}_M(x), W^{-1/p}(Mx) \mathbf{G}_M(x) \rangle_{\ell_2}\, dx \\
&\leq M^d \left(\int_{\mathbb{T}^d} |W^{1/p}(Mx) T_{\{b(m/M)\}} \mathbf{F}_M(x)|^p \, dx \right)^{1/p} \\
&\qquad\qquad \times \left(\int_{\mathbb{T}^d} |W^{-1/p}(Mx) \mathbf{G}_M(x)|^q \, dx \right)^{1/q} \\
&\leq M^d \|\{b(m/M)\}_m\|_{\mathscr{M}_p(\mathbb{T}^d;W(M\cdot))} \|\mathbf{F}_M\|_{L_p(\mathbb{T}^d;W(M\cdot))} \| \cdot \|\mathbf{G}_M\|_{L_q(\mathbb{T}^d;W^{-q/p}(M\cdot))} \\
&= C_p \|\mathbf{F}\|_{L_p(\mathbb{R}^d;W)} \|\mathbf{G}\|_{L_q(\mathbb{R}^d;W^{-q/p})}.
\end{aligned}
$$

The functions $b(\xi)\widehat{F}_i(\xi)\widehat{G}_i(\xi)$ are Riemann integrable on $\mathbb{R}^d$, so letting the integer $M \to \infty$ in (2.5), we obtain using Parseval's relation,

$$
\left| \sum_{i=1}^N \int_{\mathbb{R}^d} b(\xi)\hat{F}_i(\xi)\overline{\widehat{G}_i(\xi)}\, d\xi \right| = \left| \int_{\mathbb{R}^d} \langle T_b \mathbf{F}(x), \mathbf{G}(x) \rangle_{\ell_2}\, dx \right| \leq C_p \|\mathbf{F}\|_{L_p(\mathbb{R}^d;W)} \|\mathbf{G}\|_{L_q(\mathbb{R}^d;W^{-q/p})}.
$$

Notice that the family of vectors of compactly smooth functions are dense in both $L_p(\mathbb{R}^d;W)$ and $L_q(\mathbb{R}^d;W^{-q/p})$, respectively, since $W^{1/p}, W^{-1/p} \in L_{1,\text{loc}}$. Therefore, it follows that $b \in \mathscr{M}_p(\mathbb{R}^d;W)$ with $\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)} \leq C_p$.

□

3. Muckenhoupt matrix weights

So far we have proved two transference results, Proposition 2.4 and Proposition 2.5. However, for these results to be useful we need to have interesting examples of bounded multipliers on $L_p(\mathbb{R}^d;W)$ and/or $L_p(\mathbb{T}^d;W)$ that can be used for the transfer process. This section contains an application of Proposition 2.4 to the case of a matrix weight W that satisfies the so-called A_p condition for matrices.

The Muckenhoupt A_p-condition for matrix weights was introduced by Nazarov, Treĭl′ and Volberg in [7,10] to study boundedness properties of the vector-valued Hilbert transform. Here we follow Roudenko [9] and give an equivalent and more direct definition of matrix A_p weights. It is proved in [9] that the following definition is equivalent to the A_p condition considered in [7,10]. We let $\mathscr{B}(d)$ denote the family of all Euclidean balls in $\mathbb{R}^d$.

Definition 3.1. Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a matrix weight. For $1 < p < \infty$, let q denote the conjugate exponent to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We say that W belongs to the matrix Muckenhoupt class A_p provided

$$
A(p,W) := \sup_{B \in \mathscr{B}(d)} \int_B \left(\int_B \left\| W^{1/p}(x) W^{-1/p}(t) \right\|^p \frac{dt}{|B|} \right)^{p/q} \frac{dx}{|B|} < \infty. \tag{3.1}
$$

We notice that a simple change of variable in (3.1) reveals that A_p is dilation invariant. More precisely, for a matrix weight $W \in A_p$, and any $M > 0$, the dilated weight $W(M\cdot)$ is also is A_p with the same bound $A(p, W(M\cdot)) = A(p, W)$. This fact will be used in Section 4.

The importance of the Muckenhoupt A_p class is already apparent from the study of the Hilbert transform in [7, 10]. Later Goldberg [5] demonstrated that the Muckenhoupt A_p class is also useful for the study of general vector-valued multipliers. In fact, the following general result on vector-valued multipliers is proved in [5].

Theorem 3.2 ([5]). *Suppose $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ is a matrix weight in A_p for some $1<p<\infty$. Assume $T:L_q(\mathbb{R}^d)\to L_q(\mathbb{R}^d)$ is a bounded convolution operator for some $1<q<\infty$, with associated convolution kernel K satisfying*

$$|K(x)|\le C|x|^{-d} \quad \text{and} \quad |\nabla K(x)|\le C|x|^{-d-1}, \qquad x\in\mathbb{R}^d\backslash\{0\}.$$

Then T extends to a bounded operator on $L_p(\mathbb{R}^d;W)$.

We combine Theorem 3.2 with Proposition 2.4 to obtain the following result.

Corollary 3.3. *Suppose $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ is a matrix weight in A_p for some $1<p<\infty$. Assume that for some $1<q<\infty$, $T_b:L_q(\mathbb{R}^d)\to L_q(\mathbb{R}^d)$ is a bounded multiplier operator induced by a regulated multiplier $b:\mathbb{R}^d\to\mathbb{R}^d$. If the associated convolution kernel K satisfies*

$$|K(x)|\le C|x|^{-d} \quad \text{and} \quad |\nabla K(x)|\le C|x|^{-d-1}, \qquad x\in\mathbb{R}^d\backslash\{0\},$$

then $\{b(m)\}_m\in\mathscr{M}_p(\mathbb{T}^d;W)$, and $b\in\mathscr{M}_p(\mathbb{R}^d;W)$, with

$$\|\{b(m)\}\|_{\mathscr{M}_p(\mathbb{T}^d;W)}\le\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)}.$$

4. Bochner-Riesz summation

We now turn to a specific application of Corollary 3.3 and Proposition 2.5. As our main example, we consider vector Bochner-Riesz summation. For $f\in L_1(\mathbb{T}^d)$ we define the Bochner-Riesz partial sum operators B_R^α, $\alpha,R>0$, by

$$B_R^\alpha(f)(x):=\sum_{m\in\mathbb{Z}^d:|m|\le R}\left(1-\frac{|m|^2}{R^2}\right)^\alpha\hat{f}(m)e^{2\pi im\cdot x}, \tag{4.1}$$

with the Fourier coefficients $\hat{f}(k)$ given by the usual formula,

$$\hat{f}(m):=\int_{\mathbb{T}^d}f(x)e^{-2\pi im\cdot x}\,dx.$$

We will need a few well-know results about Bochner-Riesz summation. The reader can find these results and much more on Bochner-Riesz summation in [6]. Closely related to B_R^α is the multiplier m_α on $\mathbb{R}^d$ given by

$$m_\alpha(\xi)=(1+|\xi|^2)_+^\alpha. \tag{4.2}$$

The convolution kernel K^α associated to m_α is given by

$$(m_\alpha)^\vee(x)=K^\alpha(x):=\frac{\Gamma(\alpha+1)}{\pi^\alpha}\frac{J_{\frac{d}{2}+\alpha}(2\pi|x|)}{|x|^{\frac{d}{2}+\alpha}},$$

with J_β the Bessel function of the first kind. As is well-known, $J_\beta(r)=O(r^{-1/2})$ as $r\to\infty$ for any $\beta\ge0$. Also, $J'_\beta(r)=J_{\beta-1}(r)-J_{\beta+1}(r)$. It follows that for $\alpha\ge c_d:=\frac{d-1}{2}$, there exists a constant C such that

$$|K^{\alpha}(x)| \leq C|x|^{-d}, \qquad |\nabla K^{\alpha}(x)| \leq C|x|^{-d-1}. \tag{4.3}$$

We can now state and prove our main result on Bochner-Riesz summation.

Proposition 4.1. *Suppose $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ is a periodic matrix weight.*

(i) *If W is in A_p for some $1 < p < \infty$, then for $\alpha \geq \frac{d-1}{2}$,*

$$\sup_{M\in\mathbb{N}} \sup_{R>0} \|B_R^{\alpha}\|_{L_p(\mathbb{R}^d;W(M\cdot))\to L_p(\mathbb{R}^d;W(M\cdot))} < \infty.$$

(ii) *Conversely, suppose that for some $1 < p < \infty$,*

$$\sup_{M\in\mathbb{N}} \sup_{R>0} \|B_R^{\frac{d-1}{2}}\|_{L_p(\mathbb{R}^d;W(M\cdot))\to L_p(\mathbb{R}^d;W(M\cdot))} < \infty.$$

Then $W \in A_p$.

Proof. First we prove (i). The multiplier operators

$$T_R(f) := \big(m_{\alpha}\big(\frac{\cdot}{R}\big)\hat{f}\big)^{\vee},$$

are bounded on $L_2(\mathbb{R}^d)$ since m_α is a bounded function. Moreover, the associated kernels $R^d K^{\alpha}(R\cdot)$ satisfy the estimates (4.3) uniformly in $R > 0$. Also, $W \in A_p$ is periodic, so $W(M\cdot)$ is a periodic A_p-weight with $M(p,W(M\cdot)) = M(p,W)$ for $M \in \mathbb{N}$. We notice that m_α is continuous and thus regulated, so by Corollary 3.3, $\{B_R^{\alpha}\}_{R>0}$ extends to a uniformly bounded family of operators on $L_p(\mathbb{T}^d;W(M\cdot))$.

We turn to the proof of (ii). By Proposition 2.5, the multiplier operators

$$T_M(f) := \big(m_{\frac{d-1}{2}}\big(\frac{\cdot}{M}\big)\hat{f}\big)^{\vee},$$

are uniformly bounded on $L_p(\mathbb{R}^d;W)$. The convolution kernel for T_M is given by $M^d K^{\frac{d-1}{2}}(M\cdot)$ We use the asymptotic form,

$$\frac{\pi^{d-\frac{1}{2}}}{\Gamma(d+\frac{1}{2})}|x|^d M^d K^{\frac{d-1}{2}}(Mx) = |x|^{1/2} J_{d-\frac{1}{2}}(2\pi M|x|) \sim \sqrt{\frac{2}{\pi}}\cos\left(2\pi M|x| - \frac{d\pi}{2}\right), \quad \text{as } |x| \to \infty,$$

together with the equidistribution theorem, to conclude that there exists $C > 0$ such that for $x \in \mathbb{R}^d$, we have

$$\sup_{M\in\mathbb{N}} |x|^d M^d |K^{\frac{d-1}{2}}(Mx)| \geq C.$$

It now follows from Lemma 4.2 below that $W \in A_p$. □

The following technical lemma is used for the proof of (ii) in Proposition 4.1; the lemma gives a necessary condition for a family of multipliers to be uniformly bounded on $L_p(\mathbb{R}^d;W)$ under a mild "size" condition on the associated convolution kernels.

For notational convenience, we define for $B \in \mathscr{B}(d)$, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$A(B,p,W) := \int_B \left(\int_B \big\|W^{1/p}(x)W^{-1/p}(t)\big\|^p \frac{dt}{|B|}\right)^{p/q} \frac{dx}{|B|}. \tag{4.4}$$

We can now state and prove Lemma 4.2. The proof of the lemma is based on [5, Theorem 5.2].

Lemma 4.2. *Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a matrix weight, and let $K_m \in C^1(\mathbb{R}^d\setminus\{0\})$, $m \in \mathbb{N}$, be a sequence of convolution kernels, with T_m denoting the operator induced by K_m. Suppose there exists a uniform C such that*

$$|\nabla K_m(x)| \le C|x|^{-d-1}, \qquad x \in \mathbb{R}^d, m \in \mathbb{N}.$$

Assume that there exists a unit vector $\mathbf{u} \in \mathbb{S}^{d-1}$, and a constant $a > 0$, such that

$$\sup_m \min\{|K_m(-r\mathbf{u})|, |K_m(r\mathbf{u})|\} \ge a|r|^{-d}, \qquad r \in \mathbb{R}\setminus\{0\}.$$

If $\{T_m\}$ is a uniformly bounded family of operators on $L_p(\mathbb{R}^d;W)$, then W is in A_p.

Proof. Let $C_{d,N}$ be the constant given by Lemma 4.3 below, let $\varepsilon > 0$ be such that $2\varepsilon+\varepsilon^2 < \frac{1}{2}C_{d,N}^{-2}$, and define $t_0 := \frac{2^{d+3}C}{\varepsilon a} + 4$. We claim that for each $r > 0$ there exists $m_r \in \mathbb{N}$ such that

$$|K_{m_r}(\mathbf{v}) - K_{m_r}(\pm rt_0\mathbf{u})| \le \varepsilon|K_{m_r}(\pm rt_0\mathbf{u})|, \qquad \forall \mathbf{v} \in B(\pm rt_0\mathbf{u}, 2r). \tag{4.5}$$

To verify (4.5) we pick $m_r \in \mathbb{N}$ such that $|K_{m_r}(\pm rt_0\mathbf{u})| \ge \frac{a}{2r^d t_0^d}$. However, $|\nabla K_m(x)| \le \frac{2^{d+1}C}{t_0^{d+1}r^{d+1}}$ for $x \in B(\pm rt_0\mathbf{u}, 2r)$, so the claim follows directly from the mean value theorem.

Now, take any ball $B(y,r) \in \mathscr{B}(d)$, and let $B' = B(y+rt_0\mathbf{u}, r)$. We consider

$$S_B f := \chi_B T_{m_r}(\chi_{B'} T_{m_r}(\chi_B f)),$$

which is an integral operator with kernel

$$S_B(x,y) = \chi_{B\times B}\int_{B'} K_{m_r}(x-z)K_{m_r}(z-y)\,dz,$$

supported on $B \times B$. We clearly have the operator norm estimate $\|S_B\| \le \|T_{m_r}\|^2 \le \sup_m \|T_m\|^2$ on $L_p(\mathbb{R}^d;W)$.

We notice that the restriction $\{x,y \in B; z \in B'\}$ implies that $z-y \in B(rt_0\mathbf{u}, 2r)$ and $x-z \in B(-rt_0\mathbf{u}, 2r)$. We rewrite the kernel as

$$S_B(x,y) = |B|K_{m_r}(rt_0\mathbf{u})K(-rt_0\mathbf{u}))\chi_{B\times B} + S_1(x,y), \tag{4.6}$$

where we use (4.5), and the fact that $2\varepsilon+\varepsilon^2 < \frac{1}{2}C_{d,N}^{-2}$, to obtain the estimate

$$|S_1(x,y)| \le \frac{1}{2}C_{d,N}^{-2}|B|\cdot|K(rt_0\mathbf{u})K(-rt_0\mathbf{u})|. \tag{4.7}$$

We use Lemma 4.3 to conclude that the operator with the constant kernel

$$S_0(x,y) := |B|K_{m_r}(rt_0\mathbf{u})K(-rt_0\mathbf{u}))\chi_{B\times B}$$

has norm at least $D\cdot A(B,p,W)$ on $L_p(\mathbb{R}^d;W)$, with D proportional to $a^2 t_0^{-2d} C_{d,N}^{-1}$. The estimate (4.7) and Lemma 4.3 shows that the norm of S_1 on $L_p(\mathbb{R}^d;W)$ is at most $\frac{D}{2}A(B,p,W)$. It follows that $\|S_B\| \ge \frac{D}{2}A(B,p,W)$, so

$$A(B,p,W) \le \frac{2}{D}\|S_B\| \le \frac{2}{D}\|T_{m_r}\|^2 \le \frac{2}{D}\sup_m \|T_m\|^2 < \infty.$$

We conclude that W belongs to A_p. □

The following technical Lemma, which is used to derive Lemma 4.2, is due to M. Goldberg [5].

Lemma 4.3 ([5]). *Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a matrix weight, and let $B \in \mathscr{B}(d)$. Suppose S is an integral operator, $Sf(x) := \int_{\mathbb{R}^d} S(x,y)f(y)dy$, whose scalar kernel $S(x,y)$ is supported in $B\times B$ and satisfies the bound $|S(x,y)| \leq |B|^{-1}$ for all $(x,y) \in B\times B$.*

(i) *The norm of S as an operator on $L_p(\mathbb{R}^d;W)$ is at most $C_{d,N}A(B,p,W)$, with $C_{d,N}$ a dimensional constant independent of S.*
(ii) *In the special case $S_0 = |B|^{-1}\chi_{B\times B}$, the norm of S_0 is at least $C_{d,N}^{-1}A(B,p,W)$.*

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