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by

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On a Conjecture about Inverse Domination in Graphs

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Abstract

Let $G = (V, E)$ be a graph with no isolated vertex. A classical observation in domination theory is that if D is a minimum dominating set of G, then $V \setminus D$ is also a dominating set of G. A set D' is an inverse dominating set of G if D' is a dominating set of G and $D' \subseteq V \setminus D$ for some minimum dominating set D of G. The inverse domination number of G is the minimum cardinality among all inverse dominating sets of G. The independence number of G is the maximum cardinality of an independent set of vertices in G. Domke, Dunbar, and Markus (Ars Combin. 72 (2004), 149–160) conjectured that the inverse domination number of G is at most the independence number of G. We prove this conjecture for special families of graphs, including claw-free graphs, bipartite graphs, split graphs, very well covered graphs, chordal graphs and cactus graphs.

Keywords: dominating set, inverse domination number; independence number
AMS subject classification: 05C69

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1 Introduction

In this paper, we continue the study of domination in graphs. Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *dominating set* of G is a set D of vertices of G such that every vertex in $V \setminus D$ is adjacent to a vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. Domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [6, 7].

Let $G = (V, E)$ be a graph with no isolated vertex. A classical result in domination theory due to Ore [12] is that if D is a minimum dominating set of G, then $V \setminus D$ is also a dominating set of G. A set D' is an *inverse dominating set* of G if D' is a dominating set of G and $D' \subseteq V \setminus D$ for some $\gamma(G)$-set D. By Ore's result, every graph with no isolated vertex has an inverse dominating set. The *inverse domination number* of G, denoted $\gamma^{-1}(G)$, is the minimum cardinality among all inverse dominating sets of G. An inverse dominating set of G of cardinality $\gamma^{-1}(G)$ we call a $\gamma^{-1}(G)$-set. If D' is a $\gamma^{-1}(G)$-set and D is a $\gamma(G)$-set such that $D' \subseteq V \setminus D$, then we refer to the pair (D, D') as an *inverse dominating pair.*

A set I of vertices in G is an *independent set* if no two vertices of I are adjacent in G. The *independence number* of G, denoted $\alpha(G)$, is the maximum cardinality of an independent set of vertices in G. An *independent dominating set* of G is a set that is both an independent set and a dominating set of G. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. By definition, $\gamma(G) \le i(G)$ for all graphs G.

Inverse domination in graphs was introduced by Kulli and Sigarkant [10]. In their original paper in 1991, they include a proof that for all graphs with no isolated vertex the inverse domination number is at most the independence number. However this proof is incorrect and contains an error. In 2004, Domke, Dunbar, and Markus [4] formally stated this "result" of Kulli and Sigarkant as a conjecture:

Conjecture 1 (Domke, Dunbar, Markus [4]) *If G is a graph with no isolated vertex, then $\gamma^{-1}(G) \le \alpha(G)$.*

For notation and graph theory terminology we in general follow [6]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A vertex $w \in V$ is an *S-private neighbor of $v \in S$* if $N[w] \cap S = \{v\}$, while the *S-private neighbor set of v*, denoted $\mathrm{pn}(v, S)$, is the set of all S-private neighbors of v. Further if $w \in V \setminus S$, then w is called an *external S-private neighbor of* v and the *external S-private neighbor set of* v, denoted $\mathrm{epn}(v, S)$, is the set of all external S-private neighbors of v.

If $X, Y \subseteq V$, then the set X is said to *dominate* the set Y if $Y \subseteq N[X]$. In particular, if

X dominates V, then X is a dominating set in G. If X dominates G and no subset of X dominates G, then X is called a minimal dominating set of G. The largest cardinality of a minimal dominating set for G is the upper domination number of G, denoted $\Gamma(G)$.

For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$).

Two edges in a graph G are *independent* if they are not adjacent in G. A set M of pairwise independent edges of G is called a *matching* in G and if $2|M| = |V(G)|$ then M is called a *perfect matching*.

If G does not contain a graph F as an induced subgraph, then we say that G is F-free. In particular, we say a graph is *claw-free* if it is $K_{1,3}$-free.

A *block* of a graph G is a maximal 2-connected subgraph of G. A graph G is a *cactus* if and only if every block of G is a cycle or a K_2. A graph G is a *generalized cactus graph* (or a *generalized Gallai-tree*) if and only if every block of G is a cycle or a complete graph. An *endblock* of G is a block that contains only one cutvertex of G.

Our aim in this paper is to prove Conjecture 1 for some families of graphs. For this purpose, we recall that a *minimal dominating set* in G is a dominating set that contains no dominating set as a proper subset. We shall need the following classical result due to Ore [12] of properties of minimal dominating sets.

Lemma 1 (Ore [12]) *Let D be a dominating set of a graph G. Then, D is a minimal dominating set of G if and only if for each $v \in D$, the vertex v is isolated in $G[D]$ or $|epn(v, D)| \geq 1$.*

2 Special Families

In this section, we prove that Conjecture 1 is true for special families of graphs.

2.1 Claw-Free Graphs

First we establish that Conjecture 1 is true for claw-free graphs. For this purpose, we prove that every graph that has a minimum dominating set which is independent satisfies Conjecture 1.

Theorem 1 *If G is a graph with $\delta(G) \geq 1$ satisfying $\gamma(G) = i(G)$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Proof. Let $G = (V, E)$ and let D be a $\gamma(G)$-set that is independent. Let I be a maximal independent set of vertices in $G[V \setminus D]$. If $I = V \setminus D$, then I is an inverse dominating set of G that is independent, and so $\gamma^{-1}(G) \leq |I| \leq \alpha(G)$. Hence we may assume that

$I \subset V \setminus D$. By the maximality of I, the set I dominates $V \setminus D$. Let A be the set of all vertices of D not dominated by I in G. If $A = \emptyset$, then I is an inverse dominating set of G that is independent, and so $\gamma^{-1}(G) \leq |I| \leq \alpha(G)$. Hence we may assume that $|A| \geq 1$. Since D is an independent set and $\delta(G) \geq 1$, and since no vertex in A is dominated by the set I, we note that $N(v) \subseteq V \setminus (D \cup I)$ for each $v \in A$. For each $v \in A$, let $v' \in N(v)$. Let $A' = \cup\{v'\}$ where the union is taken over all vertices $v \in A$. Then, $|A'| \leq |A|$ and $A' \cup I$ is an inverse dominating set of G. Hence, since $A \cup I$ is an independent set of G, we have that $\gamma^{-1}(G) \leq |A' \cup I| = |A'| + |I| \leq |A| + |I| = |A \cup I| \leq \alpha(G)$. □

Since every claw-free graph G satisfies $\gamma(G) = i(G)$ ([1]), we have the following immediate consequence of Theorem 1.

Corollary 1 *If G is a claw-free graph with $\delta(G) \geq 1$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

By applying Theorem 1 to graphs satisfying $\gamma(G) = \alpha(G)$ we get the following observation.

Observation 1 *If a graph G without isolated vertices satisfies $\gamma(G) = \alpha(G)$, then G has two vertex-disjoint $\gamma(G)$-sets.*

From the proof of Theorem 1 it actually follows that in a graph G without isolated vertices satisfying $\gamma(G) = \alpha(G)$, there is for each independent $\gamma(G)$-set D a $\gamma(G)$-set contained in $V(G)\setminus D$.

2.2 Bipartite and Chordal Graphs

Since every minimal inverse dominating set in a graph without isolated vertices is a minimal dominating set in the graph, the inverse domination number is at most the upper domination number. Every maximal independent set in a graph is a minimal dominating set, and so the independence number of a graph is bounded above by its upper domination number. Hence we have the following observations.

Observation 2 *If G is a graph with $\delta(G) \geq 1$, then $\gamma^{-1}(G) \leq \Gamma(G)$ and $\alpha(G) \leq \Gamma(G)$.*

Observation 3 *If G is a graph with $\delta(G) \geq 1$ and $\alpha(G) = \Gamma(G)$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Several families of graphs G are known to satisfy $\alpha(G) = \Gamma(G)$. These include, among others, bipartite graphs ([3]), chordal graphs ([9]), circular arc graphs ([5]), permutation graphs and comparability graphs ([2]). Hence, by Observation 3, Conjecture 1 is true for every graph in one of these families.

Corollary 2 *If G is a bipartite graph, a chordal graph, a circular arc graph, a permutation graph, or a comparability graph with $\delta(G) \geq 1$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

2.3 Very Well-Covered Graphs

Recall that a graph G of order n is said to be *very well-covered* if $i(G) = \alpha(G) = n/2$. We show next that Conjecture 1 is true for every very well-covered graph. For this purpose, we first show that every graph that has a perfect matching and has independence number one-half its order satisfies Conjecture 1.

Theorem 2 *If G is a graph of order n that has a perfect matching and $\alpha(G) = n/2$, then $\Gamma(G) = n/2$.*

Proof. Let $G = (V, E)$ and let M be a perfect matching in G. By Observation 2, $\Gamma(G) \geq \alpha(G)$, and so $\Gamma(G) \geq n/2$. Let D be a $\Gamma(G)$-set and let V_i denote the vertices $v \in V$ such that $uv \in M$ and $|\{u, v\}| \cap D = i$ for $i \in \{0, 1, 2\}$. Then, $V = V_0 \cup V_1 \cup V_2$, $V_2 \subseteq D$ and $|D| = \frac{1}{2}|V_1| + |V_2|$. Since D is a minimal dominating set of G, Lemma 1 implies that $|\mathrm{epn}(x, D)| \geq 1$ for each vertex $x \in V_2$. Thus since each vertex of $V_1 \cap (V \setminus D)$ is dominated by $V_1 \cap D$, we must have that $\mathrm{epn}(x, D) \subset V_0$ for each vertex $x \in V_2$. Hence,

$$|V_0| \geq |\bigcup_{x \in V_2} \mathrm{epn}(x, D)| = \sum_{x \in V_2} |\mathrm{epn}(x, D)| \geq |V_2|.$$

Therefore, $\Gamma(G) = |D| = \frac{1}{2}|V_1| + |V_2| \leq \frac{1}{2}(|V_0| + |V_1| + |V_2|) = n/2$. As observed earlier, $\Gamma(G) \geq n/2$. Consequently, $\Gamma(G) = n/2$. $\square$

As an immediate consequence of Observation 3 and Theorem 2, we have the following result.

Corollary 3 *If G is a graph of order n that has a perfect matching and $\alpha(G) = n/2$, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Since every very well-covered graph has a perfect matching (e.g. see [13]) we obtain the following consequence of Corollary 3.

Corollary 4 *If G is a very well-covered graph of order n, then $\gamma^{-1}(G) \leq \alpha(G)$.*

2.4 Nearly Bipartite Graphs and Split Graphs

Recall that a graph G is called a *split graph* if its vertices can be partitioned into two sets X and Y such that X is an independent set and $G[Y]$ is a complete graph. As shown in Corollary 2, Conjecture 1 is true for the family of bipartite graphs. Here we establish that Conjecture 1 is true for the family of graphs that can be obtained from a bipartite graph by adding edges to one of its partite set such that each component of the subgraph induced by that set is a complete graph. As a special case of this result, we have that Conjecture 1 is true for split graphs.

Theorem 3 *If G is a graph without isolated vertices that can be obtained from a bipartite graph by adding edges to one of its partite set Y such that each component of $G[Y]$ is a complete graph, then $\gamma^{-1}(G) \le \alpha(G)$.*

Proof. Let $G = (V, E)$ be a graph without isolated vertices obtained from a bipartite graph H with partite sets X and Y by adding edges to the subgraph $H[Y]$ induced by the partite set Y in such a way that each component of $G[Y]$ is a complete graph.

Among all $\gamma(G)$-sets, let D be one such that $|N(X \setminus D) \cap D|$ is a maximum. Let $D_X = D \cap X$ and $D_Y = D \cap Y$. Note that if $v \in D$ is not isolated in $G[D]$, then by Lemma 1, $|\mathrm{epn}(v, D)| \ge 1$. Let $A = X \setminus D_X$ and let $B = N(A) \subseteq Y$. Hence, B is the set of vertices in Y dominated by the set A. Let C be the set of vertices in Y not dominated by A; that is, $C = Y \setminus B$. Let I_C be a maximum independent set in $G[C]$ such that $|I_C \cap D|$ is a minimum.

We show that $I_C \cap D = \emptyset$. Assume, to the contrary, that there exists a vertex $v \in I_C \cap D$. Since $v \in C$, we have $\mathrm{pn}(v, D) \subseteq Y$. Thus the vertex v and the vertices from $\mathrm{pn}(v, D)$ are in the same component G_v of $G[Y]$. Let $C_v = V(G_v)$. For each vertex $y \in C_v \setminus \{v\}$, the set $D' = (D \setminus \{v\}) \cup \{y\}$ is a $\gamma(G)$-set since G_v is a complete graph containing $\mathrm{pn}(v, D)$. Further if $y \in B$, then $|N(A) \cap D'| > |N(A) \cap D|$, contradicting our choice of the set D. Hence, $C_v \subseteq C$. If $C_v = \{v\}$, then since $\delta(G) \ge 1$, $N(v) \subseteq D_X$ contradicting our earlier observation that $\mathrm{pn}(v, D) \subseteq Y$. Hence, $D \cap C_v = \{v\}$ and $|C_v| \ge 2$. But then for each vertex $u \in C_v \setminus \{v\}$, $(I_C \setminus \{v\}) \cup \{u\}$ is a maximum independent set in C, contradicting our choice of the set I_C. Hence, $I_C \cap D = \emptyset$.

Let Z be the set of vertices in D_X that are not dominated by I_C in G. Then, $A \cup I_C$ dominates $V \setminus Z$ and $A \cup I_C \cup Z$ is an independent set in G. For each $v \in Z$, let $v' \in N(v) \setminus D$ and let $Z' = \cup\{v'\}$ where the union is taken over all vertices $v \in Z$. Then, $|Z'| \le |Z|$ and $A \cup I_C \cup Z'$ is an inverse dominating set of G. Hence, $\gamma^{-1}(G) \le |A \cup I_C \cup Z'| = |A| + |I_C| + |Z'| \le |A| + |I_C| + |Z| = |A \cup I_C \cup Z| \le \alpha(G)$. □

As an immediate consequence of Theorem 3, we have that Conjecture 1 is true for split graphs.

Corollary 5 *If G is a split graph with no isolated vertex, then $\gamma^{-1}(G) \le \alpha(G)$.*

Note that since split graphs are chordal this result also follows from Corollary 2.

2.5 Cactus Graphs and Generalized Cactus Graphs

We prove in this section that if a graph only has cycles and complete graphs as blocks, then it satisfies Conjecture 1. It can easily be shown that if G is a cactus such that each vertex contained in a C_3 is in no other cycle-block then G can be obtained from a bipartite graph with no isolated vertex by adding the edges of a matching to one of its partite sets. Thus, by Theorem 3, such a graph G satisfies Conjecture 1.

In the following we give an algorithm which for graphs only having cycles and complete graphs as blocks, can be used to construct an inverse dominating set with at most $\alpha(G)$ vertices. As in [6, 8] we label each vertex as Required, Bound or Free. Thus we partition $V(G)$ into three disjoint sets R, B and F. Here R denotes vertices required to be in our minimum dominating set under construction, B denotes vertices not dominated by R, but bound to be dominated later on, and F, the free vertices, are vertices which need not to be dominated, either because they from the outset are declared to need no domination or because they already have been dominated in an earlier step of the algorithm. A vertex labelled F may later have its label changed to R. The algorithm here starts with all vertices labelled B and will stepwise change labeling until R has grown into a minimum dominating set for G.

Mitchell, Cockayne, and Hedetniemi [11] presented a linear algorithm for finding a minimum (R, B, F)-dominating set D for a rooted tree T with root x and with each vertex labelled by one of R, B or F. That is, their algorithm finds a set D of minimum cardinality such that $B \subseteq N[D]$. Further the algorithm constructs the set D such that for each vertex v the subtree containing v and all its descendants contains as few vertices from D as possible. Essentially the algorithm selects a dominating set by pushing D-vertices as far up the tree as possible. We shall use this tree algorithm given in [11] (it may also be found in [6]) to find a set D of minimum cardinality such that $R \subseteq D$ and $B \subseteq N[D]$. Later we use the notation $\text{Tree}(T, x, R, B, F)$ for the vertices added to R to obtain a (R, B, F)-dominating set by using this algorithm on the tree T with root x and where (R, B, F) is a weak partition of $V(T)$. (By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty.)

In the algorithm we construct an independent set I and a set S such that S is an inverse dominating set. Further we define a function $s : S \to V(G)$ such that $s(v) \in N[v] \setminus R$ for each vertex $v \in S$.

To show correctness of the algorithm we use the following loop invariant, where H is a subgraph of G :

Loop invariant :

1. Each vertex in $R \cap V(H)$ is dominated by $S \setminus V(H)$.
2. I is an independent set and $I \subseteq V(G) \setminus V(H)$.
3. If $x \in V(H) \setminus R$ and $N(x) \cap I \neq \emptyset$ then $x \in S$.
4. $S \subseteq V(G) \setminus R$ and for each set $A \subseteq S \cap V(H)$ the set $(S \setminus A) \cup (\bigcup_{v \in A} s(v))$ dominates $V(G) \setminus V(H)$ and $s(v) \in N[v] \setminus (R \cup V(H))$ for each $v \in S$.
5. $|S| \leq |I|$.
6. R can be extended to a minimum dominating set of G by adding vertices from $V(H)$.
7. $V(G) \setminus N[R] = B$.

In the algorithm we always consider an end-block C of the current graph H (or $C := H$ if H does not have an end-block) with the three sets R, B and F given. If x denotes the cutvertex in C (or any vertex in C if H is a block), then we reduce the problem, such that we only have to consider the graph $H - (V(C) \setminus \{x\})$ (or the empty graph if $H \cong K_1$). In a step we may use auxiliary sets D and A, D to enlarge R and A to enlarge I and S.

When adding vertices to S and I we have to make sure that 2), 3), and 6) are still satisfied. For this purpose we define the operation $extendIS(A, x, K)$ for an independent set of vertices A, the cutvertex x of C and a vertex-set $K \subseteq A$, where $|K| \leq 1$. This operation adds each vertex of $A \setminus N[I]$ both to I and to S, and changes the function s such that $s(v) = v$ for each $v \in A \setminus N[I]$. Further if $K = \{y\} \subseteq I$ (note that I is changed now) and $x \notin R \cup S$ then the operation exchanges y with x in S, i.e., $S := (S \setminus \{y\}) \cup \{x\}$, sets $s(x) = y$ and cancels definition of $s(y)$.

If we add new vertices to the set R that already belong to S we have to change S since S should be a inverse dominating set and thus must have $S \cap R = \emptyset$. For this purpose the operation $addtoR(A)$ can be used if A is the set of vertices added to R. The operation adds A to the set R, changes S to $(S \setminus A) \cup (\bigcup_{v \in A \cap S} s(v))$, F to $F \cup (N(A) \cap B)$ and B to $B \setminus (N[A] \cap B)$. Further, the definition of s on vertices deleted from S is cancelled and we set $s(v) = v$ for the new vertices just added to S.

Algorithm : Inverse dominating set
Input : A generalized cactus graph G

Let $R = F = \emptyset$, $B = V(G)$, $H = G$ and $I = S = \emptyset$.

While $H \neq \emptyset$ do

If H contains a cutvertex, then let C be a end-block in H; otherwise, let C be the graph H. If $H \neq C$, then let x be the unique cutvertex contained in C. If $H = C$, let x be any vertex in C. Depending on C, F, B and R we now perform changes to I, S, F, B and R and after this we set $H := H - (V(C) \setminus \{x\})$ if $C \neq K_1$ and $H := \emptyset$ if $C = K_1$.

Case 1: $C \neq K_1$ is a complete graph.

If there exists a vertex $y \in (V(C) \setminus \{x\}) \cap B$, then perform the operations $addtoR(\{x\})$ and $extendIS(\{y\}, x, \emptyset)$. If there does not exist such a vertex and there is a vertex $y \in (V(C) \setminus \{x\}) \setminus N[S \setminus \{x\}]$, then perform the operation $extendIS(\{y\}, x, \{y\})$.

Case 2: $C \cong K_1$.

In this case, $H = C = K_1$. If $x \in B$, then perform the operation $addtoR(\{x\})$. If $N(x) \cap I = \emptyset$ then add x to I and let $S := S \cup \{v\}$ for a vertex $v \in N[x] \setminus R$ and set $s(v) := v$ (note that $N(x) \cap I \neq \emptyset$ implies $N[x] \cap S \neq \emptyset$).

Case 3: C is a cycle.

Case 3.1: $V(C) \setminus N[x] \subseteq B$ and $N[x] \cap R = \emptyset$.

Assume C: $x, v_1, v_2, \ldots, v_{3k+i}, x$ where $i \in \{0, 1, 2\}$. Further assume, without loss of generality, that $v_1 \in F$ if $v_{3k+i} \in F$. If $v_1 \notin F$, then assume v_1 is adjacent to a vertex from I if v_{3k+i} is adjacent to a vertex from I. Depending on $F' := \{v_1, v_{3k+i}\} \cap F$ and i, let D and A be the sets from Table 1 and Table 2. Now perform the operations $addtoR(D)$ and $extendIS(A, x, A \cap \{v_{3k+i}\})$.

D	$F' = \emptyset$	$F' = \{v_1\}$	$F' = \{v_1, v_{3k+i}\}$
$i = 0$	$v_2, v_5, \ldots, v_{3k-1}$	$v_3, v_6, \ldots, v_{3k}$	$v_3, v_6, \ldots, v_{3k}$
$i = 1$	$x, v_3, v_6, \ldots, v_{3k}$	$v_3, v_6, \ldots, v_{3k}$	$v_3, v_6, \ldots, v_{3k}$
$i = 2$	$x, v_3, v_6, \ldots, v_{3k}$	$x, v_3, \ldots, v_{3k}$	$v_3, v_6, \ldots, v_{3k}$

Table 1: The set D in Case 3.1

A	$F' = \emptyset$	$F' = \{v_1\}$
$i = 0$	$v_{3k}, v_1, v_4, \ldots, v_{3k-2}$	$v_2, v_5, \ldots, v_{3k-1}$
$i = 1$	$v_1, v_4, \ldots, v_{3k+1}$	$v_{3k+1}, v_2, v_5, \ldots, v_{3k-1}$
$i = 2$	$v_1, v_4, \ldots, v_{3k+1}$	$v_2, v_5, \ldots, v_{3k+2}$

A	$F' = \{v_1, v_{3k+i}\}$
$i = 0$	$v_2, v_5, \ldots, v_{3k-1}$
$i = 1$	$v_2, v_5, \ldots, v_{3k-1}, v_{3k+1}$
$i = 2$	$v_2, v_5, \ldots, v_{3k-1}, v_{3k+1}$

Table 2: The set A in Case 3.1

Case 3.2 : $V(C) \cap R \neq \emptyset$.

Let e be an edge on C incident with a vertex from R. Use the tree-algorithm to find $D := Tree(C - e, x, R \cap (V(C) \setminus \{x\}), B \cap (V(C) \setminus \{x\}), (F \cap V(C)) \cup \{x\})$. Thus the new R-dominators appointed are temporarily called D until they are added to the set R. Now let A' be the children, in the tree $C - e$ rooted at x, of the vertices from D that are not contained in $R \cup D$ (note that by the properties of D each vertex from D has such a child). Let A'' be a maximal independent set in $C - (N[A'] \cup R \cup \{x\})$ constructed upwards from the bottom in $C - e$ (keep adding a vertex at maximal distance from x and removing the closed neighborhood of that vertex) and let $A = A' \cup A''$. Now perform the operation $addtoR(D)$, and if A contains a vertex $y \in N(x) \setminus N[I]$, then perform the operation $extendIS(A, x, \{y\})$; otherwise, perform the operation $extendIS(A, x, \emptyset)$.

Case 3.3 : There is a vertex $y \in (V(C) \cap F) \setminus N[x]$.

Using the tree-algorithm on $C - N[y]$ and $C - y$ it can be determined whether $\{y\} \cup R$ can be extended to a minimum dominating set or R can be extended to a minimum dominating set not containing y.

If $\{y\} \cup R$ can be extended to a minimum dominating set, then let $D := \{y\} \cup Tree(C - y, x, (R \cap V(C)) \setminus (\{x\} \cup N[y]), (B \cap V(C)) \setminus (\{x\} \cup N[y]), (F \cap V(C)) \cup \{x\} \cup N[y])$. If R can be extended to a minimum dominating set without y, then let $D := Tree(C - y, x, (R \cap$

$V(C)) \setminus \{x\}, (B \cap V(C)) \setminus (\{x, y\}), (F \cap V(C)) \cup \{x, y\})$. Further, let A' be the children, in $C - y$ rooted at x, of the vertices from $D \setminus \{y\}$ that are not contained in $R \cup D$. Let A'' be a maximal independent set in $C - (N[A'] \cup R \cup \{x, y\})$ constructed from the bottom up in $C - y$ and let $A = A' \cup A''$. Now perform the operation *addtoR*(D).

First assume that one of three cases occur: (i) $y \in N(I)$, (ii) both vertices on C adjacent to y are in A or (iii) exactly one of these neighbors z is in A but $A \cap (N(x) \setminus N[I]) \neq \{z\}$. If there is a vertex $u \in A \cap (N(x) \setminus N[I])$, then let u be one with maximum distance to y and perform the operation $extendIS(A, x, \{u\})$. Otherwise, perform the operation $extendIS(A, x, \emptyset)$.

Next it can be assumed that none of cases (i)-(iii) occur. Assume neither neighbor of y is in A. If there is a vertex $z \in (N(x) \cap A) \setminus N(I)$, then perform the operation $extendIS(A, x, \{z\})$; otherwise, perform the operation $extendIS(A, x, \emptyset)$. Further, add the vertex y to I and a vertex $z \in N[y] \setminus R$ to S and let $s(z) = z$. Otherwise it can be assumed that exactly one of the neighbors of y is in A and this neighbor z satisfies $A \cap (N(x) \setminus N[I]) = \{z\}$. Now perform the operation $extendIS(A, x, \emptyset)$ and exchange the vertex z with y in I, setting $I := (I \setminus \{z\}) \cup \{y\}$.

Output of algorithm : An inverse dominating set S and an independent set I of G with $|S| \leq |I|$.

Theorem 4 *If $G \not\cong K_1$ is a generalized cactus graph, then $\gamma^{-1}(G) \leq \alpha(G)$.*

Proof. First assume that the loop invariant is true for the loop in the algorithm. Since the loop invariant is obviously true when reaching the while-loop in the algorithm, the loop invariant is also true just before terminating the algorithm. By 6), R is a $\gamma(G)$-set and by 4), $S \subseteq V(G) \setminus R$ and S dominates G. Thus S is an inverse dominating set, and by 2) and 5) the set I is independent and $|S| \leq |I|$. It follows that $\gamma^{-1}(G) \leq |S| \leq |I| \leq \alpha(G)$.

Thus the theorem is true if the loop invariant can be verified. In all cases of the algorithm either we only add a single vertex $v \in V(C) \setminus \{x\}$ to I not already adjacent to a vertex of I, or we add a subset of an independent set $A \subseteq V(C) \setminus \{x\}$ to I by using the extend-operation ($extendIS(A, x, K)$). Here only the vertices from A not adjacent to a vertex in I are added to I and thus the new set is independent and does not contain any vertices from the new graph H resulting from this step after deletion of $C \setminus \{x\}$. Further, it follows that when using the extend-operation we add the same number of vertices to I and S, and if vertices are added to S but not by the extend-operation, then it is only a single vertex and at the same time a vertex is added to I. From these observations 2) and 5) remain satisfied.

Since vertices are only added to R by using the operation *addtoR*(D) for a set D property 7) remains true since the add-operation changes B to $B \setminus N[D]$ when adding D to R.

When considering a block C with the vertex x in the algorithm it follows that if $D \subseteq V(C)$ is a set of minimum cardinality that contains a vertex as near to x as possible such that $(V(C) \setminus \{x\}) \cap B \subseteq N[D]$, then $D \cup R$ can be extended to a minimum dominating set of G by adding vertices from $V(H) \setminus V(C - x)$ if R can be extended to a minimum dominating set by adding vertices from $V(H)$. In Case 1 and Case 3.1, the set D is constructed such that it has this property, and in Case 3.2 and Case 3.3, the set D gets this property since it is produced from the algorithm for trees. In Case 2 it follows that R is a minimum dominating set if $x \notin B$; otherwise, $R \cup \{x\}$ is a minimum dominating set. Thus property 6) remains satisfied.

If a vertex $x \in R \cap V(H)$, then this vertex x must have been added earlier when considering the block C associated with the vertex x.

If the vertex x is added to the set R when considering a block, it can be seen that the extend-operation is used with a set containing a neighbor of x, and thus this neighbor is added to S or is already in S. This is seen directly in Case 1 and Case 3.1. In Cases 3.2 and 3.3 it follows that $x \in D$ if x is added to R and in this case a neighbor z of x is in $A' \subseteq A$.

Since no vertex in $S \setminus V(H)$ is ever removed from S property 1) remains true.

Further by looking at all cases it can be seen that if a neighbor of x is added to I when considering C, then either the add-operation is used with a set containing x or the neighbor is added by using a extend-operation $extendIS(A, x, \{u\})$, where $u \in A$ is a vertex not adjacent to a vertex from I and thus x is added to S except if x is in R or is already in S. Since a vertex is only removed from S if it is added to R, property 3) remains true.

When proving 4), let G_x be the component of $H - x$ containing $V(C - x)$. In the following we prove that after having considered C and added vertices to S, the set $S \cap V(G_x)$ dominates $C - x$ if x was not added to S, and if x was added, then $S \cap (V(G_x)) \cup \{x\}$ and $S \cap (V(G_x)) \cup \{s(x)\}$ dominates $C - x$. By showing this, property 4) follows.

By considering Case 1, Case 2 and Case 3.1, this can easily be verified. In the other cases it can be shown in a similar manner and we only consider Case 3.3. By the properties of the domination-algorithm for trees it follows that the set A' dominates D.

Since A'' is a maximal independent set in $C - (N[A'] \cup R \cup \{x, y\})$, the set $A = A' \cup A''$ dominates $V(C) \setminus (\{x, y\} \cup R)$, but all vertices from $R \cap (V(C - x))$ are already dominated from vertices from $(S \cap V(G_x)) \setminus V(C)$. Further by the choice of A'' it follows that if a neighbor u of x is in A'' then $(A \cup \{x\}) \setminus \{u\}$ also dominates $V(C) \setminus (\{x, y\} \cup R)$.

If $y \in N(I)$, both vertices on C adjacent to y are in A or exactly one of these neighbors z is in A but $A \cap (N(x) \setminus N[I]) \neq \{z\}$ then the vertex y will be dominated after using the extend operation. If this is not the case, a vertex from $N[y]$ is added to S for the sole purpose to dominate y. Further all vertices from $C \setminus \{x, y\}$ are dominated since after the operation the set S either contains A or contains A with a neighbor from A'' of x exchanged with x. Thus property 4) follows in this case.

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