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Distance domination in vertex partitioned graphs

by

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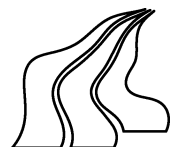# Distance domination in vertex partitioned graphs

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Abstract

We treat a variation of domination which involves a partition $V = (V_1, V_2, \ldots, V_k)$ of $V(G)$ and domination of each partition class V_i over distance d where all vertices and edges of G may be used in the domination process. Strict upper bounds and extremal graphs are presented; the results are collected in three handy tables. Further, we compare a high number of partition classes and the number of dominators needed.

Keywords: domination; distance; vertex partition; domination number; partitioned graph

AMS subject classification: 05C69

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1 Notation

By $G = (V, E)$ we denote a *graph G* with *vertex* set $V = V(G)$ and *edge* set $E = E(G)$. The *order* of G is $|V(G)| = n$ and the *size* of G is $|E(G)|$. For $x \in V(G)$ we denote by $N_G(x)$ the set of *neighbours* to x and $N_G[x] = \{x\} \cup N_G(x)$. Analogously for a subset D of $V(G)$ we define $N_G(D) = \bigcup \{N_G(x) \mid x \in D\}$ and $N_G[D] = \bigcup \{N_G[x] \mid x \in D\} = D \cup N_G(D)$. Let d be a positive integer. We let $N_{(d,G)}(x)$ denote the set of vertices in $V \setminus \{x\}$ having distance at most d to x, and we define $N_{(d,G)}[x] = N_{(d,G)}(x) \cup \{x\}$. We let $N_{(d,G)}(D) = \bigcup \{N_{(d,G)}(x) \mid x \in D\}$ and $N_{(d,G)}[D] = \bigcup \{N_{(d,G)}[x] \mid x \in D\}$. Indices may be omitted if clear from context. The *degree* of x is $d_G(x) = |N_G(x)|$, the number of neighbours to x. We let $\delta(G) = \delta$ denote the *minimum degree* in G and $\Delta(G) = \Delta$ the *maximum* degree. A *corona graph G*, denoted by $G = H \circ K_1$, has order $2n$ and is obtained from a graph H of order n and n new vertices, one corresponding to each vertex of H, by joining each vertex of H to its corresponding new vertex. Analogously $G = H \circ P_d$ denotes a *P_d-corona graph G* of order $n(d+1)$ obtained as the disjoint union of a graph H of order n and n disjoint paths P_d, each of length $d-1$, by joining each vertex of H to an end vertex of its corresponding path P_d. In G a set S of vertices is called *distance d independent* if the distance between any two vertices of S is at least $d+1$. For $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G spanned by S.

A set $S \subseteq V$ in a graph G *dominates* G if every vertex in $G \setminus S$ is adjacent to some vertex of S. The minimum number of vertices needed to dominate V is the *domination number* $\gamma(G)$.

A set $S \subseteq V$ in a graph G *distance d dominates* G if every vertex in $G \setminus S$ has distance at most d to some vertex of S, i.e. if $V \subseteq \bigcup_{x \in S} N_d[x]$.

The minimum number of vertices needed to distance d dominate V is the *distance d domination number* $\gamma_d(G)$. For $d = 1$ we have the ordinary domination, $\gamma_1(G) = \gamma(G)$.

For $V_i \subseteq V$ we define $\gamma_d(G; V_i)$ to be the minimum cardinality of a set $S \subseteq V$ such that each vertex $v \in V_i \setminus S$ satisfies that $N_{(d,G)[v]} \cap S \neq \emptyset$.

A *partition* $(V_1, V_2, \ldots, V_k)$ of $V = V(G)$ into k disjoint sets, $k \geq 2$, has $V = \bigcup_{i=1}^{k} V_i$ with $V_i \cap V_j = \emptyset$ for all $1 \leq i < j \leq k$. For an integer $k \geq 1$ and a partition $(V_1, V_2, \ldots, V_k)$ of V, we define for distance $d = 1$ the following.

$$f(G; V_1, V_2, \ldots, V_k) = \gamma(G) + \gamma(G; V_1) + \gamma(G; V_2) + \ldots + \gamma(G; V_k)$$
$$g(G; V_1, V_2, \ldots, V_k) = \gamma(G; V_1) + \gamma(G; V_2) + \ldots + \gamma(G; V_k)$$
$$f(k, G) = \max\{f(G; V_1, V_2, \ldots, V_k) \mid (V_1, V_2, \ldots, V_k) \text{ is a partition of } V\}$$
$$g(k, G) = \max\{g(G; V_1, V_2, \ldots, V_k) \mid (V_1, V_2, \ldots, V_k) \text{ is a partition of } V\}$$

We observe that $f(k, G) = \gamma(G) + g(k, G)$. For distance at most d, $d \geq 1$, definitions of $f_d(G; V_1, V_2, \ldots, V_k)$ etc. are analogous. For further notation we refer to Chartrand and Lesniak [5].

Since $\gamma_d(G; V_i) \leq \gamma_d(G)$ and hence $g_d(k, G) \leq k g_d(G)$ always holds, we

have

$$g_d(k, G) \leq \frac{k}{k+1} f_d(k, G)$$

for every graph G and all integers $k \geq 2$ and $d \geq 1$.

2 Introduction

Half a century ago Ore [15] defined domination and proved that a connected graph G of order n has $\gamma(G) \leq n/2$. Payan and Xuong [17] and Fink, Jacobson, Kinch and Roberts [6] proved that equality, $\gamma(G) = n/2$, holds precisely for C_4 and corona graphs. Obviously, for connected graphs of fixed order n the domination number will decrease with increasing size as illustrated in Table 1.

Table 1: Bounds under minimum-degree conditions.

Bounds for $\gamma(G)$ when G is connected and has order n.			
$\delta = \delta(G)$	Result	Reference	Extremal graphs known
$\delta \geq 1$	$\gamma(G) \leq n/2$	Ore [15]	yes
$\delta \geq 1$	$\gamma(G) \leq \frac{n+2-\delta(G)}{2}$	Payan [16]	
$\delta \geq 2$ $n \geq 8$	$\gamma(G) \leq 2n/5$	McCuaig and Shepherd [13]	
$\delta \geq 3$	$\gamma(G) \leq 3n/8$	Reed [18]	
$\delta \geq 1$	$\gamma(G) \leq \frac{1+\ln(\delta(G)+1)}{\delta(G)+1} \cdot n$	Arnautov [2], Payan [16]	
$\delta \geq 1$	$\gamma(G) \leq \left(1 - \delta(G)(\frac{1}{\delta(G)+1})^{1+\frac{1}{\delta}}\right) \cdot n$	Caro and Roditty [3, 4]	
$\delta \geq 1$	$\gamma(G) \leq \frac{n}{\delta(G)+1} \sum_{j=1}^{\delta(G)+1} \frac{1}{j}$	Arnautov [2], Payan [16]	

Several variants of domination in graphs have been surveyed in two books by Haynes, Hedeniemi and Slater [9, 10]. We shall here be concerned with distance domination in partitioned graphs.

A graph G has its various domination numbers bounded above by the corresponding domination number for any one of its spanning trees T, e.g. $f(2, G) \leq f(2, T)$, and if we search for an upper bound holding for all connected graphs of order n it suffices to search among all trees of order n, e.g. $f(2, G) \leq f(2, T) \leq \frac{5n}{4}$. As exhibited in Table 2, several tight results are known for 2-partitioned graphs, and in most of them the extremal graphs are characterized, too.

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Table 2: Results for 2-partitioned graphs.

T denotes a tree of order n and G a connected graph of order n			
	$\delta = \delta(G)$		Extremal graphs known
$f(2,T) \leq \frac{5}{4} \cdot n$	$d = 1, n \geq 3$	Hartnell and Vestergaard [8]	yes
$g(2,T) \leq \frac{4}{5} \cdot n$	$d = 1, n \geq 3$	Tuza and Vestergaard [20]	yes
$f(2,G) \leq n$	$d = 1, \delta \geq 2$	Seager [19]	
$g(2,G) \leq \frac{2}{3} \cdot n$	$d = 1, \delta \geq 2$	Tuza and Vestergaard [20]	yes
$g(2,G) \leq \frac{\delta+1}{2\delta} \cdot n$	$d = 1, \delta \geq 1$	Tuza and Vestergaard [20]	
$f_d(2,T) \leq \frac{6}{2d+3} \cdot n$	$d \geq 2, n \geq d+2$	Fu and Vestergaard [7]	yes
$g_d(2,T) \leq \frac{4}{2d+3} \cdot n$	$d \geq 2, n \geq d+2$	$k = 2$ in Theorem 4 below	

Our main concern in this paper is to prove tight estimates on 3-partitioned graphs, as summarized in Table 3. The graphs attaining maximum will be determined in all cases considered.

We also investigate the other extreme, where the number of partition classes is very large. The results of Section 4 show that the best possible universal upper bound on $g_d(k,G)$ is the trivial one, (namely n), for all n, d, and $k \geq (d+1)^2$; and for such large k, the best bound on $f_d(k,G)$ is $\frac{d+2}{d+1}n$.

Table 3: Results for 3-partitioned graphs.

			Extremal graphs known
$f(3,T) \leq \frac{7}{5} \cdot n$	$d = 1, n \geq 3$	Hartnell and Vestergaard [8]	yes
$f_2(3,T) \leq n$	$d = 2, n \geq 4$	Fu and Vestergaard [7]	yes
$f_2(3,T) \leq \frac{30}{31} \cdot n$	$d = 2, n \geq 5$ $T \notin \{P_6, P_7, P_8, G_{10}\}$	Theorem 2 below	yes
$g_2(3,T) \leq \frac{18}{25} \cdot n$	$d = 2, n \geq 5$ $T \neq T_8$	Theorem 2 below	yes
$f_3(3,T) \leq \frac{24}{31} \cdot n$	$d = 3, n \geq 6,$ $T \notin \{P_9, P_{10}\}$	Theorem 2 below	yes
$f_d(3,T) \leq \frac{24}{6d+13} \cdot n$	$d \geq 4, n \geq d+3,$ $T \neq P_{2d+4}$	Theorem 2 below	yes
$g_d(3,T) \leq \frac{18}{6d+13} \cdot n$	$d \geq 3, n \geq d+3$ $T \neq P_{2d+4}$	Theorem 2 below	yes

At the end of this introduction we prove that Ore's theorem $\gamma(G) \leq n/2$ generalises from $d = 1$ to $d \geq 1$.

Theorem 1 *Let $d \geq 1$ be an integer and let G be a connected graph with diameter at least d. Then $\gamma_d(G) \leq \frac{n}{d+1}$ where equality holds if and only if $n = d + 1$, $G \cong C_{2d+2}$ or $G \cong H \circ P_d$ for a connected graph H.*

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Proof. Let d be a fixed positive integer. A connected graph G with diameter less than d has $\gamma_d(G) = 1$ and $1 = \gamma_d(G) = \frac{n}{d+1}$ can occur for any connected graph with $n = d+1$. So assume G is connected and has diameter at least d. The inequality was proven by induction by Henning, Oellermann and Swart [11]. We give a short, direct proof, which also leads to a characterization of all graphs satisfying equality. Choose v to be an endvertex of a diametrical path in G and let T be a spanning tree rooted at v. Consider distance classes from v

$$A_i = \{x \in V(T) \mid d_T(x, v) = i\}, \quad i = 0, 1, 2, \dots$$

Each of the $d+1$ sets $B_i = \bigcup_{j=0}^{\infty} A_{i+j(d+1)}$, $i = 0, 1, \dots, d$, distance d dominates G and as $\bigcup_{j=0}^{d} B_i = V(T)$ we can among the $d+1$ sets choose a smallest set B_{i_0} and obtain $\gamma_d(G) \le |B_{i_0}| \le \frac{n}{d+1}$.

Henning [10, Prop. 12.4] proves that there exists a minimum distance d dominating set D such that each vertex $x \in D$ has a *private neighbour* x' at distance exactly d from x, i.e.,

$$\forall x \in D \ \exists x' \in V(G) \setminus D : N_{\le d}(x') \cap D = \{x\} \text{ and } d_G\{x, x'\} = d.$$

Assume next that $\gamma_d(G) = \frac{n}{d+1}$ and let D be such a minimum distance d dominating set with $|D| = \frac{n}{d+1}$. Let D' denote the set of distance d private neighbours to D,

$$D' = \{x' \in V(G) \mid \exists x \in D : N_{\le d}(x') \cap D = \{x\} \text{ and } d_G\{x, x'\} = d\}.$$

Thus from each $x \in D$ emanates a path $P_x : x_0 x_1 x_2 \dots x_d$ with $x_0 = x$ and $x_d = x'$. If $x, y \in D, x \ne y$, we have $V(P_x) \cup V(P_y) = \emptyset$, as x', y' are private neighbours to x, y, resp. Since $|D| = \frac{n}{d+1}$ and each path $P_x, x \in D$, has $d+1$ vertices we see that $V(G) = \bigcup_{x \in D} V(P_x)$ and $|D| = |D'| = \frac{n}{d+1}$.

If two of the paths, say P_x and P_y were joined by an edge whose endpoints are not the endpoints of both P_x and P_y, then the subgraph induced by $V(P_x) \cup V(P_y)$ would be distance d dominated by just one vertex of the connecting edge. Moreover, for $|D| > 2$, if neither D nor D' were independent, then three of those paths would induce a subgraph containing P_{3d+3}, admitting distance d domination with just two vertices. These situations contradict the assumption $\gamma_d = \frac{n}{d+1}$. Hence, for $|D| = |D'| = 2$ the circuit $G = C_{2d+2}$ may occur, but for $|D| \ne 2$ every edge either is on a path P_x or has both its ends in precisely one of the sets D, D'. That implies by connectivity of G that one of the sets D, D', say D', is independent. Consequently $G = H \circ P_d$, where $V(H) = D$. $\qquad\square$

From Theorem 1 we immediately obtain the following universal bounds on f_d and g_d.

Observation 1 *If G is a graph and $k, d \ge 1$ are integers then $g_d(k, G) \le |V(G)|$ and if G is a connected graph such that $|V(G)| \ge d+1$ then $f_d(k, G) \le \frac{d+2}{d+1}|V(G)|$.*

3   Bounds for $f_d(3,T)$ and $g_d(3,T)$

By the remark preceding Table 2, the worst-case behavior of $f_d(k,G)$ and $g_d(k,G)$ over connected graphs occurs when G is a tree. Moreover, the case $|V(G)| \leq d+1$ is trivial.

In the following we prove optimal bounds for $f_d(3,T)$ and $g_d(3,T)$ when T is a tree with at least $d+2$ vertices. First some families of graphs are defined.

For each integer $d \geq 2$ let Q_d be the family of trees consisting of P_{2d+4} and all trees with $d+2$ vertices. Let G_{10} denote P_9 with a pendent vertex attached to its center, i.e, the graph with 10 vertices illustrated in Figure 1.
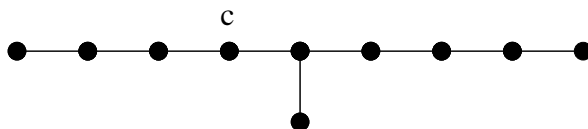

Figure 1: Illustration of the graph G_{10}.

A neighbour c to the center of the path $v_1, v_2, v_3, c, v_5, \ldots, v_9$ in G_{10} is called a connection-vertex in G_{10}. Let further $Q_2' = Q_2 \cup \{P_6, P_7, G_{10}\}$, $Q_3' = Q_3 \cup \{P_9\}$ and let $Q_d' = Q_d$ for $d \geq 4$.

For $d \geq 2$ let T_d be the tree with the smallest diameter, $2d+6$, that can be obtained from $3P_{2d+4} \cup K_1$ by adding three edges all incident with the isolated vertex which will be called the central vertex in T_d. For $d \geq 2$ we define $\mathcal{F}_d$ as the family of trees that can be obtained from graphs isomorphic to T_d by adding edges between their central vertices. Let T_2' be the tree obtained from $3G_{10} \cup K_1$ by adding three edges all incident with the isolated vertex (this vertex will be called central in T_2') and a connection-vertex from each of the three G_{10}-components. Define $\mathcal{F}_d' = \mathcal{F}_d$ for $d \geq 3$ and $\mathcal{F}_2'$ as the family of trees that can be obtained from isomorphic copies of T_2' by adding edges between central vertices.

Lemma 1 *Let $d \geq 2$ be an integer and let T be a tree with $n \geq d+2$ vertices such that for each edge $e \in E(T)$ a component of $T - e$ has fewer than $d+2$ vertices. Then*

- $g_d(3,T) = \frac{3}{d+2}n$ *if $T \in Q_d$ and if $T \notin Q_d$ then $g_d(3,T) < \frac{18}{6d+13}n$.*

- *If $d = 2$ then $f_d(3,T) = n$ if $T \in Q_2'$ and if $T \notin Q_2'$ then $f_d(3,T) < \frac{30}{31}n$.*

- *If $d \geq 3$ then $f_d(3,T) = \frac{4}{d+2}n$ if $T \in Q_d$ and if $T \notin Q_d'$ then $f_d(3,T) < \frac{24}{6d+13}n$.*

Proof. Equality for the specific graphs can be verified by the following summary of parameters. Double separation indicates the examples for $d = 2$ and the last one for $d = 3$, respectively.

graph	$\|V(T)\| = d+2$	P_{2d+4}	G_{10}	P_6	P_7	P_9
$g_d(3, T)$	3	6	7	4	5	5
$\gamma_d(T)$	1	2	3	2	2	2

To prove the estimates in general, let (V_1, V_2, V_3) be a partition of $V(T)$ such that $g_d(3, T) = \sum_{i=1}^{3} \gamma_d(V_i)$ and let $P : v_1, \ldots, v_{diam(T)+1}$ be a diametrical path in T. By the assumptions for T we have that $diam(T) \leq 2d + 2$. In the following we consider three cases.

$diam(T) \leq 2d$:

Since $\gamma_d(T) = 1$ in this case $g_d(3, T) = 3 \leq \frac{3}{d+2}n$ and $f_d(3, T) = 4 \leq \frac{4}{d+2}n$. Equality holds if and only if $n = d+2$. If $n > d+2$ then $g_d(3, T) = 3 < \frac{18}{6d+13}n$ and $f_d(3, T) = 4 < \frac{24}{6d+13}n < \frac{30}{31}n$.

$diam(T) = 2d + 1$:

In this case it can be assumed that the vertices $v_{d+2}, \ldots, v_{2d+2}$ only are adjacent to vertices from $V(P)$. Hence $\{v_{d+1}, v_{d+2}\}$ is a distance d dominating set for T and if $v_{2d+2} \notin V_i$ then $\{v_{d+1}\}$ is a distance d dominating set for V_i. Thus $g_d(3, T) = 4 \leq \frac{4}{2d+2}n < \frac{18}{6d+13}n$ and if $d \geq 3$ then $f_d(3, T) = 6 \leq \frac{6}{2d+2}n < \frac{24}{6d+13}n$. If $d = 2$ then $f_d(3, T) = n$ when $n = 6$ and otherwise $f_d(3, T) = 6 \leq \frac{6}{7}n < \frac{30}{31}n$.

$diam(T) = 2d + 2$:

In this case the condition on T implies that the vertices from $V(P) \backslash \{v_{d+2}\}$ only are adjacent to vertices from $V(P)$. Let U be the vertices in T at distance $d+1$ from v_{d+2}. Let $D_i := (V_i \cap U) \cup \{v_{d+2}\}$ then D_i is a distance d dominating set for V_i. Let $D' := \{v_{d+1}, v_{d+3}\} \cup (U \backslash \{v_1, v_{2d+3}\})$ and let $D := D'$ if no endvertex in T has distance d to v_{d+2} and otherwise let $D := D' \cup \{v_{d+2}\}$. Then D distance d dominates T and $\gamma_d(T) \leq |D| \leq 2 + \frac{n-(2d+3)}{d+1}$. We combine this with $g_d(3, T) \leq \sum_{i=1}^{3} |D_i| \leq 5 + \frac{n-(2d+3)}{d+1}$, and obtain $f_d(3, T) \leq 7 + 2\frac{n-(2d+3)}{d+1}$.

For $d \geq 2$ we have $5 < \frac{18}{6d+13}(2d + 3)$ which together with $\frac{1}{d+1} < \frac{18}{6d+13}$ gives

$$g_d(3, T) < \frac{18}{6d + 13}(2d + 3) + \frac{18}{6d + 13}(n - (2d + 3)) = \frac{18}{6d + 13}n \qquad (d \geq 2).$$

Analogously for $d \geq 4$ we have $7 < \frac{24}{6d+13}(2d + 3)$ and $\frac{1}{d+1} < \frac{24}{6d+13}$ giving

$$f_d(3, T) < \frac{24}{6d + 13}(2d + 3) + \frac{24}{6d + 13}(n - (2d + 3)) = \frac{24}{6d + 13}n \qquad (d \geq 4).$$

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Finally if $d \in \{2, 3\}$ and $n > 2d + 3$ then

$$
f_d(3, T) \le \left(\frac{7}{2d+3}(2d+3) + \frac{2}{d+1} \right) + 2\frac{n - (2d+4)}{d+1}
$$
$$
< \frac{24}{6d+13}(2d+4) + \frac{24}{6d+13}(n - (2d+4)) = \frac{24}{6d+13}n.
$$

$\square$

Remark For $d \ge 4$ we have that $Q_d = Q'_d$ and the last statement of Lemma 1 includes all trees of order $n \ge d + 2$. For $d = 3$, however, $Q'_d = Q_3 \cup \{P_9\}$, and P_9, where $f_3(3, P_9) = 7$, is not included in the statement since $7 > \frac{4}{d+2}n$ (third bullet) and consequently $f_3(3, P_9)$ will not be included in the statement of Theorem 2 below (second bullet).

Observation 2 If G is a graph from Q'_d and (V_1, V_2, V_3) is a partition such that $g_d(3, G) = \sum_{i=1}^{3} \gamma_d(G; V_i)$ then for each vertex $v \in V(G)$ there exists an index $i \in \{1, 2, 3\}$ such that $\gamma_d(G; V_i) = \gamma_d(G; V_i - N_{d-1}[v]) + 1$ and a vertex $x \in N[v]$ such that x is contained in a $\gamma_d(G)$-set. Further each vertex in $V(G_{10})$ is contained in a $\gamma_2(G_{10})$-set.

Lemma 2 Let $d \ge 2$ be a integer. If G_1 and G_2 are contained in Q'_d and G is a connected graph that can be obtained from $G_1 \cup G_2$ by adding an edge, but $G \notin Q'_d$, then $f_d(3, G) < f_d(3, G_1) + f_d(3, G_2)$.

Proof. Assume that there exists a graph G obtained by adding an edge uv between two graphs G_1 ($u \in V(G_1)$) and G_2 ($v \in V(G_2)$) from Q'_d such that $f_d(3, G) = f_d(3, G_1) + f_d(3, G_2)$. Let (V_1, V_2, V_3) be a partition of G such that $f_d(3, G) = \gamma_d(G) + \sum_{i=1}^{3} \gamma_d(G; V_i)$. It follows by the assumptions that $\gamma_d(G_1; V_i \cap V(G_1)) + \gamma_d(G_2; V_i \cap V(G_2)) = \gamma_d(G; V_i)$ for $i \in \{1, 2, 3\}$. Assume that $G_1 \in Q_d$. If $G_1 \not\cong P_{d+2}$ or u is not an endvertex then u is in a $\gamma_d(G_1; V_i \cap V(G_1))$-set in G_1 for $i \in \{1, 2, 3\}$. Thus by Observation 2 there exists an index $i \in \{1, 2, 3\}$ such that $\gamma_d(G; V_i) \le \gamma_d(G_1; V_i \cap V(G_1)) + \gamma_d(G_2; V_i \cap V(G_2)) - 1$ which gives a contradiction. It can therefore be assumed that $G_1 \notin Q_d$ (and $G_2 \notin Q_d$). This implies the theorem for $d \ge 4$ since $Q_d = Q'_d$ when $d \ge 4$. In the case where $d \in \{2, 3\}$ the lemma can easily be verified for the graphs G obtained when G_1 and G_2 belongs to $Q'_d \setminus Q_d$ by examining a small number of specific graphs. $\square$

As a consequence of Lemma 2 we note

Observation 3 Let $d \ge 2$, $\{G_1, G_2\} \subset \mathcal{F}'_d$ and $G = G_1 \cup G_2 + uv$ where $u \in V(G_1)$ and $v \in V(G_2)$. Then $G \in \mathcal{F}'_d$ if $f_d(3, G) = f_d(3, G_1) + f_d(3, G_2)$.

Theorem 2 Let $d \ge 2$ be a integer and let T be a tree with $n \ge d + 2$ vertices. Then

- If $d = 2$ then $f_d(3, T) = n$ if $T \in Q'_d$ and if $T \notin Q'_d$ then $f_d(3, T) \le \frac{30}{31}n$ where equality holds if and only if $T \in \mathcal{F}'_d$.

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- If $d \geq 3$ then $f_d(3, T) = \frac{4}{d+2}n$ if $T \in Q_d$ and if $T \notin Q'_d$ then $f_d(3, T) \leq \frac{24}{6d+13}n$ and equality holds if and only if $T \in \mathcal{F}'_d$.

- For all $d \geq 2$: $g_d(3, T) = \frac{3}{d+2}n$ if $T \in Q_d$ and if $T \notin Q_d$ then $g_d(3, T) \leq \frac{18}{6d+13}n$ and equality holds if and only if $T \in \mathcal{F}_d$.

Proof. The theorem is proven by induction on n. The assertion follows from Lemma 1 if we are in the case where T is a tree such that $T - e$ has a component with at most $d + 1$ vertices for each edge $e \in E(T)$.

Thus it can be assumed that there exists an edge $e \in E(T)$ such that both components of $T - e$ have at least $d + 2$ vertices. Let E_0 be the set of edges having this property.

Case 1. If there exists an edge $e \in E_0$ such that neither of the components T_1 and T_2 in $T - e$ is contained in Q'_d then for $i \in \{1, 2\}$ the induction hypothesis gives that

$$f_d(3, T_i) \leq \begin{cases} \frac{24}{6d+13}|V(T_i)| & \text{if } d \geq 3 \\ \frac{30}{31}|V(T_i)| & \text{if } d = 2 \end{cases} \tag{1}$$

and

$$g_d(3, T_i) \leq \frac{18}{6d+13}|V(T_i)| \quad \text{if } d \geq 2. \tag{2}$$

Further, equality holds in (1) if and only if $T_i \in \mathcal{F}'_d$ and equality holds in (2) if and only if $T_i \in \mathcal{F}_d$.

It follows that

$$f_d(3, T) \leq f_d(3, T_1) + f_d(3, T_2) \leq \begin{cases} \frac{24}{6d+13}n & \text{if } d \geq 3 \\ \frac{30}{31}n & \text{if } d = 2 \end{cases} \tag{3}$$

and

$$g_d(3, T) \leq g_d(3, T_1) + g_d(3, T_2) \leq \frac{18}{6d+13}|V(T)| \quad \text{if } d \geq 2. \tag{4}$$

If equality holds in (3) it follows from Observation 3 that $T \in \mathcal{F}'_d$ and if equality holds in (4) then analogously $T \in \mathcal{F}_d$.

Case 2. If there exists an edge $e \in E_0$ such that both $(T - e)$-components T_1 and T_2 are in Q'_d, i.e., unless $d = 3$ and one or both of T_1, T_2 equals P_9, they satisfy $f_d(3, T_i) = \frac{4}{d+2}n_i$, $i = 1, 2$, then the induction hypothesis gives that $n \leq 4d + 8$ if $d \geq 3$ and $n \leq 20$ if $d = 2$. If $d = 3$ and $\{T_1, T\} \cap \{P_9\} \neq \emptyset$ we can verify Theorem 2 by inspection. Since Theorem 2 is easily verified if $T \in Q'_d$ we may assume that $T \notin Q'_d$ and Lemma 2 then implies that $f_d(3, T) \leq f_d(3, T_1) + f_d(3, T_2) - 1$ and calculations give that

$$f_d(3, T) \leq \begin{cases} \frac{7}{4d+8}n < \frac{24}{6d+13}n & \text{if } d \geq 3. \\ \frac{19}{20}n < \frac{30}{31}n & \text{if } d = 2, \end{cases}$$

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For $d \geq 2$ we have $g_d(3,T) \leq \frac{3}{4} f_d(3,T)$. For $d = 2$ that implies $g_d(3,T) \leq \frac{3}{4} \cdot \frac{19}{20} n = \frac{57}{80} n < \frac{18}{25} n$. Also, for $d \geq 3$ analogously $g_d(3,T) \leq \frac{18}{25} n$.

Case 3. Thus it can be assumed that for each edge $e \in E_0$ exactly one of the two $(T - e)$-components belongs to Q'_d.

Let $\vec{T}$ be the (partially) directed graph such that $V(\vec{T}) = V(T)$ and the arcs of $\vec{T}$ are

$$A(\vec{T}) = \{\overrightarrow{uv} \mid uv \in E_0 \ \wedge \ \text{the component of } T - uv \text{ containing } u \text{ is in } Q'_d\}.$$

Since T is a tree and $E_0 \neq \emptyset$ it follows by taking a longest directed path of E_0-arcs in $\vec{T}$ that there must exist a vertex $x \in \vec{T}$ with in-degree at least one and out-degree zero. An edge e from $E \setminus E_0$ incident with x has a component of $T - e$ with at most $d + 1$ vertices. From these observations we see that each component in $T - x$ has at most $d + 1$ vertices or is in Q'_d. Further, $T - x$ must contain a component from Q'_d. Let $deg_{E_0}(x)$ be the number of components of $T - x$ contained in Q'_d and let H be the induced subgraph of T containing x and the vertices from these components.

Let (V_1, V_2, V_3) be a partition of $V(T)$. From Observation 2 it can be seen that there exist sets D'_i and D' such that

- $\sum_{i=1}^{3} |D'_i| = 3 - deg_{E_0}(x) + g_d(3, H - x)$,

- $V(H) \cap V_i \subseteq N_{(d,G)}[D'_i]$, $V(H) \subseteq N_{(d,G)}[D']$, $|D'| = \gamma_d(H - x)$, $x \in D'_i$ and $N_{(2,G)}(x) \cap D' \neq \emptyset$.

If $d = 2$ and at least one of the components in $H - x$ is isomorphic to G_{10} the set D' can be chosen such that $N(x) \cap D' \neq \emptyset$.

Let U_k denote the endvertices from $T - H$ at distance k from x. It follows that $D_i := D'_i \cup (U_{d+1} \cap V_i)$ distance d dominates V_i.

Further define $I := 1$ if there exists a vertex v in a component C of $T - H$ such that $d(v, x) = \max_{u \in V(C)} d(u, x) \leq d$ and $d(v, D') > d$ and otherwise let $I := 0$. If $I = 0$ then define $D := D' \cup U_{d+1}$ and if $I = 1$ then define $D := D' \cup U_{d+1} \cup \{x\}$. Now D is defined such that it distance d dominates T. Therefore

$$f_d(3,T) \leq |D| + \sum_{i=1}^{3} |D'_i| = I + \gamma_d(H - x) + |U_{d+1}| + \sum_{i=1}^{3} |D'_i|.$$

First assume that $d = 2$. By the induction hypothesis it follows that $f_d(3,T) < n$ and thus the theorem easily follows if $n \leq 30$ since $f_d(3,T) \leq n - 1$ and $n - 1 < \frac{30}{31} n$ for $n \leq 30$. Thus it can be assumed that $n \geq 31$ in this case.

If $deg_{E_0}(x) \leq 2$ we obtain that

$$
\begin{aligned}
f_d(3, T) &\leq 3 - deg_{E_0}(x) + g_d(3, H - x) + \gamma_d(H - x) + 2|U_{d+1}| + I \\
&\leq 3 + f_d(3, H - x) + 2|U_{d+1}| \\
&\leq 3 + f_d(3, H - x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq 3 + |V(H - x)| + \frac{2}{d+1}9 + \frac{2}{d+1}(n - |V(H)| - 9) \\
&\leq \frac{29}{30}30 + \frac{2}{3}(n - 30) \\
&< \frac{30}{31}n.
\end{aligned}
$$

If $deg_{E_0}(x) \geq 3$ the following is obtained:

$$
\begin{aligned}
f_d(3, T) &\leq 3 - deg_{E_0}(x) + f_d(3, H - x) + I + 2|U_{d+1}| \\
&\leq \frac{30}{31}(|V(H)| + I) + \frac{2}{d+1}(n - (|V(H)| + I)) \\
&\leq \frac{30}{31}n.
\end{aligned}
$$

Further, equality holds if and only if $n = |V(H)| = 31$ and in this case we have that $T \cong T_2$.

Assume that $d \geq 3$. If $deg_{E_0}(x) \leq 2$ and $n \geq 6d + 13$ then

$$
\begin{aligned}
f_d(3, T) &\leq 3 - deg_{E_0}(x) + g_d(3, H - x) + \gamma_d(H - x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq 2 + f_d(3, H - x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq 2 + \frac{24}{6d+12}(|V(H)| - 1) + \frac{2}{d+1}(2d + 4) \\
&\qquad + \frac{2}{d+1}(n - |V(H)| - (2d + 4)) \\
&\leq 2 + \frac{24}{6d+12}(4d + 8) + \frac{2}{d+1}(2d + 4) + \frac{2}{d+1}(n - 6d - 13) \\
&\leq 23 + \frac{2}{d+1}(n - 6d - 13) \\
&< \frac{24}{6d+13}(6d + 13) + \frac{24}{6d+13}(n - (6d + 13)) = \frac{24}{6d+13}n.
\end{aligned}
$$

If $deg_{E_0}(x) \geq 3$ then

$$
\begin{aligned}
f_d(3, T) &\leq 3 - deg_{E_0}(x) + g_d(3, H - x) + \gamma_d(H - x) + \frac{2}{d+1}(n - |V(H)|) \\
&\leq \frac{24}{6d+13}|V(H)| + \frac{2}{d+1}(n - |V(H)|) \\
&\leq \frac{24}{6d+13}n.
\end{aligned}
$$

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Here equality holds if and only if $deg_{E_0}(x) = 3$ and $n = |V(H)| = 6d + 13$. Thus it follows that equality holds if and only if $T \cong T_d$. (T_d defined in beginning of section.)

Assume that $n \leq 6d + 12$ and $deg_{E_0}(x) \leq 2$. By the choice of x it follows that the graph $G' = G - V(H - x)$ has $diam(G') \leq 2d + 2$. Assume that $deg_{E_0}(x) = 1$ then it follows from the assumptions for the set E_0 that $|V(G')| \geq d + 3$.

If $diam(G') \leq 2d$ then $f_d(3, G') = 4 \leq \frac{4}{d+2}(|V(G')| - 1)$ and $f_d(3, G) \leq f_d(3, G') + f_d(3, G - V(G')) \leq \frac{4}{d+2}(|V(G)| - 1) < \frac{24}{6d+13}|V(G)|$. Assume that $diam(G') = 2d + 1$ and $|V(G')| \geq 2d + 3$. It follows that $f_d(3, G') = 6 \leq \frac{4}{d+2}(|V(G')| - 1)$ and from this we obtain that $f_d(3, G) < \frac{24}{6d+13}|V(G)|$. If this is not the case then $G' \cong P_{2d+2}$ and x is a central vertex in G' and it follows from Observation 2 that $f_d(3, G) \leq f_d(3, H - x) + f_d(3, G') - 1 \leq \frac{4}{d+2}|V(G)| - 1 \leq \frac{4}{d+2}(|V(G)| - 1) < \frac{24}{6d+13}|V(G)|$. Thus it can be assumed that $diam(G') = 2d + 2$. In this case it was proven in Lemma 1 that for each partition (V_1', V_2', V_3') of G' where $f_d(3, G') = \gamma_d(G') + \sum_{i=1}^{3} \gamma_d(G; V_i')$ the vertex x is in a $\gamma_d(V_i')$-set. From this and Observation 2 it follows that $f_d(3, G) \leq f_d(3, H - x) + f_d(3, G') - 1 < \frac{24}{6d+13}|V(G)|$.

Thus it can be assumed that $deg_{E_0}(x) = 2$ and $|V(G)| \leq 6d + 13$. If $|V(G')| = 1$ then let C be a component from $G - x$ and let C' be the induced subgraph of G containing $V(C) \cup \{x\}$. It follows by the induction hypothesis that $f_d(3, C') = f_d(3, C)$ and thus $f_d(3, G) \leq f_d(G - x) \leq \frac{4}{d+2}(|V(G)| - 1) < \frac{24}{6d+13}|V(G)|$. Thus it can be assumed that $|V(G')| \geq 2$.

First we assume that the two components of $H - x$ are not both isomorphic to P_{d+2}. Thus it can be assumed that $D_i \cap N[x] \neq \emptyset$ for each $i \in \{1, 2, 3\}$. It follows that $f_d(3, G) \leq \frac{4}{d+2}|V(H - x)| + \frac{3}{d+1}|V(G - H)| < \frac{24}{6d+13}|V(G)|$. Thus the theorem has been proved in this subcase and it can be assumed that both components of $H - x$ are isomorphic to P_{d+2}.

If $|V(G')| \leq d - 1$ it can easily be seen that $f_d(3, G) \leq f_d(3, H - x) + 2$ and the theorem follows since $|V(G')| \geq 2$. If this is not the case we obtain that

$$
\begin{aligned}
f_d(3, T) &\leq f_d(H - x) + 1 + \frac{2}{d+1}(n - |V(H)|) \\
&= 9 + \frac{2}{d+1}(d - 1) + \frac{2}{d+1}(n - (2d + 5) - (d - 1)) \\
&< \frac{24}{6d+13}(3d + 4) + \frac{2}{d+1}(n - (3d + 4)) \\
&\leq \frac{24}{6d+13}n.
\end{aligned}
$$

The last inequality is true for $n \geq 3d + 4$ and for $n \leq 3d + 3$ we have that $|V(G')| \leq d - 1$, which was treated above.

Thus the theorem has been proved in this subcase, too.

We shall now consider $g_d(3, T)$. If $T \in Q_d'$ then the theorem easily follows by verification and if $d \geq 3$ the results follow since $g_d(3, T) \leq \frac{3}{4}f_d(3, T)$. Thus

it can be assumed that $d = 2$ and $T \notin Q'_d$. It follows that $f_d(3, T) \leq n - 1$ and since $g_d(3, T) \leq \frac{3}{4} f_d(3, T)$ we obtain that $g_d(3, T) \leq \lfloor \frac{3}{4}(n - 1) \rfloor$. From this inequality it follows that $g_d(3, T) < \frac{18}{25} n$ if $n < 25$. Thus it can be assumed that $n \geq 25$. Since D_i is a distance d dominating set for V_i we have that

$$g_2(3, T) \leq \sum_{i=1}^{3} |D_i| \leq 3 - deg_{E_0}(x) + g_2(3, H - x) + \frac{1}{d + 1}(n - |V(H)|).$$

If $deg_{E_0}(x) \leq 2$ then $|V(H)| \leq 2(2d + 4) + 1 = 17$ and it follows that

$$g_2(3, T) \leq 3 - deg_{E_0}(x) + g_2(3, H - x) + \frac{1}{d + 1}(n - |V(H)|)$$
$$\leq 2 + \frac{3}{4}17 + \frac{1}{d + 1}8 + \frac{1}{d + 1}(n - 17 - 8)$$
$$< \frac{18}{25}25 + \frac{18}{25}(n - 17 - 8) = \frac{18}{25}n.$$

If $deg_{E_0}(x) \geq 3$ then

$$g_2(3, T) \leq 3 - deg_{E_0}(x) + g_2(3, H - x) + \frac{1}{d + 1}(n - |V(H)|)$$
$$\leq \frac{18}{25}|V(H)| + \frac{18}{25}(n - |V(H)|) = \frac{18}{25}n,$$

and if equality holds then $deg_{E_0}(x) = 3$ and $n = |V(H)| = 6d + 13 = 25$. From the observations done so far we obtain that $H \cong T_2$ if equality holds. $\square$

4 Many partition classes

Lemma 3 *Let k and d be positive integers and let T be a tree with n vertices. Then $g_d(k, T) = n$ if and only if $|N_{(d,T)}[v]| \leq k$ for each vertex $v \in V(T)$.*

Proof. For $v \in V(T)$ the subgraph of T induced by $N_{(d,G)}[v]$ is denoted T_v. Let T' be the graph where $V(T') = V(T)$ and $E(T') = \{uv \mid V(T_u) \cap V(T_v) \neq \emptyset\}$. Since T is a tree T' is a chordal graph. Clearly the chromatic number of T' equals the minimum number of $2d$-independent sets into which $V(T)$ can be partitioned. Since T' is a chordal graph it is perfect and we have that its chromatic number equals its clique number, $\chi(T') = \omega(T')$.

Now let $\{v_1, \ldots, v_a\}$ be a clique in T', i.e., a subset of $V(T)$ such that $V(T_{v_i}) \cap V(T_{v_j}) \neq \emptyset$ for all i, j, $1 \leq i \leq j \leq a$. It then follows that there must exist a vertex v such that $v \in V(T_{v_i})$ for $i \in \{1, \ldots, a\}$. Let namely S denote the subtree of T spanned by the union of paths connecting the vertices of $\{v_1, \ldots, v_a\}$. Let v be a central vertex for a longest path in S, then v has distance at most d to all v_i, $1 \leq i \leq a$, and $v \in \bigcap_{i=1}^{a} T_{v_i}$.

From this observation we have the following sequence of equivalences:

$g_d(k, T) = n$

$\Updownarrow$

there exists a partition $(V_1, V_2, V_3, \ldots, V_k)$ of $V(T)$ such that for $i = 1, 2, \ldots, k$, V_i is a $2d$-independent set

$\Updownarrow$

$\chi(T') \leq k$

$\Updownarrow$

$\omega(T') \leq k$

$\Updownarrow$

$|N_d[x]| \leq k$ for each x in $V(T)$. $\hspace{2cm}$ $\square$

Lemma 4 *Let H be a connected graph with at least $2d + 1$ vertices and let $G = H \circ P_d$ be the P_d-corona graph of H. Then there exist vertices $v_H \in V(H)$ and $v_G \in V(G)$ such that*

$$|N_{(d,H)}[v_H]| \geq 2d + 1 \qquad \text{and} \qquad |N_{(d,G)}[v_G]| \geq (d + 1)^2.$$

Proof. Let H be a connected graph such that $|V(H)| \geq 2d + 1$ and let $G = H \circ P_d$, i.e., G is obtained by joining each vertex of H to an end of its own copy of a P_d, and G has order $|V(H)|(d + 1)$. Let v_H be a central vertex (a vertex with minimum eccentricity) in H. If $\gamma_d(H) = 1$ then $N_{(d,H)}[v_H] = V(H)$ and we obtain that $|N_{(d,H)}[v_H]| \geq 2d + 1$. If $\gamma_d(H) \neq 1$ then there must be a path $P : v_1, \ldots, v_d, v_H, v_{d+2}, \ldots, v_{2d+1}$ in H and since $V(P) \subseteq N_{(d,H)}[v_H]$ it can be concluded that $|N_{(d,H)}[v_H]| \geq 2d + 1$.

Let a_i denote the number of vertices in H at distance i from v_H. Then

$$|N_{(d,G)}[v_H]| = (d + 1) + \sum_{i=1}^{d} (d + 1 - i)a_i.$$

If $a_i \geq 2$ for each $i \in \{1, \ldots, d\}$ then

$$|N_{(d,G)}[v_H]| \geq (d + 1) + \sum_{i=1}^{d} 2(d + 1 - i) = (d + 1)^2.$$

If $a_k \leq 1$ for an index $k \in \{1, \ldots, d\}$ then $a_i = 0$ for $i > k$ since v_H is a central vertex. Since $\sum_{i=1}^{d} a_i \geq 2d$ it follows that

$$\begin{aligned}
|N_{(d,G)}[v_H]| &= (d + 1) + \sum_{i=1}^{d} (d + 1 - i)a_i = (d + 1) + \sum_{i=1}^{k} (d + 1 - i)a_i \\
&> (d + 1) + \sum_{i=1}^{k-1} 2(d + 1 - i) + (2d - 2(k - 1))(d + 1 - k) \\
&\geq (d + 1)^2.
\end{aligned}$$

$\hspace{14cm}$ $\square$

From Theorem 1, Observation 1, Lemma 3 and Lemma 4 the following result is easily obtained.

Corollary 1 *Let $d \geq 1$ be a integer and let T be a tree with n vertices. Then*

- $f_d((d+1)^2, P_{\frac{n}{d+1}} \circ P_d) = \frac{d+2}{d+1}n$ *if* $(d+1) \mid n$.

- $g_d(2d+1, P_n) = n$ *for each* $n \geq 1$.

- $g_d(d^2 + 2d, T) < n$ *if* T *is a* P_d-*corona graph and* $|V(T)| > 2d(d+1)$.

- $f_d(d^2 + 2d, T) < \frac{d+2}{d+1}n$ *if* $|V(T)| > 2d(d+1)$.

- $f_d(d^2 + 2d, P_{2d} \circ P_d) = \frac{d+2}{d+1}n$.

- $g_d(2d, T) < n$ *if* $|V(T)| \geq 2d+1$.

Lemma 5 *If T is a tree, $X \subseteq V(T)$ and $d \geq 1$ is an integer then there exists a set $X' \subseteq X$ such that X' is $2d$-independent in T and $|X'| = \gamma_d(T; X)$.*

Proof. This result is proven by induction on $\gamma_d(X)$. If $\gamma_d(T; X) \leq 1$ the theorem is trivially true. Assume that $\gamma_d(T; X) \geq 2$. Let $P = x_1, v_1, \ldots, v_a, x_2$ be a path in T of maximum length when $\{x_1, x_2\} \subseteq X$.

Since $\gamma_d(T; X \backslash N_{(d,T)}[v_d]) \geq \gamma_d(T; X) - 1$ the induction hypothesis gives that there exists a $2d$-independent set $X' \subseteq X \backslash N_{(d,T)}[v_d]$ in $T - N_{(d,T)}[v_d]$ with cardinality $\gamma_d(X) - 1$. By the choice of P it must hold that $X' \cup \{x_1\}$ is a $2d$-independent set in T and the result follows. $\square$

Theorem 3 *Let $d \geq 1$ be a integer and let T be a tree with $n > 2d^2 + 2d$ vertices. Then*

$$f_d(d^2 + 2d, T) < \frac{d+2}{d+1}n - \frac{n}{2(d+1)^5}.$$

Proof. Let T be a tree with $n > 2d^2 + 2d$ vertices. If $diam(T) \leq 2d$ then $\gamma_d(T) = 1$ and so $f_d(d^2 + 2d, T) = d^2 + 2d + 1 \leq \frac{d+2}{d+1}n - \frac{n}{(d+1)^2}$. Thus it can be assumed that $diam(T) \geq 2d+1$ and further we assume that the theorem does not hold for T. Denoting $k = 2(d+1)^5$ it means

$$f_d(d^2 + 2d, T) \geq \frac{d+2}{d+1}n - \frac{n}{k}.$$

From the inequality above $g_d \leq n$ implies $\gamma_d \geq \frac{n}{d+1} - \frac{n}{k}$ and there must exist a constant $a \in [0,1]$ such that $\gamma_d(T) \geq \frac{n}{d+1} - a\frac{n}{k}$ and $g_d(d^2 + 2d, T) \geq n - (1-a)\frac{n}{k}$.

Let $X_1, \ldots, X_{d^2+2d}$ be a partition of $V(T)$ such that

$$g_d(d^2 + 2d, T) = \sum_{i=1}^{d^2+2d} \gamma_d(T; X_i).$$

From Lemma 5 it follows that for each $i \in \{1, \ldots, d^2 + 2d\}$ there exists a set $X_i' \subseteq X_i$ such that $\gamma_d(T; X_i) = |X_i'|$ and X_i' is a $2d$-independent set in T. Hence

$$\sum_{i=1}^{d^2+2d} |X_i'| = g_d(d^2 + 2d, T) \geq n - (1-a)\frac{n}{k}.$$

Let $H := \{v \in V(T) | deg(v) \geq (d+1)^2\}$. Then we shall prove below that

$$\sum_{v \in H} (deg(v) - (d+1)^2) \leq (1-a)\frac{n}{k}.$$

Root the tree T at a vertex $x \in V(T)$ and let $c(v)$ denote the set of children of a vertex v in the rooted tree T. For $v \in H$ then $c(v)$ contains at least $deg(v) - 1$ vertices and since each set X_i' can contain at most one vertex from $c(v)$ we have that

$$|c(v)\backslash(\bigcup_{i=1}^{d^2+2d} X_i')| \geq deg(v) - 1 - (d^2 + 2d) = deg(v) - (d+1)^2.$$

Since $c(v_1) \cap c(v_2) = \emptyset$ if $v_1 \neq v_2$ and $\sum_{i=1}^{d^2+2d} |X_i'| \geq n - (1-a)\frac{n}{k}$ it must hold that

$$(1-a)\frac{n}{k} \geq \sum_{v \in H} |c(v)\backslash(\bigcup_{i=1}^{d^2+2d} X_i')| \geq \sum_{v \in H} (deg(v) - (d+1)^2).$$

This proves the inequality.

Let S be a maximum $2d$-independent set in T. From Lemma 5 it follows that $|S| = \gamma_d(T) \geq \frac{n}{d+1} - a\frac{n}{k}$. Consider for each $s \in S$ the tree $T_s := T[N_d[s]]$ spanned in T by $N_d[s]$. Since S is $2d$-independent no vertex from T is in more than one of these trees. From the assumption that $diam(T) \geq 2d+1$ it follows that there must be a path $P_s = s, v_2, \ldots, v_{d+1}$ in T_s. It follows that $\sum_{s \in S} |V(P_s)| = |S|(d+1) \geq n - (d+1)a\frac{n}{k}$. Let $F' := \bigcup_{s \in S} V(P_s)$ and let $A = V(T) - V(F')$. Let B be all vertices from those paths P_s for which $P_s \nsubseteq \bigcup_{i=1}^{d^2+2d} X_i'$ and let B' be the set of endvertices not in S from these paths.

In the following we examine the number of vertices and components in the induced subgraph of T with vertex set $F := F' - B$. We observe that both $T[F]$ and $T[F']$ are P_d-corona graphs, as they are obtained by adding edges between some endvertices, not in S, of the P_s-paths. Each $P_s \in B$ contains a vertex of $V(T) \setminus \bigcup_{i=1}^{d^2+2d} X_i'$, so from $|V(T) \setminus \bigcup_{i=1}^{d^2+2d} X_i'| \leq (1-a)\frac{n}{k}$ and $|V(P_s)| = d+1$ we get $|B| \leq (d+1)(1-a)\frac{n}{k}$.

By the assumptions we have that
$|F| = |F'| - |B| \geq |F'| - (d+1)(1-a)\frac{n}{k} \geq (n - (d+1)a\frac{n}{k}) - (d+1)(1-a)\frac{n}{k} = n - (d+1)\frac{n}{k}$.

From the following calculations we obtain an upper bound on the number of components in F:

$$\omega(F) \leq \sum_{v \in A \cup B'} deg(v) = \sum_{v \in (A \cup B') \setminus H} deg(v) + \sum_{v \in (A \cup B') \cap H} deg(v)$$

$$\leq (d+1)^2 |(A \cup B') \setminus H| + (d+1)^2 |(A \cup B') \cap H| + \sum_{v \in H} (deg(v) - (d+1)^2)$$

$$\leq (d+1)^2 |A \cup B'| + (1-a)\tfrac{n}{k} \leq \tfrac{n}{k}((1-a) + (1-a)(d+1)^2 + a(d+1)^3),$$

where the last step follows from $|A| \leq (d+1)a\tfrac{n}{k}$ and $|B'| \leq \tfrac{|B|}{d+1} \leq (1-a)\tfrac{n}{k}$.

Since $V(F) \subseteq \bigcup_{i=1}^{d^2+2d} X'_i$ we have that F is a P_d-corona graph which satisfies $g_d(d^2 + 2d, F) = |V(F)|$. By Corollary 1 this can only hold if each component of F has at most $2d(d+1)$ vertices and we obtain that

$$\frac{n - (d+1)\tfrac{n}{k}}{\tfrac{n}{k}((1-a) + (1-a)(d+1)^2 + a(d+1)^3)} \leq 2d(d+1).$$

From this equation we easily obtain the contradiction that $k < 2(d+1)^5$. $\square$

The following result generalizes Theorem 1.

Theorem 4 *Let G be a tree with $n \geq d + \tfrac{k+1}{2}$ vertices. Then*

$$g_d(k, G) \leq n \frac{2k}{2d + k + 1}.$$

Proof. The theorem easily follows when $k \geq 2d + 1$ since $g_d(k, G) \leq n \leq n\frac{2k}{2d+k+1}$ in this case. For $k < 2d+1$ the theorem is proven by induction on n. Note that the case $n = \lceil d + \tfrac{k+1}{2} \rceil$ is immediate because then $\gamma_d(G) = 1$ and hence $g_d(k, G) \leq k$. If the graph G has an edge e such that both components, G_1 and G_2, of $G - e$ have at least $d + \tfrac{k+1}{2}$ vertices, the induction hypothesis can be used on both components to obtain the inequality.

Thus it can be assumed that the removal of each edge in G gives a component G_1 with fewer than $d + \tfrac{k+1}{2}$ vertices, i.e., with at most $2d$ vertices, such that $g_d(G_1) = 1$ and $g_d(k, G_1) \leq k$.

Let e be an edge in G such that one of the components, G_1, in $G - e$ has a maximum number of vertices when the other G_2 must contain $d + \tfrac{k+1}{2}$ vertices or more. Let u be the vertex from G_2 incident to e. By the choice of e it follows that the maximum distance from u to a vertex in G_2 is at most $d + \tfrac{k-1}{2}$. By using the induction hypothesis on G_2 it can be observed that to each partition $V_1, \ldots, V_k$ of G_2 there are related dominating sets $D_1, \ldots, D_k$ such that $g_d(k, G) \leq \sum_{i=1}^{k} |D_i| \leq |V(G_2)|\frac{2k}{2d+k+1}$ and each set D_i contains a vertex from $N_{\lfloor \frac{k-1}{2} \rfloor}[u]$; moreover, if $A := \bigcap_{i=1}^{k} N_{(d,G)}[D_i] \cap V(G_1)$ then

$|A| \geq \lceil d - \frac{k-1}{2} \rceil$ and $g_d(k, G) \leq \sum_{i=1}^{k} |D_i| + |V(G_1)| - |A|$. Calculation now gives:

$$\frac{|V(G_1)| - |A|}{|V(G_1)|} \leq \frac{|V(G_1)| - \lceil d - \frac{k-1}{2} \rceil}{|V(G_1)|} < \frac{d + \frac{k+1}{2} - (d - \frac{k-1}{2})}{d + \frac{k+1}{2}} = \frac{2k}{2d+k+1}.$$

Thus it follows that $g_d(k, G) \leq g_d(k, G_2) + |V(G_1)| - |A| \leq |V(G_2)|\frac{2k}{2d+k+1} + |V(G_1)|\frac{2k}{2d+k+1} = n\frac{2k}{2d+k+1}$. $\qquad\qquad\square$

From Theorem 4 we obtain

Corollary 2 *A graph G with $n \geq 2d + 1$ vertices satisfies that $g_d(2d, G) \leq n - \frac{n}{4d+1}$.*

In [20] it has been proven that this bound is optimal when $d = 1$.

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