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SUMMATION OF MULTIPLE FOURIER SERIES IN MATRIX WEIGHTED L_p-SPACES

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Abstract. This paper is concerned with rectangular summation of multiple Fourier series in matrix weighted L_p-spaces. We introduce a product Muckenhoupt A_p condition for matrix weights W and prove that rectangular Fourier partial sums converge in the corresponding matrix weighted space $L_p(\mathbb{T}^d;W)$, $1<p<\infty$, if and only if the weight satisfies the product Muckenhoupt A_p condition. The same result is shown to hold true for other summation methods such as Cesàro and summation with the Jackson kernel.

1. Introduction

Let $\mathcal{M}$ be the family of non-negative-definite $m\times m$ complex-valued matrices. A (periodic) matrix weight is by definition an integrable map $W:\mathbb{T}^d\to\mathcal{M}$. For a measurable vector-valued function $\mathbf{f}=(f_1,\dots,f_m)^T:\mathbb{T}^d\to\mathbb{C}^m$, let

$$\|f\|_{L_p(\mathbb{T}^d;W)} := \left(\int_{\mathbb{T}^d} |W^{1/p}(t)\mathbf{f}(t)|^p\,dt\right)^{1/p}, \qquad 1\le p<\infty, \tag{1.1}$$

where $|\cdot|$ denote the usual norm on $\mathbb{C}^m$. We let $L_p(\mathbb{T}^d;W)$ denote the family $\{\mathbf{f}:\mathbb{T}^d\to\mathbb{C}^m:\|\mathbf{f}\|_{L_p(\mathbb{T}^d;W)}<\infty\}$, and $L_p(\mathbb{T}^d;W)$ becomes a Banach space when we factorize over $\mathcal{N}=\{\mathbf{f}:\|\mathbf{f}\|_{L_p(\mathbb{T}^d;W)}=0\}$.

In this paper we are interested in convergence properties of multiple trigonometric series in $L_p(\mathbb{T}^d;W)$ and how specific convergence properties of trigonometric series can be related to properties of the weight W. To be more specific, let $D_n(t)=\sum_{|k|\le n} e^{-2\pi ikt}$ denote the univariate Dirichlet kernel, and for $\mathbf{N}\in\mathbb{N}^d$ we define the *rectangular* kernel $D_{\mathbf{N}}(t):=\prod_{j=1}^d D_{N_j}(t_j)$. Then

$$S_{\mathbf{N}}f := f*D_{\mathbf{N}} := \int_{\mathbb{T}^d} f(t)D_{\mathbf{N}}(\cdot-t)\,dt, \qquad f\in L_1(\mathbb{T}^d),$$

defines the rectangular partial sum operator for the trigonometric system. We define the action of $S_{\mathbf{N}}$ on vector-valued functions $\mathbf{f}$ by letting it act separately on each coordinate function, i.e.,

$$[S_{\mathbf{N}}(\mathbf{f})]_j = S_{\mathbf{N}}(f_j), \qquad j=1,2,\dots m. \tag{1.2}$$

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It is well-known (see, e.g., [3, Theorem 3.5.7]) that $\|f - S_{\mathbf{N}}(f)\|_{L_p(\mathbb{T}^d)} \to 0$, as $\min_j N_j \to +\infty$, for $1 < p < \infty$. An immediate corollary is that we have convergence of the partial sums $S_{\mathbf{N}}(\mathbf{f})$, in $L_p(\mathbb{T}^d;\mathrm{Id})$, for $1 < p < \infty$ in the vactor-valued case. However, it is not obvious what can be said about convergence of $S_{\mathbf{N}}(\mathbf{f})$ in $L_p(\mathbb{T}^d;W)$ for a general matrix weight W. The main result of the preset paper completely characterizes the special class of weights that allow convergence; $S_{\mathbf{N}}(\mathbf{f})$ converges if and only if the weight W satisfies a certain matrix Muckenhoupt A_p product condition. Moreover, the characterization relies soly on certain localization properties of the Dirichlet kernels shared by many other summation kernels. So, in addition, we prove that the rectangular Cesàro means and approximation using the Jackson kernels converge in $L_p(\mathbb{T}^d;W)$ if and only if the weight W satisfies the mentioned matrix Muckenhoupt A_p product condition.

The structure of this paper is as follows. In Section 2 we introduced a product A_p condition for matrix weights. Then necessary and sufficient conditions for a convolution operator of product type (such as $S_{\mathbf{N}}$) to be bounded on $L_p(\mathbb{T}^d;W)$ are given. Section 3 contains applications of the results in Section 2 to convolution operators induced by rectangular Dirichlet, Fejér, and Jackson kernels.

2. The Muckenhoupt condition and operators on $L_p(\mathbb{T}^d;W)$

In this section we introduce a matrix Muckenhoupt A_p product condition suitable for dealing with convolution operators of product type such as the partial sum operator $S_{\mathbf{N}}$ defined by (1.2). A sufficient condition for convolution operators of product type bounded on $L_p(\mathbb{T}^d;W)$ is given in Propositions 2.5, while a converse type result is considered in Proposition 2.7. We prove that convolution operators with "nicely localized" kernels can only be uniformly bounded on $L_p(\mathbb{T}^d;W)$ when W satisfies the product A_p condition.

The scalar A_p condition was introduced by Muckenhoupt [5], and it was proved by Hunt, Muckenhoupt, and Wheeden in their seminal paper [4] that the A_p condition on a weight w is necessary and sufficient for the Hilbert transform to be bounded on the weighted space $L_p(\mathbb{T};w)$.

More recently, Hunt-Muckenhoppt-Wheeden type results for matrix weights have been considered. The matrix A_p condition was introduced by Nazarov, Treil, and Volberg [6,7,11] and they showed that it is the right condition for "standard" singular integral operators to be bounded on $L_p(\mathbb{T}^d;W)$. The A_p condition ($1 < p < \infty$) for weights $W : \mathbb{R}^d \to \mathcal{M}$ was originally stated in terms of dual matrics and averagings, but it was shown by Roudenko [8] to be equivalent to

$$\sup_B \left(\int_B \left(\int_B \|W^{1/p}(x) W^{-1/p}(y)\|^{p'} \frac{dy}{|B|} \right)^{p/p'} \frac{dx}{|B|} \right)^{p'/p}, \tag{2.1}$$

where the sup is taken over all open balls in $\mathbb{R}^d$ and $p' = p/(p-1)$ is the conjugate exponent.

Since our goal is to study operators of product type related to rectangular trigonometric partial sums, the condition given by (2.1) is not the appropriate one. The periodic weights satisfying (2.1) are well-behaved when it comes to the study of square or spherical partial sum operators for trigonometric series. Let us therefore introduce a new and slightly modified Muckenhoupt condition. Inspired by (2.1), we let $\mathcal{R}(d)$ denote the family of all rectangles in $\mathbb{R}^d$ of the form $R = I_1 \times I_2 \times \cdots \times I_d$, with I_j a bounded open interval in $\mathbb{R}$. Then we consider the following more restrictive subclass of matrix weights.

Definition 2.1. Let $W : \mathbb{T}^d \to \mathcal{M}$ be a periodic matrix weight. For $1 < p < \infty$, let $p' = p/(p-1)$ denote the conjugate exponent to p. We say that W belongs to the matrix Muckenhoupt (product) class PA_p provided there exists a uniform constant c_W such that

$$A_W(R,p) := \int_R \left(\int_R \left\| W^{1/p}(x) W^{-1/p}(y) \right\|^{p'} \frac{dy}{|R|} \right)^{p/p'} \frac{dx}{|R|} \le c_W, \tag{2.2}$$

for any $R \in \mathcal{R}(d)$.

Remark 2.2. For $d = 1$, Definition 2.1 reduces to the standard matrix A_p condition on $\mathbb{T}$, which we denote $A_p(\mathbb{T})$. In the scalar case (i.e., $m = 1$), Definition 2.1 reduces to the known product A_p condition for scalar weights, which has a long history, see [1] and references therein.

The similarity of conditions (2.2) and (2.1) implies that many results for matrix A_p weights have straightforward analogs in the product case; the proofs can be "translated verbatim". Let us state the following lemma which will be needed below.

Lemma 2.3. *Let $W : \mathbb{T}^d \to \mathcal{M}$ be a matrix weight. Then the following statements are equivalent for $1 < p < \infty$,*

(i) $W \in PA_p$;

(ii) $W^{-p'/p} \in PA_{p'}$;

(iii) $\displaystyle\int_R \left(\int_R \left\| W^{1/p}(x) W^{-1/p}(y) \right\|^{p} \frac{dx}{|R|} \right)^{p'/p} \frac{dy}{|R|} \le c_W, \quad \forall R \in \mathcal{R}(d).$

We refer the reader to Roudenko [8] for the proof of Lemma 2.3 in the non-product case.

The following Lemma reveals why one can expect AP_p to be useful for product operators. A weight in AP_p is uniformly in $A_p(\mathbb{T})$ in each of its d variables.

Lemma 2.4. *Let $W : \mathbb{T}^d \to \mathcal{M}$ be a matrix weight, and let $1 < p < \infty$. Then the following holds.*

(a) *For any rectangles $R \subseteq \tilde{R} \subset \mathcal{R}(d)$,*

$$A_W(R,p) \le \left(\frac{|\tilde{R}|}{|R|} \right)^p A_W(\tilde{R},p).$$

(b) *Suppose $W \in PA_p$, then the univariate weight $\xi_j \to W(\xi)$, obtained by fixing the variables ξ_k, $k \neq j$, is uniformly in $\mathbb{A}_p(\mathbb{T})$ for a.e. $(\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_d) \in \mathbb{T}^{d-1}$.*

Proof. For (a), we notice that whenever $R \subseteq \tilde{R} \subset \mathcal{R}(d)$,

$$\begin{aligned} A_W(R,p) &= \int_R \Big(\int_R \big\| W^{1/p}(x) W^{-1/p}(t) \big\|^{p'} \frac{dt}{|R|} \Big)^{p/p'} \frac{dx}{|R|} \\ &\le \Big(\frac{|\tilde{R}|}{|R|} \Big)^p \int_{\tilde{R}} \Big(\int_{\tilde{R}} \big\| W^{1/p}(x) W^{-1/p}(t) \big\|^{p'} \frac{dt}{|\tilde{R}|} \Big)^{p/p'} \frac{dx}{|\tilde{R}|} \\ &= \Big(\frac{|\tilde{R}|}{|R|} \Big)^p A_W(\tilde{R},p). \end{aligned}$$

Now we turn to the proof of (b). It suffices to consider $\tilde{W}(t) := W(t,\xi_2,\dots,\xi_d)$ for $(\xi_2,\dots,\xi_d) \in \mathbb{T}^{d-1}$ fixed. Given an interval $I \subset \mathbb{R}$, we form $R_\varepsilon = I_\varepsilon(\xi_2) \times \cdots \times I_\varepsilon(\xi_d)$, where $I_\varepsilon(\xi_j)$ is an interval of length 2ε centered at ξ_j. First suppose $p \le p'$. Since $W \in PA_p$ there exists a constant c_W independent of $I \times R_\varepsilon$ such that

$$\begin{aligned} &\frac{1}{|R_\varepsilon|^2} \int_{R_\varepsilon} \int_{R_\varepsilon} \Big[\int_I \Big(\int_I \|W^{1/p}(t,\mathbf{u}) W^{-1/p}(w,\mathbf{v})\|^{p'} \frac{dw}{|I|} \Big)^{p/p'} \frac{dt}{|I|} \Big] d\mathbf{u}\, d\mathbf{v} \\ &\qquad \le \int_{R_\varepsilon} \int_I \Big(\int_{R_\varepsilon} \int_I \|W^{1/p}(t,\mathbf{u}) W^{-1/p}(w,\mathbf{v})\|^{p'} \frac{dw\, d\mathbf{v}}{|I| \cdot |R_\varepsilon|} \Big)^{p/p'} \frac{dt\, d\mathbf{u}}{|I| \cdot |R_\varepsilon|} \\ &\qquad = A_W(R,p) \le c_W, \end{aligned}$$

where we have used the continuous embedding $L_1(R_\varepsilon; \frac{d\mathbf{v}}{|R_\varepsilon|}) \hookrightarrow L_{p/p'}(R_\varepsilon; \frac{d\mathbf{v}}{|R_\varepsilon|})$. Hence, by Lebesgue's differentiation theorem, for almost every $(\xi_2,\dots,\xi_d) \in \mathbb{T}^{d-1}$,

$$\begin{aligned} c_W &\ge \lim_{\varepsilon \to 0^+} A_W(I \times R_\varepsilon, p) \\ &= \int_I \Big(\int_I \|\tilde{W}^{1/p}(t) \tilde{W}^{-1/p}(w)\|^{p'} \frac{dw}{|I|} \Big)^{p/p'} \frac{dt}{|I|} = A_{\tilde{W}}(I,p), \end{aligned}$$

where the constant is independent of I and $(\xi_2,\dots,\xi_d)$. It follows that $\tilde{W}$ is uniformly in $\mathbb{A}_p(\mathbb{T})$ for a.e. $(\xi_2,\dots,\xi_d) \in \mathbb{T}^{d-1}$. In the case $p' < p$, we use Lemma 2.3 to conclude that $W^{-p'/p} \in PA_{p'}$ which implies the following estimate

$$\int_{R_\varepsilon} \int_I \Big(\int_{R_\varepsilon} \int_I \|W^{1/p}(t,\mathbf{u}) W^{-1/p}(w,\mathbf{v})\|^{p} \frac{dt\, d\mathbf{u}}{|I| \cdot |R_\varepsilon|} \Big)^{p'/p} \frac{dw\, d\mathbf{v}}{|I| \cdot |R_\varepsilon|} \le c_W.$$

By repeating the argument from the $p \le p'$ case, we conclude that $\tilde{W}^{-p'/p}$ is uniformly in $A_{p'}(\mathbb{T})$ which again by Lemma 2.3 implies that $\tilde{W}$ is uniformly in $A_p(\mathbb{T})$. □

We can now prove the following result that explains how to get from a bounded convolution operator on $L_p(\mathbb{T}; W)$ to a bounded convolution operator on $L_p(\mathbb{T}^d; W)$, for $W \in AP_p$, simply by forming the natural product kernel.

Proposition 2.5. *Suppose that $\{K_N\}_{N \ge 0}$ is a sequence of convolution kernels defined on $\mathbb{T}$ for which the corresponding operators*

$$T_N f := \int_{\mathbb{T}} f(t) K_N(\cdot - t)\, dt$$

are uniformly bounded on $L_p(\mathbb{T};W)$ whenever $W \in A_p(\mathbb{T})$. Then the associated product convolution kernels

$$K_{\mathbf{N}}(\xi) = \prod_{j=1}^{d} K_{N_j}(\xi_j), \qquad \mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{N}_0^d, \quad \xi \in \mathbb{T}^d,$$

induce a uniformly bounded family of operators on $L_p(\mathbb{T}^d;W)$ for $W \in PA_p$.

Proof. Suppose that $W \in PA_p$. In the case $d = 1$, there is nothing to prove. We focus on the case $d = 2$; the reader can easily verify that the argument below generalizes to any $d \geq 3$.

According to Lemma 2.4.(b), $W_{\xi_1} := W(\xi_1, \cdot)$ and $W_{\xi_2} := W(\cdot, \xi_2)$ satisfy uniform Muckenhoupt A_p-conditions a.e. on $\mathbb{T}$. Pick any $f \in L_p(\mathbb{T}^2, W)$. By Fubini's theorem, $f_{\xi_1} := f(\xi_1, \cdot) \in L_p(\mathbb{T}, W_{\xi_1})$ and $f_{\xi_2} := f(\cdot, \xi_2) \in L_p(\mathbb{T}, W_{\xi_2})$ for a.e. $[\xi_1]$ and $[\xi_2]$, respectively.

We define

$$T_N^1 f := K_N * f_{\xi_2} := \int_{\mathbb{T}} f_{\xi_2}(t) K_N(\cdot - t)\, dt, \qquad T_M^2 f := K_M * f_{\xi_1} := \int_{\mathbb{T}} f_{\xi_1}(t) K_M(\cdot - t)\, dt.$$

Notice that $T_{N,M} f = T_N^1 T_M^2 f$. By assumption,

$$\int_{\mathbb{T}} |W_{\xi_1}^{1/p}(\xi_2) T_M^2 f_{\xi_1}(\xi_2)|^p d\xi_2 \leq C \int_{\mathbb{T}} |W_{\xi_1}^{1/p}(\xi_2) f_{\xi_1}(\xi_2)|^p d\xi_2, \qquad \text{a.e.}[\xi_1].$$

An integration yields,

$$\int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/p}(\xi_1,\xi_2) T_M^2 f(\xi_1,\xi_2)|^p d\xi_2 d\xi_1 \leq C \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/p}(\xi_1,\xi_2) f(\xi_1,\xi_2)|^p d\xi_2 d\xi_1. \tag{2.3}$$

Similarly,

$$\begin{aligned}
\|T_{N,M} f\|_{L_p(\mathbb{T}^2,W)}^p &= \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/p}(\xi_1,\xi_2) T_N^1 T_M^2 f(\xi_1,\xi_2)|^p d\xi_1 d\xi_2 \\
&\leq C \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/p}(\xi_1,\xi_2) T_M^2 f(\xi_1,\xi_2)|^p d\xi_1 d\xi_2 \\
&\leq C^2 \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/p}(\xi_1,\xi_2) f|^p d\xi_1 d\xi_2.
\end{aligned}$$

It follows that the family $\{T_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}_0^2}$ is uniformly bounded on $L_p(\mathbb{T}^2;W)$. □

We now turn to a converse type result to Proposition 2.5. Proposition 2.7 below will show that well-localized trigonometric convolution kernels of product type can only be uniformly bounded on $L_p(\mathbb{T}^d;W)$ when $W \in AP_p$.

We need the following Lemma which gives an estimate of the norm of integral operators on $L_2(\mathbb{T}^d;W)$ with nice compactly supported kernels.

Lemma 2.6. *Suppose $Sf(\xi) = \int_{\mathbb{T}^d} S(\xi,\eta) f(\eta) d\eta$ is an integral operator with a scalar kernel $S(\xi,\eta)$ that satisfies $|S(\xi,\eta)| \leq \alpha |R|^{-1} \chi_{R \times R}$ for some bounded rectangle $R \subset \mathbb{R}^d$. For $1 < p <$*

∞, there exists a constant C_d independent of the particular choice of S such that the norm of S on $L_p(\mathbb{T}^d;W)$ is at most $C_d \cdot \alpha \cdot A_W(R,p)$, with $A_W(R,p)$ given by (2.2)*. Moreover, the kernel $\alpha|R|^{-1}\chi_R \times \chi_R$ induces an operator with norm at least $C_d^{-1} \cdot \alpha \cdot A_W(R,p)$ on $L_p(\mathbb{T}^d;W)$.*

The proof of Lemma 2.6 for non-product A_p-weights can be found in Goldberg [2]. We leave the straightforward adaptation of the proof in [2] to the product case for the reader.

We can now give a proof of Proposition 2.7. For $K \in \mathbb{N}$, we let $\mathcal{P}_K = \operatorname{span}\{e^{2\pi i k \cdot} : k \in \mathbb{Z}; |k| \leq K\}$.

Proposition 2.7. *Let $W : \mathbb{T}^d \to \mathcal{M}$ be a periodic matrix weight, and let $\{K_n\}_{n\geq 1}$ be a sequence of real-valued trigonometric convolution kernels defined on $\mathbb{T}$. Assume there exist constants c, C such that $K_n \in \mathcal{P}_{c\cdot n}$, with $C^{-1}n \leq \|K_n\|_\infty = K_n(0) \leq Cn$, for $n \in \mathbb{N}$. Suppose that the corresponding product kernels*

$$K_{\mathbf{N}}(\xi) = \prod_{j=1}^{d} K_{N_j}(\xi_j), \qquad \mathbf{N} \in \mathbb{N}_0^d,$$

induce a uniformly bounded family $\{T_{K_{\mathbf{N}}}\}$ of convolution operators on $L_p(\mathbb{T}^d;W)$. Then $W \in PA_p$.

Proof. We have to estimate $A_W(R,p)$ for an arbitrary rectangle $R \in \mathcal{R}(d)$. The idea is to form a suitable product kernel $K_{\mathbf{N}}$ that is "large" on R in the sense that the corresponding operator can be well approximated by an integral operator of the type considered in Lemma 2.6.

By assumption, the kernel $K_n \in \mathcal{P}_{c\cdot n}$ is real and $\|K_n\|_\infty = K_n(0) \leq Cn$ so by Bernstein's inequality, $\|K_n'\|_\infty \leq cCn^2$. We can thus find an integer M (independent of n) such that for $t \in [-\frac{1}{Mn}, \frac{1}{Mn}]$ we have $K_n(t) \geq \left(1 - \frac{1}{2C_d^2}\right)^{1/d} \|K_n\|_\infty$, where C_d is the constant from Lemma 2.6.

Let a rectangle $R = I_1 \times I_2 \times \cdots \times I_d$ be given. For $j = 1, 2, \ldots, d$, with $|I_j| > \frac{1}{2M}$, we define $N_j = 0$ and replace I_j with $[-1/2, 1/2)$, and obtain a possibly larger rectangle $\tilde{R}$. By Lemma 2.4.(b), there exists a universal constant b such that $A_W(R,p) \leq bA_W(\tilde{R},p)$ since $|\tilde{R}| \leq (2M)^d|R|$. Next, for each $j = 1, 2, \ldots, d$ with $|I_j| \leq \frac{1}{2M}$, we choose an integer $N_j \geq 1$ such that

$$\frac{1}{4M} \cdot \frac{1}{N_j} \leq |I_j| \leq \frac{1}{2M} \cdot \frac{1}{N_j}. \tag{2.4}$$

Notice that for $t, u \in I_j$, we have $t - u \in I_j - I_j \subset [-\frac{1}{MN_j}, \frac{1}{MN_j}]$ so

$$K_{N_j}(t-u) \geq \left(1 - \frac{1}{2C_d^2}\right)^{1/d} \|K_{N_j}\|_\infty. \tag{2.5}$$

For notational convenience we put $K_0 := 1$, and form the product kernel

$$K_{\mathbf{N}}(\xi) = \prod_{j=1}^{d} K_{N_j}(\xi_j).$$

The plan of attack is to use the simple fact that $f \to \chi_{\tilde{R}} T_{K_{\mathbf{N}}}(\chi_{\tilde{R}} f)$ is uniformly bounded in both $\tilde{R}$ and $\mathbf{N} \in \mathbb{N}_0^d$. We notice that $f \to \chi_{\tilde{R}} T_{K_{\mathbf{N}}}(\chi_{\tilde{R}} f)$ has integral kernel

$$S_2(\xi,\eta) := \chi_{\tilde{R}}(\eta)\chi_{\tilde{R}}(\xi)K_{\mathbf{N}}(\eta-\xi).$$

We wish to estimate the operator norm of S_2 from below. For that purpose we first consider the operator with kernel

$$S(\xi,\eta) := S_1(\xi,\eta) - S_2(\xi,\eta) := \|K_{\mathbf{N}}\|_\infty \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) - \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta)K_{\mathbf{N}}(\xi-\eta).$$

Notice that the estimate (2.5), together with the fact that $\|K_{\mathbf{N}}\|_\infty = \prod_{j=1}^d K_{N_j}(0)$, imply the following size estimate

$$\begin{aligned}
|S(\xi,\eta)| &= \Big| \|K_{\mathbf{N}}\|_\infty \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) - \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta)K_{\mathbf{N}}(\xi-\eta) \Big| \\
&\le \frac{\|K_{\mathbf{N}}\|_\infty}{2C_d^2} \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) \\
&= \frac{|\tilde{R}| \cdot \|K_{\mathbf{N}}\|_\infty}{2C_d^2} |\tilde{R}|^{-1} \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta).
\end{aligned}$$

According to Lemma 2.6, the kernel S induces an operator of norm at most $\frac{1}{2}C_d^{-1}|\tilde{R}| \cdot \|K_{\mathbf{N}}\|_\infty A_W(\tilde{R},p)$ on $L_p(\mathbb{T}^d;W)$. At the same time, Lemma 2.6 shows that the operator with kernel $S_1(\xi,\eta) = |\tilde{R}|\|K_{\mathbf{N}}\|_\infty \cdot |\tilde{R}|^{-1}\chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta)$ has norm at least $C_d^{-1}|\tilde{R}| \cdot \|K_{\mathbf{N}}\|_\infty A_W(\tilde{R},p)$ on $L_p(\mathbb{T}^d;W)$. The triangle inequality for operator norms now implies that

$$\frac{1}{2C_d}|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty M(\tilde{R},W) \ge \|S_1 - S_2\| \ge \big| \|S_1\| - \|S_2\| \big| \ge C_d^{-1}|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty M(\tilde{R},W) - \|S_2\|,$$

so $\|S_2\| \ge \frac{1}{2}C_d^{-1}|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty A_W(\tilde{R},p)$. Moreover, by (2.4), we see that $|\tilde{R}| \cdot \|K_{\mathbf{N}}\|_\infty \ge C(4M)^{-d}$, so we may conclude that

$$\begin{aligned}
A_W(R,p) &\le bA_W(\tilde{R},p) \\
&\le 2bCC_d(4M)^d\|S_2\| \\
&= C' \sup_{\|f\|_{L_p(\mathbb{T}^d;W)}=1} \|\chi_{\tilde{R}} T_{K_{\mathbf{N}}}(\chi_{\tilde{R}} f)\|_{L_p(\mathbb{T}^d;W)} \\
&\le C' \sup_{\|f\|_{L_p(\mathbb{T}^d;W)}=1} \|T_{K_{\mathbf{N}}} f\|_{L_p(\mathbb{T}^d;W)} \\
&\le C''.
\end{aligned}$$

with constant C'' independent of R. We may finally conclude that $W \in PA_p(d)$. □

3. Summation of multiple trigonometric series

This section contains applications of the results of Section 2 to convolution operators induced by rectangular Dirichlet, Fejér, and Jackson kernels. The Dirichlet kernels correspond to standard rectangular trigonometric summation while the Fejér kernels generate the corresponding Cesàro means. The Jackson kernels are (normalized) squares of the Fejér kernels, and they induce the well-known Jackson approximation by trigonometric polynomials.

We begin by studying the univariate Dirichlet kernel. The Hilbert transform H is defined on $L_p(\mathbb{T})$, $1 < p < \infty$, by

$$H(f)(x) := \text{p.v.} \int_{\mathbb{T}} f(t) \cot(\pi(x-t))\, dt.$$

We lift H to a linear operator on $L_p(\mathbb{T}; W)$, for any matrix weight $W : \mathbb{T} \to \mathcal{M}$, by letting it act coordinatewise.

Treil and Volberg completely characterized when the Hilbert transform H is bounded in the matrix case on $\mathbb{T}$ when $p = 2$, see [10]. Later, Nazarov and Treil introduced in a new "Bellman function" method [7] to extend the theory to $1 < p < \infty$. Volberg presented a different solution to the matrix weighted L_p boundedness of the Hilbert transform via Littlewood-Paley theory [11]. The fundamental result is the following.

Theorem 3.1 ([7, 10, 11]). *Let $W : \mathbb{T} \to \mathcal{M}$ be a matrix weight. Suppose $1 < p < \infty$. Then the Hilbert transform is bounded on $L_p(\mathbb{T}; W)$ if and only if $W \in A_p(\mathbb{T})$.*

We recall that the univariate Dirichlet kernel D_N is given by

$$D_N(t) = \frac{\sin 2\pi(N+1/2)t}{\sin \pi t}, \qquad N \geq 1, \tag{3.1}$$

and for $f \in L_p(\mathbb{T})$ we define the associated partial sum operators,

$$S_N(f) := \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k \cdot} = f * D_N := \int_{\mathbb{T}} f(t) D_N(\cdot - t)\, dt.$$

We have the following lemma which follows easily from Theorem 3.1.

Lemma 3.2. *Let $W : \mathbb{T} \to \mathcal{M}$ be a matrix weight in $A_p(\mathbb{T})$. Then the partial sum operators $f \to f * D_N$ are uniformly bounded on $L_p(\mathbb{T}; W)$.*

Proof. We let $P_+ = \frac{1}{2}(I + iH + S_0)$ denote the Riesz projection onto H^p for $f \in L_p(\mathbb{T}; W)$, where $S_0 f := \int_{\mathbb{T}} f(y)\, dy$ is the 0-order partial sum operator. It follows that P_+ is bounded on $L_p(\mathbb{T}; W)$ since H is bounded according to Theorem 3.1, and S_0 is bounded according to [9, Lemma 1.5]. Notice that $f \to f e^{2\pi i M \cdot}$ is a norm preserving operator on $L_p(\mathbb{T}; W)$, just as in the scalar case. Then we observe that

$$f * D_N = e^{-2\pi i N \cdot} P_+(e^{2\pi i N \cdot} f) - e^{2\pi i (N+1) \cdot} P_+(e^{-2\pi i (N+1) \cdot} f),$$

and the result follows. □

For $\mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{N}_0^d$, we form the product kernel $D_{\mathbf{N}}(\xi) := \prod_{j=1}^{d} D_{N_j}(\xi_j)$.

$$S_{\mathbf{N}}(\mathbf{f}) := \int_{\mathbb{T}^d} \mathbf{f}(t) D_{\mathbf{N}}(\cdot - t)\, dt. \tag{3.2}$$

We notice that $D_N \in \mathcal{P}_N$ and $D_N(0) = \|D_N\|_\infty = 2N + 1$, so the following Corollary follows directly from Propositions 2.5 and 2.7 and Lemma 3.2.

Corollary 3.3. *Let $W : \mathbb{T}^d \to \mathcal{M}$ be a matrix weight. For $1 < p < \infty$, the operators $\{S_{\mathbf{N}} : \mathbf{N} \in \mathbb{N}^d\}$ are uniformly bounded on $L_p(\mathbb{T}^d; W)$ if and only if $W \in AP_p$.*

Remark 3.4. It is easy to verify that vectors of trigonometric polynomials are dense in $L_p(\mathbb{T}^d; W)$, $1 < p < \infty$, whenever W is a matrix weight (since $W \in L_1$ so each entry in W is in $L_1(\mathbb{T})$). It therefore follows by standard techniques that the family $\{S_{\mathbf{N}} : \mathbf{N} \in \mathbb{N}^d\}$ is uniformly bounded on $L_p(\mathbb{T}^d; W)$ if and only if $\|\mathbf{f} - S_{\mathbf{N}}(\mathbf{f})\|_{L_p(\mathbb{T};W)} \to 0$, as $\min_j N_j \to +\infty$, for all $\mathbf{f} \in L_p(\mathbb{T}; W)$.

Corollary 3.3 relies on basic localization properties of the Dirichlet kernel. However, many well-known summation kernels share the necessary properties needed to apply Propositions 2.5 and 2.7. Let us illustrate this fact by considering two specific examples.

The rectangular Cesàro summation is given by

$$\sigma_{\mathbf{N}}(\mathbf{f}) = \frac{1}{(N_1+1)\cdots(N_d+1)} \sum_{k_1=0,\dots,k_d=0}^{N_1,\dots,N_d} S_{(k_1,k_2,\dots,k_d)}(\mathbf{f}) = \int_{\mathbb{T}^d} \mathbf{f}(t) F_{\mathbf{N}}(\cdot - t)\, dt \tag{3.3}$$

with the product Féjer kernel given by $F_{\mathbf{N}}(\xi) = \prod_{j=1}^d F_{N_j}(\xi_j)$, where the scalar Féjer kernel is defined by

$$F_n(t) = \frac{1}{n}\left(\frac{\sin(n\pi t)}{\sin(\pi t)}\right)^2.$$

Notice that $F_b \in \mathcal{P}_n$ and $F_n(0) = \|F_n\|_\infty = n$. The scalar Jackson kernel is the normalized square of the Féjer kernel and given by

$$J_n(t) = \frac{3}{n(2n^2+1)}\left(\frac{\sin(n\pi t)}{\sin(\pi t)}\right)^4.$$

The correspondig product kernel is $J_{\mathbf{N}}(\xi) = \prod_{j=1}^d J_{N_j}(\xi_j)$, $\mathbf{N} \in \mathbb{N}_0^d$, and the rectangular Jackson summation operator is given by $\mathcal{J}_{\mathbf{N}}(f) := f * J_{\mathbf{N}}$. Notice that $J_n \in \mathcal{P}_{2n}$ and $J_n(0) = \|J_n\|_\infty = n$.

We now conclude by stating the main result, which summarizes the results obtained in the present paper. The theorem shows that uniform boundedness of the rectangular operators $S_{\mathbf{N}}$, $\sigma_{\mathbf{N}}$ and $\mathcal{J}_{\mathbf{N}}$ on $L_p(\mathbb{T}^d; w)$ are all equivalent to the condition $W \in AP_p$.

Theorem 3.5. *Let $W : \mathbb{T}^d \to \mathcal{M}$ be a matrix weight. For $1 < p < \infty$, the following conditions are equivalent*

(i) $W \in AP_p$;
(ii) *The operators $\{S_{\mathbf{N}} : \mathbf{N} \in \mathbb{N}_0^d\}$ are uniformly bounded on $L_p(\mathbb{T}^d; W)$;*
(iii) *The operators $\{\sigma_{\mathbf{N}} : \mathbf{N} \in \mathbb{N}_0^d\}$ are uniformly bounded on $L_p(\mathbb{T}^d; W)$;*
(iv) *The operators $\{\mathcal{J}_{\mathbf{N}} : \mathbf{N} \in \mathbb{N}_0^d\}$ are uniformly bounded on $L_p(\mathbb{T}^d; W)$.*

Proof. We first notice that each of the univariate kernels D_n, F_n, and J_n satisfies the hypothesis of Proposition 2.7, so (ii), (iii) and (iv) each implies that $W \in AP_p$. Now, suppose that $W \in AP_p$. Then (ii) holds by Corollary 3.3. To conclude, we just need to recall that Cesàro and Jackson summation are both regular summation methods so (ii) implies both (iii) and (iv). ☐

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