A NEW APPROACH TO THE REGULARITY OF SOLUTIONS FOR PARABOLIC EQUATIONS

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Dedicated to Jerry Goldstein on the occasion of his 60th birthday

Abstract. In this note we describe a new approach to establish regularity properties for solutions of parabolic equations. It is based on maximal regularity and the implicit function theorem.

1. Introduction

In this note we describe a new approach to establish regularity properties for a wide array of parabolic evolution equations. It is based on the theory of maximal regularity. The thrust of this approach is manifold.

- It allows to solve a given partial differential equation without loss of derivatives, thus permitting to handle fully nonlinear equations.
- It allows to resort to the implicit function theorem to study further properties of solutions, such as smooth dependence on given data.
- It allows to study the regularity of solutions by merely applying scaling arguments in conjunction with the implicit function theorem.

In order to explain the main idea of our approach, let us consider the model problem of a family of graphs $\{\Gamma(t) = \operatorname{graph}(u(\cdot,t))\,;\, 0 \le t \le T\}$ over $\mathbb{R}^n$, evolving according to the mean curvature flow

$$\partial_t u - \left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\right)\partial_i\partial_j u = 0, \qquad u(0) = u_0, \tag{1.1}$$

where $1 \le i,j \le n$, and where δ_{ij} denotes the Kronecker delta. Equation (1.1) is a quasilinear parabolic evolution equation of second order. To economize notation we set

$$F(u) := -\left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\right)\partial_i\partial_j u$$

and restate equation (1.1) as

$$\partial_t u + F(u) = 0, \quad u(0) = u_0. \tag{1.2}$$

Let $E_j := buc^{2j+s}(\mathbb{R}^n)$, $j = 0,1$, be the little Hölder spaces defined in (2.8). The mapping F is real analytic, that is,

$$F \in C^\omega(E_1, E_0). \tag{1.3}$$

Given that F is differentiable, one can consider the linearized problem

$$\partial_t v + F'(u)v = f, \quad v(0) = v_0, \tag{1.4}$$

where $F'(u)$ is the Fréchet derivative of F at $u \in E_1$. Next we introduce the anisotropic spaces

$$\mathbb{E}_0(I) := C(I, E_0), \quad \mathbb{E}_1(I) := C^1(I, E_0) \cap C(I, E_1),$$

where $I = [0, T]$ is a fixed interval. Clearly, the trace operator $\gamma_0 : \mathbb{E}_1(I) \to E_1$, $v \mapsto v(0)$ is linear and continuous. It can be shown, and this is the essential part of the analysis, that the linear problem (1.4) enjoys the property of *maximal regularity*. By definition, this means that

$$(\partial_t + F'(u), \gamma_0) \in \mathrm{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_1) \tag{1.5}$$

for any function $u \in E_1$. That is, the linear mapping $(\partial_t + F'(u), \gamma_0)$ is a topological isomorphism between the indicated spaces. It is here where maximal regularity begins to unfold. It implies that the linear problem (1.4) has a unique solution $v \in \mathbb{E}_1(I)$ for any given right hand side $(f, v_0) \in \mathbb{E}_0(I) \times E_1$. The solution v has optimal regularity, and therefore, no loss of regularity can occur for the linearized problem. Existence of a unique solution in $\mathbb{E}_1(I)$ to the nonlinear problem (1.2) can now be obtained by a reiteration argument and the contraction principle. As an immediate outcome, one sees that there is also no 'loss of derivatives' for the nonlinear problem. (This is also true if F is fully nonlinear). It should be noted that iteration techniques based on the Nash-Moser implicit function theorem usually result in a loss of derivatives.

We give a brief account on how the property of maximal regularity in conjunction with a scaling argument (or a parameter trick) will show that the solution $u \in \mathbb{E}_1(I)$ of (1.2) is real analytic in space and time for any positive time.

Let u be the unique solution of (1.2) defined on a maximal interval of existence $[0, t^+(u_0))$. Let $T \in (0, t^+(u_0))$ be a fixed number and set $I := [0, T]$. For any given parameters $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^n$ with $\lambda \in (-\varepsilon_0, \varepsilon_0)$ one can set

$$u_{\lambda,\mu}(t, x) := u(t + t\lambda, x + t\mu), \quad (t, x) \in I \times \mathbb{R}^n. \tag{1.6}$$

It is easy to see that $u_{\lambda,\mu} \in \mathbb{E}_1(I)$ for all (λ, μ), provided ε_0 is sufficiently small. Since the mapping F commutes with translations, that is,

$$\tau_a F(u) = F(\tau_a u), \quad u \in E_1, \quad a \in \mathbb{R}^n, \tag{1.7}$$

one finds that $v := u_{\lambda,\mu} \in \mathbb{E}_1(I)$ satisfies the parameter dependent equation

$$\partial_t v + (1 + \lambda)F(v) - (\mu|\nabla v) = 0, \quad v(0) = u_0,$$

or equivalently, that $v := u_{\lambda,\mu}$ solves

$$\Phi(v, (\lambda, \mu)) = 0 \tag{1.8}$$

where $\Phi(v, (\lambda, \mu)) := (\partial_t v + (1 + \lambda)F(v) - (\mu|\nabla v), \gamma_0 v - u_0)$. It follows from (1.3) that the mapping

$$\Phi : \mathbb{E}_1(I) \times ((-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^n) \to \mathbb{E}_0(I) \times E_1$$

is real analytic. Moreover, $\Phi(\bar{u},(0,0))=(0,0)$, where $\bar{u}:=u|_I$. It is a consequence of the maximal regularity property (1.5) that the Fréchet derivative $D_1\Phi(\bar{u},(0,0))$ of Φ with respect to v satisfies

$$D_1\Phi(\bar{u},(0,0))=(\partial_t+F'(\bar{u}),\gamma_0)\in \mathrm{Isom}(\mathbb{E}_1(I),\mathbb{E}_0(I)\times E_1). \tag{1.9}$$

The implicit function theorem now allows to solve equation (1.8) for v in terms of (λ,μ) in an open neighborhood U of $(0,0)\in\mathbb{R}\times\mathbb{R}^n$. One concludes that

$$[(\lambda,\mu)\mapsto u_{\lambda,\mu}]\in C^\omega(U,\mathbb{E}_1(I)). \tag{1.10}$$

Consequently, the mapping

$$[(\lambda,\mu)\mapsto u_{\lambda,\mu}(t_0,x_0)=u(t_0+t_0\lambda,x_0+t_0\mu)]:U\to\mathbb{R} \tag{1.11}$$

is real analytic for any fixed $(t_0,x_0)\in I\times\mathbb{R}^n$ with $t_0>0$. Hence, the solution u of the mean curvature flow (1.1) is analytic in space and time for any positive time $t\in(0,t^+(u_0))$.

It is now clear that the only properties needed to carry through the arguments are (1.3), (1.7), and the crucial maximal regularity property (1.5). The nature of the mapping F is completely immaterial: it can be fully nonlinear, can act as a nonlocal mapping, and it can be of any order.

The idea of using parameters to prove regularity properties of solutions goes back to Angenent [3, 4]. The strategy of using translations to show analyticity in space was first employed in [8] for a free boundary problem for the flow of an incompressible fluid in a porous medium of infinite extent. In that context the mapping F happens to be fully nonlinear, nonlocal, and of first order. Translations were also used in [7] for the Stefan problem with surface tension in the case where the free interface is represented as the graph of a function over $\mathbb{R}^n$.

The advantage of applying maximal regularity lies in the fact that one can resort to the implicit function theorem. The difficulty, of course, lies in establishing maximal regularity for a given partial differential equation.

Our approach described so far relies on the fact that we can use translations on $\mathbb{R}^n$, and that the mapping F is equivariant with respect to translations. The approach can be generalized in two directions. First, it can be generalized to parabolic equations on a symmetric Riemannian manifold M, where one assumes that the nonlinear mapping F is equivariant with respect to the Lie group which acts as a transformation group on M. This has been done in [9]. In this note we show how the translation-parameter trick can be localized. In order to do so, we pick $(t_0,x_0)\in J\times\mathbb{R}^n$ and choose smooth cut-off functions $\chi\in\mathcal{D}(\mathbb{R}^n)$ and $\zeta\in\mathcal{D}(J)$ with

$$\mathrm{supp}\,(\chi)\subset\mathbb{B}(x_0,\varepsilon_0),\qquad \mathrm{supp}\,(\zeta)\subset(t_0-\varepsilon_0,t_0+\varepsilon_0), \tag{1.12}$$

where ε_0 can be chosen as small as we wish for. Instead of (1.6) we can now consider the parameter-dependent function

$$u_{\lambda,\mu}(t,x):=u(t+\zeta(t)\lambda,x+\zeta(t)\chi(x)\mu),\qquad (t,x)\in J\times\mathbb{R}^n. \tag{1.13}$$

The function $v := u_{\lambda,\mu}$ also satisfies a parameter-dependent equation

$$\partial_t v + F_{\lambda,\mu}(v) = 0, \qquad v(0) = u_0. \tag{1.14}$$

The new difficulty now lies in showing that the mapping $[(v,(\lambda,\mu)) \mapsto F_{\lambda,\mu}(v)]$ is analytic. This will be done in the following sections. The current note will serve as the basis to establish regularity results for free boundary problems, such as the Stefan problem with surface tension, and the Navier-Stokes equations with surface tension.

2. Parameter-dependent diffeomorphisms

In the following, we assume that X is an open set in $\mathbb{R}^n$. Moreover, we assume that $x_0 \in X$ is fixed. Let $\varepsilon_0 > 0$ be chosen such that $\overline{\mathbb{B}}(x_0, 3\varepsilon_0) \subset X$ and let $\chi \in \mathcal{D}(\mathbb{B}(x_0, 2\varepsilon_0), \mathbb{R})$ be a smooth cut-off function with $\chi \equiv 1$ on $\overline{\mathbb{B}}(x_0, \varepsilon_0)$ and with $0 \le \chi \le 1$. We define the parameter dependent mapping

$$\Theta_\mu(x) := x + \chi(x)\mu, \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{C}^n. \tag{2.1}$$

Here and in the following, $\mathbb{B}(x_0, r)$ denotes the ball of radius r and center x_0 with respect to the Euclidean norm in $\mathbb{R}^n$, and $\mathbb{B}_{\mathbb{C}^n}(x_0, r)$ stands for the corresponding ball in $\mathbb{C}^n$.

Lemma 2.1. *There exists a positive number r_0 such that*

(a) $\Theta_\mu(\mathbb{B}(x_0, 3\varepsilon_0)) \subset \mathbb{B}_{\mathbb{C}^n}(x_0, 3\varepsilon_0)$ *for any* $\mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$.

(b) $\Theta_\mu(\overline{\mathbb{B}}(x_0, 3\varepsilon_0)) \subset \overline{\mathbb{B}}_{\mathbb{C}^n}(x_0, 3\varepsilon_0)$ *for any* $\mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$.

(c) $|\Theta_\mu(x) - \Theta_{\mu_0}(y)| \le 3/2|x-y| + |\mu - \mu_0|, \ \forall x, y \in X, \forall \mu, \mu_0 \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$.

Proof. (a) Choose $r_0 < \varepsilon_0$ and let $x \in \mathbb{B}(x_0, 2\varepsilon_0)$ and $\mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$ be given. Then we have

$$|\Theta_\mu(x) - x_0| \le |x - x_0| + \chi(x)\,|\mu| < 3\varepsilon_0,$$

showing that $\Theta_\mu(\mathbb{B}(x_0, 2\varepsilon_0)) \subset \mathbb{B}_{\mathbb{C}^n}(x_0, 3\varepsilon_0)$. Since

$$\Theta_\mu(x) = x \text{ for } x \in \mathbb{B}(x_0, 3\varepsilon_0) \setminus \mathbb{B}(x_0, 2\varepsilon_0)$$

we obtain the assertion in (a).
(b) is a consequence of (a).
(c) It follows from the mean value theorem that $|\chi(x) - \chi(y)| \le \|\nabla\chi\|_\infty |x - y|$ for $x, y \in X$. A simple computation then yields

$$\begin{aligned}|\Theta_\mu(x) - \Theta_{\mu_0}(y)| &\le |x-y| + |\chi(x) - \chi(y)|\,|\mu| + |\chi(y)|\,|\mu - \mu_0| \\ &\le (1 + \|\nabla\chi\|_\infty r_0)|x-y| + |\mu - \mu_0|.\end{aligned}$$

We can assume that r_0 is already chosen small enough such that $\|\nabla\chi\|_\infty r_0 \le 1/2$ and this implies (c). □

Proposition 2.2. *There exists a positive number r_0 such that*

$$\Theta_\mu \in \mathit{Diff}^\infty(X), \qquad \mu \in \mathbb{B}(0, r_0).$$

Proof. Let $\mu \in \mathbb{R}^n$ be given. Clearly, the mapping Θ_μ is smooth in x. Its derivative is given by

$$D\Theta_\mu = I + R_\mu \quad \text{with} \quad R_\mu(x) = [\nabla\chi(x) \otimes \mu]. \tag{2.2}$$

Let r_0 be the number of Lemma 2.1. We can assume that

$$\sup_{x\in X} \|R_\mu(x)\| \le 1/2, \qquad \mu \in \mathbb{B}(0, r_0). \tag{2.3}$$

Note that equations (2.2)–(2.3) imply that the derivative $D\Theta_\mu(x)$ is invertible for $x \in X$. Lemma 2.1(a) shows that $W_\mu := \Theta_\mu(\mathbb{B}(x_0, 3\varepsilon_0)) \subset \mathbb{B}(x_0, 3\varepsilon_0)$ for any $\mu \in \mathbb{B}(0, r_0)$. We can then infer from the inverse function theorem, applied to the mapping $\Theta_\mu : \mathbb{B}(x_0, 3\varepsilon_0) \to \mathbb{B}(x_0, 3\varepsilon_0)$, that

$$W_\mu \subset \mathbb{B}(x_0, 3\varepsilon_0) \text{ is open}, \quad \mu \in \mathbb{B}(0, r_0). \tag{2.4}$$

We claim that $W_\mu = \mathbb{B}(x_0, 3\varepsilon_0)$ and that Θ_μ is injective. Since $\mathbb{B}(x_0, 3\varepsilon_0)$ is convex, we may apply the mean value theorem, yielding

$$x - y = \Theta_\mu(x) - \Theta_\mu(y) - \int_0^1 R_\mu(y + \tau(x-y))\, d\tau\, (x-y) \tag{2.5}$$

for $x, y \in \mathbb{B}(x_0, 3\varepsilon_0)$. It follows from (2.3) that

$$|x - y| \le 2\, |\Theta_\mu(x) - \Theta_\mu(y)| \tag{2.6}$$

for every $x, y \in \mathbb{B}(x_0, 3\varepsilon_0)$ and every $\mu \in \mathbb{B}(0, r_0)$. We conclude that Θ_μ is injective and that W_μ is closed in $\mathbb{B}(x_0, 3\varepsilon_0)$. Since $\mathbb{B}(x_0, 3\varepsilon_0)$ is connected, (2.4) implies that W_μ coincides with $\mathbb{B}(x_0, 3\varepsilon_0)$. It follows from the inverse function theorem that

$$\Theta_\mu \in \mathrm{Diff}^\infty(\mathbb{B}(x_0, 3\varepsilon_0)), \qquad \mu \in \mathbb{B}(x_0, r_0).$$

Since $\Theta_\mu(x) = x$ for $x \in X \setminus \mathbb{B}(x_0, 2\varepsilon_0)$ and $\Theta_\mu \in C^\infty(X)$ we obtain $\Theta_\mu \in \mathrm{Diff}^\infty(X)$, and the proof is now complete. □

Remarks 2.3. (a) It follows from Proposition 2.2 and the definition of Θ_μ that

$$\Theta_\mu \in \mathrm{Diff}^\infty(\mathbb{B}(x_0, 2\varepsilon_0)), \qquad \mu \in \mathbb{B}(0, r_0).$$

(b) It is clear that $\Theta_\mu \in \mathrm{Diff}^\infty(U)$ for any open set U with $\overline{\mathbb{B}}(x_0, 3\varepsilon_0) \subset U$.

In the following we assume that U is an open set in $\mathbb{R}^n$ such that

$$\begin{aligned} &\bullet\ \overline{\mathbb{B}}(x_0, 3\varepsilon_0) \subset U, \\ &\bullet\ U \text{ is either bounded and has a smooth boundary, or } U = \mathbb{R}^n. \end{aligned} \tag{2.7}$$

Let $s \ge 0$. The *little Hölder spaces* are defined by

$$buc^s(U) := \begin{cases} BUC^s(U), & \text{if } s \in \mathbb{N} \\ \text{the closure of } BUC^{[s]+1}(U) \text{ in } BUC^s(U), & \text{if } s \notin \mathbb{N} \end{cases} \tag{2.8}$$

where $[s]$ denotes the integer part of s, and where $BUC^s(U)$ are the classical Hölder spaces. Moreover, for $1 < p < \infty$ let $W_p^s(U)$ denote the *Sobolev-Slobodecki spaces*, and let $H_p^s(U)$ be the *Bessel-potential spaces*.

Let $m \in \mathbb{N}$ be given and let $s \in (0, m)$. The following interpolation results are well-known, see [11, 13, 14], and also [1, Section I.2] for a short account of interpolation theory,

$$\begin{aligned} (BUC(U), BUC^m(U))_{s/m,\infty} &= BUC^s(U), && s \notin \mathbb{N}, \\ (BUC(U), BUC^m(U))^0_{s/m,\infty} &= buc^s(U), && s \notin \mathbb{N}, \\ (L_p(U), W_p^m(U))_{s/m,p} &= W_p^s(U), && s \notin \mathbb{N}, \\ [L_p(U), W_p^m(U)]_{s/m} &= H_p^s(U). \end{aligned} \tag{2.9}$$

Moreover, we have the interpolation inequalities

$$\|u\|_{\mathfrak{F}^s} \le c(s)\|u\|_{\mathfrak{F}^0}^{1-s/m}\|u\|_{\mathfrak{F}^m}^{s/m}, \qquad s \in (0,m), \quad u \in \mathfrak{F}^m, \tag{2.10}$$

where $\mathfrak{F} \in \{buc(U), BUC(U), W_p(U), H_p(U)\,;\, 1 < p < \infty\}$. Our notation indicates that we choose one of the symbols in $\{buc(U), BUC(U), W_p(U), H_p(U)\}$, and then use this symbol exclusively throughout formula (2.10). We recall that

$$buc^0(U) := buc(U) = BUC(U) =: BUC^0(U), \quad W_p^0(U) = H_p^0(U) = L_p(U).$$

We also recall that

$$\mathfrak{F}^m \subset \mathfrak{F}^s \quad \text{is dense for} \quad \mathfrak{F} \in \{buc(U), W_p(U), H_p(U)\,;\, 1 < p < \infty\}. \tag{2.11}$$

It is well-known that

$$\partial_j \in \mathcal{L}(\mathfrak{F}^{s+1}, \mathfrak{F}^s), \quad \mathfrak{F} \in \{buc(U), BUC(U), W_p(U), H_p(U)\}, \quad s \ge 0. \tag{2.12}$$

Moreover, point-wise multiplication $[(a,u) \mapsto au]$ is bilinear and continuous for the spaces

$$\begin{aligned} &BUC^\rho(U) \times \mathfrak{F}^s(U) \to \mathfrak{F}^s(U), \quad \mathfrak{F} \in \{buc, W_p, H_p\}, \quad 0 \le s < \rho, \\ &BUC^m(U) \times \mathfrak{F}^m(U) \to \mathfrak{F}^m(U), \quad \mathfrak{F} \in \{BUC, W_p\}, \quad m \in \mathbb{N}, \\ &BUC^s(U) \times BUC^s(U) \to BUC^s(U), \quad s \ge 0, \\ &buc^s(U) \times buc^s(U) \to buc^s(U), \qquad s \ge 0. \end{aligned} \tag{2.13}$$

Given a function $u \in L_{1,\mathrm{loc}}(U)$ we define the pull-back and the push-forward operator, respectively, induced by the diffeomorphism Θ_μ:

$$\begin{aligned} &\Theta_\mu^* u := u \circ \Theta_\mu, \\ &\Theta_*^\mu u := u \circ (\Theta_\mu)^{-1}, \qquad \mu \in \mathbb{B}(0, r_0). \end{aligned} \tag{2.14}$$

In the following Proposition we collect some useful properties for the operators Θ_μ^*. We show that Θ_μ^* induces an isomorphism on all the function spaces introduced above, and we study the dependence on the parameter μ.

For future reference, the results are stated in a more general form than actually needed in the present note.

Proposition 2.4. *Let $m \in \mathbb{N}$ and $s \in [0,m]$.*

(a) *Suppose $\mathfrak{F} \in \{buc(U), BUC(U), W_p(U), H_p(U)\}$. Then*

$$\Theta_\mu^* \in Isom(\mathfrak{F}^s), \qquad [\Theta_\mu^*]^{-1} = \Theta_*^\mu, \qquad \mu \in \mathbb{B}(0,r_0).$$

Moreover, there exists a positive constant $M = M(m)$ such that

$$\|\Theta_\mu^*\|_{\mathcal{L}(\mathfrak{F}^s)} \le M, \qquad \mu \in \mathbb{B}(0,r_0). \tag{2.15}$$

(b) *Suppose $\mathfrak{F} \in \{buc(U), W_p(U), H_p(U)\}$. Then*

$$[\mu \mapsto \Theta_\mu^* u] \in C(\mathbb{B}(0,r_0), \mathfrak{F}^s) \quad \text{for any} \ \ u \in \mathfrak{F}^s. \tag{2.16}$$

(c) *Suppose $\mathfrak{F} \in \{buc(U), W_p(U), H_p(U)\}$. Then*

$$[\mu \mapsto \Theta_\mu^* u] \in C^1(\mathbb{B}(0,r_0), \mathfrak{F}^s) \quad \text{for any} \ \ u \in \mathfrak{F}^{s+1}. \tag{2.17}$$

The partial derivatives are given by

$$\partial_{\mu_j}[\Theta_\mu^* u] = \chi[\Theta_\mu^* \partial_j u], \qquad u \in \mathfrak{F}^{s+1}, \quad j \in \{1,\cdots,n\}. \tag{2.18}$$

Proof. (a) (i) Pick $\mu \in \mathbb{B}(0,r_0)$ and $u \in BUC(U)$. We conclude from Lemma 2.1(c) and from Proposition 2.2 that

$$\Theta_\mu^* u \in BUC(U), \qquad \|\Theta_\mu^* u\|_{BUC(U)} \le \|u\|_{BUC(U)}, \quad \mu \in \mathbb{B}(0,r_0). \tag{2.19}$$

Next, let $u \in BUC^m(U)$. It is evident that $\Theta_\mu^* u \in C^m(U)$, and a straightforward computation shows that

$$\partial^\beta[\Theta_\mu^* u] = \sum_{|\gamma|\le|\beta|} b_{\beta,\gamma}(\mu,\cdot)\,[\Theta_\mu^* \partial^\gamma u], \qquad |\beta| \le m, \tag{2.20}$$

where $b_{\beta,\gamma} \in BUC(\mathbb{B}(0,r_0)\times U)$. (We have, in fact, $b_{\beta,\gamma} \in BUC^\infty(\mathbb{B}(0,r_0)\times U)$). We conclude from (2.19) and (2.20) that

$$\Theta_\mu^* u \in BUC^m(U), \qquad \|\Theta_\mu^* u\|_{BUC^m(U)} \le M\|u\|_{BUC^m(U)}, \quad \mu \in \mathbb{B}(0,r_0),$$

for an appropriate constant M. Clearly, Θ_μ^* is linear for every fixed $\mu \in \mathbb{B}(0,r_0)$, and it follows from (2.19)-(2.20) that

$$\Theta_\mu^* \in \mathcal{L}(BUC^l(U)), \qquad \|\Theta_\mu^*\|_{\mathcal{L}(BUC^l(U))} \le M, \quad l \in [0,m]\cap\mathbb{N}. \tag{2.21}$$

It is clear that $[\Theta_\mu^*]^{-1} = \Theta_*^\mu$, and the open mapping theorem yields $\Theta_\mu^* \in BUC^l(U)$ for $l \in [0,m]\cap\mathbb{N}$. The case $\mathfrak{F}^s \in \{buc^s(U), BUC^s(U)\}$ for $s \in (0,m)\setminus\mathbb{N}$ follows from (2.9) and (2.21) by interpolation.

(ii) It is a consequence of the transformation rule, Remark 2.3, and equations (2.2)–(2.3) that

$$\Theta_\mu^* \in \mathcal{L}(L_p(U)), \quad \|\Theta_\mu^*\|_{\mathcal{L}(L_p(U))} \le M_1, \quad \mu \in \mathbb{B}(0,r_0). \tag{2.22}$$

It is not difficult to show (by approximating) that formula (2.20) remains valid for $u \in W_p^m(U)$. One can then conclude that

$$\Theta_\mu^* \in \mathcal{L}(W_p^m(U)), \quad \|\Theta_\mu^*\|_{\mathcal{L}(W_p^m(U))} \le M_2, \quad \mu \in \mathbb{B}(0,r_0). \tag{2.23}$$

As in (i) we obtain the assertion for $\mathfrak{F}^s \in \{W_p^s(U), H_p^s(U)\}$ by interpolation.

(b) (i) We first consider $u \in BUC(U)$. Since u is uniformly continuous we find for every $\varepsilon > 0$ a number $\delta > 0$ such that $|u(y) - u(z)| < \varepsilon$ whenever $y, z \in U$ and $|y - z| < \delta$. Lemma 2.1(c) then shows that

$$|(\Theta_\mu^* u)(x) - (\Theta_{\mu_0}^* u)(x)| < \varepsilon,$$

whenever $x \in U$, $\mu, \mu_0 \in \mathbb{B}(0, r_0)$ and $|\mu - \mu_0| < \delta$. We have, thus, proved that

$$[\mu \mapsto \Theta_\mu^* u] \in C(\mathbb{B}(0, r_0), BUC(U)), \qquad u \in BUC(U). \tag{2.24}$$

The assertion in (b) follows now from (2.20) and (2.24) for $\mathfrak{F}^l = BUC^l$, $l \in \{0, m\}$. Suppose that $s \in (0, m)$ and let $u \in buc^s(U)$. Let $\varepsilon > 0$ be given. According to (2.11) we find a function v such that

$$v \in BUC^m(U), \quad \|u - v\|_s < \varepsilon / 3M, \tag{2.25}$$

where M is the constant of equation (2.15). Equations (2.15) and (2.10) yield

$$\begin{aligned}
\|\Theta_\mu^* u - \Theta_{\mu_0}^* u\|_s &\leq \|\Theta_\mu^*(u - v)\|_s + \|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_s + \|\Theta_{\mu_0}^*(u - v)\|_s \\
&\leq 2M\|u - v\|_s + c\|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_m^{s/m} \|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_0^{1 - s/m} \\
&\leq 2M\|u - v\|_s + c(2M\|v\|_m)^{s/m} \|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_0^{1 - s/m}
\end{aligned}$$

for any $\mu, \mu_0 \in \mathbb{B}(0, r_0)$, where we use $\|\cdot\|_s := \|\cdot\|_{BUC^s(U)}$. The case $\mathfrak{F}^s = buc^s(U)$ is now a consequence of (2.24) and (2.25).

(ii) Let $u \in L_p(U)$ and let $\varepsilon > 0$ be given. There exists a function v with

$$v \in C_c(U), \qquad \|u - v\|_p \leq \varepsilon / 3M. \tag{2.26}$$

Using Lemma 2.1 and Proposition 2.2 it is easy to see that there exists a compact set K contained in U such that $\operatorname{supp}(\Theta_\mu^* v) \subset K$ for any $\mu \in \mathbb{B}(0, r_0)$. We conclude that

$$\|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_p \leq (\lambda_n(K))^{1/p} \|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_{BUC(U)} \tag{2.27}$$

where $\lambda_n(K)$ denotes the Lebesgue measure of K. It follows from (2.15) that

$$\begin{aligned}
\|\Theta_\mu^* u - \Theta_{\mu_0}^* u\|_p &\leq \|\Theta_\mu^*(u - v)\|_p + \|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_p + \|\Theta_{\mu_0}^*(u - v)\|_p \\
&\leq 2M\|u - v\|_p + \|\Theta_\mu^* v - \Theta_{\mu_0}^* v\|_p
\end{aligned}$$

and we infer from (2.24) and (2.26)-(2.27) that

$$[\mu \mapsto \Theta_\mu^* u] \in C(\mathbb{B}(0, r_0), L_p(U)), \qquad u \in L_p(U). \tag{2.28}$$

The case $\mathfrak{F}^s \in \{W_p^s(U), H_p^s(U)\}$ follows in the same way as in step (b)(i).

(c) Pick $u \in \mathfrak{F}^{s+1}$. We infer from (2.12)–(2.13) and from part (b) that

$$[\mu \mapsto \chi \Theta_\mu^* \partial_j u] \in C(\mathbb{B}(0, r_0), \mathfrak{F}^s), \qquad j \in \{1, \cdots, n\}. \tag{2.29}$$

Let $\mu \in \mathbb{B}(0, r_0)$ be fixed, and choose $\varepsilon > 0$ small enough such that $\mu + h e_j \in \mathbb{B}(0, r_0)$ for $h \in (-\varepsilon, \varepsilon)$.

(i) Let us temporarily assume that $u \in C^\infty(U) \cap \mathfrak{F}^{s+1}$. It follows from (3.7) that

$$\frac{1}{h}[\Theta^*_{\mu+he_j}u - \Theta^*_\mu u] - \chi\Theta^*_\mu \partial_j u = \int_0^1 \big(\chi\Theta^*_{\mu+\tau he_j}\partial_j u - \chi\Theta^*_\mu \partial_j u\big)\, d\tau \quad \text{in } \mathfrak{F}^s. \tag{2.30}$$

(ii) An approximation argument shows that the statement in (2.30) is also valid for $u \in \mathfrak{F}^{s+1}$. The assertion in (c) can now be obtained from (2.29) by analogous arguments as in step (i) of the proof of Proposition 3.2. □

Remarks 2.5.
(a) The assumption that U be a 'smooth' open set is not indispensable. It is only required for the interpolation and and multiplier results (2.9)–(2.13) which are used in the proof of Proposition 2.4.
(b) The proof of Proposition 2.4 shows that the assertions are valid in the case

$$\mathfrak{F}^l \in \{BUC^l(U), W^l_p(U)\}, \quad l \in \mathbb{N},$$

for any open set U with $\overline{\mathbb{B}}(x_0, 3r_0) \subset U$.
(c) The assertions of Proposition 2.4 also remain valid if

$$\mathfrak{F}^s \in \{buc^s(U), BUC^s(U), W^s_p(U)\}$$

for any open set U with $\overline{\mathbb{B}}(x_0, 3r_0) \subset U$.

3. Higher Regularity

In this section we show that the mapping $[\mu \mapsto \Theta^*_\mu u]$ enjoys more regularity than stated in Proposition 2.4, provided the function u has better regularity properties. In the following, U and X are open sets as considered in section 2. We begin with a technical Lemma.

Lemma 3.1. *Let $m \in \mathbb{N}$ and $k \in \mathbb{N}^* \cup \{\infty\}$. Suppose that $a \in C^{m+k}(X)$. Then*

$$[\mu \mapsto \chi^{|\alpha|}\Theta^*_\mu \partial^\alpha a] \in C(\mathbb{B}(0, r_0), BUC^m(U)), \qquad 0 < |\alpha| \le k.$$

Proof. Since $\operatorname{supp}(\chi) \subset \mathbb{B}(x_0, 3\varepsilon_0) \subset U \cap X$, the assertion of the Lemma is meaningful. Let $v \in C(X)$ be given, and let $\zeta \in \mathcal{D}((\mathbb{B}(x_0, 2\varepsilon_0), \mathbb{R})$. Since v is uniformly continuous on $\overline{\mathbb{B}}(x_0, 3\varepsilon_0)$ we find for every given $\varepsilon > 0$ a number $\delta > 0$ such that

$$|v(y) - v(z)| < \varepsilon, \quad y, z \in \overline{\mathbb{B}}(x_0, 3\varepsilon_0), \quad |y - z| < \delta.$$

We can now deduce from Lemma 2.1(b)–(c) that

$$|(\zeta\Theta^*_\mu v)(x) - (\zeta\Theta^*_{\mu_0} v)(x)| = |\zeta(x)|\, |v(\Theta_\mu(x)) - v(\Theta_{\mu_0}(x))| < \varepsilon,$$

whenever $x \in U$, $\mu, \mu_0 \in \mathbb{B}(0, r_0)$ and $|\mu - \mu_0| < \delta$. We have shown that

$$[\mu \mapsto \zeta\Theta^*_\mu v] \in C(\mathbb{B}(0, r_0), BUC(U)). \tag{3.1}$$

Now let $a \in C^{m+k}(X)$. Let $\eta \in \mathbb{N}^n$ be a fixed multi-index with $|\eta| \le m$. If follows from Leibniz' rule and from (2.20) that

$$\partial^\eta(\chi^{|\alpha|}\, [\Theta^*_\mu \partial^\alpha a]) = \sum_{\beta \le \eta} \sum_{|\gamma| \le |\beta|} \binom{\eta}{\beta} b_{\beta,\gamma}(\mu, \cdot)(\partial^{\eta-\beta}\chi^{|\alpha|})[\Theta^*_\mu \partial^{\alpha+\gamma} a]. \tag{3.2}$$

Note that $v := \partial^{\alpha+\gamma} a \in C(U)$ and $\zeta := \partial^{\eta-\gamma}\chi^{|\alpha|} \in \mathcal{D}((\mathbb{B}(x_0, 2\varepsilon_0), \mathbb{R})$. The claim in (c) follows from (3.1) and (3.2). $\square$

Proposition 3.2. *Let $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Suppose that*

$$a \in C^{m+k}(X) \cap BUC^m(U).$$

Then we have

$$[\mu \mapsto \Theta_\mu^* a] \in C^k(\mathbb{B}(0, r_0), BUC^m(U)) \tag{3.3}$$

and

$$\partial_\mu^\alpha [\Theta_\mu^* a] = \chi^{|\alpha|}[\Theta_\mu^* \partial^\alpha a], \qquad |\alpha| \leq k.$$

Proof. Lemma 2.4 shows that the mapping

$$g := [\mu \mapsto \Theta_\mu^* a] \in C(\mathbb{B}(0, r_0), BUC^m(U))$$

is well-defined. Moreover, Lemma 3.1 shows that

$$[\mu \mapsto \chi^{|\alpha|}\Theta_\mu^* \partial^\alpha a] \in C(\mathbb{B}(0, r_0, BUC^m(U)), \qquad 0 < |\alpha| \leq k. \tag{3.4}$$

(i) Let $\mu \in \mathbb{B}(0, r_0)$ be fixed, and choose $\varepsilon > 0$ small enough such that $\mu + he_j \in \mathbb{B}(0, r_0)$ for $h \in (-\varepsilon, \varepsilon)$. Let $x \in X$ be given. Then the mean value theorem yields

$$\frac{1}{h}[(\Theta_{\mu+he_j}^* a)(x) - (\Theta_\mu^* a)(x)] = \int_0^1 (\chi\Theta_{\mu+\tau he_j}^* \partial_j a)(x)\, d\tau. \tag{3.5}$$

Note that both sides of equation (3.5) vanish if $x \notin \operatorname{supp}(\chi)$. Consequently, formula (3.5) is also valid for any $x \in U$. It is not difficult to verify (by resorting to Riemann sums, for instance) that

$$\int_0^1 (\chi\Theta_{\mu+\tau he_j}^* \partial_j a)(x)\, d\tau = \Big(\int_0^1 \chi\Theta_{\mu+\tau he_j}^* \partial_j a\, d\tau\Big)(x), \quad x \in U, \tag{3.6}$$

where the integral $\int_0^1 (\chi\Theta_{\mu+\tau he_j}^* \partial_j a)\, d\tau$ exists in $BUC^m(U)$. We conclude that

$$\frac{1}{h}[g(\mu + he_j) - g(\mu)] - [\chi\Theta_\mu^* \partial_j a] = \int_0^1 \big(\chi\Theta_{\mu+\tau he_j}^* \partial_j a - \chi\Theta_\mu^* \partial_j a\big)\, d\tau \tag{3.7}$$

in $BUC^m(U)$. Lemma 3.1 implies that

$$\big(\chi\Theta_{\mu+\tau he_j}^* \partial_j a - \chi\Theta_\mu^* \partial_j a\big) \to 0 \quad \text{in } BUC^m(U) \text{ as } \varepsilon \to 0,$$

uniformly in $\tau \in [0, 1]$. It follows that

$$\int_0^1 \big(\chi\Theta_{\mu+\tau he_j}^* \partial_j a - \chi\Theta_\mu^* \partial_j a\big)\, d\tau \to 0 \quad \text{in } BUC^m(U) \text{ as } \varepsilon \to 0. \tag{3.8}$$

Consequently, also the left side of equation (3.7) converges to 0 in $BUC^m(U)$ as $\varepsilon \to 0$. We have, thus, proved that the partial derivative $\partial_{\mu_j} g(\mu)$ exists and is given by $\partial_{\mu_j} g(\mu) = \chi\Theta_\mu^* \partial_j a$. In addition, Lemma 3.1 shows that

$$g \in C^1(\mathbb{B}(0, r_0), BUC^m(U)).$$

(ii) We can now repeat the steps above with a replaced by $\chi\Theta_\mu^* \partial_j a$ to obtain

$$\partial_{\mu_i}\partial_{\mu_j}[\Theta_\mu^* a] = \chi^2[\Theta_\mu^* \partial_i \partial_j a].$$

An induction argument yields

$$g \in C^k(\mathbb{B}(0,r_0), BUC^m(U)) \quad \text{and} \quad \partial_\mu^\alpha g(\mu) = \chi^{|\alpha|}\Theta_\mu^* \partial^\alpha a$$

for $|\alpha| \le k$, where $k \in \mathbb{N} \cup \{\infty\}$.

(iii) Suppose now that $k = \omega$. Since a is (real) analytic, there exists an open neighborhood $U_{\mathbb{C}}$ of U in $\mathbb{C}^n$ and a (unique) mapping

$$a_{\mathbb{C}} \in C^\omega(U_{\mathbb{C}}, \mathbb{C}) \quad \text{such that} \quad U_{\mathbb{C}} \cap \mathbb{R}^n = U \quad \text{and} \quad a_{\mathbb{C}}|U = a. \tag{3.9}$$

We can assume without loss of generality that ε_0 is small enough such that $\mathbb{B}_{\mathbb{C}^n}(x_0, 3\varepsilon_0) \subset U_{\mathbb{C}}$. It follows from Lemma 2.1(a) and the definition of Θ_μ that $\Theta_\mu(U) \subset U_{\mathbb{C}}$ for any $\mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$, and consequently, the mapping

$$g_{\mathbb{C}}(\mu)(x) := (\Theta_\mu^* a_{\mathbb{C}})(x) := a_{\mathbb{C}}(\Theta_\mu(x)), \tag{3.10}$$

is well-defined for $x \in U$ and $\mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$.

(iv) Since $a_{\mathbb{C}} \in C^\omega(U_{\mathbb{C}}, \mathbb{C})$, it is clear that $\Theta_\mu^* a_{\mathbb{C}} \in C^m(U, \mathbb{C})$. We claim that

$$[\mu \mapsto \Theta_\mu^* a_{\mathbb{C}}] \in C(\mathbb{B}_{\mathbb{C}^n}(0, r_0), BUC^m(U, \mathbb{C})). \tag{3.11}$$

Let $\zeta \in \mathcal{D}(\mathbb{B}(x_0, 3\varepsilon_0))$ be a smooth cut-off function with $\zeta \equiv 1$ on $\operatorname{supp}(\chi)$. As in the proof of Lemma 3.1 one shows that

$$[\mu \mapsto \zeta\Theta_\mu^* a_{\mathbb{C}}] \in C(\mathbb{B}_{\mathbb{C}^n}(0, r_0), BUC^m(U, \mathbb{C})). \tag{3.12}$$

In more detail, we have

$$\partial_x^\beta [\Theta_\mu^* a_{\mathbb{C}}](x) = \sum_{|\gamma| \le |\beta|} b_{\beta,\gamma}(\mu, x)\, [\Theta_\mu^* \partial_x^\gamma a_{\mathbb{C}}](x), \qquad x \in U,\ |\beta| \le m, \tag{3.13}$$

with appropriate functions $b_{\mu,\gamma} \in BUC(\mathbb{B}_{\mathbb{C}^n}(0, r_0) \times U, \mathbb{C})$. Let $\gamma \in \mathbb{N}^n$ be a fixed multi-index with $|\gamma| \le m$. We know that the real partial derivatives $\partial_x^\gamma a_{\mathbb{C}}$ are continuous on $U_{\mathbb{C}}$, and therefore are uniformly continuous on the compact set $\overline{\mathbb{B}}_{\mathbb{C}^n}(x_0, 3\varepsilon_0)$. That is to say that for any $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$|\partial_x^\gamma a_{\mathbb{C}}(z_1) - \partial_x^\gamma a_{\mathbb{C}}(z_2)| < \varepsilon, \quad z_1, z_2 \in \overline{\mathbb{B}}_{\mathbb{C}^n}(x_0, 3\varepsilon_0), \quad |z_1 - z_2| < \delta.$$

Lemma 2.1(b)-(c) then implies that

$$|\Theta_\mu^* \partial_x^\gamma a_{\mathbb{C}}(x) - \Theta_\mu^* \partial_x^\gamma a_{\mathbb{C}}(y)| < \varepsilon, \quad x, y \in \overline{\mathbb{B}}(x_0, 3\varepsilon_0),\ |x - y| < (2/3)\delta, \tag{3.14}$$

and $\mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$, as well as

$$|\Theta_\mu^* \partial_x^\gamma a_{\mathbb{C}}(x) - \Theta_{\mu_0}^* \partial_x^\gamma a_{\mathbb{C}}(x)| < \varepsilon, \quad \mu, \mu_0 \in \mathbb{B}_{\mathbb{C}^n}(x_0, r_0),\ |\mu - \mu_0| < \delta, \tag{3.15}$$

uniformly in $x \in \overline{\mathbb{B}}(x_0, 3\varepsilon_0)$. Equation (3.12) follows now from Leibniz' rule and from (3.13)–(3.15). Recall that $\Theta_\mu(x) = x$ for $x \notin \operatorname{supp}(\chi)$. If follows from the fact that $\zeta \equiv 1$ on $\operatorname{supp}(\chi)$ and from (3.9) that

$$(1 - \zeta)\Theta_\mu^* a_{\mathbb{C}} = (1 - \zeta)a, \qquad \mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0). \tag{3.16}$$

Since $a \in BUC^m(U)$ by assumption, we evidently have

$$[\mu \mapsto (1 - \zeta)\Theta_\mu^* a_{\mathbb{C}}] \in C(\mathbb{B}_{\mathbb{C}^n}(0, r_0), BUC^m(U)) \tag{3.17}$$

and the assertion in (3.11) follows from (3.12) and (3.17).

(v) Let $\partial_{z_j} a_{\mathbb{C}}$ denote a complex partial derivative of $a_{\mathbb{C}}$, where we use the notation $z = (z_1, \cdots, z_j, \cdots, z_n) \in \mathbb{C}^n$. Since $\partial_{z_j} a_{\mathbb{C}} \in C^m(U_{\mathbb{C}}, \mathbb{C})$ we can conclude as in (i) that

$$[\mu \mapsto \chi \Theta_\mu^* \partial_{z_j} a_{\mathbb{C}}] \in C(\mathbb{B}_{\mathbb{C}^n}(0, r_0), BUC^m(U, \mathbb{C})). \tag{3.18}$$

(vi) Next we show that

$$g_{\mathbb{C}} \in C^1(\mathbb{B}_{\mathbb{C}^n}(0, r_0), BUC^m(U, \mathbb{C})) \quad \text{and} \quad \partial_{\mu_j} g_{\mathbb{C}} = \chi \Theta_\mu^* \partial_{z_j} a_{\mathbb{C}}. \tag{3.19}$$

In fact, the assertion follows by the same arguments as in step (i) of the proof. We now have $\mu \in \mathbb{B}_{\mathbb{C}^n}(0, r_0)$, $h \in \mathbb{B}_{\mathbb{C}}(0, \varepsilon)$, and we replace g and a by $g_{\mathbb{C}}$ and $a_{\mathbb{C}}$, respectively.

(vii) We infer from step (iii) – and the well-known fact that a holomorphic function is complex analytic – that $g_{\mathbb{C}} \in C^\omega(\mathbb{B}_{\mathbb{C}^n}(0, r_0), BUC^m(U, \mathbb{C}))$. Consequently,

$$g = g_{\mathbb{C}} | \mathbb{B}(0, r_0) \in C^\omega(\mathbb{B}(0, r_0), BUC^m(U))$$

and the proof is now complete. □

In order to be able to treat differential operators in various function spaces, we present the following result, which generalizes Proposition 3.2.

Theorem 3.3. *Let $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Suppose that*

$$a \in C^{m+k}(X) \cap \mathfrak{F}^s$$

where $\mathfrak{F} \in \{buc(U), W_p(U), H_p(U)\,;\, 1 < p < \infty\}$, $s \in [0, m]$. Then we have

$$[\mu \mapsto \Theta_\mu^* a] \in C^k(\mathbb{B}(0, r_0), \mathfrak{F}^s).$$

Proof. (i) Let $k \in \mathbb{N} \cup \{\infty\}$. Lemma 2.4 asserts that the mapping

$$g := [\mu \mapsto \Theta_\mu^* a] \in C(\mathbb{B}(0, r_0), \mathfrak{F}^s)$$

is well-defined. Next, observe that equation (3.8) remains valid in the present context, since the proof only relies on the property that $a \in C^{m+k}(X)$. We conclude from

$$\operatorname{supp}\Big(\int_0^1 \big(\chi \Theta_{\mu+\tau h e_j}^* \partial_j a - \chi \Theta_\mu^* \partial_j a\big)\, d\tau\Big) \subset \operatorname{supp}(\chi), \qquad \mu \in \mathbb{B}(0, r_0),$$

and from (3.8) that

$$\int_0^1 \big(\chi \Theta_{\mu+\tau h e_j}^* \partial_j a - \chi \Theta_\mu^* \partial_j a\big)\, d\tau \to 0 \quad \text{in } \mathfrak{F}^s \text{ as } \varepsilon \to 0 \tag{3.20}$$

and the assertions follow from (3.7) and (3.20).

(ii) Suppose $k = \omega$. An inspection of step (iii) in the proof of Proposition 3.2 shows that equations (3.12) and (3.17) are also satisfied for the spaces $\mathfrak{F}^s(U, \mathbb{C})$. The proof proceeds now along the lines of steps (v)–(vii) of the proof of Proposition 3.2. □

The following result shows that our method can be used to characterize smoothness.

Theorem 3.4. *Let $X \subset \mathbb{R}^n$ be an open set and let $k \in \mathbb{N} \cup \{\infty, \omega\}$. Suppose that $u \in BUC(X)$. Then $u \in C^k(X)$ iff for any $x_0 \in X$ there exists $r_0 := r_0(x_0) > 0$ such that*

$$[\mu \mapsto \Theta_\mu^* u] \in C^k(\mathbb{B}(0, r_0), BUC(X)).$$

Proof. (i) Assume that $u \in C^k(X)$. Let x_0 be fixed and choose $\varepsilon_0 > 0$ such that $\mathbb{B}(x_0, 3\varepsilon_0) \subset X$. The assertion follows now from Remark 2.5 and Theorem 3.3.

(ii) Let $x_0 \in X$ be fixed and suppose that

$$[\mu \mapsto \Theta_\mu^* u] \in C^k(\mathbb{B}(0, r_0), BUC(X)), \tag{3.21}$$

for some number $r_0 > 0$, where Θ_μ is defined in (2.1). Let $\mathcal{E} : BUC(X) \to \mathbb{R}$, $\mathcal{E}v := v(x_0)$, and observe that $\mathcal{E} \in \mathcal{L}(BUC(X), \mathbb{R})$. Hence $\mathcal{E} \in C^\omega(BUC(X), \mathbb{R})$ and we conclude from (3.21) that

$$[\mu \mapsto (\Theta_\mu^* u)(x_0) = u(x_0 + \mu)] \in C^k(\mathbb{B}(0, r_0), \mathbb{R}). \tag{3.22}$$

Equation (3.22) means that $u \in C^k(\mathbb{B}(x_0, r_0), \mathbb{R})$. Since this is true for any point $x_0 \in X$ we have proved that $u \in C^k(X)$, and the proof is now complete. □

4. Differential operators

For later use we study how differential operators transform under a change of coordinates induced by Θ_μ.

We will first consider differential operators with constant coefficients and we set

$$A^\alpha(\mu) := \Theta_\mu^*\big(\partial^\alpha\big(\Theta_*^\mu\ \cdot\big)\big), \quad \alpha \in \mathbb{N}^n, \quad \mu \in \mathbb{B}(0, r_0). \tag{4.1}$$

Proposition 4.1. *Suppose that $\mathfrak{F} \in \{buc(U), BUC(U), W_p(U), H_p(U)\}$. Let $l \in \mathbb{N}$. Then*

$$[\mu \mapsto A^\alpha(\mu)] \in C^\omega(\mathbb{B}(0, r_0), \mathcal{L}(\mathfrak{F}^{s+l}, \mathfrak{F}^s)), \qquad |\alpha| \le l.$$

Proof. (i) Let $A_j(\mu) := \Theta_\mu^*\big(\partial_j\big(\Theta_*^\mu\ \cdot\big)\big)$. An easy computation shows that

$$A_j(\mu)u = ((D\Theta_\mu)^{-1} e_j | \nabla u), \quad u \in C^1(U), \quad \mu \in \mathbb{B}(0, r_0), \tag{4.2}$$

where e_j is the j-th canonical basis vector in $\mathbb{R}^n$, and where $(\cdot\,|\,\cdot)$ denotes the inner product in $\mathbb{R}^n$. It is not difficult to see that formula (4.2) holds true for any $u \in \mathfrak{F}^t$ with $t \ge 1$. In fact, this is evident for $\mathfrak{F} \in \{buc, BUC\}$, and follows by approximation in the other cases. It follows from (2.2)–(2.3) and from Cramer's rule (for instance) that

$$[\mu \mapsto (D\Theta_\mu)^{-1}] \in C^\omega(\mathbb{B}(0, r_0), BUC^m(U, \mathcal{L}(\mathbb{R}^n))) \tag{4.3}$$

where we take $m = [s] + l$. We conclude from (2.12) and (2.13) that

$$[B \mapsto (Be_j | \nabla\, \cdot)] \in C^\omega(BUC^m(U, \mathcal{L}(\mathbb{R}^n)), \mathcal{L}(\mathfrak{F}^t, \mathfrak{F}^{t-1})), \quad 1 \le t \le s + l, \tag{4.4}$$

since all the operations involved are linear [or bilinear] and continuous. It is now a straightforward consequence of (4.2)–(4.4) that

$$[\mu \mapsto A_j(\mu)] \in C^\omega(\mathbb{B}(0, r_0), \mathcal{L}(\mathfrak{F}^t, \mathfrak{F}^{t-1})), \quad 1 \le t \le s + l. \tag{4.5}$$

(ii) Suppose $\alpha = e_j + e_k$. Then we have

$$A^\alpha(\mu)u = \Theta_\mu^*\big(\partial_k\partial_j\big(\Theta_*^\mu u\big)\big) = \Theta_\mu^*\big(\partial_k\Theta_*^\mu\Theta_\mu^*\partial_j\big(\Theta_*^\mu u\big)\big) = \big(A_k(\mu)A_j(\mu)\big)u. \tag{4.6}$$

The mapping

$$\mathcal{L}(\mathfrak{F}^t, \mathfrak{F}^{t-1}) \times \mathcal{L}(\mathfrak{F}^{t-1}, \mathfrak{F}^{t-2}) \to \mathcal{L}(\mathfrak{F}^t, \mathfrak{F}^{t-2}), \qquad (A, B) \mapsto BA \tag{4.7}$$

is bilinear and continuous, and hence it is analytic. We infer from (4.5)–(4.7) that

$$[\mu \mapsto A^\alpha(\mu)] \in C^\omega(\mathbb{B}(0, r_0), \mathcal{L}(\mathfrak{F}^t, \mathfrak{F}^{t-2})), \quad 2 \le t \le s + l.$$

(iii) Let $\alpha = (\alpha_1, \cdots, \alpha_n)$. Then

$$A^\alpha(\mu) = (A_1(\mu))^{\alpha_1} \cdots (A_n(\mu))^{\alpha_n}$$

and the claim follows from (4.5) and (4.7) by induction. □

We will now consider differential operators with variable coefficients. That is, we consider the differential operator

$$A := \sum_{|\alpha| \le l} a_\alpha\, \partial^\alpha, \qquad a_\alpha \in C(U, \mathbb{R}), \qquad |\alpha| \le l, \tag{4.8}$$

where l is a positive integer. The parameter-dependent family of diffeomorphisms $\{\Theta_\mu\,;\, \mu \in \mathbb{B}(0, r_0)\}$ generate a parameter-dependent family $\{A_\mu\,;\, \mu \in \mathbb{B}(0, r_0)\}$ of differential operators (the transformed differential operators), given by

$$A_\mu := \sum_{|\alpha| \le l} (\Theta_\mu^* a_\alpha)\Theta_\mu^*(\partial^\alpha(\Theta_*^\mu\, \cdot)), \qquad \mu \in \mathbb{B}(0, r_0). \tag{4.9}$$

We shall show that regularity properties of the coefficients a_α translate into regularity properties for the map $[\mu \mapsto A_\mu]$.

Theorem 4.2. *Let $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$.*

(a) *Suppose that $a_\alpha \in C^{m+k}(X) \cap BUC^m(U)$ for every $|\alpha| \le l$. Then*

$$[\mu \mapsto A_\mu] \in C^k(\mathbb{B}(0, r_0), \mathcal{L}(\mathfrak{F}^{s+l}(U), \mathfrak{F}^s(U))),$$

where $\mathfrak{F} \in \{buc, BUC, W_p, H_p\}$ and $s \in [0, m]$.

(b) *Suppose that $a_\alpha \in C^{m+k}(X) \cap buc^s(U)$ for every $|\alpha| \le l$, where $s \in [0, m]$ is fixed. Then*

$$[\mu \mapsto A_\mu] \in C^k(\mathbb{B}(0, r_0), \mathcal{L}(\mathfrak{F}^{s+l}(U), \mathfrak{F}^s(U))),$$

where $\mathfrak{F}^s(U) = buc^s(U)$.

Proof. (a) It follows from (2.13) that the mapping

$$\begin{aligned} &f : BUC^m(U) \times \mathcal{L}(\mathfrak{F}^{s+l}(U), \mathfrak{F}^s(U)) \to \mathcal{L}(\mathfrak{F}^{s+l}(U), \mathfrak{F}^s(U)), \\ &f(a, T)(u) := a(Tu) \end{aligned} \tag{4.10}$$

is bilinear and continuous, and hence analytic. We can now conclude from (3.3) and Proposition 4.1 that

$$[\mu \mapsto \sum_{|\alpha|\le l} f(\Theta_\mu^* a_\alpha, A^\alpha(\mu))] \in C^k(\mathbb{B}(0,r_0), \mathcal{L}(\mathfrak{F}^{s+l}(U), \mathfrak{F}^s(U))), \tag{4.11}$$

and this is exactly the assertion in (a).

The proof of (b) follows from (2.13) and Theorem 3.3 by analogous arguments. □

5. Time dependence

We will now consider the situation where time is an additional variable.

In the following we use the notation $I := [0,T]$, where T is a fixed positive number. Let J be an open interval in $(0,T)$. Let $t_0 \in J$ be fixed and choose ε_0 such that $[t_0 - 3\varepsilon_0, t_0 + 3\varepsilon_0] \subset J$. Moreover, let $\zeta \in \mathcal{D}(t_0 - 2\varepsilon_0, t_0 + 2\varepsilon_0)$ be a smooth cut-off function with $\zeta \equiv 1$ on $[t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ and with $0 \le \zeta \le 1$. Of course, we can – and we will – assume that the number ε_0 also satisfies the assumptions stated at the beginning of section 2.

It turns out to be convenient to introduce the mapping

$$\theta_\lambda(t) := t + \zeta(t)\lambda, \quad t \in I, \quad \lambda \in \mathbb{R}. \tag{5.1}$$

Proposition 2.2 shows that there is a positive number r_0 such that

$$\theta_\lambda \in \mathrm{Diff}^\infty(J), \qquad \lambda \in (-r_0, r_0). \tag{5.2}$$

In order to obtain regularity results in time and space for parabolic equations, we define the parameter-dependent mapping

$$\Phi_{\lambda,\mu}(t,x) := (t + \zeta(t)\lambda, x + \zeta(t)\chi(x)\mu), \quad (t,x) \in J \times U, \quad (\lambda,\mu) \in \mathbb{R}^{n+1}. \tag{5.3}$$

A straightforward modification of the proof of Proposition 2.2 shows that there exists a number $r_0 > 0$ such that

$$\Phi_{\lambda,\mu} \in \mathrm{Diff}^\infty(J \times U), \qquad (\lambda,\mu) \in \mathbb{B}^{n+1}(0,r_0). \tag{5.4}$$

We can assume that all the results of sections 2 and 3 remain valid for the same number r_0. Given a function $u : I \times U \to \mathbb{R}$ we set

$$u_{\lambda,\mu} := \Phi_{\lambda,\mu}^* u, \qquad (\lambda,\mu) \in \mathbb{B}^{n+1}(0,r_0). \tag{5.5}$$

The parameter-dependent function $u_{\lambda,\mu}$ can also be written as

$$u_{\lambda,\mu}(t) = T_\mu(t)\theta_\lambda^* u(t,\cdot) \quad \text{where} \quad T_\mu(t) := \Theta_{\zeta(t)\mu}^*, \quad t \in I. \tag{5.6}$$

It is important to note that

$$u_{\lambda,\mu}(0,\cdot) = u(0,\cdot) \quad \text{for any function } u \text{ and any } (\lambda,\mu). \tag{5.7}$$

We will first prove the following useful extension result.

Lemma 5.1. *Let E be a Banach space. Suppose that*

$$[\mu \mapsto f(\mu)] \in C^k(\mathbb{B}(0,r_0),E), \qquad k \in \mathbb{N}^* \cup \{\infty,\omega\}.$$

Let $F(\mu)(t) := f(\zeta(t)\mu)$ for $\mu \in \mathbb{B}(0,r_0)$ and $t \in I$. Then we have

$$[\mu \mapsto F(\mu)] \in C^k(\mathbb{B}(0,r_0),C(I,E)).$$

Proof. To shorten the notation we set $W := \mathbb{B}(0,r_0)$. Since $0 \le \zeta \le 1$ we see that $F(\mu) \in C(I,E)$ for each $\mu \in W$. We will focus on the case $k=\omega$. It will be clear from the proof how to proceed for $k \in \mathbb{N}^* \cup \{\infty\}$.

Let us assume that $f \in C^\omega(W,E)$. Then there exists an open neighborhood $W_{\mathbb{C}}$ of W in $\mathbb{C}^n$ and a unique mapping

$$f_{\mathbb{C}} \in C^\omega(W_{\mathbb{C}},E_{\mathbb{C}}) \quad \text{such that} \quad W_{\mathbb{C}} \cap \mathbb{R}^n = W \quad \text{and} \quad f_{\mathbb{C}}|W = f,$$

where $E_{\mathbb{C}}$ is the complexification of E. We can assume without loss of generality that $W_{\mathbb{C}} = \mathbb{B}_{\mathbb{C}^n}(0,r_0)$. This implies that $\zeta(t)\mu \in W_{\mathbb{C}}$ whenever $\mu \in W_{\mathbb{C}}$ and $t \in I$. It is then clear that

$$F_{\mathbb{C}}(\mu) := f_{\mathbb{C}}(\zeta(\cdot)\mu) \in C(I,E_{\mathbb{C}}), \qquad \mu \in W_{\mathbb{C}}.$$

Let $\mu \in W_{\mathbb{C}}$ be fixed and choose $\varepsilon > 0$ such that $\mu + he_j \in W_{\mathbb{C}}$ for all $h \in \mathbb{B}_{\mathbb{C}}(0,\varepsilon)$. It follows from the mean value theorem that

$$\frac{1}{h}[F_{\mathbb{C}}(\mu+he_j) - F_{\mathbb{C}}(\mu)](t) = \int_0^1 \zeta(t)\partial_{z_j} f_{\mathbb{C}}(\zeta(t)(\mu+\tau he_j))\,d\tau.$$

It is easy to see that the quotient on the left side converges to $\zeta(t)\partial_{z_j} f_{\mathbb{C}}(\zeta(t)\mu)$ uniformly in $t \in I$ as $h \to 0$, and we conclude that the partial derivatives $\partial_{\mu_j} F_{\mathbb{C}}$ exist in $C(I,E_{\mathbb{C}})$ and are given by

$$\partial_{\mu_j} F_{\mathbb{C}} = \zeta(\cdot)\partial_{z_j} f_{\mathbb{C}}(\zeta(\cdot)\mu) \quad \mu \in W_{\mathbb{C}}, \quad j \in \{1,\dots,n\}.$$

A moment of reflection shows that

$$[\mu \mapsto \zeta(\cdot)\partial_{z_j} f_{\mathbb{C}}(\zeta(\cdot)\mu)] \in C(W,C(I,E_{\mathbb{C}})).$$

Therefore, the mapping $F_{\mathbb{C}}$ is (continuously) complex differentiable and we obtain

$$[\mu \mapsto F_{\mathbb{C}}(\mu)] \in C^\omega(W_{\mathbb{C}},C(I,E_{\mathbb{C}})).$$

We conclude that $F = F_{\mathbb{C}}|W \in C^\omega(W,C(I,E))$, and this completes the proof. $\square$

Proposition 5.2.

(a) *Let $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Suppose that*

$$a \in C^{m+k}(X) \cap \mathfrak{F}^s \tag{5.8}$$

where $\mathfrak{F} \in \{buc(U), W_p(U), H_p(U)\,;\, 1 < p < \infty\}$, $s \in [0, m]$. Then we have

$$[\mu \mapsto T_\mu a] \in C^k(\mathbb{B}(0, r_0), C(I, \mathfrak{F}^s)).$$

Moreover, $\partial_\mu^\alpha [T_\mu a] = (\zeta\chi)^{|\alpha|}[T_\mu \partial^\alpha a]$ for every $|\alpha| \leq k$.

(b) *Suppose $\mathfrak{F} \in \{buc(U), BUC(U), W_p(U), H_p(U)\}$. Then*

$$[\mu \mapsto T_\mu \partial^\alpha T_\mu^{-1}] \in C^\omega\big(\mathbb{B}(0, r_0), C(I, \mathcal{L}(\mathfrak{F}^{s+l}, \mathfrak{F}^s))\big), \quad |\alpha| \leq l.$$

Proof. (a) Let $E = \mathfrak{F}^s$ and define $f(\mu) := \Theta_\mu^* a$. The assertions follow from Theorem 3.3 and Lemma 5.1.

(b) Let $E := \mathcal{L}(\mathfrak{F}^{s+l}, \mathfrak{F}^s)$ and let $f(\mu) := A^\alpha(\mu)$, where $A^\alpha(\mu)$ is defined in (4.1). We can now apply Proposition 4.1 and Lemma 5.1. $\square$

Proposition 5.3. *Let $l \in \mathbb{N}^*$ be fixed and let $\mathfrak{F} \in \{buc(U), W_p(U), H_p(U)\}$.*

(a) *Suppose $u \in C^1(I, \mathfrak{F}^s) \cap C(I, \mathfrak{F}^{s+l})$. Then $u_{\lambda,\mu} \in C^1(I, \mathfrak{F}^s) \cap C(I, \mathfrak{F}^{s+l})$.*

(b) *Suppose $u \in W_p^1(I, \mathfrak{F}^s) \cap L_p(I, \mathfrak{F}^{s+l})$. Then $u_{\lambda,\mu} \in W_p^1(I, \mathfrak{F}^s) \cap L_p(I, \mathfrak{F}^{s+l})$.*

In both cases, the time derivative is given by

$$\partial_t u_{\lambda,\mu} = (1 + \zeta'\lambda) T_\mu \theta_\lambda^* \partial_t u + B_\mu u_{\lambda,\mu}\,, \tag{5.9}$$

where

$$[\mu \mapsto B_\mu] \in C^\omega(\mathbb{B}(0, r_0), C(I, \mathcal{L}(\mathfrak{F}^{s+l}, \mathfrak{F}^s))). \tag{5.10}$$

Proof. (i) We first observe that the parameter-dependent mapping θ_λ has the same properties as the mapping Θ_μ of section 2, where the set U is now replaced by the interval I. The fact that I is closed does not create any additional difficulties. It is clear that the proof of Proposition 2.4(a) also works for the vector-valued spaces $BUC^m(I, E)$ and $W_p^m(I, E)$, where E is some Banach space. Note that we have $C^m(I, E) = BUC^m(I, E)$ due to the fact that I is compact. We conclude that

$$\theta_\lambda^* u \in \mathbb{E}_1(I) \quad \text{and} \quad \partial_t \theta_\lambda^* u = (1 + \zeta'\lambda)\theta_\lambda^* \partial_t u \quad \text{for any} \quad u \in \mathbb{E}_1(I), \tag{5.11}$$

where $\mathbb{E}_1(I) \in \{C^1(I, \mathfrak{F}^s) \cap C(I, \mathfrak{F}^{s+l}), W_p^1(I, \mathfrak{F}^s) \cap L_p(I, \mathfrak{F}^{s+l})\}$.

(ii) Let $v \in C^1(I, \mathfrak{F}^s) \cap C(I, \mathfrak{F}^{s+l})$.

We obtain from (2.15)–(2.18) that $T_\mu v \in C^1(I, \mathfrak{F}^s) \cap C(I, \mathfrak{F}^{s+l})$, and also that

$$\partial_t T_\mu v = T_\mu \partial_t v + \sum_j \zeta' \chi\, \mu_j\, T_\mu \partial_j v. \tag{5.12}$$

(iii) Let $v \in L_p(I, \mathfrak{F}^{s+l})$. It follows from (2.16) that $T_\mu : I \to \mathcal{L}(\mathfrak{F}^{s+l})$ is strongly continuous. A well-known property then asserts that $T_\mu v : I \to \mathfrak{F}^{s+l}$ is measurable and (2.15) implies that $T_\mu v \in L_p(I, \mathfrak{F}^{s+l})$.

(iv) Suppose that $v \in W_p^1(I,\mathfrak{F}^s) \cap L_p(I,\mathfrak{F}^{s+l})$. Based on the property that v is absolutely continuous and has a derivative in $L_p(I,\mathfrak{F}^s)$ almost everywhere, we obtain by similar arguments as in (ii) that

$$T_\mu v \in W_p^1(I,\mathfrak{F}^s), \qquad \partial_t T_\mu v = T_\mu \partial_t v + \sum_j \zeta' \chi \mu_j T_\mu \partial_j v. \tag{5.13}$$

(v) Let

$$B_\mu v := \sum_{j=1}^n \zeta' \, \chi \, \mu_j \, [T_\mu \partial_j T_\mu^{-1}] v, \quad \mu \in \mathbb{B}(0,r_0), \quad v \in \mathbb{E}_1(I). \tag{5.14}$$

Proposition 5.3 follows now from (5.11)–(5.14) and Proposition 5.2. □

Proposition 5.4. *Let $k \in \mathbb{N}^* \cup \{\infty,\omega\}$.*
Suppose that $a \in C^{m+k}(J \times X) \cap BUC^m(I \times U)$. Then

$$[(\lambda,\mu) \mapsto a_{\lambda,\mu}] \in C^k(\mathbb{B}^{n+1}(0,r_0), BUC^m(I \times U)). \tag{5.15}$$

Proof. The proof follows by the same arguments as in the proof of Proposition 3.2. □

6. Examples

In this section we collect four simple examples which show the flexibility and power of our approach.

In our first example we show how the results in sections 2 and 3 can be used to prove regularity properties for elliptic equations.
Let us consider the second order elliptic equation

$$a \, \Delta u = f \quad \text{in} \quad X, \tag{6.1}$$

where X is an open set in $\mathbb{R}^n$. We assume that the differential operator $A := a\,\Delta$ is uniformly strongly elliptic, that is, we assume that there exists a positive number δ such that $a(x) \geq \delta$ for every $x \in X$.

Example 6.1. Suppose that $(a,f) \in C^\omega(X)$ and that $u \in C^2(X)$ is a solution of (6.1). Then $u \in C^\omega(X)$.

Proof. Pick $x_0 \in X$. Choose $\varepsilon_0 > 0$ with $\overline{\mathbb{B}}(x_0, 3\varepsilon_0) \subset X$. Let $\{\Theta_\mu \,;\, \mu \in \mathbb{B}(0,r_0)\}$ be the family of diffeomorphisms introduced in section 2. Let $U := \mathbb{B}(x_0, 3\varepsilon_0)$ and observe that

$$u \in W_p^2(U), \quad a \in C^\omega(U) \cap BUC(U), \quad f \in C^\omega(U) \cap L_p(U). \tag{6.2}$$

Next set $g := \gamma u$, where $\gamma \in \mathcal{L}(W_p^2(U), W_p^{2-1/p}(\Gamma))$ denotes the trace operator for $\Gamma := \partial U$, see [13, 14]. Clearly, u solves the elliptic boundary value problem

$$Au = f \quad \text{in} \quad U, \qquad \gamma u = g \quad \text{on} \quad \Gamma.$$

For later use we note that

$$(A, \gamma) \in \mathrm{Isom}(W_p^2(U), L_p(U) \times W_p^{2-1/p}(\Gamma)) \tag{6.3}$$

see [14, Theorem 4.3.3.(ii)], for instance. We introduce the transformed quantities

$$u_\mu := \Theta_\mu^* u, \quad A_\mu := \Theta_\mu^*(A\,\Theta_*^\mu\,\cdot), \quad f_\mu := \Theta_\mu^* f, \qquad \mu \in \mathbb{B}(0, r_0). \tag{6.4}$$

It follows from (6.2), from Theorem 3.3 and from Theorem 4.2(a) that

$$[\mu \mapsto (A_\mu, f_\mu)] \in C^\omega(\mathbb{B}(0, r_0), \mathcal{L}(W_p^2(U), L_p(U)) \times L_p(U)). \tag{6.5}$$

Moreover, we know from Proposition 2.4 that $u_\mu \in W_p^2(U)$ for $\mu \in \mathbb{B}(0, r_0)$. We note that equation (6.4) yields $A_\mu u_\mu = \Theta_\mu^*(Au) = \Theta_\mu^* f$, and that $\gamma u_\mu = \gamma u = g$. We conclude that u_μ solves the transformed elliptic problem

$$(A_\mu v, \gamma v) = (f_\mu, g), \qquad \mu \in \mathbb{B}(0, r_0). \tag{6.6}$$

We finally introduce the function

$$\begin{aligned} &\Phi : W_p^2(U) \times \mathbb{B}(0, r_0) \to L_p(U) \times W_p^{2-1/p}(\Gamma), \\ &\Phi(v, \mu) := (A_\mu v - f_\mu, \gamma v - g). \end{aligned}$$

It is a consequence of (6.5) that

$$\Phi \in C^\omega(W_p^2(U) \times \mathbb{B}(0, r_0), L_p(U) \times W_p^{2-1/p}(\Gamma)) \tag{6.7}$$

and it follows from equations (6.3) and (6.6) that

$$\begin{aligned} \Phi(u_\mu, \mu) &= (0, 0), \quad \mu \in \mathbb{B}(0, r_0), \\ D_1\Phi(u_0, 0) &= (A, \gamma) \in \mathrm{Isom}(W_p^2(U), L_p(U) \times W_p^{2-1/p}(\Gamma) \end{aligned} \tag{6.8}$$

where, of course, $u_0 = u$. We conclude from (6.7)–(6.8) and the implicit function theorem that there exists a number $r = r(x_0) \in (0, r_0)$ such that

$$[\mu \mapsto u_\mu] \in C^\omega(\mathbb{B}(0, r), W_p^2(U)).$$

Let us assume that p is chosen large enough such that $W_p^2(U) \hookrightarrow BUC(U)$. Since x_0 can be taken arbitrary we obtain that $u \in C^\omega(X)$ from Theorem 3.4. □

In our second example we consider the linear parabolic equation

$$\partial_t u - a\,\Delta u = 0 \quad \text{in} \quad \mathbb{R}^n, \qquad u(0) = u_0. \tag{6.9}$$

We assume that X is an open subset of $\mathbb{R}^n$, that

$$a \in C^\omega(X) \cap BUC(\mathbb{R}^n), \tag{6.10}$$

and that the differential operator $A := a\,\Delta$ is uniformly strongly elliptic.

Let $T > 0$ be fixed and set $I := [0, T]$ and $J := (0, T)$. Finally, let $p \in (1+n/2, \infty)$.

Example 6.2. *Let $u_0 \in W_p^{2-2/p}(\mathbb{R}^n)$ be given. Then equation (6.9) has a unique solution $u \in W_p^1(I, L_p(\mathbb{R}^n)) \cap L_p(I, W_p^2(\mathbb{R}^n))$ with $u \in C^\omega(J \times X)$.*

Proof. The proof is based on the maximal regularity result

$$(\partial_t + (\nu - A), \gamma_0) \in \operatorname{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times W_p^{2-2/p}(\mathbb{R}^n)) \tag{6.11}$$

where $\nu > 0$ is an appropriate constant, where $\gamma_0 v := v(0)$, and where

$$\mathbb{E}_1(I) := W_p^1(I, L_p(\mathbb{R}^n)) \cap L_p(I, W_p^2(\mathbb{R}^n)), \quad \mathbb{E}_0(I) := L_p(I, L_p(\mathbb{R}^n)),$$

see [6, Corollary 6.2] and [1, Theorem III.4.10.7].

Let $v \in \mathbb{E}_1(I)$ be the (unique) solution of

$$(\partial_t v + (\nu - A)v, \gamma_0 v) = (0, u_0). \tag{6.12}$$

Pick $(t_0, x_0) \in J \times X$ and let

$$v_{\lambda,\mu}(t,x) := v(t + \zeta(t)\lambda, x + \zeta(t)\chi(x)\mu), \qquad (t,x) \in I \times \mathbb{R}^n.$$

It follows from Proposition 5.3 that $v_{\lambda,\mu} \in \mathbb{E}_1(I)$. We conclude from (5.9) and (6.12) that

$$\begin{aligned} \partial_t v_{\lambda,\mu} &= -(1 + \zeta'\lambda)\,[\nu - (T_\mu a)T_\mu \Delta T_\mu^{-1}]T_\mu \theta_\lambda^* v + B_\mu v_{\lambda,\mu} \\ &= -(1 + \zeta'\lambda)\,[\nu - (T_\mu a)T_\mu \Delta T_\mu^{-1}]v_{\lambda,\mu} + B_\mu v_{\lambda,\mu}. \end{aligned} \tag{6.13}$$

Consequently, $v_{\lambda,\mu}$ is a solution of the parameter-dependent equation

$$(\partial_t w + A_{\lambda,\mu} w, \gamma_0 w) = (0, u_0),$$

where

$$A_{\lambda,\mu} w := (1 + \zeta'\lambda)\,[\nu - (T_\mu a)T_\mu \Delta T_\mu^{-1}]w - B_\mu w.$$

We infer from (5.10) and Proposition 5.2 that

$$[(\lambda,\mu) \mapsto A_{\lambda,\mu}] \in C^\omega(\mathbb{B}^{n+1}(0, r_0), \mathcal{L}(\mathbb{E}_1(I), \mathbb{E}_0(I))).$$

The implicit function theorem shows that

$$[(\lambda,\mu) \mapsto v_{\lambda,\mu}] \in C^\omega(\mathbb{B}^{n+1}(0, r_0), \mathbb{E}_1(I)).$$

Since $\mathbb{E}_1(I) \hookrightarrow BUC(I \times \mathbb{R}^n)$ we conclude that

$$[(\lambda,\mu) \mapsto v_{\lambda,\mu}(t_0, x_0) = v(t_0 + \lambda, x_0 + \mu)] \in C^\omega(\mathbb{B}^{n+1}(0, r_0), \mathbb{R}),$$

showing that v is in fact analytic on a neighborhood of (t_0, x_0). Since (t_0, x_0) can be chosen anywhere in $J \times X$ we have shown that $v \in C^\omega(J \times X)$. It remains to observe that $u(t) := e^{\nu t} v(t)$ solves the parabolic equation (6.9) and that u has the same regularity properties as v. $\square$

Remarks 6.3. (a) By relying on maximal regularity results in little Hölder spaces of negative order [2], the regularity assumptions on the initial value u_0 can be considerably relaxed (at the expense of imposing slightly more regularity on the coefficient a).

(b) It is clear that we can also treat much more general parabolic systems which satisfy the condition of normal ellipticity, see [2, 6]. In addition, we can also admit time dependent coefficients and time dependent source terms.

In our next example we presuppose existence of a classical solution for the non-autonomous parabolic equation

$$\partial_t u - a\,\Delta u = f \quad \text{in} \quad J \times X, \tag{6.14}$$

where X is an arbitrary open subset of $\mathbb{R}^n$ and $J = (0, T)$ for some $T > 0$, and we will be concerned with the regularity properties of u. We assume that

$$(a, f) \in C^\omega(J \times X), \tag{6.15}$$

and that the differential operator $A := a\,\Delta$ is uniformly strongly elliptic.

Example 6.4. *Suppose that u is a classical solution of the parabolic equation (6.14). Then $u \in C^\omega(J \times X)$.*

Proof. Let $(t_0, x_0) \in J \times X$ be fixed and choose $\varepsilon_0 > 0$ such that $\overline{\mathbb{B}}(x_0, 3\varepsilon_0) \subset U$ as well as $[t_0 - 3\varepsilon_0, t_0 + 3\varepsilon_0] \subset J$. In addition, choose τ small enough such that $\tau \notin \operatorname{supp}(\chi)$ and define $I := [0, T_0 - \tau]$ where T_0 is slightly smaller than T. Let $U := \mathbb{B}(x_0, 3\varepsilon_0)$ and let $\Gamma := \partial U$. Moreover, let $p \in (1 + n/2, \infty)$. The basic maximal regularity result for the present situation is

$$(\partial_t - b\Delta, \gamma_\Gamma, \gamma_0) \in \operatorname{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I)), \tag{6.16}$$

where we set $b(t, x) = a(\tau + t, x)$ for $t \in I$, and where

$$\begin{aligned}
&\mathbb{E}_1(I) := W_p^1(I, L_p(U)) \cap L_p(I, W_p^2(U)),\\
&\mathbb{E}_0(I) := \{(g_1, g_2, w_0) \in \mathbb{F}_0(I) \cap W_p^{2-2/p}(U)\,;\; g_2(0) = w_0|_\Gamma\},\\
&\mathbb{F}_0(I) := L_p(I, L_p(U)) \times \big(W_p^{1-1/2p}(I, L_p(\Gamma))\big) \cap L_p(I, W_p^{2-1/p}(\Gamma))
\end{aligned}$$

see [10, Section IV.9], and also [5, 12]. Next we set

$$v_0 := u(\tau), \quad f_1(t) := f(t + \tau)|_U, \quad f_2(t) := u(t + \tau)|_\Gamma, \quad t \in I.$$

Since u is classical solution of equation (6.14), we obtain $(f_1, f_2, v_0) \in \mathbb{E}_0(I)$. Let $v(t) := u(t + \tau)$. We conclude that $v \in \mathbb{E}_1(I)$, and that v is the (unique) solution of $(\partial_t - b\Delta, \gamma_\Gamma, \gamma_\tau)v = (f_1, f_2, v_0)$. Let

$$v_{\lambda,\mu}(t, x) := v(t + \zeta(t)\chi(x)\lambda, x + \zeta(t)\chi(x)\mu), \qquad (t, x) \in I \times U.$$

It can be shown that the pertinent results of section 5 do also hold for the transformation $\hat{\Phi}(t, x) := (t + \zeta(t)\chi(x)\lambda, x + \zeta(t)\chi(x)\mu)$ and we can, once again, conclude that

$$[(\lambda, \mu) \mapsto v_{\lambda,\mu}] \in C^\omega(\mathbb{B}^{n+1}(0, r_0), \mathbb{E}_1(I)).$$

The assertion follows as in the previous example. □

In our last example we consider the nonlinear parabolic equation

$$\partial_t u - a\left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1 + |\nabla u|^2}\right)\partial_i \partial_j u = 0 \quad \text{on} \quad \mathbb{R}^n, \qquad u(0) = u_0. \tag{6.17}$$

We assume that the coefficient a satisfies the assumptions

$$a \in buc^s(\mathbb{R}^n) \cap C^\omega(X), \qquad a(x) \geq \delta, \quad x \in X, \tag{6.18}$$

where X is an open subset of $\mathbb{R}^n$, and where $\delta > 0$. Equation (6.17) coincides with the mean curvature flow described in the introduction in case that $a \equiv 1$.

Example 6.5. *Let $u_0 \in buc^{2+s}(\mathbb{R}^n)$ be given. Then there exists a number $T > 0$ such that equation* (6.17) *has a unique solution*

$$u \in C^1([0,T], buc^s(\mathbb{R}^n)) \cap C([0,T], buc^{2+s}(\mathbb{R}^n)). \tag{6.19}$$

The solution has the additional regularity property $u \in C^\omega((0,T) \times X)$.

Proof. Let

$$F(v) := -a\left(\delta_{ij} - \frac{\partial_i v \partial_j v}{1+|\nabla u|^2}\right)\partial_i\partial_j v.$$

One can show that (1.5) remains valid for our new function F. Based on equation (1.5) we obtain a unique solution u in the class (6.19) for the parabolic equation (6.17). Let $(t_0, x_0) \in (0,T) \times X$ be fixed and set

$$u_{\lambda,\mu}(t,x) := u(t + \zeta(t)\lambda, x + \zeta(t)\chi(x)\mu), \qquad (t,x) \in I \times \mathbb{R}^n,$$

where $I := [0,T]$. It follows from Proposition 5.3 that $v = u_{\lambda,\mu}$ satisfies the parameter dependent equation

$$\partial_t v + F_{\lambda,\mu}(v) = 0, \quad v(0) = u_0,$$

where

$$F_{\lambda,\mu}(v) := (1 + \zeta'\lambda)T_\mu F(T_\mu^{-1} v) - B_\mu v.$$

Based on Proposition 5.2 we conclude that

$$[(v,(\lambda,\mu)) \mapsto F_{\lambda,\mu}(v)] \in C^\omega(\mathbb{E}_1(I) \times \mathbb{B}^{n+1}(0,r_0), \mathbb{E}_0(I))),$$

where the spaces $\mathbb{E}_1(I)$ and $\mathbb{E}_0(I)$ have the same meaning as in the introduction. The implicit function theorem lets us once more conclude that

$$[(\lambda,\mu) \mapsto u_{\lambda,\mu}] \in C^\omega(\mathbb{B}^{n+1}(0,r_0), \mathbb{E}_1(I))$$

and we obtain as in the previous examples that $u \in C^\omega(J \times X)$. □

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