On Convex Functions with Values in Semi–linear Spaces

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Abstract

The following result of convex analysis is well–known [2]: *If the function $f: X \to [-\infty, +\infty]$ is convex and some $x_0 \in \operatorname{core}(\operatorname{dom} f)$ satisfies $f(x_0) > -\infty$, then f never takes the value $-\infty$.* From a corresponding theorem for convex functions with values in semi–linear spaces a variety of results is deduced, among them the mentioned theorem, a theorem of Deutsch and Singer on the single–valuedness of convex set–valued maps as well as a result on the compact–valuedness of convex set–valued maps. We also discuss the possibility of embedding the image points of such a convex function into a linear space.

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1. Introduction

Semi–linear structures naturally occur in optimization and analysis. In many cases, semi–linear structures can be considered as convex cones in linear spaces. However, the concept of a convex cone is not appropriated in important cases, since it is not possible to find a linear space in which the semi–linear structure is a convex cone. Therefore, we start introducing the concept of a semi–linear space. We define convexity and convex functions with values in partially ordered semi–linear spaces and prove a basic principle for such functions. Then we show that this principle is the common basis for a variety of well–known assertions. Some other conclusions of the principle seem to be new.

We state a simple condition implying that the embedding of a semi–linear space into a linear space is not possible. However, the principle tells us that in the special case of a convex set–valued map the image points are, essentially, part of a linear structure. This could be the basis of a duality theory of convex set–valued maps, different to Tanino's [13] approach. As in [6] we understand the map as a function into a semi–linear space rather than a set–valued map into a linear space.

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2. Preliminaries

The concept of a semi–linear space and similar concepts were already considered, for instance, in [4] (almost linear spaces) and in [5]. In some of the cited references the axioms slightly differ from ours.

Let X be a set. On X let an addition $+ : \; X \times X \to X$, a multiplication $\cdot : \; \mathbb{R}_+ \times X \to X$ with non–negative reals and some neutral element $0_X \in X$ be defined such that for all $x, u, z \in X$ and real $\alpha, \beta \geq 0$ the following axioms are satisfied:

(S1) $(x + u) + z = x + (u + z)$;

(S2) $0_X + x = x$;

(S3) $x + u = u + x$;

(S4) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$;

(S5) $1 \cdot x = x$;

(S6) $\alpha \cdot (x + u) = \alpha \cdot x + \alpha \cdot u$;

(S7) $0 \cdot x = 0_X$.

Then, X is called a *semi–linear space*. Compare [1, page 141] for a special case and note that the concept of almost linear spaces of [4] additionally involves the multiplication with negative reals. The axioms imply that the neutral element is unique and $\alpha \cdot 0_X = 0_X$ for all $\alpha \geq 0$.

In the remainder of this section let X be a semi–linear space. A subset $C \subset X$ is said to be *convex* if $x, u \in C$ implies $\lambda \cdot x + (1 - \lambda) \cdot u \in C$ for all $\lambda \in [0, 1]$ and a subset $K \subset X$ is said to be a *cone* if $x \in K$ implies $\alpha \cdot x \in K$ for all $\alpha > 0$.

Proposition 2.1 *A subset $\{x\} \subset X$, consisting of exactly one element $x \in X$, is convex if and only if the "second distributive law" holds, i.e.*

(S8) $\forall \alpha, \beta \in \mathbb{R}_+ : \; \alpha \cdot x + \beta \cdot x = (\alpha + \beta) \cdot x.$

Proposition 2.2 *Let $X_c \subset X$ be the set of all points of X for which the second distributive law (S8) holds. Then X_c is a convex cone in X with $0_X \in X_c$.*

Examples of semi–linear spaces. (1) Every linear space V.

(2) Every convex cone $C \subset X$ of a semi–linear space with $0_X \in C$.

(3) The collection $\hat{\mathcal{P}}(X)$ ($\mathcal{P}(X)$) of all (nonempty) subsets of X with the following operations: $A, B \in \mathcal{P}(X)$, $\alpha \in \mathbb{R}_+$, $A + B := \{a + b | \; a \in A, \; b \in B\}$, $\alpha \cdot A := \{\alpha \cdot a | \; a \in A\}$, $\alpha \cdot \emptyset := \emptyset$ if $\alpha > 0$, $0 \cdot \emptyset := 0$.

(4) Let V be a topological linear space. The space $\hat{\mathcal{F}}(V)$ ($\mathcal{F}(V)$) of all (nonempty) closed subsets of V, where the addition is defined as $A + B := \operatorname{cl} \{a + b | \; a \in A, \; b \in B\}$ and the multiplication as in the previous example.

(5) The spaces $\hat{\mathcal{P}}_c(X)$, $\mathcal{P}_c(X)$, $\hat{\mathcal{F}}_c(V)$ and $\mathcal{F}_c(V)$ (compare (3), (4), and Proposition 2.2).

(6) Let V be a separated topological linear space. The spaces $\hat{\mathcal{C}}(V) \subset \hat{\mathcal{F}}(V)$, ($\mathcal{C}(V) \subset \mathcal{F}(V)$) of all (nonempty) compact subsets of V where the operations are defined as in (3).

(7) The spaces $\hat{\mathcal{C}}_c(V) \subset \hat{\mathcal{C}}(V)$, $(\mathcal{C}_c(V) \subset \mathcal{C}(V))$ of all (nonempty) convex compact subsets of V where the operations are defined as in (3).

(8) The space $\mathcal{K}(X)$ of all cones $K \subset X$ with $0_X \in K$, and the space $\hat{\mathcal{K}}(X) := \mathcal{K}(X) \cup \{\emptyset\}$ where the operations are defined as in (3).

(9) The space of extended reals $\mathbb{R}^\star := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ with the extended operations: $x+(-\infty) = (-\infty)+x = -\infty$ for all $x \in \mathbb{R}^\star \setminus \{\infty\}$, $x+\infty = \infty+x = \infty$ for all $x \in \mathbb{R}^\star$, $\alpha \cdot \pm\infty = \pm\infty$ for all $\alpha > 0$ and $0 \cdot \pm\infty = 0$ (compare [12]).

(10) The space of extended reals $\mathbb{R}^\diamond := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ with the extended operations: $x+\infty = \infty+x = \infty$ for all $x \in \mathbb{R}^\diamond \setminus \{-\infty\}$, $x+(-\infty) = (-\infty)+x = -\infty$ for all $x \in \mathbb{R}^\diamond$ and the multiplication as above.

Note that a subset $\{x\} \subset X$ consisting of exactly one element $x \in X$ can be a cone in X, even if $x \neq 0_X$. Such an element $x \in X$ with $x = \alpha \cdot x$ for all $\alpha > 0$ is called a *vertex*. Of course, the neutral element 0_X is a vertex in every semi–linear space X, therefore a vertex $x \neq 0_X$ is called a *nontrivial vertex*. Let A and B be two nonempty subsets of a semi–linear space X. We say A is *stronger* than B, in short $A \succ B$, if $a \in A$, $b \in B$ implies $a + b \in A$. If there is some $\hat{x} \in X$ such that $\{\hat{x}\} \succ X$, then $\hat{x}$ is called the *strongest* element of X. It can be shown that the strongest element of a semi–linear space X, if it exists, is a vertex and is uniquely defined. Moreover, the union of all vertexes is a convex cone in X. The following proposition underlines the advantage of considering semi–linear spaces instead of convex cones of linear spaces.

Proposition 2.3 *A semi–linear space having a nontrivial vertex cannot be embedded into a linear space.*

Proof. Suppose the contrary, i.e. there exists a linear space L such that X is a convex cone in L with a vertex $\hat{x} \neq 0_X$. Then there must be an inverse element $\bar{x}$ and we have $\hat{x} + \bar{x} = 0_L$. It follows $0_L = \bar{x} + \hat{x} = \bar{x} + 2 \cdot \hat{x} = (\bar{x} + \hat{x}) + \hat{x} = \hat{x}$ which contradicts the assumption. □

The preceding proposition shows that a lot of important examples of semi–linear spaces, for instance the spaces of the Examples (3) to (10), cannot be treated as convex cones in a linear space. A sufficient condition for embedding a semi–linear space into a linear space is discussed in Radström [10, Theorem 1].

In the following, let the semi–linear space X be equipped with a partial ordering $\leq$ (i.e. a reflexive, transitive and antisymmetric relation on X). We say $(X, \leq)$ (shortly X) is a *partially ordered semi–linear space* if it holds

$$x_1 \leq x_2, x_3 \leq x_4 \quad \Rightarrow \quad \alpha \cdot (x_1 + x_3) \leq \alpha \cdot (x_2 + x_4) \tag{1}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $\alpha \geq 0$.

Proposition 2.4 *Let X be a partially ordered semi–linear space. Then the largest (smallest) element of X, if it exists, is a vertex.*

Proof. Let $\hat{x}$ be the largest element of X, i.e. $x \leq \hat{x}$ for all $x \in X$. For given $\alpha > 0$, condition (1) yields $\alpha \cdot x \leq \alpha \cdot \hat{x}$ for all $x \in X$. Given any $u \in X$, we have $x := 1/\alpha \cdot u \in X$. Hence for all $\alpha > 0$ and all $u \in X$ it holds $u \leq \alpha \cdot \hat{x}$, i.e. $\alpha \cdot \hat{x}$ is the largest element of X. Since the largest element of a partially ordered set is uniquely defined we get $\alpha \cdot \hat{x} = \hat{x}$ for all $\alpha > 0$. The proof for the smallest element is analogous. $\square$

Since a linear space cannot have any nontrivial vertex, the preceding result means that a partially ordered linear space cannot be order complete (a partially ordered set is said to be order complete if every subset has supremum and infimum [14]). However, every Dedekind complete partially ordered semi–linear space (a partially ordered set is said to be Dedekind complete if every subset which is bounded above (below) has a supremum (infimum) [14]) can be extended to an order complete partially ordered semi–linear space. To see this extend the space by a new element defined to be the largest (smallest) and strongest one and, after this, extend the space by a second new element defined to be the smallest (largest) and strongest one (compare Examples (9) and (10)).

Examples of partially ordered semi–linear spaces.

(11) Every partially ordered linear space.

(12) The spaces of the Examples (3) to (8) equipped with the partial orderings $\subset$ and $\supset$ of set inclusion.

(13) The extended reals of Examples (9) and (10) with the usual $\leq$ relation.

Now we are able to give the definition of a convex function. Let $(Y, \leq)$ be a partially ordered semi–linear space and $C \subset X$. The set $\operatorname{epi} f := \{(x, y) \in C \times Y | \ f(x) \leq y\}$ is called *epigraph* of f. A function $f : C \to (Y, \leq)$ is said to be *convex* if its epigraph $\operatorname{epi} f$ is a convex subset of $X \times Y$. In this case, C must be convex. It is an easy task to show that a function $f : C \to (Y, \leq)$ is convex if and only if for all $\lambda \in [0, 1]$ and all $x, u \in C$ it holds

$$f(\lambda \cdot x + (1 - \lambda) \cdot u) \leq \lambda \cdot f(x) + (1 - \lambda) \cdot f(u).$$

A convex function $f : C \to Y$, defined on a subset $C \subset X$, can be extended to the whole space X, if $(Y, \leq)$ has a largest element $\hat{y}$ which is simultaneously the strongest element of Y. In this case, the extension $\hat{f} : X \to Y$, defined by $\hat{f}(x) := f(x)$ if $x \in C$ and $\hat{f}(x) = \hat{y}$ elsewhere, is convex. Moreover, the set $\operatorname{dom} f := \{x \in C | \ f(x) \neq \hat{y}\}$ is called the *effective domain* of f. In the spaces of the Examples (3) to (8), equipped with the relation $\supset$, we have $\hat{y} = \emptyset$, if the empty set belongs to the space. If we take instead the relation $\subset$, we have $\hat{y} = X$ (respectively $\hat{y} = V$) if the empty set does not belong to the space.

It is well–known that in the special case of a function $f : U \to (\mathcal{P}(V), \supset)$, where U and V are linear spaces, f is convex if and only if its "graph" $G(f) := \{(u, v) \in U \times V | \ v \in f(u)\}$ is a convex subset of $U \times V$.

3. A basic principle and its conclusions

The following theorem is the essential part of a lot of assertions concerning convex functions (and maps). It states that under certain assumptions to the semi–linear structure and to the ordering structure, a convex function cannot attain values in a certain cone of its partially ordered semi–linear image space.

In this section, let X be a linear space and $C \subset X$. The core or the algebraic interior of a subset $A \subset X$ is denoted by $\operatorname{core} A$ (compare [7]). As usual, for $f : C \to (Y, \leq)$ and $A \subset C$ we define $f(A) := \{y \in Y | \; \exists x \in A : \;\; y = f(x)\}$.

Theorem 3.1 *Let $(Y, \leq)$ be a partially ordered semi–linear space, $S \subset Y$ a cone, $f : \; C \to (Y, \leq)$ a convex function and $A \subset C$ such that $S \succ f(A)$. If there exists $x_0 \in \operatorname{core} A$ such that $f(x_0) \not\leq s$ for all $s \in S$, then $f(x) \notin S$ for all $x \in C$.*

Proof. Assume $f(x) \in S$. Since $x_0 \in \operatorname{core} A$, we find some $x' \in A$ such that $x_0 = \lambda x' + (1 - \lambda)x$ for some $\lambda \in (0, 1)$. The convexity of f yields $f(x_0) \leq \lambda \cdot f(x') + (1 - \lambda) \cdot f(x) =: s$. Since S is a cone in Y and $\lambda > 0$, $S \succ f(A)$ implies $S \succ \lambda \cdot f(A)$. Consequently, we have $s \in S$. Hence $f(x_0) \leq s$ and $s \in S$ contradicting the assumption. This means $f(x) \notin S$ for all $x \in C$. $\square$

The first corollary is a classical result for convex functions with values in the extended reals $\mathbb{R}^\star$ of Example (9). Note that, for instance, in [11] other calculus rules in the extended reals are used, but the same result is valid.

Corollary 3.2 *Let $f : \; C \to (\mathbb{R}^\star, \leq)$ be a convex function. If some point $x_0 \in \operatorname{core}(\operatorname{dom} f)$ satisfies $f(\bar{x}) > -\infty$, then f never takes the value $-\infty$.*

Proof. $S = \{-\infty\}$, $A = \operatorname{dom} f$. $\square$

In the following result, we set $\ker f = \{x \in C | \; f(x) = 0_Y\}$.

Corollary 3.3 *Let $f : \; C \to \mathbb{R}$ be a convex function. If $x_0 \in \operatorname{core} \ker f$, then $f(x) \geq 0$ for all $x \in C$.*

Proof. $S = \{y \in \mathbb{R} | \; y < 0\}$, $A = \ker f$. $\square$

With aid of the principle it is easy to obtain a vector–valued variant of the preceding assertion. Therein, $\operatorname{bd} K = \operatorname{cl} K \setminus \operatorname{int} K$ denotes the boundary of K.

Corollary 3.4 *Let $(Y, \leq_K)$ be a separated topological linear space partially ordered by a closed pointed convex cone $K \subset Y$ containing 0_Y and having a nonempty interior, $f : \; C \to (Y, \leq_K)$ a convex function. If f takes values in $-\operatorname{bd} K$ on an algebraically open subset of C, then f never takes values in $-\operatorname{int} K$.*

Proof. $S = -\operatorname{int} K$, $A = f^{-1}(-\operatorname{bd} K)$ $\square$

In vector optimization optimality conditions of the following type occur [8, Theorem 7.6]: If $\bar{x} \in S$ is a weakly minimal solution of the vector optimization problem $\min_{x \in S} f(x)$ of [8, page 153] and if $f : S \to (Y, \leq_K)$ has a directional variation $f'(\bar{x}) : S - \{\bar{x}\} \to (Y, \leq_K)$ with respect to $-\operatorname{core} K$ [8, Definition 2.14], then

$$\forall x \in S : \quad f'(\bar{x})(x - \bar{x}) \notin -\operatorname{core} K. \tag{2}$$

If the directional variation $f'(\bar{x}) : S - \{\bar{x}\} \to (Y, \leq_K)$ is convex and takes values in $-\operatorname{bd} K$ on an algebraically open set, and if $\operatorname{int} K \neq \emptyset$ (in particular this implies $\operatorname{core} K = \operatorname{int} K$), then, by Corollary 3.4, the optimality condition (2) is satisfied.

The following corollary is a result of Deutsch and Singer [3] on the single–valuedness of a convex set–valued map. In [3] a further conclusion, namely f must be affine on $\operatorname{dom} f$, is drawn and applications to metric projections and adjoints of set–valued maps are discussed.

Corollary 3.5 *Let V be a linear space and let $f : C \to (\hat{\mathcal{P}}(V), \supset)$ be convex. If f is single–valued in some point of $x_0 \in \operatorname{core}(\operatorname{dom} f)$, then f is single–valued everywhere in* $\operatorname{dom} f$.

Proof. $S = \{\text{"nonsingletons"}\}$, $A = \operatorname{dom} f$. □

Corollary 3.6 *Let V be a separated topological linear space and $f : C \to (\hat{\mathcal{F}}(V), \supset)$ be convex. If f is compact–valued at some point $x_0 \in \operatorname{core}(\operatorname{dom} f)$, then f is compact–valued everywhere in* $\operatorname{dom} f$.

Proof. $S = \{\text{"noncompacts"}\}$, $A = \operatorname{dom} f$. □

The following result of Zamfirescu [15] was published in the framework of a generalization of the mentioned result of Deutsch and Singer to so–called *star–shaped* functions. The same generalization could be done for all the assertions given here.

Corollary 3.7 *Let $V = \mathbb{R}^n$ and let $f : C \to (\hat{\mathcal{P}}(V), \supset)$ be convex. Then* $\dim f(x)$*, as a function of x, is constant on* $\operatorname{core}(\operatorname{dom} f)$ *and not larger elsewhere.*

Proof. Let $x_0 \in \operatorname{core}(\operatorname{dom} f)$ with $\dim f(x_0) = k$, $S = \{v \subset V \mid \dim v > k\}$, $A = \operatorname{dom} f$. Then the theorem yields $\dim f(x) \leq k$ for all $x \in C$. Now suppose there is some $x_1 \in \operatorname{core}(\operatorname{dom} f)$ such that $\dim f(x_1) = m < k$. Applying the theorem again we obtain $\dim f(x) \leq m < k$ for all $x \in C$ contradicting $\dim f(x_0) = k$. □

Corollary 3.8 *Let Z be a semi–linear space and let $f : C \to (\hat{\mathcal{K}}(Z), \supset)$ be convex. Then f is constant on* $\operatorname{core}(\operatorname{dom} f)$.

Proof. Let $x_0 \in \operatorname{core}(\operatorname{dom} f)$ with $f(x_0) = k_0$, $S = \{k \in \mathcal{K}(Z) \mid k \not\subset k_0\}$, $A = \operatorname{dom} f$. Then the theorem yields $f(x) \subset k_0$ for all $x \in C$. Now suppose there is some $x_1 \in \operatorname{core}(\operatorname{dom} f)$ with $f(x_1) = k_1 \subsetneq k_0$. Applying the theorem again we obtain $f(x) \subset k_1 \subsetneq k_0$ for all $x \in C$ contradicting $f(x_0) = k_0$. □

Let V be a locally convex space and V^* its topological dual. As usual, $\delta^*(\,\cdot\,|\,A\,) : V^* \to \mathbb{R}^\star$, $\delta^*(v^*|\,A\,) = \sup\{\langle v^*, a\rangle \mid a \in A\}$ is the support function of a convex set $A \subset V$. For $A, B \in \mathcal{F}_c(V)$ (compare Example (5), in particular $A, B \neq \emptyset$) it holds

$$\forall v^* \in V^* : \quad \delta^*(v^*|\,A + B\,) = \delta^*(v^*|\,A\,) + \delta^*(v^*|\,B\,) \tag{3}$$

and for $A \in \mathcal{F}_c(V)$ and $\alpha \leq 0$ we have

$$\forall v^* \in V^* : \quad \delta^*(v^*|\,\alpha \cdot A\,) = \alpha \cdot \delta^*(v^*|\,A\,)$$

Hence, the map which assigns every $A \in \mathcal{F}_c(V)$ its support functions is a homomorphism into the semi–linear space Ψ of all functions $\psi : V^* \to \mathbb{R} \cup \{\infty\}$, where the semi–linear operations are defined pointwise. Using a separation theorem, for instance [9, page 25], it can easily be seen that this homomorphism is injective, i.e. we have an embedding. Moreover, it is clear that functions $\psi : V^* \to \mathbb{R} \cup \{\infty\}$ having the same effective domain can considered to be a linear space L. Let $\mathcal{A} \subset \mathcal{F}_c(V)$ having the following property: The support functions of all $A \in \mathcal{A}$ have the same effective domain. Then $\mathcal{A}$ can be embedded into a linear space.

The following corollary tells us that the values of a convex function $f : C \to (\hat{\mathcal{F}}(V), \supset)$ have essentially the same effective domain.

Corollary 3.9 *Let $f : C \to (\hat{\mathcal{F}}(V), \supset)$ be convex. Then $x \longmapsto \operatorname{dom}\delta^*(\,\cdot\,|\,f(x)\,)$ is constant on* $\operatorname{core}(\operatorname{dom} f)$.

Proof. The convexity of f implies $f(x) \supset \lambda f(x) + (1-\lambda) f(x)$ for all $\lambda \in [0,1]$. Hence $f(x)$ is a convex subset of V for all $x \in \operatorname{dom} f$. Define $K_0 := \operatorname{dom}\delta^*(\,\cdot\,|\,f(x_0)\,)$ for some $x_0 \in \operatorname{core}(\operatorname{dom} f)$, $S := \{c \in \mathcal{F}_c(V) |\, \operatorname{dom}\delta^*(\,\cdot\,|\,c\,) \not\supset K_0\}$ and $A := \operatorname{dom} f$. Obviously, S is a cone in $\mathcal{F}(Y)$.

Let $s \in S$ and $c \in f(A)$. Then we have $\operatorname{dom}\delta^*(\,\cdot\,|\,s\,) \not\supset K_0$ and since $c \neq \emptyset$ we have $\delta^*(\,\cdot\,|\,c\,) > -\infty$. Since s and c are nonempty, (3) is valid and it follows $s + c \in S$, i.e. the assumption $S \succ f(A)$ is satisfied.

For all $s \in S$ we have $f(x_0) \not\supset s$. Indeed, assuming that $\bar{s} \subset f(x_0)$ for some $\bar{s} \in S$ we obtain $\delta^*(\,\cdot\,|\,\bar{s}\,) \leq \delta^*(\,\cdot\,|\,f(x_0)\,)$ and hence $\operatorname{dom}\delta^*(\,\cdot\,|\,\bar{s}\,) \supset \operatorname{dom}\delta^*(\,\cdot\,|\,f(x_0)\,) = K_0$. This contradicts the definition of S. The theorem yields $f(x) \notin S$ for all $x \in C$. This means $\operatorname{dom}\delta^*(\,\cdot\,|\,f(x)\,) \supset K_0$ for all $x \in C$.

Assuming that $\operatorname{dom}\delta^*(\,\cdot\,|\,f(x_1)\,) = K_1 \supsetneq K_0$ for some $x_1 \in \operatorname{core}(\operatorname{dom} f)$ and applying the same procedure we get $\operatorname{dom}\delta^*(\,\cdot\,|\,f(x)\,) \supset K_1$ for all $x \in C$, in particular, $\operatorname{dom}\delta^*(\,\cdot\,|\,f(x_0)\,) \supset K_1 \supsetneq K_0$ contradicting the definition of K_0. Hence $x \longmapsto \operatorname{dom}\delta^*(\,\cdot\,|\,f(x)\,)$ is constant on $\operatorname{core}(\operatorname{dom} f)$. $\square$

From the previous corollary and the considerations above, we may conclude that the set $f(\operatorname{core}(\operatorname{dom} f)) \subset \mathcal{F}(V)$ can be embedded into a linear space L, even though $\mathcal{F}(V)$ cannot be embedded (compare Proposition 2.3). Note that for different functions $f : C \to \hat{\mathcal{F}}(V)$ with the same X, $C \subset X$ and V the linear space L can be different, in particular, the neutral element of L does not coincides with the neutral element of $\hat{\mathcal{F}}(V)$, in general.

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