Necessary and sufficient conditions for approximate saddle points

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Abstract

The aim of the paper is to derive approximate saddle point assertions for a general class of vector-valued approximation problems using a generalized Lagrangean. We derive necessary and sufficient conditions for approximate saddle points, estimate the approximation error and study the relations between the original problem and saddle point assertions under regularity assumptions.

1 Introduction

In our paper we consider a general class of vector-valued approximation problems which contains many practically important special cases and introduce the concept of approximately efficient elements of this problem. Approximate solutions of optimization problems are of interest from the numerical as well as the theoretical point of view. Especially, numerical algorithms only generate approximate solutions if we stop them after a finite number of steps. Moreover, the solution set may be empty in the general noncompact case whereas approximate solutions exist under very weak assumptions.

Valyi [8], [9] developed Hurwiz-type saddle point theorems for different types of approximately efficient solutions of vector optimization problems. The aim of the present paper is to derive approximate saddle point assertions for vector-valued location and approximation problems using a generalized Lagrangean.

We introduce a generalized saddle function for the vector-valued approximation problem and different concepts of approximate saddle points. Furthermore, we derive necessary and sufficient conditions for approximate saddle points, estimate the approximation error and study the relations between the original problem and saddle point assertions under regularity assumptions.

2 Terminology and notations

All topological linear spaces that will occur throughout the paper are over the field $\mathbb{R}$ of real numbers. If U and W are topological linear spaces, then $\mathcal{L}(U,W)$ denotes the set of all continuous linear mappings from U into W. The topological dual spaces of a topological linear space U is denoted by U^*. If U is a locally convex Hausdorff space, then we write U_σ and U_τ for U under the weak topology $\sigma(U,U^*)$ and the Mackey topology $\tau(U,U^*)$, respectively. Furthermore, $\sigma - \operatorname{int} S$ denotes the interior of a set $S \subset U_\sigma$ etc.

A nonempty subset K_W of a linear space W is said to be a *convex cone* if

$$K_W + K_W \subseteq K_W \quad \text{and} \quad \alpha K_W \subseteq K_W \quad \text{for all} \quad \alpha \in [0, \infty[.$$

A convex cone K_W is called *pointed* if $K_W \cap (-K_W) = \{0\}$. If K_W is a convex cone in a topological linear space W, then the set

$$K_W^* := \{\lambda \in W^* \mid \forall k \in K_W : \ \lambda(k) \geq 0\}$$

is called the *dual cone* of K_W.

Given a convex cone K_W in a linear space W, a binary relation $\leq$ can be introduced in W by setting $x \leq y$ if $y - x \in K_W$. It is easy to see that this relation $\leq$ is an ordering on W with respect to which W is an ordered linear space. A mapping f of a linear space U into the ordered linear space W is called *sublinear* if

$$f(\alpha u_1) = \alpha f(u_2) \quad \text{and} \quad f(u_1 + u_2) \leq f(u_1) + f(u_2)$$

for all $u_1, u_2 \in U$ and all $\alpha \in [0, \infty[$.

Given two topological linear spaces U and W, where W is ordered by a convex cone K_W, we say that the pair (U, W) has the *Hahn-Banach extension property* if for each sublinear mapping $f : U \to W$ the following assertion holds: For each $u_0 \in U$ there exists an $Y \in \mathcal{L}(U, W)$ such that

$$\forall u \in U \ : \ f(u) \in Y(u) + K_W \text{ and } \quad f(u_0) = Y(u_0).$$

In the following we list some sufficient conditions ensuring that the pair (U, W) has the Hahn-Banach extension property (cf. [13]): Assume that U and W are locally convex Hausdorff spaces and

(A1) U is a weakly compactly generated space,

(A2) the convex cone K_W of W is closed and $\tau - \operatorname{int} K_W^* \neq \emptyset$,

(A3) the sublinear map f is continuous at 0 as a map from U to W_σ.

Remark. (i) The assumption (A1) holds if, for example, U is a separable Banach space or a reflexive normed space. Furthermore, it is easily verified that (A3) holds if (A2) holds and f is order bounded above in some 0-neighborhood of U.

(ii) For example, the assumptions (A1), (A2) and (A3) are satisfied in the special case that U and W are finite-dimensional Euclidean spaces and K_W is any pointed closed convex cone in W. Indeed, if K_W is pointed, then the dual cone K_W^* has nonempty interior and, since each sublinear functional $\lambda \circ f$ with $\lambda \in K_W^*$, is continuous on the finite dimensional Euclidean space U and $\operatorname{int} K_W^*$ is nonempty, f must be continuous itself.

Let M be a subset of a topological linear space W, and let w_0 be a point in W. Given a subset K of W and an element $e \in W$, the point w_0 is called a (K, e)–*minimal* (resp. (K, e)–*maximal*) *point* of M if $w_0 \in M$ and

$$(w_0 - e - K) \cap M \subseteq \{w_0 - e\} \quad (\text{resp. } (w_0 + e + K) \cap M \subseteq \{w_0 + e\}).$$

The set consisting of all (K,e)–minimal (resp. (K,e)–maximal) points of M is denoted by $\mathrm{Min}(M,K,e)$ (resp. $\mathrm{Max}(M,K,e)$). If e is the origin of W, then the (K,e)–minimal (resp. (K,e)–maximal) points of M are simply called *K–minimal* (resp. *K–maximal*) *points* of M and their set is denoted by $\mathrm{Min}(M,K)$ (resp. $\mathrm{Max}(M,K)$).

Given a function $\lambda \in W^*$ and an element $e \in W$, the point w_0 is called a *(λ,e)–minimal* (resp. *(λ,e)– maximal*) *point* of M if $w_0 \in M$ and for all $w \in M$ it holds $\lambda(w_0) - \lambda(e) \leq \lambda(w)$ (resp. $\forall w \in M : \lambda(w) \leq \lambda(w_0) + \lambda(e)$).

The set consisting of all (λ,e)–minimal (resp. (λ,e)–maximal) points of M is denoted by $\mathrm{Min}(M,\lambda,e)$ (resp. $\mathrm{Max}(M,\lambda,e)$). If e is the origin of W, then the (λ,e)–minimal (resp. (λ,e)– maximal) points of M are simply called *λ–minimal* (resp. *λ–maximal*) *points* of M and their set is denoted by

$$\mathrm{Min}(M,\lambda) \quad (\text{resp. } \mathrm{Max}(M,\lambda)).$$

Let M and N be nonempty sets, let W be a topological linear space, and let Φ be a mapping from $M \times N$ into W. Given a subset K of W and an element $e \in W$, a point $(x_0,y_0) \in M \times N$ is said to be a *(K,e)–saddle point of* Φ *with respect to* $M \times N$ if the following conditions are satisfied:

$$\Phi(x_0,y_0) \in \mathrm{Min}\left(\{\Phi(x,y_0) \mid x \in M\}, K, e\right); \tag{1}$$

$$\Phi(x_0,y_0) \in \mathrm{Max}\left(\{\Phi(x_0,y) \mid y \in N\}, K, e\right). \tag{2}$$

Given a function $\lambda \in W^*$ and an element $e \in W$, a point $(x_0,y_0) \in M \times N$ is said to be a *(λ,e)– saddle point of* Φ *with respect to* $M \times N$ if the following conditions are satisfied:

$$\Phi(x_0,y_0) \in \mathrm{Min}\left(\{\Phi(x,y_0) \mid x \in M\}, \lambda, e\right); \tag{3}$$

$$\Phi(x_0,y_0) \in \mathrm{Max}\left(\{\Phi(x_0,y) \mid y \in N\}, \lambda, e\right). \tag{4}$$

3 Formulation of the vector-valued approximation problem

In the whole paper we suppose that W is a locally convex Hausdorff space partially ordered by the closed, pointed and convex cone $K_W \subseteq W$; X, U and V are reflexive Banach spaces; the pair (U,W) has the Hahn-Banach extension property; $A : X \to U$, $B : X \to V$ and $C : X \to W$ are continuous linear mappings; $f : U \to W$ is a continuous sublinear mapping; $\mathcal{A} \subseteq U$, $\mathcal{X} \subseteq X$ and $K_V \subseteq V$ are closed, pointed and convex cones and $b \in V$. Defining $F : \mathcal{A} \times \mathcal{X} \to W$ by

$$F(a,x) := C(x) + f(a - A(x)),$$

and

$$S := \{(a,x) \in U \times X \mid a \in \mathcal{A}, x \in \mathcal{X}, B(x) - b \in K_V\},$$

we consider the following vectorial approximation problem

$$(\mathrm{P}(K_W)) \qquad \text{Compute the set} \quad \mathrm{Min}(F[S], K_W).$$

4 Approximate saddle point theorems

Proceeding as in [6] and [1], we define the mapping $L : U \times X \times \mathcal{L}(U,W) \times \mathcal{L}(V,W) \to W$ by

$$L(a, x, Y, Z) := C(x) + Y(a - A(x)) + Z(b - B(x)).$$

When three of the four variables $a \in U$, $x \in X$, $Y \in \mathcal{L}(U,W)$ and $Z \in \mathcal{L}(X,W)$ are fixed, then the corresponding partial mappings

$$L(., x, Y, Z), L(a, ., Y, Z), L(a, x, ., Z), L(a, x, Y, .)$$

are affine. This property distinguishes our mapping L from the Lagrangean mapping usually associated with the problem $(P(K_W))$ (see e.g. [9]).

In what follows we consider L as a function of two variables (a, x) and (Y, Z), and investigate approximate saddle points of L with respect to $(\mathcal{A} \times \mathcal{X}) \times (\mathcal{Y} \times \mathcal{Z})$, where $\mathcal{Y}$ and $\mathcal{Z}$ are given by

$$\mathcal{Y} := \{Y \in \mathcal{L}(U,W) \mid \forall u \in U \ : \ f(u) \in Y(u) + K_W\}. \tag{5}$$

and

$$\mathcal{Z} \ := \ \{Z \in \mathcal{L}(V,W) \mid Z[K_V] \subseteq K_W\},$$

respectively. For short, we set $\mathcal{D} := (\mathcal{A} \times \mathcal{X}) \times (\mathcal{Y} \times \mathcal{Z})$.

Theorem 1 *Let λ be a functional in $K_W^* \setminus (-K_W^*)$, let e be an element in K_W, and let (a_0, x_0, Y_0, Z_0) be an element in $\mathcal{D}$. Then (a_0, x_0, Y_0, Z_0) is a (λ, e)–saddle point of L with respect to $\mathcal{D}$ if and only if the following conditions are satisfied:*

(i) $L(a_0, x_0, Y_0, Z_0) \in \mathrm{Min}(\{L(a, x, Y_0, Z_0) \mid (a, x) \in \mathcal{A} \times \mathcal{X}\}, \lambda, e)$;

(ii) $B(x_0) - b \in K_V$;

(iii) $\lambda \circ Y_0(a_0 - A(x_0)) + \lambda \circ Z_0(b - B(x_0)) \geq \lambda \circ f(a_0 - A(x_0)) - \lambda(e)$.

The proof can be found in [1]. ■

Theorem 2 *Let $(W, \|\cdot\|)$ be a Banach space, $\mathcal{L}(U,W)$ be reflexive, $\lambda \in K_W^* \setminus (-K_W^*)$, $e \in K_W$, and let $(a_0, x_0) \in S$ be a (λ, e)–minimal point of $F[S]$. If $\{(a, x) \in S \mid B(\bar{x}) - b \in \mathrm{int}\, K_V\} \neq \emptyset$ and $\mathcal{Y}$ is bounded, then there exist mappings $Y_0 \in \mathcal{Y}$ and $Z_0 \in \mathcal{Z}$ such that (a_0, x_0, Y_0, Z_0) is a (λ, e)–saddle point of L with respect to $\mathcal{D}$.*

Proof. We consider the scalarized Lagrangean defined by

$$L_\lambda(a, x, Y, v^*) := \lambda \circ C(x) + \lambda \circ Y(a - A(x)) + v^*(b - B(x))$$

over $\mathcal{D}' := (\mathcal{A} \times \mathcal{X}) \times (\mathcal{Y} \times K_V^*)$. We show that for this scalarized Lagrangean the assumptions (H1), (H2) and (H3*) of Theorem 49.B.(3)(ii) in [12] are fulfilled. Obviously, the assumptions (H1) and (H2) of this theorem are true. In order to show (H3*), we consider a sequence $\{(Y^n, v_n^*)\} \subset \mathcal{Y} \times K_V^*$ with $\|(Y^n, v_n^*)\| \to \infty$ if $n \to \infty$. Since $Y^n \in \mathcal{Y}$ for all n and $\mathcal{Y}$ is

bounded, there is a constant $\alpha > 0$ such that $\|Y^n\| \leq \alpha$ for all n. Thus we have $\|v_n^*\| \to \infty$ if $n \to \infty$. Next we choose a point $(\bar{a}, \bar{x}) \in S$ with $B(\bar{x}) - b \in \operatorname{int} K_V$.
From $B(\bar{x}) - b \in \operatorname{int} K_V$ it follows the existence of a $\delta \in (0, \infty)$ such that $v^*(b - B(\bar{x})) \leq -\delta$ for all $v^* \in K_V^*$ with $\|v^*\| = 1$. So, we have

$$\begin{aligned} L_\lambda(\bar{a}, \bar{x}, Y^n, v_n^*) &= \lambda \circ C(\bar{x}) + \lambda \circ Y^n(\bar{a} - A(\bar{x})) + v_n^*(b - B(\bar{x})) \\ &\leq \lambda \circ C(\bar{x}) + \alpha\|\lambda\|\|\bar{a} - A(\bar{x})\| - \delta\|v_n^*\| \\ &\overset{n\to\infty}{\longrightarrow} -\infty, \end{aligned}$$

which proves (H3*). By applying the above mentioned theorem, we conclude that there exist $Y_0 \in \mathcal{Y}$ and $v_0^* \in K_V^*$ satisfying

$$\inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} L_\lambda(a, x, Y_0, v_0^*) = \sup_{(Y,v^*)\in\mathcal{Y}\times K_V^*} \inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} L_\lambda(a, x, Y, v^*),$$

and

$$\inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} \sup_{(Y,v^*)\in\mathcal{Y}\times K_V^*} L_\lambda(a, x, Y, v^*) = \sup_{(Y,v^*)\in\mathcal{Y}\times K_V^*} \inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} L_\lambda(a, x, Y, v^*).$$

With aid of the Hahn-Banach extension property we conclude that

$$\sup_{(Y,v^*)\in\mathcal{Y}\times K_V^*} L_\lambda(a, x, Y, v^*) = \lambda \circ F(a, x) \quad \text{whenever} \quad (a, x) \in S.$$

From this we deduce that

$$\begin{aligned} \sup_{(Y,v^*)\in\mathcal{Y}\times K_V^*} L_\lambda(a_0, x_0, Y, v^*) &= \lambda \circ F(a_0, x_0) \\ &\leq \inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} \lambda \circ F(a, x) + \lambda(e) \\ &= \inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} \sup_{(Y,v^*)\in\mathcal{Y}\times K_V^*} L_\lambda(a, x, Y, v^*) + \lambda(e) \\ &= \sup_{(Y,v^*)\in\mathcal{Y}\times K_V^*} \inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} L_\lambda(a, x, Y, v^*) + \lambda(e) \\ &= \inf_{(a,x)\in\mathcal{A}\times\mathcal{X}} L_\lambda(a, x, Y_0, v_0^*) + \lambda(e). \end{aligned}$$

This yields the following inequalities:

$$\forall (Y, v^*) \in \mathcal{Y} \times K_V^* : \qquad L_\lambda(a_0, x_0, Y, v^*) - \lambda(e) \leq L_\lambda(a_0, x_0, Y_0, v_0^*); \tag{6}$$

$$\forall (a, x) \in \mathcal{A} \times \mathcal{X} : \qquad L_\lambda(a_0, x_0, Y_0, v_0^*) \leq L_\lambda(a, x, Y_0, v_0^*) + \lambda(e). \tag{7}$$

Finally, we have to show that for $v_0^* \in K_V^*$, $v \in V$ and $\lambda \in K_W^* \setminus (-K_W^*)$ there is a mapping $Z \in \mathcal{Z}$ such that $v_0^*(v) = \lambda \circ Z(v)$. Since $\lambda \in K_W^* \setminus (-K_W^*)$ we can choose $k \in K_W$ such that $\lambda(k) > 0$. Define the mapping $Z_0 : V \to W$ by

$$Z_0(v) := \frac{v_0^*(v)}{\lambda(k)}\, k.$$

Then we have $Z_0 \in \mathcal{L}(V,W)$. Since $v_0^*(v) \geq 0$ for all $v \in K_V$, we conclude that $Z_0[K_V] \subseteq K_W$, i.e., $Z_0 \in \mathcal{Z}$. Furthermore, we obtain

$$\forall v \in V: \quad \lambda \circ Z_0(v) = \frac{v_0^*(v)}{\lambda(k)}\, \lambda(k) = v_0^*(v).$$

Consequently, (6) and (7) yield the desired assertion. ■

Remark. In Theorem 2 we assume that $\mathcal{L}(U,W)$ is reflexive. In order to give sufficient conditions for this assumption we use the following assertion given by [2, VIII,4,Theorem 4]: *If U and W are Banach spaces and one of them has the approximation property (cf. [2, VIII,3, Definition 1], then $\mathcal{L}(U,W)$ is reflexive if and only if U and W are reflexive and each member of $\mathcal{L}(U,W)$ is compact.* For example, $\mathcal{L}(U,W)$ is reflexive in the following two cases:

(i) U is a reflexive Banach space and W is a finite dimensional Euclidian space;

(ii) $U = l_q$ and $W = l_p$, where $1 < p < q < \infty$ (compare [2, VIII,4,Corollary 5]).

Let us discuss sufficient conditions for the boundedness of the set $\mathcal{Y}$ in Theorem 2. Therefore we recall the concept of *absolute-monotone norms* (cf. [11]). Let $(W, \leq)$ be an ordered normed space. The norm $\|.\|_a$ is said to be absolute-monotone on W if for $w, w' \in W$, $-w \leq w' \leq w$ implies $\|w'\| \leq \|w\|$. The set $\mathcal{Y}$ is bounded (with respect to the norm in $\mathcal{L}(U,W)$) if the following conditions are satisfied:

(i) The norm $\|.\|$ of W is equivalent to an absolute-monotone norm $\|.\|_a$ on W.

(ii) There exists some continuous sublinear map $g : U \to W$ with

$$\forall u \in U, \alpha \in \mathbb{R}: \qquad g(\alpha u) = |\alpha| g(u), \tag{8}$$

such that $f(u) \leq g(u)$ for all $u \in U$.

Indeed, let us show that the assumptions of the Banach-Steinhaus Theorem (e.g. [Werner(1995)]) are satisfied. Since U is a Banach space and W is a normed space it remains to show that the following pointwise boundedness assumption is satisfied

$$\forall u \in U: \ \sup_{Y \in \mathcal{Y}} \|Y(u)\| < \infty. \tag{9}$$

Indeed, for arbitrary $u \in U$ we have $-g(u) \leq -f(u) \leq -Y(u) = Y(-u) \leq f(-u) \leq g(-u) = g(u)$. Hence, $\|Y(u)\|_a \leq \|g(u)\|_a$ for all $Y \in \mathcal{Y}$. Since g is continuous in zero, we have $\|g(u)\| < \infty$, hence (9) holds. The Banach-Steinhaus theorem yields the boundedness of $\mathcal{Y}$.

For instance, if W has finite dimension and the ordering cone $K_W \subset W$ is closed convex and pointed, then the first condition (i) is satisfied. Indeed, in this setting we can find a cone $\bar{K}_W \supset K_W$, which is again closed convex and pointed but additionally has nonempty interior. Let B be a base of K_W, in particular, B is closed, bounded, convex, $0 \notin B$ and $K_W = [0,\infty[\cdot B$. Thus we can find $\varepsilon > 0$ such that by adding the closed (and compact) ball $B(0,\varepsilon)$ we obtain a closed, bounded and convex set $\bar{B}_\varepsilon := B + B(0,\varepsilon)$ with $0 \notin B$. Define $\bar{K}_W := [0,\infty[\cdot\bar{B}_\varepsilon$.

Taking $k \in \operatorname{int} \bar{\mathrm{K}}_{\mathrm{W}}$ we can construct a neighborhood U of zero by $U := (k - \bar{K}_W) \cap (-k + \bar{K}_W)$. The norm $\|.\|_a$ defined by $\|w\|_a := \inf\{\alpha > 0 : \ w \in \alpha U\}$ is absolute-monotone. Indeed, let $-w \leq w' \leq w$ be given. Then we can find some $\alpha > 0$ such that $w \in \alpha U$. It follows $w' \in -w + K_W \subset \alpha(-k + \bar{K}_W) + K_W \subset -\alpha k + \bar{K}_W + \bar{K}_W \subset -\alpha k + \bar{K}_W \subset \alpha(-k + \bar{K}_W)$. Analogously we obtain $w' \in \alpha(k - \bar{K}_W)$. Hence $w' \in \alpha U$. Taking the infimum over $\alpha > 0$ yields $\|w'\|_a \leq \|w\|_a$. Of course in a finite dimensional space the norm $\|.\|_a$ is equivalent to the norm $\|.\|$ of W.

Condition (ii) is satisfied in many applications because f itself often satisfies property (8).

Theorem 3 *Let $\tilde{K}_W$ be a pointed convex cone in W satisfying $\tilde{K}_W \supseteq K_W$, let e be an element in K_W, and let (a_0, x_0, Y_0, Z_0) be an element in $\mathcal{D}$. Then (a_0, x_0, Y_0, Z_0) is an $(\tilde{K}_W, e)$–saddle point of L with respect to $\mathcal{D}$ if and only if the following conditions are satisfied:*

(i) $L(a_0, x_0, Y_0, Z_0) \in \operatorname{Min}(\{L(a, x, Y_0, Z_0) \mid (a, x) \in \mathcal{A} \times \mathcal{X}\}, \tilde{K}_W, e)$;

(ii) $B(x_0) - b \in K_V$;

(iii) $Y_0(a_0 - A(x_0)) + Z_0(b - B(x_0)) \notin f(a_0 - A(x_0)) - e - (\tilde{K}_W \setminus \{0\})$.

Proof. *Necessity.* Condition (i) follows from (1) in the definition of a $(\tilde{K}_W, e)$–saddle point. In order to prove (ii) and (iii), we argue similarly as in the proof of necessity of Theorem 1. First, we suppose that $B(x_0) - b \notin K_V$. Then we apply the strict separation theorem and conclude that there is a functional $\mu \in K_V^*$ such that $\mu(B(x_0) - b) < 0$. Let k be a point chosen from $K_W \setminus \{0\}$. By means of μ and k we define the mapping $Z : V \to W$ by

$$Z(v) := \frac{\mu(v)}{\mu(b - B(x_0))}(e + k) + Z_0(v).$$

Obviously, Z belongs to $\mathcal{Z}$. Taking in account that $K_W \subseteq \tilde{K}_W$, we also see

$$(Z - Z_0)(b - B(x_0)) = e + k \in e + (K_W \setminus \{0\}) \subset e + (\tilde{K}_W \setminus \{0\}).$$

This result implies that

$$\begin{aligned} L(a_0, x_0, Y_0, Z) &= L(a_0, x_0, Y_0, Z_0) + (Z - Z_0)(b - B(x_0)) \\ &\subset L(a_0, x_0, Y_0, Z_0) + e + (\tilde{K}_W \setminus \{0\}), \end{aligned}$$

which contradicts

$$L(a_0, x_0, Y_0, Z_0) \in \operatorname{Max}(\{L(a_0, x_0, Y, Z) \mid (Y, Z) \in \mathcal{Y} \times \mathcal{Z}\}, \tilde{K}_W, e). \tag{10}$$

Therefore, condition (ii) must be satisfied. Next we apply (10) again and conclude that

$$L(a_0, x_0, Y, Z) \notin L(a_0, x_0, Y_0, Z_0) + e + (\tilde{K}_W \setminus \{0\}) \tag{11}$$

for every $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$. By specializing (Y, Z) in (11), we obtain (iii). Indeed, from (11) it follows that for any mapping $Y \in \mathcal{Y}$ with the property $Y(a_0 - A(x_0)) = f(a_0 - A(x_0))$ and for $Z = 0$ the relation

$$f(a_0 - A(x_0)) \notin Y_0(a_0 - A(x_0)) + Z_0(b - B(x_0)) + e + (\tilde{K}_W \setminus \{0\})$$

holds. This means that (iii) is true.

Sufficiency. (i) is equivalent to (1) in the definition of a $(\tilde{K}_W, e)$– saddle point. We have to proof that (2) also holds. To this end we suppose that there is a pair $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$ such that

$$L(a_0, x_0, Y, Z) \in L(a_0, x_0, Y_0, Z_0) + e + (\tilde{K}_W \setminus \{0\}).$$

Then we have

$$Y(a_0 - A(x_0)) + Y(b - B(x_0)) \in Y_0(a_0 - A(x_0)) + Z_0(b - B(x_0)) + e + (\tilde{K}_W \setminus \{0\})$$

which implies that

$$\begin{aligned} & Y(a_0 - A(x_0)) + Z(b - B(x_0)) + K_W \\ & \quad \subseteq \; Y_0(a_0 - A(x_0)) + Z_0(b - B(x_0)) + e + K_W + (\tilde{K}_W \setminus \{0\}) \\ & \quad \subseteq \; Y_0(a_0 - A(x_0)) + Z_0(b - B(x_0)) + e + (\tilde{K}_W \setminus \{0\}). \end{aligned}$$

But on the other hand, from (5) $f(a_0 - A(x_0)) \in Y(a_0 - A(x_0)) + K_W$ and (ii) $Z(B(x_0) - b) \in K_W$ it follows that

$$\begin{aligned} f(a_0 - A(x_0)) & \in Y(a_0 - A(x_0)) + Z(b - B(x_0)) + Z(B(x_0) - b) + K_W \\ & \subseteq Y(a_0 - A(x_0)) + Z(b - B(x_0)) + K_W + K_W \\ & \subseteq Y(a_0 - A(x_0)) + Z(b - B(x_0)) + K_W. \end{aligned}$$

Consequently, we have

$$f(a_0 - A(x_0)) \in Y_0(a_0 - A(x_0)) + Z_0(b - B(x_0)) + e + (\tilde{K}_W \setminus \{0\}),$$

which contradicts (iii). ■

Corollary 4 *Let $\tilde{K}_W$ be a pointed convex cone in W satisfying $\tilde{K}_W \supseteq K_W$, let e be an element in K_W, and let $(a_0, x_0, Y_0, Z_0) \in \mathcal{D}$ be a $(\tilde{K}_W, e)$–saddle point of L with respect to $\mathcal{D}$. Then the following assertions are true:*

(j) $(a_0, x_0) \in S$;

(jj) $Y_0(a_0 - A(x_0)) \notin f(a_0 - A(x_0)) - e - (\tilde{K}_W \setminus \{0\})$;

(jjj) $Z_0(b - B(x_0)) \notin -e - (\tilde{K}_W \setminus \{0\})$.

Proof. Obviously (j) results from (ii) in Theorem 3. In order to prove (jj) and (jjj), we note that (11) implies

$$\begin{aligned} & \forall Y \in \mathcal{Y} \quad \forall Z \in \mathcal{Z}: \; Y(a_0 - A(x_0)) + Z(b - B(x_0)) \\ & \notin Y_0(a_0 - A(x_0)) + Z_0(b - B(x_0)) + e + (\tilde{K}_W \setminus \{0\}). \end{aligned} \tag{12}$$

By setting in (12) a $Y \in \mathcal{Y}$ with the property $Y(a_0 - A(x_0)) = f(a_0 - A(x_0))$ and $Z = Z_0$, we obtain (jj); while by setting $Y = Y_0$ and $Z = 0$ in (12), we obtain (jjj). ■

Remark. Item (jjj) in Corollary 4 can be interpreted as a condition of approximate complementary slackness for Z_0 and $b - B(x_0)$. Namely, we have $Z_0(b - B(x_0)) \in -K_W$ and $Z_0(b - B(x_0)) \notin -e - (\tilde{K}_W \setminus \{0\})$. Hence, $Z_0(b - B(x_0))$ is contained in the set $(-K_W \setminus [-e - \tilde{K}_W]) \cup \{-e\}$. In the case of a finite dimensional space W this is a bounded set if we claim $\tilde{K}_W \supset (\neq) K_W$. Putting $e = 0$, this relation implies the well-known condition

$$Z_0(b - B(x_0)) = 0.$$

Theorem 5 *Let $\tilde{K}_W$ be a pointed convex cone in W satisfying $\tilde{K}_W \supseteq K_W$, let e be an element in K_W, and let $(a_0, x_0, Y_0, Z_0) \in \mathcal{D}$ be a $(\tilde{K}_W, e)$–saddle point of L with respect to $\mathcal{D}$. Then (a_0, x_0) is a $(\tilde{K}_W, \bar{e})$–minimal point of $F[S]$, where*

$$\bar{e} := e + f(a_0 - A(x_0)) - Y_0(a_0 - A(x_0)) - Z_0(b - B(x_0))$$

is the approximation error.

Proof. According to property (j) in Corollary 4, we have $(a_0, x_0) \in S$. We suppose that there is a $(a, x) \in S$ such that

$$F(a, x) \in F(a_0, x_0) - \bar{e} - (\tilde{K}_W \setminus \{0\}).$$

This means that

$$F(a, x) \in L(a_0, x_0, Y_0, Z_0) - e - (\tilde{K}_W \setminus \{0\}).$$

Hence we have

$$F(a, x) - K_W \subseteq L(a_0, x_0, Y_0, Z_0) - e - (\tilde{K}_W \setminus \{0\}).$$

But, on the other hand, in view of

$$f(a - A(x)) - Y_0(a - A(x)) - Z_0(b - B(x)) \in K_W + K_W \subseteq K_W,$$

we have

$$\begin{aligned} L(a, x, Y_0, Z_0) &= F(a, x) - [f(a, x) - Y_0(a - A(x)) - Z_0(b - B(x))] \\ &\in F(a, x) - K_W. \end{aligned}$$

Consequently, it results that

$$L(a, x, Y_0, Z_0) \in L(a_0, x_0, Y_0, Z_0) - e - (\tilde{K}_W \setminus \{0\}),$$

which contradicts condition (i) in Theorem 3. ∎

Theorem 6 *Let $(W, \|\cdot\|)$ be a Banach space and let $\mathcal{L}(U, W)$ be reflexive. We assume the existence of a feasible point $(\bar{a}, \bar{x}) \in S$ with $B(\bar{x}) - b \in \operatorname{int} K_V$. Moreover, we suppose that $\mathcal{Y}$ is bounded. Let $\tilde{K}_W$ be a pointed convex cone in W satisfying $\operatorname{int}(\tilde{K}_W) \cup \{0\} \supseteq \operatorname{cl} K_W$, let e be an element in K_W, and let $(a_0, x_0) \in S$ be a $(\tilde{K}_W, e)$–minimal point of $F[S]$. Then there exist operators $Y_0 \in \mathcal{Y}$ and $Z_0 \in \mathcal{Z}$, such that (a_0, x_0, Y_0, Z_0) is a (K_W, e)–saddle point of L with respect to $\mathcal{D}$.*

Proof. Under the given assumptions there exists some $\lambda \in \operatorname{int} K_W^*$ (cf. Theorem 5.11 in [3]) such that (a_0, x_0) belongs to $\operatorname{Min}(F[S], \lambda, e)$. Theorem 2 implies the existence of a pair $(Y_0, Z_0) \in \mathcal{Y} \times \mathcal{Z}$ such that (a_0, x_0, Y_0, Z_0) is a (λ, e)–saddle point of L with respect to $\mathcal{D}$. From the strict K_W–monotonicity of λ, we can conclude that (a_0, x_0, Y_0, Z_0) is also a (K_W, e)–saddle point of L with respect to $\mathcal{D}$. ∎

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